Online Appendix, "Artificial Intelligence in Research and Development," Benjamin F. Jones. In *The Economics of Transformative AI*, edited by Ajay Agrawal, Erik Brynjolfsson, and Anton Korinek. Chicago: University of Chicago Press.

Proposition 1: The equilibrium rate of progress

Proof.

1. Set-up of the optimization problem

The idea production function is

$$\dot{Z} = \zeta Z^{\varphi}\left[\int_0^1 r(j)^{\theta} dj\right]^{1/\theta}$$

as given by (1). Note that, for notational simplicity, we drop time subscripts here and throughout this proof.

Given the task level production functions (2) and prices (the wage w and capital price μ), the unit cost of producing one unit of task output is

$$c(j) = \begin{cases} \mu/m(j), & j < \gamma \quad \text{(machine task)}, \\ w/H, & j \geq \gamma \quad \text{(human task)}. \end{cases}$$

The budget constraint (7) can thus equivalently be written

$$D = \int_0^1 c(j)\, r(j)\, dj.$$

The goal is to maximize the rate of progress, $\dot{Z}$, subject to the budget constraint and the task specific costs.

2 The Lagrangian and the first-order conditions

Because ζZ^{φ} is a positive multiplicative constant, maximizing $\dot{Z}$ is equivalent to maximizing

$$R \equiv \left[\int_0^1 r(j)^{\theta} dj\right]^{1/\theta}.$$

We therefore solve the Lagrangian

$$\max_{\{r(j)\}} \mathcal{L} = \left[\int r(j)^{\theta} dj\right]^{1/\theta} + \lambda\Big(D - \int c(j) r(j)\, dj\Big).$$

Taking the first order condition for any task, $r(j)$, we have:

$$R^{1-\theta}r(j)^{\theta-1} = \lambda c(j)$$

which we can rearrange as:

$$r(j) = \lambda^{\frac{1}{\theta-1}} c(j)^{\frac{1}{\theta-1}} R. \tag{17}$$

3. Impose the budget constraint at the optimum $r(j)$

Plugging the $r(j)$ into the budget constraint and simplifying we have:

$$D = \frac{R}{\lambda^{\frac{1}{1-\theta}}} \int_0^1 c(j)^{\frac{\theta}{\theta-1}} dj \tag{18}$$

Define the cost index (a weighted mean of the task-level unit costs) as

$$J \equiv \int_0^1 c(j)^{\frac{\theta}{\theta-1}} dj \tag{19}$$

4. Evaluate the objective function at the optimum $r(j)$

Plugging the $r(j)$ into the objective function, R, and simplifying we have:

$$1 = \lambda^{\frac{1}{\theta-1}} J^{1/\theta} \tag{20}$$

It then follows from (18), (19), and (20) that $\lambda = R/D$ and

$$R = \frac{D}{J^{(\theta-1)/\theta}}. \tag{21}$$

5. Compute the cost index J

Split the integral into the machine and human parts:

$$\begin{aligned} J &= \int_0^\gamma \left(\tfrac{\mu}{m(j)}\right)^{\frac{\theta}{\theta-1}} dj + \int_\gamma^1 \left(\tfrac{w}{H}\right)^{\frac{\theta}{\theta-1}} dj \\ &= \gamma\, \mu^{\frac{\theta}{\theta-1}} \left[\frac{1}{\gamma}\int_0^\gamma m(j)^{\frac{\theta}{\theta-1}} dj\right] + (1-\gamma)\left(\tfrac{w}{H}\right)^{\frac{\theta}{\theta-1}}. \end{aligned} \tag{22}$$

Now recall the definition of the machine-productivity index, M

$$M = \left[\frac{1}{\gamma}\int_0^\gamma m(j)^{\frac{\theta}{1-\theta}} dj\right]^{\frac{1-\theta}{\theta}}$$

Substituting this above we then have:

$$J = \gamma\Big(\frac{\mu}{M}\Big)^{\frac{\theta}{\theta-1}} + (1-\gamma)\Big(\frac{w}{H}\Big)^{\frac{\theta}{\theta-1}}. \tag{23}$$

Thus, all micro-level heterogeneity $\{m(j)\}$ collapses to the single scalar M.

6. Optimal rate of progress

The rate of progress is $\dot{Z} = \zeta Z^{\varphi} R$. Insert R from (21). Then plug in J from (23). Divide by Z, and bring back the time subscript. We have

$$\frac{\dot{Z}_t}{Z_t} = \frac{\zeta\, Z_t^{\varphi-1}\, D_t}{\Big[\gamma_t\big(\frac{\mu_t}{M_t}\big)^{\frac{\theta}{\theta-1}} + (1-\gamma_t)\big(\frac{w_t}{H}\big)^{\frac{\theta}{\theta-1}}\Big]^{\frac{\theta-1}{\theta}}}$$

which is exactly equation (8) in Proposition 1.

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Corollary 1: Capital and labor shares

Proof.

Total R&D expenditure is

$$D = \int_0^1 c(j)r(j)\, dj$$

Note that we will suppress time subscripts for notational simplicity. The expenditure share on capital inputs is then

$$s^X = \frac{1}{D}\int_0^{\gamma} c(j)r(j)\, dj$$

For any input, given the first-order condition (17), we have

$$c(j)r(j) = R\lambda^{\frac{1}{\theta-1}} c(j)^{\frac{\theta}{\theta-1}}$$

The unit cost for a capital input is given by $c(j) = \mu/m(j)$, and from the proof of Proposition 1, $\lambda = R/D$. Using these expressions, we can rewrite the capital expenditure share as:

$$s^X = \left(\frac{R}{D}\right)^{\frac{\theta}{\theta-1}} \mu^{\frac{\theta}{\theta-1}} \int_0^{\gamma} m(j)^{\frac{\theta}{1-\theta}}\, dj \tag{24}$$

Recalling the definition of the technology index, M (see (3)), we can thus rewrite the above as:

$$s^X = \gamma \left(\frac{\mu R}{MD} \right)^{\frac{\theta}{\theta-1}} \tag{25}$$

From Proposition 1, we have proved that:

$$\frac{R}{D} = \left[\gamma \left(\frac{\mu}{M} \right)^{\frac{\theta}{\theta-1}} + (1-\gamma) \left(\frac{w}{H} \right)^{\frac{\theta}{\theta-1}} \right]^{\frac{1-\theta}{\theta}}$$

Thus we can write the capital share in terms of exogenous parameters as:

$$s^X = \gamma \left(\frac{\mu}{M} \left[\gamma \left(\frac{\mu}{M} \right)^{\frac{\theta}{\theta-1}} + (1-\gamma) \left(\frac{w}{H} \right)^{\frac{\theta}{\theta-1}} \right]^{\frac{1-\theta}{\theta}} \right)^{\frac{\theta}{\theta-1}}$$

Simplifying, and adding back the time subscripts, this is equivalently:

$$s_t^X = \frac{1}{1 + \left(\frac{1-\gamma_t}{\gamma_t} \right) \left(\frac{M_t w_t}{H_t \mu_t} \right)^{\frac{\theta}{\theta-1}}}$$

as was to be shown.

The labor share can easily be confirmed, noting that:

$$s_t^L = 1 - s_t^X$$

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Corollary 2: Small increase in machine intelligence

Proof.

Consider equation (8), the result of Proposition 1. Multiply both sides by Z_t and take logs. We have:

$$\log \dot{Z}_t = \log \left(\xi Z_t^{\phi} D_t \right) - \frac{\theta - 1}{\theta} \log J_t \tag{26}$$

where we have used the definition of J from (23).

Now differentiate with respect to the technology index M_t. We seek the instantaneous increase in the rate of progress, $\dot{Z}_t$, holding the initial level of the outcome, Z_t, fixed. We have:

$$\frac{\partial \log \dot{Z}_t}{\partial \log M_t} = \frac{1-\theta}{\theta} \frac{\partial \log J_t}{\partial \log M_t} \tag{27}$$

$$= \frac{1-\theta}{\theta} \frac{1}{J_t} \frac{\theta}{\theta - 1} \gamma \left(\frac{\mu_t}{M_t} \right)^{\frac{1}{\theta-1}} \left(-\frac{\mu_t}{M_t^2} \right) \frac{\partial M_t}{\partial \log M_t} \tag{28}$$

Simplifying, we have:

$$\frac{\partial \log \dot{Z}_t}{\partial \log M_t} = \frac{1}{J_t} \gamma \left(\frac{\mu_t}{M_t} \right)^{\frac{\theta}{\theta-1}} \tag{29}$$

Substituting back in for the definition of J_t and simplifying, we obtain:

$$\frac{\partial \log \dot{Z}_t}{\partial \log M_t} = \left[1 + \left(\frac{1-\gamma_t}{\gamma_t} \right) \left(\frac{M_t w_t}{H_t \mu_t} \right)^{\frac{\theta}{\theta-1}} \right]^{-1} \tag{30}$$

which we recognize from (9) as s_t^X.

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Corollary 3, **Corollary 5**, and **Corollary 6**:

Discrete Jumps in Machine Intelligence, Automation, or Both

Proof.

Corollaries 3, 5, and 6 are presented in the text in that order for expositional clarity. Corollaries 3 and 5 can of course be nested within Corollary 6, so the following proof will focus on the general case, Corollary 6, and then prove the others as special cases.

1. Start with the rate of progress and rewrite it in a cost share form

Specifically, write the equilibrium rate of progress, (8), in the form:

$$\dot{Z} = \frac{\xi Z^{\phi} D}{\gamma^{1/b} \left(\frac{\mu}{M} \right) \left[1 + \left(\frac{1-\gamma}{\gamma} \right) \left(\frac{wM}{\mu H} \right)^b \right]^{1/b}} \tag{31}$$

where we define $b = \frac{\theta}{\theta-1}$ and suppress time subscripts for simplicity. Noting from (9) that

$$1 + \left(\frac{1-\gamma}{\gamma} \right) \left(\frac{wM}{\mu H} \right)^b = \frac{1}{s^X} \tag{32}$$

where s^X is the capital share of R&D expenditure, we can write the rate of progress as

$$\dot{Z} = \xi Z^{\phi} D \cdot \frac{M}{\mu} \left(\frac{s^X}{\gamma} \right)^{1/b} \tag{33}$$

2. Consider instantaneous shifts in M and γ

Specifically, we ask what happens when simultaneously (i) machine intelligence increases by a multiple λ (i.e., $M' = \lambda M$) and (ii) the share of R&D tasks performed by labor declines proportionally by a multiple ρ (i.e., $1 - \gamma' = \rho(1 - \gamma)$). Define the resulting multiple in the rate of progress as ϑ (i.e., $\dot{Z}' = \vartheta \dot{Z}$). Define the resulting multiple in the expenditure share as κ (i.e., $(s^X)' = \kappa s^X$). Using the rate of progress expression above, (33), and simplifying ratios:

$$\vartheta = \frac{\dot{Z}'}{\dot{Z}} = \lambda \left(\kappa \frac{\gamma}{1 - \rho(1 - \gamma)} \right)^{1/b} \tag{34}$$

Now recall that we can write the capital expenditure share as:

$$s^X = \frac{1}{1 + \left(\frac{1-\gamma}{\gamma} \right) C^b} \tag{35}$$

where $C = \frac{wM}{\mu H}$. Thus, the ratio of expenditure shares becomes:

$$\kappa = \frac{(s^X)'}{s^X} = \frac{1 + \left(\frac{1-\gamma}{\gamma} \right) C^b}{1 + \left(\frac{\rho(1-\gamma)}{1-\rho(1-\gamma)} \right) (\lambda C)^b} \tag{36}$$

To simplify this, note that $\left(\frac{1-\gamma}{\gamma} \right) C^b = \frac{1-s^X}{s^X}$. Then, with some manipulation, we can write:

$$\kappa = \frac{1}{s^X + \lambda^b \cdot \left(\frac{\rho\gamma}{1-\rho(1-\gamma)} \right) (1 - s^X)} \tag{37}$$

Plugging this into the rate of progress equation and simplifying, we have:

$$\vartheta = \frac{\dot{Z}'}{\dot{Z}} = \left[\lambda^{-b} \left(\frac{1 - \rho(1 - \gamma)}{\gamma} \right) s^X + \rho(1 - s^X) \right]^{-1/b} \tag{38}$$

which was to be shown.

Taking the limit of infinite machine intelligence, and noting that $b > 0$ when $\theta < 0$, we have:

$$\lim_{\lambda \to \infty} \frac{\dot{Z}'}{\dot{Z}} = \left[\rho(1 - s^X) \right]^{-1/b} \tag{39}$$

This proves Corollary 6.

3. Special Cases

For Corollary 3, we set $\rho = 1$. Simplifying (38), the multiple becomes:

$$\frac{\dot{Z}'}{\dot{Z}} = \left(\lambda^{-b} s^X + 1 - s^X\right)^{-1/b} \tag{40}$$

Taking the limit for large λ and noting $b > 0$, we obtain:

$$\lim_{\lambda \to \infty} \frac{\dot{Z}'}{\dot{Z}} = (1 - s^X)^{-1/b} \tag{41}$$

This proves Corollary 3.

For Corollary 5, we set $\lambda = 1$. Simplifying (38), the multiple becomes:

$$\frac{\dot{Z}'}{\dot{Z}} = \left[\left(\frac{1 - \rho(1-\gamma)}{\gamma}\right) s^X + \rho(1 - s^X)\right]^{-1/b} \tag{42}$$

$$= \left(\rho + \frac{1-\rho}{\gamma} s^X\right)^{-1/b} \tag{43}$$

Next, note that the multiple is decreasing in s^X (since $b > 0$). Recall:

$$s^X = \frac{1}{1 + \left(\frac{1-\gamma}{\gamma}\right) C^b} \tag{44}$$

and that $s^X \in [0, \gamma]$. It follows by inspection that the lower bound of the multiple (when $s^X = \gamma$) is:

$$\left(\rho + \frac{1-\rho}{\gamma}\gamma\right)^{-1/b} = 1 \tag{45}$$

And the upper bound of the multiple (when $s^X = 0$) is:

$$\rho^{-1/b} \tag{46}$$

Finally, taking the limit in (43) for small ρ, we directly find:

$$\lim_{\rho \to 0} \frac{\dot{Z}'}{\dot{Z}} = \left(\frac{s^X}{\gamma}\right)^{-1/b} \tag{47}$$

This proves Corollary 5.

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Corollary 4: Large increase in machine intelligence for a share of machine tasks

Proof.

1. The Change in Machine Intelligence

First focus on the multiple in overall machine productivity. Recall the definition:

$$M = \left(\frac{1}{\gamma} \int_0^\gamma m(j)^{\frac{\theta}{1-\theta}} dj \right)^{\frac{1-\theta}{\theta}} \tag{48}$$

which is the generalized mean of the machine productivities (and for simplicity we have suppressed time subscripts).

Let's assume that the distributions of the $m(j)$ are initially the same for both AI-machine tasks and other capital input tasks. Then we equivalently have the same generalized means for these subsets of tasks,

$$M = \left(\frac{1}{\nu\gamma} \int_0^{\nu\gamma} m(a)^{\frac{\theta}{1-\theta}} da \right)^{\frac{1-\theta}{\theta}} = \left(\frac{1}{(1-\nu)\gamma} \int_0^{(1-\nu)\gamma} m(b)^{\frac{\theta}{1-\theta}} db \right)^{\frac{1-\theta}{\theta}} \tag{49}$$

Now let's have all the $m(a)$ in the first set jump up by a multiple κ. The resulting average for this set is κM. Averaging these increased productivities together with the productivities at the other machine tasks, we have

$$M' = \left(\frac{1}{\gamma} \left(\int_0^{\nu\gamma} [\kappa m(a)]^{\frac{\theta}{1-\theta}} da + \int_0^{(1-\nu)\gamma} m(b)^{\frac{\theta}{1-\theta}} db \right) \right)^{\frac{1-\theta}{\theta}}$$

This is equivalently

$$M' = \left(\nu\kappa^{\frac{\theta}{1-\theta}} M^{\frac{\theta}{1-\theta}} + (1-\nu) M^{\frac{\theta}{1-\theta}} \right)^{\frac{1-\theta}{\theta}} \tag{50}$$

or,

$$\lambda_\nu = \frac{M'}{M} = \left(\nu\kappa^{\frac{\theta}{1-\theta}} + (1-\nu) \right)^{\frac{1-\theta}{\theta}} \tag{51}$$

as was to be shown.

If $\kappa \to \infty$, then we have:

$$\lim_{\kappa\to\infty} \left(\nu\kappa^{\frac{\theta}{1-\theta}} + (1-\nu) \right)^{\frac{1-\theta}{\theta}} = (1-\nu)^{\frac{1-\theta}{\theta}} \tag{52}$$

where we note that $\frac{\theta}{1-\theta} < 0$, as was to be shown.

2. Implication for the Rate of Progress

From Corollary 3, we know that a given multiple, λ, in the machine productivity index M_t, causes the rate of progress to go up according to:

$$\eta = \left(\lambda^{\frac{\theta}{1-\theta}} s^X + 1 - s^X\right)^{\frac{1-\theta}{\theta}} \tag{53}$$

The relevant multiple in machine intelligence is now λ_ν as given in (51). We therefore have:

$$\begin{aligned} \eta &= \left(\left(\nu\kappa^{\frac{\theta}{1-\theta}} + (1-\nu)\right)^{\frac{1-\theta}{\theta}\cdot\frac{\theta}{1-\theta}} s^X + 1 - s^X\right)^{\frac{1-\theta}{\theta}} \\ &= \left(\nu\kappa^{\frac{\theta}{1-\theta}} s^X + 1 - \nu s^X\right)^{\frac{1-\theta}{\theta}} \end{aligned} \tag{54}$$

as was to be shown.

Taking the limit as $\kappa \to \infty$, and noting that $\frac{\theta}{1-\theta} < 0$ for $\theta < 0$, we see directly that

$$\lim_{\kappa\to\infty} \eta = (1 - \nu s^X)^{\frac{1-\theta}{\theta}} \tag{55}$$

as was to be shown. ■