ONLINE APPENDIX

Artificial Intelligence, Competition, and Welfare

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1 Benchmark: Open Economy, Two Traded Goods

This is a baseline model with two traded goods, a small open economy where consumption is not modeled.

1.1 Setup, accounting, and objects

Goods, prices, and factors. The economy consists of two traded final goods A, B sold at exogenous world prices $(\bar{p}_A, \bar{p}_B)$ (small open economy) and produced with two primary factors: skilled labor S and unskilled labor U, with endowments (L_S, L_U) and factor prices (w_S, w_U). A foreign input X is used *only* by sector B at price p_X set abroad. Technologies are CRS; markets are competitive.

Unit costs, unit input coefficients, and zero profit. Let $c_A(w_S, w_U)$ and $c_B(w_S, w_U, p_X)$ be unit-cost functions. Zero-profit (unit-cost) equalities:

$$\bar{p}_A = c_A(w_S, w_U), \qquad \bar{p}_B = c_B(w_S, w_U, p_X). \tag{ZP}$$

Define *unit input coefficients* (Hicksian, i.e., per unit of output):

$$a_S^A := \frac{\partial c_A}{\partial w_S}, \quad a_U^A := \frac{\partial c_A}{\partial w_U}; \qquad a_S^B := \frac{\partial c_B}{\partial w_S}, \quad a_U^B := \frac{\partial c_B}{\partial w_U}, \quad a_X^B := \frac{\partial c_B}{\partial p_X}.$$

Define cost shares: $\theta_S^j = \frac{w_S a_S^j}{\bar{p}_j}$, $\theta_U^j = \frac{w_U a_U^j}{\bar{p}_j}$, and in B also $\theta_X^B = \frac{p_X a_X^B}{\bar{p}_B}$, with $\theta_S^A + \theta_U^A = 1$ and $\theta_S^B + \theta_U^B + \theta_X^B = 1$.

Full employment.

$$a_S^A Q_A + a_S^B Q_B = L_S, \qquad a_U^A Q_A + a_U^B Q_B = L_U. \tag{FE}$$

Imports and national income (GNP at world prices). The quantity of X imported is $M_X = a_X^B Q_B$; the import bill is $p_X M_X$.

$$Y \;=\; \bar{p}_A Q_A + \bar{p}_B Q_B \;-\; p_X M_X \;=\; w_S L_S + w_U L_U. \tag{NI}$$

The first equality is value-added (final output value minus imported intermediates); the second uses zero profits in A, B. *Envelope in p_X:*

$$\frac{dY}{dp_X} \;=\; -\,M_X \;\leq 0. \tag{Y–p_X}$$

Hence a fall in p_X raises national income (aggregate welfare at traded-goods prices), while a rise lowers it.

Trade and consumption. We do not model home demand; production and factor prices are determined by (ZP)–(FE). The trade-balance identity is $\bar{p}_A(C_A - Q_A) + \bar{p}_B(C_B - Q_B) + p_X M_X = 0$; the country can export B ($Q_B > C_B$). Welfare statements use Y (equivalently, any homothetic utility at prices $(\bar{p}_A, \bar{p}_B)$).

1.2 Comparative statics

The relative factor intensities and sign patterns for wage changes are stated in the main text. For completeness, define the *relative factor-intensity indices*:

$$R_A := \frac{\theta_S^A}{\theta_U^A}, \qquad R_B(p_X) := \frac{\theta_S^B}{\theta_U^B}.$$

We say B is *skill-intensive (relative to A)* if $R_B(p_X) > R_A$, and *unskilled-intensive* if $R_B(p_X) < R_A$. Totally differentiating the zero-profit conditions at fixed $(\bar{p}_A, \bar{p}_B)$ yields the sign pattern

$$\begin{cases} R_B(p_X) > R_A \text{ (B skill-intensive)} \;\Rightarrow\; \dfrac{dw_S}{dp_X} < 0,\; \dfrac{dw_U}{dp_X} > 0, \\ R_B(p_X) < R_A \text{ (B unskilled-intensive)} \Rightarrow\; \dfrac{dw_S}{dp_X} > 0,\; \dfrac{dw_U}{dp_X} < 0. \end{cases} \tag{Signs}$$

Aggregate income moves monotonically: $dp_X \downarrow$ raises Y, $dp_X \uparrow$ lowers Y.

Hicksian substitution. We say X is a Hicksian substitute for U in B if the Hicksian unit demand satisfies

$$\frac{\partial a_U^B(w_S, w_U, p_X)}{\partial p_X} > 0.$$

This ensures that when p_X rises, demand for unskilled labor a_U^B also rises. We do not assume smoothness at technique switches, if the production function allows firms to select among different techniques with different unit costs; at the point of a switch, a_U^B may jump upward.

Single intensity reversal. A *single intensity reversal* occurs at some $p_X^* \in (0, \infty)$ if

$$R_B(p_X) \begin{cases} < R_A, & p_X > p_X^*, \\ = R_A, & p_X = p_X^*, \\ > R_A, & p_X < p_X^*. \end{cases} \tag{Rev}$$

1.3 The introduction of AI and divergence near a reversal

Fix a low input price p_X^ℓ and compare to autarky in X ($p_X = \infty$). The *level* change in the unskilled wage is

$$w_U(p_X^\ell) - w_U(\infty) \;=\; -\int_{p_X^\ell}^{\infty} \frac{dw_U}{dp_X}\, dp_X,$$

where the integrand's sign at each p_X is given by (Signs). The *local* effect at p_X^ℓ is sign $\left(\frac{dw_U}{dp_X}\big|_{p_X^\ell}\right)$ and depends only on the current ordering $R_B(p_X^\ell)$ vs. R_A.

Proposition 1 (Single intensity reversal)**.** *Suppose p_X^* satisfies* (Rev)*. Then in neighborhoods of p_X^*:*

- *If $p_X^\ell \downarrow p_X^*$ from above (so $R_B(p_X^\ell) < R_A$), then*

$$w_U(p_X^\ell) - w_U(\infty) \;>\; 0 \quad \textit{and} \quad \frac{dw_U}{dp_X}\Big|_{p_X^\ell} \;<\; 0.$$

  Thus introducing a cheap X is good *for unskilled, and a marginal increase in p_X at that point is* bad *for unskilled.*

- *If $p_X^\ell \uparrow p_X^*$ from below (so $R_B(p_X^\ell) > R_A$), then $w_U(p_X^\ell) - w_U(\infty) > 0$ while $\frac{dw_U}{dp_X}\big|_{p_X^\ell} > 0$: opening to cheap X is* good *for unskilled, and a marginal increase in p_X is* good *for them. That implies that the opening is* bad *for skilled, and a marginal increase in p_X is also bad for skilled workers.*

In all cases, aggregate income moves with p_X by (Y–p_X).

Proof. By (Signs), $\operatorname{sign}\big(dw_U/dp_X\big) = \operatorname{sign}\big(R_B - R_A\big)$ in a neighborhood of p_X^*. If $p_X^\ell \downarrow p_X^*$ from above, then $R_B < R_A$ so $dw_U/dp_X < 0$ locally, while the level change satisfies $w_U(p_X^\ell) - w_U(\infty) = \int_\infty^{p_X^\ell} (dw_U/dp_X)\, dp_X > 0$ because the integrand is positive on most of the path before the crossing. The case approaching from below is analogous with $R_B > R_A$. □

Proposition 2 (Double-harm in the all-traded model with non-monotone skill intensities)**.** *Suppose there exist thresholds $p_X^{(1)} > p_X^{(2)} > 0$ such that a double-reversal holds*

$$R_B(p_X) \begin{cases} < \ R_A, & p_X > p_X^{(1)} \qquad \textit{(very high } p_X\textit{: } B \textit{ unskilled-intensive)}, \\ > \ R_A, & p_X^{(2)} < p_X < p_X^{(1)} \qquad \textit{(intermediate } p_X\textit{: } B \textit{ skill-intensive)}, \\ < \ R_A, & p_X < p_X^{(2)} \qquad \textit{(very low } p_X\textit{: } B \textit{ unskilled-intensive again)}. \end{cases} \tag{TwoX}$$

Then it is possible that for $p_X^\ell \in (0, p_X^{(2)})$ sufficiently close to $p_X^{(2)}$,

$$\underbrace{w_U(p_X^\ell) - w_U(\infty)}_{\textit{level vs. autarky}} \ < \ 0 \qquad \textit{and} \qquad \underbrace{\frac{dw_U}{dp_X}\Big|_{p_X^\ell}}_{\textit{local at } p_X^\ell} \ < \ 0.$$

Hence introducing (very) cheap X makes unskilled worse off in levels and *a marginal increase in p_X at p_X^ℓ also makes them worse off locally.*

Sketch. By (Signs), sign $\left(\frac{dw_U}{dp_X}\right)$ equals the sign of $R_B(p_X) - R_A$. Under (TwoX), the derivative is positive on the intermediate band $(p_X^{(2)}, p_X^{(1)})$ and negative on the tails $(p_X^{(1)}, \infty)$ and $(0, p_X^{(2)})$. The level change is the path integral $w_U(p_X^\ell) - w_U(\infty) = \int_\infty^{p_X^\ell} \frac{dw_U}{dp_X}\, dp_X$, whose sign is the net of those regions. For p_X^ℓ close to $p_X^{(2)}$, the (large) positive-derivative region in $(p_X^{(2)}, p_X^{(1)})$ dominates, making the integral negative; at $p_X^\ell < p_X^{(2)}$, we have $R_B < R_A$ so the local derivative is negative. □

Example of Double Reversal We provide an example demonstrating that the conditions of Proposition 2 are feasible.

For production functions, we let Sector A be CRS and let B choose among three techniques T_k, $k \in \{1, 2, 3\}$: (i) T_1 uses no AI and is unskilled-intensive; (ii) T_2 uses moderate AI and is skill-intensive; (iii) T_3 uses heavy AI and reverts to unskilled-intensive. Standard duality results imply the equilibrium wage ratio $\omega(p_X) := w_S/w_U$ lies in a compact interval as p_X varies. The technique-switch prices $p_{12}(\omega)$ and $p_{23}(\omega)$ are determined by cost equalization; by appropriate choice of AI input coefficients, we can ensure $p_{12}(\omega) > p_{23}(\omega)$ throughout. As p_X falls from ∞, firms switch from T_1 to T_2 (B becomes skill-intensive) and then to T_3 (B becomes unskilled-intensive again), generating the double crossing in Figure 1.

Numerical example. With $\Omega = [0.8, 1.5]$, Leontief A with $(a_S^A, a_U^A) = (0.50, 0.45)$, and techniques $T_1 : (0.40, 1.20, 0)$, $T_2 : (1.00, 0.40, 0.60)$, $T_3 : (0.15, 0.45, 3.50)$, the intensity ordering $R_B^{(1)} < R_A < R_B^{(2)}$ and again $R_B^{(3)} < R_A$ holds on Ω, with switch prices $p_{12} \approx 0.333 > p_{23} \approx 0.276$ at unit wages. Proposition 2 then applies.

1.4 Output responses in the general comparative statics of usage fees

Stack the full-employment system (FE) as $AY = \bar{F}$ with

$$A := \begin{bmatrix} a_S^A & a_S^B \\ a_U^A & a_U^B \end{bmatrix}, \qquad Y := \begin{bmatrix} Q_A \\ Q_B \end{bmatrix}, \qquad \bar{F} := \begin{bmatrix} L_S \\ L_U \end{bmatrix}.$$

Totally differentiate at fixed endowments:

$$A\,dY \;=\; -\,(dA)\,Y \quad\Longrightarrow\quad dY \;=\; -\,A^{-1}(dA)\,Y. \tag{Mix}$$

With $A^{-1} = \frac{1}{\Delta}\begin{bmatrix} a_U^B & -a_S^B \\ -a_U^A & a_S^A \end{bmatrix}$ where $\Delta := a_S^A a_U^B - a_S^B a_U^A > 0$ (distinct factor intensities), the B-output response is

$$dQ_B \;=\; \frac{1}{\Delta}\Big\{\, Q_A\big(a_U^A\, da_S^A - a_S^A\, da_U^A\big) \;+\; Q_B\big(a_U^A\, da_S^B - a_S^A\, da_U^B\big) \,\Big\}. \tag{$\star$}$$

Here each da_i^j is a Hicksian coefficient response induced by the endogenous wage changes from (Signs) and the direct p_X-effect in B.

What can be signed in general. Let $R_B \lessgtr R_A$ denote the local intensity ordering. Under standard Hicksian regularity (negative own-price effects, symmetric substitution) and mild cross-substitution assumptions, we obtain:

Proposition 3 (Output of B as p_X changes)**.** *Fix a point p_X and suppose: (i) B is unskilled-intensive locally, $R_B < R_A$; (ii) X is a (weak) Hicksian substitute for U in B, so $\partial a_U^B/\partial p_X > 0$ (equivalently, lowering p_X reduces a_U^B); (iii) own-price effects are negative for A and B (e.g., $\partial a_U^A/\partial w_U < 0$, $\partial a_S^A/\partial w_S < 0$). Then, for a small change in p_X,*

$$\frac{dQ_B}{dp_X} \;<\; 0.$$

If instead $R_B > R_A$ (skill-intensive B), the sign of $\frac{dQ_B}{dp_X}$ is a priori ambiguous *without further*

structure (e.g., on $\partial a_S^B/\partial p_X$ and cross-substitution magnitudes).

Proof sketch. Plug the wage responses from (Signs) (for $R_B < R_A$, we have $\frac{dw_U}{dp_X} < 0$, $dw_S/dp_X > 0$) into the Hicksian responses da_i^j and evaluate ($\star$). Under (iii), $da_U^A/dp_X > 0$ and $da_S^A/dp_X < 0$ via the induced wage changes. Under (ii), $da_U^B/dp_X > 0$; da_S^B/dp_X is not needed to be signed if it does not overturn the two summands in ($\star$). Because $a_U^A/a_S^A < R_A^{-1} < R_B^{-1} = a_U^B/a_S^B$ (unskilled intensity of B), the bracketed terms deliver a negative total in ($\star$). □

Interpretation. When B is unskilled-intensive, a rise in p_X makes B less competitive and, via (Signs), lowers w_U and raises w_S. Hicksian coefficients then adjust so that each unit uses *more* U in B and (through wages) more U in A as well; to clear factor markets at fixed endowments, the output mix must tilt *away* from B (hence $dQ_B/dp_X < 0$). Conversely, for $dp_X < 0$, Q_B expands ($dQ_B > 0$). If B is skill-intensive, the induced coefficient changes pull in opposing directions and the sign becomes model-specific.

2 Benchmark Model with One Nontraded Good

We now make sector A nontraded and close the model with Cobb-Douglas preferences over (A, B). This endogenizes the domestic relative price $p_A/\bar{p}_B$ and introduces a demand/price-index channel that interacts with the income and output-mix effects analyzed above.

2.1 Setup and market closure

Preferences and demand. A representative household has utility

$$U(C_A, C_B) = C_A^{1-\alpha} C_B^{\alpha}, \qquad \alpha \in (0, 1),$$

facing the price vector $(p_A, \bar{p}_B)$. Given nominal income Y, Cobb-Douglas demand implies

$$C_A = \frac{(1-\alpha)Y}{p_A}, \qquad C_B = \frac{\alpha Y}{\bar{p}_B}.$$

The exact consumption price index is $P = p_A^{1-\alpha} \bar{p}_B^{\alpha}$.

Nontraded market clearing. Because A is nontraded, $C_A = Q_A$. Hence the equilibrium price of the nontraded good is

$$p_A = \frac{(1-\alpha)\,Y}{Q_A} \qquad \Longleftrightarrow \qquad d\ln p_A = d\ln Y - d\ln Q_A. \tag{NT-P}$$

Intuitively, demand pressure from higher income ($Y \uparrow$) and production scarcity in A ($Q_A \downarrow$) both bid up p_A.

2.2 Equilibrium

Indirect utility and a two-channel welfare map. With CD, indirect utility satisfies

$$\ln U = \text{const} + \ln Y - (1-\alpha)\ln p_A - \alpha \ln \bar{p}_B.$$

Using (NT-P), the welfare differential is

$$d\ln U = \alpha\, d\ln Y + (1-\alpha)\, d\ln Q_A. \tag{W}$$

Let M_X denote domestic spending on input X (Hicksian demand times price). Then the envelope theorem gives $dY/dp_X = -M_X$.

For upstream shocks to p_X,

$$\frac{d\ln U}{dp_X} = -\alpha\frac{M_X}{Y} \;+\; (1-\alpha)\frac{1}{Q_A}\frac{dQ_A}{dp_X}, \tag{W–p_X}$$

combining the effect on income (applying the envelope theorem, $dY/dp_X = -M_X$) with the output response of A.

Real wages. Let $i \in \{S, U\}$. The real consumption wage is w_i/P. Because $d\ln P = (1-\alpha)\, d\ln p_A = (1-\alpha)(d\ln Y - d\ln Q_A)$,

$$d\ln\!\left(\tfrac{w_i}{P}\right) = d\ln w_i - (1-\alpha)\, d\ln p_A = d\ln w_i - (1-\alpha)\,(d\ln Y - d\ln Q_A). \tag{RW}$$

Thus nominal Stolper–Samuelson–type changes from (Signs) are channeled through a cost-of-living term driven by the nontraded price.

Outputs and factor markets. Production, factor prices, and Hicksian coefficients continue to be determined by (ZP)–(FE). Using the standard full-employment mix mapping,

$$dY = -A^{-1}(dA)Y, \qquad dQ_B \text{ as in } (\star),$$

while dQ_A follows from factor balance and the Hicksian responses. Signs can thus be tracked using the same local intensity and substitution conditions as in Prop. 3.

2.3 Comparative statics: entry, reversals, and incidence

We interpret "entry" as a drop in the foreign upstream price p_X to a low level due to the arrival of an efficient supplier.

Low-price entry. Under the conditions of Proposition 3 with $R_B < R_A$ (sector B locally unskilled-intensive) and X a Hicksian substitute for U in B:

$$dp_X < 0 \;\Rightarrow\; dY > 0, \quad dQ_B > 0, \quad dQ_A < 0, \quad dp_A > 0.$$

Welfare then moves according to (W–p_X): the income gain $\alpha(-M_X/Y) > 0$ is partially offset by the nontraded scarcity channel $(1-\alpha)\, d\ln Q_A < 0$ (a "Dutch disease" effect).

Real-wage dominance of the cost-of-living channel. Nominally, at fixed traded-good prices one wage rises and the other falls (by (Signs)). With a nontraded good, however, *both* real wages can decline locally if the cost-of-living effect dominates:

$$\frac{d}{dp_X}\ln\Big(\frac{w_S}{P}\Big) \;>\; 0, \qquad \frac{d}{dp_X}\ln\Big(\frac{w_U}{P}\Big) \;>\; 0, \tag{BH}$$

that is, as the upstream price p_X falls (AI becomes cheaper), both w_S/P and w_U/P fall. Sufficient conditions include a large α and a sharp contraction of Q_A (hence a strong rise in p_A) relative to the nominal gain of the winning factor.

Reversals at very low entry prices. Two reversals are possible as p_X falls:

1. An *intensity reversal* $R_B(p_X)$ crossing R_A (as in (Rev)), flipping the *nominal* incidence in (Signs).
2. A *welfare reversal* from the nontraded channel: even though $dY/dp_X = -M_X < 0$ everywhere, the sign of $d\ln U/dp_X$ in (W–p_X) can reverse depending on dQ_A/dp_X.

Proposition 4 (Effects of changes in p_X)**.** *1.* $dY/dp_X = -M_X < 0$.

2. $d\ln p_A/dp_X = d\ln Y/dp_X - d\ln Q_A/dp_X$.

3. If $R_B < R_A$ *and* X *(weakly) substitutes for* U *in* B *(so* $dQ_B/dp_X < 0$ *by Prop. 3), then factor reallocation raises* Q_A*, i.e.* $dQ_A/dp_X > 0$*. Therefore*

$$\frac{d\ln p_A}{dp_X} \;<\; 0, \qquad \frac{d\ln P}{dp_X} = (1-\alpha)\,\frac{d\ln p_A}{dp_X} \;<\; 0.$$

4. Consequently, the welfare effect decomposes as

$$\frac{d\ln U}{dp_X} = -\alpha\frac{M_X}{Y} \;+\; \underbrace{(1-\alpha)\frac{1}{Q_A}\frac{dQ_A}{dp_X}}_{\textit{cost-of-living}} \;\in\; \Big(-\tfrac{M_X}{Y},\; 0\Big).$$

That is, the income contraction is partially cushioned *by cheaper nontradables.*

If instead $R_B > R_A$*, the sign of* dQ_A/dp_X *is ambiguous and so is the direction of* dp_A*; the cushioning may vanish or become amplification.*

Corollary 1 (Both workers can be hurt locally)**.** *Under* $R_B < R_A$ *and weak substitution, if*

$$(1-\alpha)\Big(\Big|\tfrac{d\ln Y}{d\ln p_X}\Big| + \Big|\tfrac{d\ln Q_A}{d\ln p_X}\Big|\Big) \;>\; \max\Big\{\Big|\tfrac{d\ln w_S}{d\ln p_X}\Big|,\; \Big|\tfrac{d\ln w_U}{d\ln p_X}\Big|\Big\},$$

then $\frac{d}{dp_X}\ln\big(\frac{w_S}{P}\big) > 0$ *and* $\frac{d}{dp_X}\ln\big(\frac{w_U}{P}\big) > 0$*, i.e. a local decrease in* p_X *lowers both real wages.*

Remark. The two elasticities $\frac{M_X}{Y}$ and $\frac{1}{Q_A}\frac{dQ_A}{dp_X}$ are sufficient statistics for the sign and size of $d\ln U/dp_X$ in (W–p_X). Likewise, $(1-\alpha)(d\ln Y - d\ln Q_A)$ governs whether both real wages fall in (RW)–(BH).

Equation (16) decomposes *levels* from autarky to p_X^ℓ, while (18) characterizes the *local* derivative at p_X^ℓ. We use these with Propositions 4–5 to isolate when CPI effects dominate.

Proposition 5 (Local–level divergence with a nontraded good: sufficient-statistic conditions)**.** *Suppose* p_X^* *satisfies an intensity reversal* (Rev)*, and let* U *denote indirect utility under the Cobb-Douglas closure* (W)*. Fix* p_X^ℓ *near* p_X^*.

1. If $p_X^\ell \downarrow p_X^*$ *from above (so* $R_B(p_X^\ell) < R_A$*), and along the autarky* $\to p_X^\ell$ *path the* level CPI dominance *condition holds*

$$(1-\alpha)\,\big(\Delta\ln Y - \Delta\ln Q_A\big) \;>\; \Delta\ln w_U,$$

while at p_X^ℓ wage effect dominance *holds*

$$\left|\tfrac{d\ln w_U}{dp_X}\right| \;>\; (1-\alpha)\left|\tfrac{d\ln p_A}{dp_X}\right|,$$

then the unskilled real wage satisfies

$$\tfrac{w_U(p_X^\ell)}{P(p_X^\ell)} \;<\; \tfrac{w_U(\infty)}{P(\infty)} \qquad \textit{and} \qquad \frac{d}{dp_X}\Big(\tfrac{w_U}{P}\Big)\Big|_{p_X^\ell} \;<\; 0.$$

2. *If* $p_X^\ell \uparrow p_X^*$ *from below (so* $R_B(p_X^\ell) > R_A$*), and the analogous* level CPI dominance *and* wage effect dominance *conditions hold for the skilled group,*

$$(1-\alpha)\big(\Delta\ln Y - \Delta\ln Q_A\big) \;>\; \Delta\ln w_S, \qquad \left|\tfrac{d\ln w_S}{dp_X}\right| \;>\; (1-\alpha)\left|\tfrac{d\ln p_A}{dp_X}\right|,$$

then

$$\tfrac{w_S(p_X^\ell)}{P(p_X^\ell)} \;<\; \tfrac{w_S(\infty)}{P(\infty)} \qquad \textit{and} \qquad \frac{d}{dp_X}\Big(\tfrac{w_S}{P}\Big)\Big|_{p_X^\ell} \;<\; 0.$$

Moreover, under the primitives in Proposition 4 with $R_B < R_A$ *and weak Hicksian substitution (so* $dQ_A/dp_X > 0$*), the level dominance* (17) *holds whenever the CPI channel is sufficiently strong along the autarky*$\to p_X^\ell$ *path (large* α *and elastic* p_A*), and the local condition* (19) *follows from the one-group specialization of* (BH)*. Thus the conditions are implied by parameter regions already characterized in this section.*

Proof. From (RW) and (NT-P), we have

$$d\ln\Big(\tfrac{w_i}{P}\Big) = d\ln w_i - (1-\alpha)\big(d\ln Y - d\ln Q_A\big).$$

Integrating from autarky $p_X = \infty$ to p_X^ℓ delivers the level decomposition

$$\Delta\ln\Big(\tfrac{w_U}{P}\Big) = \Delta\ln w_U - (1-\alpha)\big(\Delta\ln Y - \Delta\ln Q_A\big). \tag{16}$$

A sufficient condition for the real wage to fall in levels even when the nominal level rises ($\Delta\ln w_U > 0$) is

$$(1-\alpha)\big(\Delta\ln Y - \Delta\ln Q_A\big) \;>\; \Delta\ln w_U, \tag{17}$$

which ties directly to (W–p_X) and Proposition 4: along the entry path ($dp_X < 0$) we typically have $\Delta\ln Y > 0$ by the envelope theorem and $\Delta\ln Q_A < 0$ when $R_B < R_A$, so the CPI term can dominate.

For the local effect at p_X^ℓ with $R_B(p_X^\ell) < R_A$, (Signs) implies $d\ln w_U/dp_X < 0$, and

Proposition 4 implies $d\ln p_A/dp_X < 0$. Thus

$$\frac{d}{dp_X}\ln\Big(\tfrac{w_U}{P}\Big) = \frac{d\ln w_U}{dp_X} - (1-\alpha)\frac{d\ln p_A}{dp_X}, \tag{18}$$

which is negative whenever

$$\big|\tfrac{d\ln w_U}{dp_X}\big| \;>\; (1-\alpha)\,\big|\tfrac{d\ln p_A}{dp_X}\big|. \tag{19}$$

Condition (19) is the one-group specialization of the "both types of labor hurt" condition (BH). The case with $R_B > R_A$ follows for skilled workers by symmetry. □

Lemma 1 (Crossing identities: CES in A, Leontief in B)**.** *Assume sector A has CES unit cost $c_A(w_S, w_U) = \big[\beta\, w_S^{1-\rho} + (1-\beta)\, w_U^{1-\rho}\big]^{\frac{1}{1-\rho}}$ with $\beta \in (0,1)$, $\rho > 0$, and sector B (traded) is Leontief with $c_B(w_S, w_U, p_X) = a_S^B w_S + a_U^B w_U + a_X^B p_X$, $(a_S^B, a_U^B, a_X^B) > 0$. Let $\tau := \frac{a_S^B}{a_U^B}$ and $r^* := \frac{1}{\tau}$. At the unique intensity crossing p_X^* where $R_B = R_A$, the wage ratio $\omega^* := w_S/w_U$ solves*

$$\frac{\theta_S(\omega)}{\theta_U(\omega)} = \omega\,\tau, \qquad \theta_S(\omega) = \frac{\beta\,\omega^{\,1-\rho}}{\beta\,\omega^{\,1-\rho} + (1-\beta)}, \quad \theta_U(\omega) = 1 - \theta_S(\omega).$$

Equivalently,

$$\omega^* = \Big[\tfrac{\beta}{\tau(1-\beta)}\Big]^{1/\rho}, \quad k := \frac{\bar\theta_S}{\bar\theta_U} = \tau^{\,1-\frac{1}{\rho}}\left(\frac{\beta}{1-\beta}\right)^{\frac{1}{\rho}}, \quad \bar\theta_S = \frac{k}{1+k}, \quad \bar\theta_U = \frac{1}{1+k}.$$

Lemma 2 (Thresholds at the crossing)**.** *Maintain the definitions and setup from Lemma 1 and let $\alpha \in (0,1)$. The condition*

$$(1-\alpha)\big(\bar\theta_U + \bar\theta_S\, r^*\big) < 1$$

holds if and only if

$$k\Big(\frac{1-\alpha}{\tau} - 1\Big) < \alpha, \qquad k = \tau^{\,1-\frac{1}{\rho}}\left(\frac{\beta}{1-\beta}\right)^{\frac{1}{\rho}}.$$

In particular:

(a) If $\tau \geq 1-\alpha$, the inequality holds automatically.

(b) If $\tau < 1-\alpha$, it suffices that

$$\beta \;<\; \frac{\alpha^\rho\,\tau}{\alpha^\rho\,\tau + (1-\alpha-\tau)^\rho}.$$

Corollary 2 (Both workers hurt locally at the crossing)**.** *Assume CES in A and Leontief in B as in Lemma 1, and R_B decreasing in p_X with a unique crossing p_X^*. Fix any $p'_X \in (0, p_X^*)$ (approaching from the left, so $R_B < R_A$). Let $(\bar{\theta}_S, \bar{\theta}_U)$ and r^* be as in Lemma 1. If*

$$r^* \;>\; \max\left\{ \frac{1-(1-\alpha)\bar{\theta}_U}{(1-\alpha)\bar{\theta}_S} \;,\; \frac{(1-\alpha)\bar{\theta}_U}{1-(1-\alpha)\bar{\theta}_S} \right\}, \tag{20}$$

then,

$$\frac{d}{dp_X} \ln\!\left(\tfrac{w_U}{P}\right)\Big|_{(p'_X)^-} < 0 \qquad \textit{and} \qquad \frac{d}{dp_X} \ln\!\left(\tfrac{w_S}{P}\right)\Big|_{(p'_X)^-} < 0.$$

Moreover, (20) *is implied by the primitive thresholds in Lemma 2 together with*

$$r^* > \frac{(1-\alpha)\bar{\theta}_U}{1-(1-\alpha)\bar{\theta}_S} \iff 1+\alpha\,\frac{\bar{\theta}_S}{\bar{\theta}_U} \;>\; \frac{1}{r^*}(1-\alpha) \iff k \;>\; \frac{\tau(1-\alpha)-1}{\alpha},$$

which holds automatically if $\tau(1-\alpha) \le 1$, and otherwise is equivalent to

$$\beta \;>\; \frac{\big(\tau(1-\alpha)-1\big)^{\rho}\,\tau^{\,1-\rho}}{\alpha^{\rho} + \big(\tau(1-\alpha)-1\big)^{\rho}\,\tau^{\,1-\rho}}.$$

Proposition 6 (Double harm with a single reversal)**.** *Assume the setup of Lemma 1. Let p_X^* be the unique crossing and pick any $p'_X \in (0, p_X^*)$ with*

$$\Delta \ln\!\left(\tfrac{w_U}{P}\right)\Big|_{\infty \to p'_X} < 0.$$

If the following condition from Lemma 2 holds:

$$(1-\alpha)\big(\bar{\theta}_U + \bar{\theta}_S\, r^*\big) \;<\; 1, \tag{21}$$

then

$$\frac{d}{dp_X} \ln\!\left(\tfrac{w_U}{P}\right)\Big|_{(p'_X)^-} < 0.$$

At a point where unskilled are already worse off in levels relative to autarky, a marginal increase in p_X (from the left toward the crossing) makes them even worse off locally. Condition (21) *is implied by the primitive thresholds in Lemma 2.*

3 Main Model Analysis

This section focuses on the main model outlined in the text, specifically considering the case of CES demand inside sector B. We derive equilibrium identities and comparative statics.

Unless otherwise noted, all objects are equilibrium functions of (ϕ, p_X) in the AI regime.

Let $Z = (\alpha, F, s_B, L_S, L_U)$ denote primitives. In the AI regime, adoption is discrete and skilled labor is always required in B:

$$m_B \;=\; s_B\, w_S \;+\; p_X, \qquad s_B > 0. \tag{22}$$

Domestic nominal income is

$$Y \;=\; w_S L_S \;+\; w_U L_U \;+\; FN, \tag{23}$$

and foreign outflows equal $\phi N + p_X Q_B$. Sector A is competitive with CRS and $p_A \equiv 1$.

3.1 Primitives, Demand, Markups, and Free Entry

CES in sector B (variety demand and markup). Inside B, preferences are CES with elasticity $\sigma > 1$, so the common markup is constant:

$$\mu \;=\; \frac{\sigma}{\sigma - 1} \;>\; 1. \tag{24}$$

$$p_B \;=\; \mu\, m_B. \tag{25}$$

With symmetry across active adopters, the sector-B price index is

$$P_B \;=\; \mu\, m_B\, N^{\frac{1}{1-\sigma}}. \tag{26}$$

Core pricing and expenditure identities Using the pricing first order condition (25) and the sector-B expenditure identity $E_B = \alpha Y$, symmetry implies that if $Q_B \equiv \sum_{i=1}^{N} q_i = Nq$ denotes the *physical* sum of variety quantities (not the CES aggregator), then

$$m_B\, Q_B \;=\; m_B\, \frac{E_B}{p_B} \;=\; \frac{\alpha}{\mu}\, Y,$$

so

$$Q_B \;=\; \frac{\alpha}{\mu}\, \frac{Y}{m_B}. \tag{27}$$

Notation. Here $Q_B \equiv \sum_{i=1}^{N} q_i = Nq$ is the physical sum of symmetric variety quantities (it is not the CES aggregator). The identity follows directly from $E_B = \alpha Y$ and (25).

Free entry and variety. Zero profit per firm, $(p_B - m_B)q = F + \phi$, together with symmetry and $E_B = \alpha Y$, gives

$$N \;=\; \kappa(\mu)\,\frac{Y}{F+\phi}, \qquad \kappa(\mu) \equiv \frac{\mu-1}{\mu}\,\alpha. \tag{28}$$

Combining (23) and (28) yields

$$\Big(1-\vartheta(\phi)\Big)\,dY \;=\; L_S\,dw_S + L_U\,dw_U - \Xi(\phi,Y)\,d\phi, \qquad \vartheta(\phi) \equiv \frac{F\,\kappa(\mu)}{F+\phi}, \quad \Xi(\phi,Y) \equiv \frac{F\,\kappa(\mu)}{(F+\phi)^2}\,Y. \tag{29}$$

Remark. Because N satisfies (28), either a higher usage fee p_X (which depresses Y) or a higher access fee ϕ (which raises the denominator) reduces the number of active firms and thus variety.

CES technology in sector A. In order to characterize primitive conditions for comparative statics, we sometimes assume a CES unit-cost function in A. In such cases, we refer to the following:

$$c_A(w_S, w_U) \;=\; \begin{cases} \big[\,\beta\,w_S^{1-\rho} + (1-\beta)\,w_U^{1-\rho}\,\big]^{\frac{1}{1-\rho}}, & \rho \neq 1, \\ w_S^{\beta}\,w_U^{1-\beta}, & \rho = 1 \quad \text{(CD limit)}, \end{cases} \tag{30}$$

with share parameter $\beta \in (0,1)$ and elasticity of substitution $\rho > 0$. Hicksian unit demands are $a_A^S = \partial c_A/\partial w_S$ and $a_A^U = \partial c_A/\partial w_U$, and the corresponding factor-cost shares in A are $\theta_S \equiv w_S a_A^S$ and $\theta_U \equiv w_U a_A^U$, with $\theta_S + \theta_U = 1$. Standard CES properties imply that the Hicksian semi-elasticities are given by

$$\frac{\partial \ln a_A^S}{\partial \ln w_S} = -\,\rho\,\theta_S, \quad \frac{\partial \ln a_A^U}{\partial \ln w_U} = -\,\rho\,\theta_U, \quad \frac{\partial \ln a_A^S}{\partial \ln w_U} = +\rho\,\theta_U, \qquad \frac{\partial \ln a_A^U}{\partial \ln w_S} = +\rho\,\theta_S. \tag{31}$$

With Cobb–Douglas across sectors, $Q_A = (1-\alpha)Y$, so labor usage in A is

$$a_A^U(w)\,(1-\alpha)\,Y \;=\; L_U, \tag{32}$$

$$a_A^S(w)\,(1-\alpha)\,Y \;+\; s_B\,Q_B \;=\; L_S, \tag{33}$$

where Q_B is given by (27).

3.2 Welfare Factorization

Under the assumptions that utility is Cobb–Douglas across sectors and $p_A \equiv 1$, indirect utility satisfies $W \propto Y\, P_B^{-\alpha}$. Using (26),

$$\frac{W^{AI}}{W^0} \;=\; \left(\frac{m_B^0}{m_B^{AI}}\right)^{\alpha} \left(\frac{N^{AI}}{N^0}\right)^{\frac{\alpha}{\sigma-1}} \left(\frac{Y^{AI}}{Y^0}\right). \tag{34}$$

This decomposition isolates the *unit-cost* channel (m_B), the *variety* channel (N), and the *aggregate-income* channel (Y).

3.3 Comparative Statics

We derive how Q_B, w_S, and Y respond to p_X and ϕ by differentiating (27)–(33) together with the income identity (38).

3.3.1 Preliminaries.

Differentiating (27) and using (22):

$$dQ_B \;=\; \frac{\alpha}{\mu}\left(\frac{dY}{m_B} - \frac{Y}{m_B^2}\left(s_B\, dw_S + dp_X\right)\right). \tag{35}$$

Differentiating (32) gives

$$(1-\alpha)\, a_A^U\, dY + (1-\alpha)\, Y\, da_A^U \;=\; 0 \;\;\Longrightarrow\;\; dY \;=\; -\,Y\,\frac{da_A^U}{a_A^U}. \tag{36}$$

Differentiating (33) and substituting (35) yields

$$(1-\alpha)\big(a_A^S\, dY + Y\, da_A^S\big) \;+\; s_B \frac{\alpha}{\mu}\left(\frac{dY}{m_B} - \frac{Y}{m_B^2}\left(s_B\, dw_S + dp_X\right)\right) \;=\; 0. \tag{37}$$

Finally, combining (23) and (28) implies

$$\Big(1 - \vartheta(\phi)\Big)\, dY \;=\; L_S\, dw_S \;+\; L_U\, dw_U \;-\; \Xi(\phi, Y)\, d\phi, \tag{38}$$

Equations (36)–(38) form a linear system in (dY, dw_S, dw_U) in terms of $(dp_X, d\phi)$.

3.3.2 Comparative statics of usage fee

Proposition 7 (Usage-fee incidence)**.** *Set $d\phi = 0$. In the CES environment described above (with $s_B > 0$ and CES in A with parameters (β, ρ)), the factor-price responses satisfy*

$$\frac{\partial w_U}{\partial p_X} > 0 \qquad \textit{and} \qquad \frac{\partial w_S}{\partial p_X} < 0. \tag{39}$$

Proof. Let $x := d\ln w_S$ and $y := d\ln w_U$. From (22), $dm_B = s_B\, dw_S + dp_X = s_B w_S\, x + dp_X$. Differentiate the A-sector factor-balance equations (32)–(33) and use the CES Hicksian semi-elasticities in (31):

For notational compactness, write $h_{SS} = -\rho\theta_S,\ h_{UU} = -\rho\theta_U,\ h_{SU} = +\rho\theta_U,\ h_{US} = +\rho\theta_S$ (as in (31)).

$$(1-\alpha)\, a_A^U\, dY + (1-\alpha)\, Y\, da_A^U = 0 \iff \frac{dY}{Y} = -\big(h_{US}x + h_{UU}y\big) = -\rho\big(\theta_S x - \theta_U y\big). \tag{40}$$

Divide (33) after differentiation by Y, substitute (40) and $dm_B = s_B w_S x + dp_X$:

$$\begin{aligned} 0 &= (1-\alpha)\left(\frac{a_A^S}{Y}\, dY + da_A^S\right) + \frac{s_B\alpha}{\mu}\left(\frac{1}{m_B}\frac{dY}{Y} - \frac{1}{m_B^2}(s_B w_S x + dp_X)\right) \\ &= (1-\alpha)\Big[- a_A^S\big(h_{US}x + h_{UU}y\big) + a_A^S\big(h_{SS}x + h_{SU}y\big)\Big] \qquad (41) \\ &+ \frac{s_B\alpha}{\mu m_B}\big(- h_{US}x - h_{UU}y\big) - \frac{s_B\alpha}{\mu m_B^2}(s_B w_S x + dp_X). \qquad (42) \end{aligned}$$

Using $h_{SS} = -\rho\theta_S$, $h_{UU} = -\rho\theta_U$, $h_{SU} = +\rho\theta_U$, $h_{US} = +\rho\theta_S$, the $(1-\alpha)$ bracket simplifies to $2(1-\alpha)\rho\, a_A^S\big(- \theta_S x + \theta_U y\big)$. .

Collecting coefficients on x and y delivers the linear system

$$\underbrace{\Big(- 2(1-\alpha)\rho\, a_A^S\theta_S \ - \ \tfrac{s_B\alpha}{\mu m_B}\rho\theta_S \ - \ \tfrac{s_B^2\alpha w_S}{\mu m_B^2}\Big)}_{:=\Gamma_1} x + \underbrace{\Big(2(1-\alpha)\rho\, a_A^S\theta_U \ + \ \tfrac{s_B\alpha}{\mu m_B}\rho\theta_U\Big)}_{:=\Gamma_2} y \ = \ \underbrace{\tfrac{s_B\alpha}{\mu m_B^2}}_{:=\Gamma_3} dp_X. \tag{43}$$

By construction $\Gamma_1 < 0$, $\Gamma_2 > 0$, and $\Gamma_3 > 0$. Solving (43) together with (40) (Cramer's rule) yields the semi-elasticities

$$\frac{\partial \ln w_S}{\partial p_X} = -\frac{s_B}{D}\cdot\frac{s_B\,\alpha}{\mu\, m_B^2} \ < \ 0, \qquad \frac{\partial \ln w_U}{\partial p_X} = \ \frac{s_B}{D}\cdot\frac{s_B\,\alpha}{\mu\, m_B^2}\cdot\frac{\theta_S}{\theta_U} \ > \ 0, \tag{44}$$

where the positive denominator is

$$D \ \equiv \ (1-\alpha)\,\rho\,\theta_S \ + \ \frac{s_B\,\alpha}{\mu\, m_B}\cdot\frac{s_B}{w_S} \ > \ 0. \tag{45}$$

The strict positivity of the right-hand side of $\partial \ln w_U / \partial p_X$ in (44) establishes (39). $\square$

Intuition. Sector B uses only skilled labor, so it is skill-intensive relative to A. A higher p_X raises m_B, contracts B, and releases skilled labor. General equilibrium reallocation pushes production toward A, raising relative demand for unskilled labor; hence w_U rises while w_S falls.

Corollary 3. *From* (38) *with* $d\phi = 0$ *and* (44),

$$\frac{\partial Y}{\partial p_X} \;<\; 0. \tag{46}$$

Finally, (27) *and* (22) *give*

$$\frac{\partial m_B}{\partial p_X} \;=\; 1 + s_B \,\frac{\partial w_S}{\partial p_X} \;\in\; (0,1), \qquad \frac{\partial Q_B}{\partial p_X} \;=\; \frac{\alpha}{\mu}\left(\frac{1}{m_B}\frac{\partial Y}{\partial p_X} - \frac{Y}{m_B^2}\frac{\partial m_B}{\partial p_X}\right) \;<\; 0. \tag{47}$$

Special cases for the A-sector CES parameter ρ

(i) $\rho \to 0$ (Leontief in A) When A has no substitution, the Hicksian responses vanish and the denominator in (44) simplifies to

$$D_{\mathrm{L}} \;=\; \frac{s_B\,\alpha}{\mu\, m_B} \cdot \frac{s_B}{w_S}.$$

Substituting into (44) gives

$$\frac{\partial \ln w_S}{\partial p_X} = -\,\frac{s_B}{D_{\mathrm{L}}}\,\frac{s_B\alpha}{\mu m_B^2} = -\,\frac{w_S}{m_B} \;<\; 0,$$
$$\frac{\partial \ln w_U}{\partial p_X} = \frac{s_B}{D_{\mathrm{L}}}\,\frac{s_B\alpha}{\mu m_B^2}\,\frac{\theta_S}{\theta_U} = \frac{w_S}{m_B}\,\frac{\theta_S}{\theta_U} \;>\; 0.$$

$$\frac{d\ln Y}{dp_X} = \frac{1}{1-\vartheta(\phi)}\,\frac{w_S}{m_B}\Big(-\theta_S^Y + \theta_U^Y\frac{\theta_S}{\theta_U}\Big) = \frac{1}{1-\vartheta(\phi)}\,\frac{w_S}{m_B}\,\theta_S\Big(\frac{\theta_U^Y}{\theta_U} - \frac{\theta_S^Y}{\theta_S}\Big).$$

Substituting in:

$$\frac{d\ln Y}{dp_X} = \frac{1}{1-\vartheta(\phi)}\,\frac{w_S}{m_B}\,\frac{-\,w_S L_S + \frac{a_A^S}{a_A^U}\frac{w_S}{w_U}\,w_U L_U}{Y} = \frac{1}{1-\vartheta(\phi)}\,\frac{w_S^2}{m_B\,Y}\Big(\frac{a_A^S}{a_A^U}L_U - L_S\Big).$$

(ii) $\rho = 1$ (Cobb-Douglas in A) At $\rho = 1$, the denominator in (44) becomes

$$D_{\mathrm{CD}} \;=\; (1-\alpha)\,\theta_S \;+\; \frac{s_B\,\alpha}{\mu\, m_B} \cdot \frac{s_B}{w_S} \;>\; 0.$$

Substituting into (44) yields

$$\frac{\partial \ln w_S}{\partial p_X} = -\frac{s_B^2 \alpha/(\mu m_B^2)}{(1-\alpha)\theta_S + \frac{s_B \alpha}{\mu m_B}\frac{s_B}{w_S}},$$
$$\frac{\partial \ln w_U}{\partial p_X} = \frac{s_B^2 \alpha/(\mu m_B^2)}{(1-\alpha)\theta_S + \frac{s_B \alpha}{\mu m_B}\frac{s_B}{w_S}}\frac{\theta_S}{\theta_U}.$$

$$\frac{d \ln Y}{dp_X} = \frac{1}{1-\vartheta(\phi)}\frac{s_B^2 \alpha}{\mu m_B^2}\frac{-w_S L_S + \frac{\beta}{1-\beta} w_U L_U}{Y\left[(1-\alpha)\beta + \frac{s_B \alpha}{\mu m_B}\frac{s_B}{w_S}\right]}.$$

(iii) $\rho \to \infty$ (perfect substitutes in A) As $\rho \to \infty$, the first term in D dominates:

$$D \;=\; (1-\alpha)\,\rho\,\theta_S \;+\; \frac{s_B\,\alpha}{\mu\, m_B}\cdot\frac{s_B}{w_S} \;\sim\; (1-\alpha)\,\rho\,\theta_S \;\to\; \infty.$$

Therefore the semi-elasticities shrink to zero:

$$\frac{\partial \ln w_S}{\partial p_X} \approx -\frac{s_B^2 \alpha}{\mu m_B^2}\frac{1}{(1-\alpha)\rho\theta_S}, \qquad \frac{\partial \ln w_U}{\partial p_X} \approx \frac{s_B^2 \alpha}{\mu m_B^2}\frac{1}{(1-\alpha)\rho\theta_U}.$$

Hence both responses approach zero at rate $O(1/\rho)$, and $\frac{\partial m_B}{\partial p_X} \to 1^-$. Further, substitution gives:

$$\frac{d \ln Y}{dp_X} = \frac{1}{1-\vartheta(\phi)}\frac{s_B^2 \alpha}{\mu m_B^2}\frac{-w_S L_S + \frac{\beta}{1-\beta}\left(\frac{w_S}{w_U}\right)^{1-\rho} w_U L_U}{Y\left[(1-\alpha)\rho\,\frac{\beta\, w_S^{1-\rho}}{\beta\, w_S^{1-\rho} + (1-\beta)\, w_U^{1-\rho}} + \frac{s_B \alpha}{\mu m_B}\frac{s_B}{w_S}\right]}.$$

Local incidence of a usage–price increase with discrete adoption We next characterize the CPI and welfare consequences of a marginal increase in p_X when adoption is discrete and unchanged. In this case, there is no substitution back toward unskilled labor in B.

Lemma 3 (Signs for m_B, N, P_B)**.** *For $dp_X > 0$ at fixed adoption:*

$$0 < \frac{dm_B}{dp_X} < 1, \qquad \frac{d \ln N}{dp_X} = \frac{d \ln Y}{dp_X} < 0, \qquad \frac{d \ln P_B}{dp_X} > 0.$$

Proof. The first inequality follows from (47). The second uses free entry $N \propto Y$ and $\partial Y/\partial p_X < 0$ (see (46)). For the third, using (26), $\ln P_B = \ln m_B + \frac{1}{1-\sigma}\ln N + \ln \mu$. Since $\sigma > 1$, $\frac{1}{1-\sigma} < 0$ and $d \ln N/dp_X < 0$, so that term is positive; and $d \ln m_B/dp_X > 0$. □

Recall that we have normalized $p_A = 1$ so that Sector A dynamics show up through factor markets and wages. Then, we have

Proposition 8 (CPI increase and welfare loss)**.** *In the CES environment with $p_A \equiv 1$ (sector A is the numéraire), the CPI is $P = P_B^{\alpha}$ and welfare satisfies $W \propto Y/P$. Then (within the regime that AI is adopted),*

$$\frac{d\ln P}{dp_X} \;=\; \alpha\left(\frac{d\ln m_B}{dp_X} \;-\; \frac{1}{\sigma-1}\frac{d\ln N}{dp_X}\right) \;=\; \alpha\left(\frac{d\ln m_B}{dp_X} \;-\; \frac{1}{\sigma-1}\frac{d\ln Y}{dp_X}\right) \;>\; 0,$$

and hence

$$\frac{d\ln W}{dp_X} \;=\; \frac{d\ln Y}{dp_X} - \frac{d\ln P}{dp_X} \;<\; 0.$$

Proof. Within the regime where AI is adopted, $0 < \frac{dm_B}{dp_X} < 1$ and $\frac{d\ln Y}{dp_X} < 0$ (Cor. 3). Since $N \propto Y$ by (28), $d\ln N = d\ln Y$. With $\sigma > 1$, $-\frac{1}{\sigma-1}\frac{d\ln Y}{dp_X} > 0$, so the bracketed term is strictly positive, implying $d\ln P/dp_X > 0$ and therefore $d\ln U/dp_X < 0$. □

Corollary 4 (Real wages)**.** ***(i)*** *Comparative statics for real wages are:*

$$\frac{d\ln(w_S/P)}{dp_X} = \frac{d\ln w_S}{dp_X} - \frac{d\ln P}{dp_X} < 0, \qquad \frac{d\ln(w_U/P)}{dp_X} = \frac{d\ln w_U}{dp_X} - \alpha\left(\frac{d\ln m_B}{dp_X} - \frac{1}{\sigma-1}\frac{d\ln Y}{dp_X}\right).$$

A sufficient *condition for $\frac{d\ln(w_U/P)}{dp_X} < 0$ is*

$$\alpha\left(\frac{d\ln m_B}{dp_X} - \frac{1}{\sigma-1}\frac{d\ln Y}{dp_X}\right) \;\geq\; \frac{d\ln w_U}{dp_X}. \tag{48}$$

(ii) *Using* (44) *in* (48) *gives*

$$\alpha\left(\frac{d\ln m_B}{dp_X} - \frac{1}{\sigma-1}\frac{d\ln Y}{dp_X}\right) \;\geq\; \frac{s_B^2\alpha}{\mu m_B^2 D}\cdot\frac{\theta_S}{\theta_U}.$$

A sufficient (conservative) condition, since $d\ln m_B/dp_X > 0$, is

$$\frac{\alpha}{\sigma-1}\left|\frac{d\ln Y}{dp_X}\right| \;\geq\; \frac{s_B^2\alpha}{\mu m_B^2 D}\cdot\frac{\theta_S}{\theta_U}.$$

(iii) Limiting cases with CES production in sector A.

(a) ***Leontief in*** *A* ***($\rho \to 0$).***

$$\frac{d\ln(w_U/P)}{dp_X} =$$

$$\underbrace{\frac{w_S}{m_B}\frac{a_A^S}{a_A^U}\frac{w_S}{w_U}}_{\text{wage}} - \underbrace{\alpha}_{\text{unit cost}} + \underbrace{\alpha\frac{s_B\, w_S^2}{m_B}}_{\text{offset in } dm_B/dp_X} + \underbrace{\frac{\alpha}{\sigma-1}\frac{1}{1-\vartheta(\phi)}\frac{w_S^2}{m_B\, Y}\left(\tfrac{a_A^S}{a_A^U}L_U - L_S\right)}_{\text{variety via } Y}.$$

(b) ***Cobb–Douglas in*** A ***($\rho = 1$).*** *Let*

$$H_{\mathrm{CD}} \;\equiv\; \frac{s_B^2\,\alpha}{\mu\, m_B^2\left[(1-\alpha)\,\beta \;+\; \dfrac{s_B\,\alpha}{\mu\, m_B}\,\dfrac{s_B}{w_S}\right]}.$$

Then:

$$\frac{d\ln(w_U/P)}{dp_X}$$

$$= \underbrace{-\,\alpha}_{\textit{unit cost}} \;+\; \underbrace{H_{\mathrm{CD}}\left[\frac{\beta}{1-\beta} \;+\; \alpha s_B w_S \;+\; \frac{\alpha}{\sigma-1}\,\frac{1}{1-\vartheta(\phi)}\,\frac{-\,w_S L_S + \frac{\beta}{1-\beta} w_U L_U}{Y}\right]}_{\textit{wage + offset + variety}}.$$

(c) ***Perfect substitutes in*** A ***($\rho \to \infty$).***

$$\frac{d\ln(w_U/P)}{dp_X} = \underbrace{\frac{d\ln w_U}{dp_X}}_{=O(1/\rho)} - \alpha\left[1 - \frac{s_B^3\,\alpha\, w_S}{\mu\, m_B^2 D}\right] \;+\; \underbrace{\frac{\alpha}{\sigma-1}\,\frac{d\ln Y}{dp_X}}_{=O(1/\rho)}.$$

Using the expression for D,

$$\lim_{\rho\to\infty} \frac{d\ln(w_U/P)}{dp_X} = \lim_{\rho\to\infty} \frac{d\ln(w_S/P)}{dp_X} = -\,\alpha\,\frac{1}{m_B}.$$

3.3.3 Access fee: Income and Skilled Wage

Throughout this subsection set $dp_X = 0$. Recall

$$\Xi(\phi, Y) = \frac{F\,\kappa(\mu)\,Y}{(F+\phi)^2}, \qquad \kappa(\mu) = \frac{\mu-1}{\mu}\,\alpha, \qquad \vartheta(\phi) = \frac{F\,\kappa(\mu)}{F+\phi} \in (0,1),$$

and let $x := d\ln w_S$, $y := d\ln w_U$.

Step 1: Express (x, y) in terms of dY. From (36) and the CES Hicksian elasticities (31),

$$\frac{dY}{Y} = -\rho\,\theta_U(x-y) \quad\Longrightarrow\quad y = x + \frac{1}{\rho\,\theta_U}\,\frac{dY}{Y}. \tag{49}$$

With $dm_B = s_B\,dw_S = s_B w_S\,x$, differentiate (33), divide by Y, and substitute (49):

$$\begin{aligned} 0 &= (1-\alpha)\Big(\tfrac{a_A^S}{Y}\,dY + a_A^S(-\rho\theta_S x + \rho\theta_U y)\Big) + \frac{s_B\alpha}{\mu}\Big(\tfrac{1}{m_B}\tfrac{dY}{Y} - \tfrac{s_B w_S}{m_B^2}x\Big) \\ &= (1-\alpha)a_A^S\left[2\,\frac{dY}{Y} - \rho(1-2\theta_U)\,x\right] + \frac{s_B\alpha}{\mu}\left(\frac{1}{m_B}\frac{dY}{Y} - \frac{s_B w_S}{m_B^2}x\right). \end{aligned} \tag{50}$$

Rearranging gives

$$x \;=\; \Lambda_S \,\frac{dY}{Y}, \qquad \Lambda_S := \frac{2(1-\alpha)a_A^S + \frac{s_B\alpha}{\mu}\frac{1}{m_B}}{(1-\alpha)a_A^S\,\rho(1-2\theta_U) + \frac{s_B\alpha}{\mu}\frac{s_B w_S}{m_B^2}}. \tag{51}$$

Step 2: Apply the income identity. Divide (38) by Y and use $dw_S = w_S x$, $dw_U = w_U y$ with (49)–(51). Writing $\theta_i^Y := \frac{w_i L_i}{Y}$,

$$\big(1-\vartheta(\phi)\big)\,\frac{dY}{Y} = \theta_S^Y\,x + \theta_U^Y\,y \;-\; \frac{\Xi(\phi,Y)}{Y}\,d\phi, \tag{52}$$

and substituting $y = x + \frac{1}{\rho\theta_U}\frac{dY}{Y}$ and $x = \Lambda_S\frac{dY}{Y}$ yields

$$\underbrace{\Big[1-\vartheta(\phi) \;-\; \theta_S^Y\,\Lambda_S \;-\; \theta_U^Y\Big(\Lambda_S + \frac{1}{\rho\theta_U}\Big)\Big]}_{:=\,\Delta_\phi}\,\frac{dY}{Y} \;=\; -\,\frac{\Xi(\phi,Y)}{Y}\,d\phi. \tag{53}$$

Hence:

$$\frac{dY}{Y} = -\,\frac{\Xi(\phi,Y)/Y}{\Delta_\phi}\,d\phi, \qquad \frac{d\ln w_S}{d\phi} = \Lambda_S\,\frac{1}{Y}\,\frac{dY}{d\phi}. \tag{54}$$

The determinant. Using $\theta_S^Y + \theta_U^Y = (w_S L_S + w_U L_U)/Y = 1-\vartheta(\phi)$ (free entry and rebates), (53) can be written as

$$\Delta_\phi = \big(1-\vartheta(\phi)\big)\,(1-\Lambda_S) \;-\; \frac{\theta_U^Y}{\rho\,\theta_U}. \tag{55}$$

Proposition 9 (Income and skilled-wage comparative statics)**.** *Suppose $\Delta_\phi > 0$ and $\Lambda_S > 0$. Then for $d\phi > 0$,*

$$\frac{\partial Y}{\partial\phi} < 0 \qquad \textit{and} \qquad \frac{\partial \ln w_S}{\partial\phi} < 0.$$

Proof. From (54), $\Xi(\phi,Y)/Y > 0$ and $\Delta_\phi > 0$ imply $dY/d\phi < 0$. Then $d\ln w_S/d\phi = \Lambda_S(1/Y)(dY/d\phi) < 0$ whenever $\Lambda_S > 0$. □

Primitive sufficient restrictions. A convenient set of primitives ensuring $\Delta_\phi > 0$ and $\Lambda_S > 0$ is:

(i) Either $\theta_U \le \frac{1}{2}$ or $\Big(\theta_U > \frac{1}{2}$ and $\rho < \rho_{\max}\Big)$,

(ii) Determinant bound: $\quad \Lambda_S + \dfrac{1}{\rho\,\theta_U} \;<\; 1-\vartheta(\phi)$,

where (for $\theta_U > \frac{1}{2}$)

$$\rho_{\max} = \frac{\alpha\, s_B^2\, w_S}{\mu\, m_B^2\,(1-\alpha)\, a_A^S\,(2\theta_U - 1)}. \tag{56}$$

Condition (i) guarantees the denominator of Λ_S in (51) is positive (hence $\Lambda_S > 0$); (ii) is a conservative way to ensure $\Delta_\phi > 0$ via (55).

Lower bound. Define

$$\epsilon_Y^\phi := -\frac{1}{Y}\frac{\partial Y}{\partial \phi} = \frac{\Xi(\phi, Y)/Y}{\Delta_\phi}.$$

Since the wage–price term in (55) is nonnegative, $\Delta_\phi \le 1 - \vartheta(\phi)$ and

$$\epsilon_Y^\phi \;\ge\; \underline{\epsilon}_Y^\phi := \frac{\Xi(\phi, Y)/Y}{1-\vartheta(\phi)} = \frac{F\,\kappa(\mu)}{(F+\phi)\left[(F+\phi) - F\,\kappa(\mu)\right]}. \tag{57}$$

Hence $\underline{\epsilon}_Y^\phi$ is strictly decreasing in ϕ, strictly increasing in $\kappa(\mu)$, and single–peaked in F with maximizer $F^* = \phi/\sqrt{1-\kappa(\mu)}$.

Upper bound. If there exists $\overline{\Lambda} < 1$ with $\Lambda_S \le \overline{\Lambda}$ and $(1-\vartheta(\phi))(1-\overline{\Lambda}) > \frac{1}{\rho\,\theta_U}$ (positivity of the denominator), then (55) implies

$$\epsilon_Y^\phi \;\le\; \overline{\epsilon}_Y^\phi := \frac{\frac{F\kappa(\mu)}{(F+\phi)^2}}{(1-\vartheta(\phi))(1-\overline{\Lambda}) - \frac{1}{\rho\,\theta_U}}. \tag{58}$$

When $\theta_U \le \frac{1}{2}$, we can use the bound

$$\overline{\Lambda} = \frac{2(1-\alpha)a_A^S + \frac{s_B\alpha}{\mu}\frac{1}{m_B}}{\frac{s_B\alpha}{\mu}\frac{s_B w_S}{m_B^2}} = \frac{m_B}{s_B w_S} + \frac{2(1-\alpha)\,\mu\, m_B^2}{\alpha\, s_B^2\, w_S}\, a_A^S,$$

and in the Cobb–Douglas case ($\rho = 1$), $a_A^S = \beta/w_S$ gives $\overline{\Lambda}_{\mathrm{CD}} = \frac{m_B}{s_B w_S} + \frac{2(1-\alpha)\,\beta\,\mu\, m_B^2}{\alpha\, s_B^2\, w_S^2}$.

Conditions for $dY/d\phi < 0$ and $d\ln w_S/d\phi < 0$ in special cases of ρ.

- **Leontief in A ($\rho \to 0$).** The denominator of Λ_S is dominated by the Sector B term; $\Lambda_S > 0$ and (55) reduces to $(1-\vartheta)(1-\Lambda_S)$ minus a large positive term $1/(\rho\theta_U)$, so the sign of Δ_ϕ is ambiguous. However, whenever $\Delta_\phi > 0$ holds, $dY/d\phi < 0$ and $d\ln w_S/d\phi < 0$ follow immediately by Proposition 9.
- **Cobb-Douglas in A ($\rho = 1$).** Then $\theta_U = 1-\beta$, $a_A^S = \beta/w_S$, and $\rho_{\max}$ simplifies to

$$\rho_{\max}\Big|_{\rho=1} = \frac{\alpha\, s_B^2\, w_S^2}{\mu\, m_B^2\,(1-\alpha)\,\beta\,(1-2\beta)}.$$

If $\beta \geq \frac{1}{2}$, the Primitive Sufficient Condition (i) holds automatically; if $\beta < \frac{1}{2}$, it suffices that $1 < \rho_{\max}$.

- **Perfect substitutes in A ($\rho \to \infty$).** As $\rho \to \infty$, $\Lambda_S \to 0$ and $\Delta_\phi \to 1 - \vartheta(\phi) > 0$. Thus $dY/d\phi < 0$ and $d \ln w_S/d\phi < 0$ unconditionally for sufficiently large ρ.

3.3.4 Access fees: Incidence on the Unskilled Wage

Throughout this subsection set $dp_X = 0$. From (51)–(53)

$$\Lambda_S = \frac{2(1-\alpha)a_A^S + \frac{s_B\alpha}{\mu}\frac{1}{m_B}}{(1-\alpha)a_A^S\,\rho(1-2\theta_U) + \frac{s_B\alpha}{\mu}\frac{s_B w_S}{m_B^2}}, \qquad \Delta_\phi = 1 - \vartheta(\phi) - \theta_S^Y\,\Lambda_S - \theta_U^Y\Big(\Lambda_S + \frac{1}{\rho\theta_U}\Big).$$

Define the income semi-elasticity and the skilled-technology threshold:

$$\epsilon_Y^\phi := -\frac{1}{Y}\frac{\partial Y}{\partial \phi} = \frac{\Xi(\phi, Y)/Y}{\Delta_\phi}, \qquad T := \frac{\alpha}{\mu\, m_B} \cdot \frac{s_B}{w_S} \cdot \frac{\theta_S}{\theta_U}.$$

Proposition 10 (Unskilled wage response to access fees)**.** *Assume $s_B > 0$, $\sigma > 1$ (so $\mu = \sigma/(\sigma-1) > 1$), CES in A with (β, ρ) and $\rho > 0$, and interior factor shares. Then*

$$\frac{\partial w_U}{\partial \phi} > 0 \quad \Longleftrightarrow \quad \epsilon_Y^\phi < T. \tag{59}$$

Proof. Differentiate the A-sector factor-balance equations and the income identity (cf. (36)–(38)) with $dm_B = s_B\, dw_S$ (since $dp_X = 0$). Solving the linear system yields the semi-elasticity (as in the derivation recorded previously):

$$\frac{\partial \ln w_U}{\partial \phi} = \underbrace{\frac{s_B}{D} \cdot \frac{s_B\,\alpha}{\mu\, m_B^2} \cdot \frac{\theta_S}{\theta_U}}_{\text{(i) substitution in } B} - \underbrace{\frac{\rho}{D} \cdot \theta_U \cdot \epsilon_Y^\phi}_{\text{(ii) income}}, \qquad D \;=\; (1-\alpha)\,\rho\,\theta_S \;+\; \frac{s_B\,\alpha}{\mu\, m_B} \cdot \frac{s_B}{w_S} \;>\; 0. \tag{60}$$

Since $w_U > 0$, $\operatorname{sign}(\partial w_U/\partial\phi) = \operatorname{sign}(\partial \ln w_U/\partial\phi)$. Setting (60) to zero and solving for ϵ_Y^ϕ gives $\epsilon_Y^\phi = \frac{\alpha}{\mu\, m_B}\frac{s_B}{w_S}\frac{\theta_S}{\theta_U} = T$, establishing (59). □

Interpretation. Term (i) in (60) captures the reallocation toward A when higher ϕ tightens entry and contracts B; this pushes up w_U. Term (ii) reflects the reduction in domestic rebates FN; a larger ϵ_Y^ϕ pulls w_U down. The sign turns on which effect dominates.

Bounds and one-sided conclusions. Since the GE wage–price effect in (53) is nonnegative, $\Delta_\phi \le 1 - \vartheta(\phi)$, yielding the lower bound

$$\epsilon_Y^\phi \;\ge\; \underline{\epsilon}_Y^\phi := \frac{\Xi(\phi, Y)/Y}{1 - \vartheta(\phi)} = \frac{F\,\kappa(\mu)}{(F+\phi)\left[(F+\phi) - F\,\kappa(\mu)\right]}\,. \tag{61}$$

Hence a conservative *exclusion* condition is $\underline{\epsilon}_Y^\phi \ge T \;\Rightarrow\; \partial w_U/\partial\phi \le 0$. Moreover, $\underline{\epsilon}_Y^\phi$ is strictly decreasing in ϕ, strictly increasing in $\kappa(\mu)$, and single-peaked in F (max at $F^* = \phi/\sqrt{1-\kappa(\mu)}$).

An *upper* bound arises if we impose the positivity restriction

$$(1 - \vartheta(\phi))(1 - \overline{\Lambda}) > \frac{1}{\rho\,\theta_U} \quad \text{for some} \quad \Lambda_S \le \overline{\Lambda} < 1. \tag{62}$$

Using $\Delta_\phi = (1-\vartheta)(1-\Lambda_S) - \frac{\theta_U^Y}{\rho\theta_U}$ and $\theta_U^Y \le 1$,

$$\epsilon_Y^\phi \;\le\; \overline{\epsilon}_Y^\phi := \frac{\frac{F\kappa(\mu)}{(F+\phi)^2}}{(1-\vartheta(\phi))(1-\overline{\Lambda}) - \frac{1}{\rho\theta_U}}\,. \tag{63}$$

Then $\overline{\epsilon}_Y^\phi \le T$ is *sufficient* for $\partial w_U/\partial\phi > 0$.

Remark 1 (Primitive bound for $\overline{\Lambda}$)**.** *When $\theta_U \le \frac{1}{2}$ the denominator of Λ_S is greater than the B-sector term, so*

$$\overline{\Lambda} = \frac{2(1-\alpha)a_A^S + \frac{s_B\alpha}{\mu}\frac{1}{m_B}}{\frac{s_B\alpha}{\mu}\frac{s_B w_S}{m_B^2}} = \frac{m_B}{s_B w_S} + \frac{2(1-\alpha)\,\mu\,m_B^2}{\alpha\,s_B^2\,w_S}\,a_A^S$$

is a convenient primitive upper bound. In the Cobb–Douglas case ($\rho = 1$), using $a_A^S = \beta/w_S$ yields $\overline{\Lambda}_{\mathrm{CD}} = \frac{m_B}{s_B w_S} + \frac{2(1-\alpha)\,\beta\,\mu\,m_B^2}{\alpha\,s_B^2\,w_S^2}$.

Corollary 5 (Access-fee incidence on w_U: special cases by ρ)**.** *Let $T = \frac{\alpha}{\mu m_B}\frac{s_B}{w_S}\frac{\theta_S}{\theta_U}$ and ϵ_Y^ϕ as above.*

(a) ***Leontief in*** *A ($\rho \to 0$)**.*** *$D \to D_{\mathrm{L}} = \frac{s_B\alpha}{\mu m_B}\frac{s_B}{w_S}$ and the income term in* (60) *vanishes, so*

$$\frac{\partial \ln w_U}{\partial\phi} = \frac{s_B}{D_{\mathrm{L}}}\cdot\frac{s_B\alpha}{\mu m_B^2}\cdot\frac{\theta_S}{\theta_U} > 0 \quad\Rightarrow\quad \frac{\partial w_U}{\partial\phi} > 0 \text{ for any } s_B > 0.$$

(b) ***Cobb-Douglas in*** *A ($\rho = 1$)**.*** *With $\theta_S = \beta$, $\theta_U = 1-\beta$,*

$$T_{\mathrm{CD}} = \frac{\alpha}{\mu m_B}\cdot\frac{s_B}{w_S}\cdot\frac{\beta}{1-\beta}.$$

A conservative sufficient condition for $\partial w_U/\partial\phi \le 0$ *is*

$$\underline{\epsilon}_Y^{\phi} \;\ge\; T_{\mathrm{CD}} \quad\Longleftrightarrow\quad \frac{F\kappa(\mu)}{(F+\phi)\big[(F+\phi)-F\kappa(\mu)\big]} \;\ge\; \frac{\alpha}{\mu m_B}\cdot\frac{s_B}{w_S}\cdot\frac{\beta}{1-\beta}.$$

Under (62) *with* $\overline{\Lambda}=\overline{\Lambda}_{\mathrm{CD}}$ *above, a sufficient condition for* $\partial w_U/\partial\phi > 0$ *is*

$$\overline{\epsilon}_Y^{\phi} = \frac{\frac{F\kappa(\mu)}{(F+\phi)^2}}{(1-\vartheta(\phi))(1-\overline{\Lambda}_{\mathrm{CD}})-\frac{1}{1-\beta}} \;\le\; T_{\mathrm{CD}}.$$

(c) ***Perfect substitutes in*** A ***(*** $\rho\to\infty$ ***).*** $D=(1-\alpha)\rho\,\theta_S+O(1)$, *hence*

$$\frac{\partial \ln w_U}{\partial\phi} = -\,\frac{\theta_U}{(1-\alpha)\theta_S}\,\epsilon_Y^{\phi}+O\Big(\frac{1}{\rho}\Big).$$

From (53), $\Delta_\phi \to 1-\vartheta(\phi)>0$, *so*

$$\epsilon_Y^{\phi} \;\to\; \frac{\dfrac{F\kappa(\mu)}{(F+\phi)^2}}{1-\dfrac{F\kappa(\mu)}{F+\phi}} = \frac{F\,\kappa(\mu)}{(F+\phi)\big[(F+\phi)-F\kappa(\mu)\big]} \;>0.$$

Therefore, for large ρ, $\partial\ln w_U/\partial\phi<0$ *and* $\partial w_U/\partial\phi<0$ unconditionally.

Proof of Corollary 5. (a) Take the limit $\rho\to 0$ in (60); since $D\to D_{\mathrm{L}}$ and $\rho/D\to 0$, the income term vanishes and the substitution term is strictly positive.

(b) Substitute $\theta_S=\beta$, $\theta_U=1-\beta$ in (59) to obtain the CD threshold T_{CD}. The lower-bound exclusion follows by comparing $\underline{\epsilon}_Y^{\phi}$ in (61) to T_{CD}. Under (62) with the primitive $\overline{\Lambda}_{\mathrm{CD}}$, (63) gives $\overline{\epsilon}_Y^{\phi}$, and the sufficiency claim is immediate.

(c) As stated, $D=(1-\alpha)\rho\theta_S+O(1)$ makes the substitution term $O(1/\rho)$ in (60). Since $\Delta_\phi\to 1-\vartheta(\phi)>0$, the limit of ϵ_Y^{ϕ} is strictly positive, and the leading term $-\frac{\theta_U}{(1-\alpha)\theta_S}\epsilon_Y^{\phi}$ determines the negative sign for large ρ. □

3.3.5 Sector B Output and Number of Firms

Decomposition. Using $Q_B=\frac{\alpha}{\mu}\frac{Y}{m_B}$ and $m_B=s_Bw_S+p_X$,

$$\frac{\partial Q_B}{\partial\phi} = \frac{\alpha}{\mu}\,\frac{\partial Y}{\partial\phi}\,\left[\frac{1}{m_B}-\frac{s_Bw_S}{m_B^2}\,\Lambda_S\right].$$

Assuming $\Delta_\phi > 0$ so that $\frac{\partial Y}{\partial \phi} < 0$, we have

$$\frac{\partial Q_B}{\partial \phi} \begin{cases} < 0, & \text{if } \Lambda_S < \dfrac{m_B}{s_B w_S} \quad \text{(income contraction dominates)}, \\ > 0, & \text{if } \Lambda_S > \dfrac{m_B}{s_B w_S} \quad \text{(unit-cost decrease dominates)}. \end{cases}$$

Note. The unit-cost decrease referred to above arises indirectly via the skilled wage: $\frac{dm_B}{d\phi} = s_B \frac{dw_S}{d\phi} = -\dfrac{s_B w_S \Lambda_S}{\Delta_\phi} \dfrac{F\kappa(\mu)}{(F+\phi)^2} < 0$ whenever $\Delta_\phi > 0$ and $\Lambda_S > 0$.

Number of firms. From $N = \kappa(\mu) \dfrac{Y}{F+\phi}$,

$$\frac{\partial N}{\partial \phi} = \kappa(\mu) \left(\frac{1}{F+\phi} \frac{\partial Y}{\partial \phi} - \frac{Y}{(F+\phi)^2} \right) \; < \; 0 \qquad (\text{since } \; \Delta_\phi > 0 \Rightarrow \partial Y/\partial \phi < 0).$$

Per-firm quantity (always rises under $\Delta_\phi > 0$). With symmetry,

$$q := \frac{Q_B}{N} = \frac{\frac{\alpha}{\mu} \frac{Y}{m_B}}{\kappa(\mu) \frac{Y}{F+\phi}} = \frac{F+\phi}{(\mu-1)\, m_B},$$

so

$$\frac{\partial q}{\partial \phi} = \frac{1}{(\mu-1)\, m_B} - \frac{F+\phi}{(\mu-1)\, m_B^2} \frac{\partial m_B}{\partial \phi} \quad \text{with} \quad \frac{\partial m_B}{\partial \phi} = s_B \frac{\partial w_S}{\partial \phi} < 0.$$

Hence $\partial q/\partial \phi > 0$: the direct $(F+\phi)$ effect is positive, and $\partial m_B/\partial \phi < 0$ (via $\partial w_S/\partial \phi < 0$ under $\Delta_\phi > 0, \Lambda_S > 0$) amplifies it. Thus $Q_B = Nq$ can rise even though N falls, if q rises enough— exactly the case $\Lambda_S > \frac{m_B}{s_B w_S}$.

Special cases.

$$\begin{cases} \text{Leontief in } A \ (\rho \to 0): & \Lambda_S > \dfrac{m_B}{s_B w_S} \;\Rightarrow\; \dfrac{\partial Q_B}{\partial \phi} > 0, \\ \text{Cobb-Douglas in } A \ (\rho = 1): & \dfrac{\partial Q_B}{\partial \phi} < 0 \iff 2 < (2\beta - 1) \dfrac{m_B}{s_B w_S} \quad (\text{requires } \beta > \tfrac{1}{2}), \\ \text{Perfect substitutes in } A \ (\rho \to \infty): & \Lambda_S \to 0 \;\Rightarrow\; \dfrac{\partial Q_B}{\partial \phi} < 0 \;\; \text{(unconditionally)}. \end{cases}$$

3.4 Adoption Frontier

Consider an AI equilibrium (Z, ϕ, p_X) (using (25) and (22)). A single deviator (no adoption, hence no access fee) uses the baseline technology that employs both factors. Let (s_0, u_0) denote the baseline (non-AI) unit inputs; the deviator's marginal cost is

$$m_{\text{dev}} \;=\; s_0\, w_S \;+\; u_0\, w_U, \tag{64}$$

with (w_S, w_U) evaluated at the AI equilibrium. Under CES, the "no-deviation" condition $\pi_{\text{dev}} \leq 0$ reduces to the adoption cap

$$p_X \;\leq\; m_{\text{dev}} \left(\frac{F}{F+\phi} \right)^{\frac{1}{\sigma-1}} \;-\; s_B \, w_S, \tag{65}$$

which defines a downward-sloping frontier $p_{X,\max}(\phi)$.

Monotonicity of the cap in ϕ Differentiating (65) *holding the AI-equilibrium wage vector fixed for the local comparison* gives

$$\frac{dp_{X,\max}}{d\phi} \;=\; -\,\frac{1}{\sigma-1}\, m_{\text{dev}} \left(\frac{F}{F+\phi} \right)^{\frac{1}{\sigma-1}} \frac{1}{F+\phi} \;<\; 0, \tag{66}$$

so the adoption cap is strictly decreasing in ϕ.

3.4.1 Joint optimality on the adoption frontier (CES): theorem and primitives

We study the constrained profit maximization problem

$$\max_{\phi \geq 0,\; p_X \leq p_{X,\max}(\phi)} \Pi(\phi, p_X) \qquad \text{with} \qquad \Pi(\phi, p_X) \;=\; \phi \, N(\phi, p_X) \;+\; p_X \, Q_B(\phi, p_X),$$

where $p_{X,\max}(\phi)$ is the adoption cap from (65) and, under CES, $\mu = \sigma/(\sigma-1)$ is constant, $\kappa(\mu) = \frac{\mu-1}{\mu}\alpha$.

Standing assumptions.

(A1) *(CES, interior)* Inside B: CES with elasticity $\sigma > 1$; in A: CES unit cost (30) with $\rho > 0$ and $0 < \beta < 1$; adoption requires $s_B > 0$ (cf. (22)). Factor endowments (L_S, L_U) and primitives (α, F) are strictly positive.

(A2) *(Regularity)* The equilibrium correspondences $(Y, N, Q_B, m_B, w_S, w_U)$ are single-valued and C^1 in (ϕ, p_X) on the feasible set $\mathcal{F} = \{(\phi, p_X) : \phi \geq 0,\; 0 \leq p_X \leq p_{X,\max}(\phi)\}$.

(A3) *(Cap monotonicity)* The adoption cap is C^1 and strictly decreasing: $p'_{X,\max}(\phi) \equiv \frac{dp_{X,\max}}{d\phi} < 0$ (cf. (66)).

(A4) *(Compactness)* For any $\bar{\phi} < \infty$, the truncated feasible set $\{(\phi, p_X) : 0 \leq \phi \leq \bar{\phi},\; 0 \leq p_X \leq p_{X,\max}(\phi)\}$ is compact.

Lemma 4 (Profit derivatives under CES)**.** *Fix $\phi \geq 0$. With $\epsilon_Y^X \equiv -\frac{1}{Y}\frac{dY}{dp_X} \geq 0$ and $0 < \frac{dm_B}{dp_X} < 1$, the derivative of Π in p_X is*

$$\frac{d\Pi}{dp_X} = Y\Big[-\epsilon_Y^X \underbrace{\Big(\frac{\phi\,\kappa(\mu)}{F+\phi} + \frac{\alpha}{\mu}\frac{p_X}{m_B}\Big)}_{=:\ \mathcal{C}_Y(\phi,p_X)} + \underbrace{\frac{\alpha}{\mu}\Big(\frac{1}{m_B} - \frac{p_X}{m_B^2}\frac{dm_B}{dp_X}\Big)}_{=:\ \mathcal{C}_M(p_X)} \Big], \tag{67}$$

and

$$\frac{\alpha}{\mu}\frac{s_B w_S}{m_B^2} \;\leq\; \mathcal{C}_M(p_X) \;\leq\; \frac{\alpha}{\mu}\frac{1}{m_B}. \tag{68}$$

Proof. Differentiate $\Pi(\phi, p_X) = \phi N(\phi, p_X) + p_X Q_B(\phi, p_X)$: $\frac{d\Pi}{dp_X} = \phi\frac{dN}{dp_X} + Q_B + p_X\frac{dQ_B}{dp_X}$. Using (28) and (27), we have $\frac{dN}{dp_X} = \frac{\kappa(\mu)}{F+\phi}\frac{dY}{dp_X}$ and

$$\frac{dQ_B}{dp_X} = \frac{\alpha}{\mu}\left(\frac{1}{m_B}\frac{dY}{dp_X} - \frac{Y}{m_B^2}\frac{dm_B}{dp_X}\right).$$

Collecting terms yields (67). Since (22) and $0 < \frac{dm_B}{dp_X} < 1$ (cf. (47)), we have

$$\frac{1}{m_B} - \frac{p_X}{m_B^2}\frac{dm_B}{dp_X} \in \Big[\frac{1}{m_B} - \frac{p_X}{m_B^2},\ \frac{1}{m_B}\Big] = \Big[\frac{s_B w_S}{m_B^2},\ \frac{1}{m_B}\Big],$$

which implies (68). □

Theorem 1 (Boundary optimality at the joint optimum (CES))**.** *Under (A1)–(A4), the constrained problem admits a maximizer $(\phi^\star, p_X^\star)$. Define the* boundary objective $\widehat{\Pi}(\phi) \equiv \Pi\big(\phi,\, p_{X,\max}(\phi)\big)$ *and the* reduced objective $\widetilde{\Pi}(\phi) \equiv \max_{0 \leq p_X \leq p_{X,\max}(\phi)} \Pi(\phi, p_X)$*. Then:*

(i) (KKT at a maximizer) *Any maximizer $(\phi^\star, p_X^\star)$ admits a multiplier $\lambda^\star \geq 0$ s.t.*

$$\frac{\partial \Pi}{\partial p_X}(\phi^\star, p_X^\star) - \lambda^\star = 0, \tag{69}$$

$$\frac{\partial \Pi}{\partial \phi}(\phi^\star, p_X^\star) + \lambda^\star\, p'_{X,\max}(\phi^\star) = 0, \tag{70}$$

$$p_X^\star \leq p_{X,\max}(\phi^\star), \quad \lambda^\star \geq 0, \quad \lambda^\star\big(p_{X,\max}(\phi^\star) - p_X^\star\big) = 0. \tag{71}$$

(ii) (Boundary iff condition) *$(\phi^\star, p_X^\star)$ is* binding *($p_X^\star = p_{X,\max}(\phi^\star)$) if and only if*

$$\left.\frac{\partial \Pi}{\partial p_X}\right|_{(\phi^\star,\, p_{X,\max}(\phi^\star))} \;\geq\; 0 \quad\Longleftrightarrow\quad \left.\frac{\partial \Pi}{\partial \phi}\right|_{(\phi^\star,\, p_{X,\max}(\phi^\star))} \;\geq\; 0, \tag{72}$$

where the equivalence uses $p'_{X,\max}(\phi^\star) < 0$ from (A3).

(iii) (Boundary selection via the inner problem) *Fix ϕ. If $p \mapsto \Pi(\phi, p)$ is nondecreasing on $[0, p_{X,\max}(\phi)]$, then the inner maximizer is $p_X^{\text{in}}(\phi) = p_{X,\max}(\phi)$ and $\widetilde{\Pi}(\phi) = \widehat{\Pi}(\phi)$.*

(iv) (Boundary optimality at the joint optimum) *If $\phi^\star \in \arg\max_{\phi \geq 0} \widehat{\Pi}(\phi)$ and, in addition, $p \mapsto \Pi(\phi^\star, p)$ is nondecreasing on $[0, p_{X,\max}(\phi^\star)]$, then $(\phi^\star,\, p_{X,\max}(\phi^\star))$ solves the joint problem.*

Outline. Existence follows from (A2)–(A4) and Weierstrass on truncated compact sets. By standard KKT for a smooth inequality constraint, there is $\lambda^\star \geq 0$ satisfying (69)–(71). If the constraint binds ($p_X^\star = p_{X,\max}(\phi^\star)$), then (69) implies $\lambda^\star = \partial\Pi/\partial p_X\big|_{\text{frontier}} \geq 0$; if interior, $\lambda^\star = 0$ and $\partial\Pi/\partial p_X(\phi^\star, p_X^\star) = 0$. Combining (70) with $p'_{X,\max}(\phi^\star) < 0$ yields the boundary condition (72). If, for fixed ϕ, the inner map $p \mapsto \Pi(\phi, p)$ is nondecreasing on $[0, p_{X,\max}(\phi)]$, the inner maximizer is the boundary $p_{X,\max}(\phi)$; maximizing $\widehat{\Pi}(\phi)$ over ϕ then delivers the joint optimum at $(\phi^\star,\, p_{X,\max}(\phi^\star))$. □

Remark 2 (On part (ii))**.** *At a boundary optimum, $\lambda^\star = \partial\Pi/\partial p_X\big|_{\textit{frontier}}$ and (72) is the boundary FOC using $p'_{X,\max}(\phi^\star) < 0$.*

We next provide *primitive sufficient conditions* that guarantee the monotonicity required in Theorem 1(iii)–(iv), expressed solely in terms of CES parameters and cost-share objects.

Corollary 6 (Primitive sufficient conditions for boundary optimality)**.** *Maintain (A1)–(A4). For a given $\phi \geq 0$, define the worst-case income semi-elasticity bound*

$$\overline{\epsilon}_Y^X(\phi) \;:=\; \sup_{p \in [0,\, p_{X,\max}(\phi)]} \epsilon_Y^X(\phi, p). \tag{73}$$

$$\mathcal{R}(\phi) \;:=\; \left. \frac{\dfrac{s_B\, w_S}{m_B}}{p_X \;+\; (\mu - 1)\, m_B\, \dfrac{\phi}{F + \phi}} \right|_{p_X = p_{X,\max}(\phi)} . \tag{74}$$

If

$$\overline{\epsilon}_Y^X(\phi) \;\leq\; \mathcal{R}(\phi), \tag{75}$$

then $p \mapsto \Pi(\phi, p)$ is nondecreasing on $[0, p_{X,\max}(\phi)]$, hence the inner maximizer is $p_{X,\max}(\phi)$ and $\widetilde{\Pi}(\phi) = \widehat{\Pi}(\phi)$. In particular, if $\phi^\star \in \arg\max_\phi \widehat{\Pi}(\phi)$ and (75) holds at $\phi^\star$, then $(\phi^\star,\, p_{X,\max}(\phi^\star))$ solves the joint problem.

Proof. By Lemma 4,

$$\frac{1}{Y}\frac{d\Pi}{dp_X} = -\,\epsilon_Y^X(\phi, p_X)\,\mathcal{C}_Y(\phi, p_X) + \mathcal{C}_M(p_X) \;\geq\; -\,\overline{\epsilon}_Y^X(\phi)\,\mathcal{C}_Y(\phi, p_X) \;+\; \frac{\alpha}{\mu}\frac{s_B w_S}{m_B^2}.$$

For the usage–price shock we have $0 < \frac{dm_B}{dp_X} < 1$ and $\frac{dw_S}{dp_X} < 0$, hence $\frac{d}{dp_X}(p_X/m_B) > 0$ and $\frac{d}{dp_X}(s_B w_S/m_B^2) \le 0$. Therefore the right–hand side is minimized at $p_X = p_{X,\max}(\phi)$, and we obtain

$$\frac{1}{Y}\frac{d\Pi}{dp_X} \;\ge\; -\,\overline{\epsilon}_Y^X(\phi)\,\mathcal{C}_Y\big(\phi, p_X\big) + \frac{\alpha}{\mu}\frac{s_B w_S}{m_B^2} \;\ge\; -\,\overline{\epsilon}_Y^X(\phi)\,\mathcal{C}_Y\big(\phi, p_{X,\max}(\phi)\big) + \frac{\alpha}{\mu}\frac{s_B w_S}{m_B^2}\bigg|_{p_X = p_{X,\max}(\phi)} \;\ge\; 0,$$

where the last inequality is exactly (75). Hence $\frac{d\Pi}{dp_X} \ge 0$ on $[0, p_{X,\max}(\phi)]$, so the inner maximizer is $p_{X,\max}(\phi)$ and $\widetilde{\Pi}(\phi) = \widehat{\Pi}(\phi)$. □

Special cases for substitutability in A (ρ). The boundary condition (75) compares a worst–case income feedback $\overline{\epsilon}_Y^X(\phi)$ to the technological/market ratio $\mathcal{R}(\phi)$ in (74).

- **Leontief in A ($\rho \to 0$).** With limited factor substitution in A, the usage–price income feedback ϵ_Y^X is typically larger; thus (75) is *harder* to satisfy. Boundary optimality can still hold if $\mathcal{R}(\phi)$ is large (e.g., high s_B/w_S, low m_B, high σ).
- **Cobb–Douglas in A ($\rho = 1$).** The criterion (75) applies directly by computing $\overline{\epsilon}_Y^X(\phi)$ over $[0, p_{X,\max}(\phi)]$ and comparing it to $\mathcal{R}(\phi)$.
- **Perfect substitutes in A ($\rho \to \infty$).** In the CES A-sector, wage semi-elasticities and the usage–price income feedback scale as $O(1/\rho)$, hence $\epsilon_Y^X(\phi, p) \to 0$ uniformly on compact feasible sets. Therefore, for sufficiently large ρ, (75) is automatically satisfied and the inner problem selects the boundary $p_{X,\max}(\phi)$.

Proposition 11 (High–ρ boundary optimality)**.** *Fix $\sigma > 1$ and $\phi \ge 0$. There exists $\bar{\rho} < \infty$ (depending on primitives and ϕ) such that for all $\rho \ge \bar{\rho}$ the inner objective $p \mapsto \Pi(\phi, p)$ is nondecreasing on $[0, p_{X,\max}(\phi)]$. Consequently, any joint maximizer satisfies $p_X^\star = p_{X,\max}(\phi^\star)$.*

Proof sketch. For the usage–price shock, the CES A-sector linear system yields wage semi-elasticities and dY/dp_X of order $O(1/\rho)$ (the Sector A determinant is proportional to $(1-\alpha)\rho\theta_S$). Hence $\epsilon_Y^X(\phi, p) = -(1/Y)\, dY/dp_X = O(1/\rho)$ uniformly in $p \in [0, p_{X,\max}(\phi)]$. Since the frontier ratio $\mathcal{R}(\phi)$ in (74) does not scale with ρ through the A-sector elasticities, while $\epsilon_Y^X(\phi, p) = O(1/\rho)$ uniformly in p, choose $\bar{\rho}$ large enough that $\sup_p \epsilon_Y^X(\phi, p) \le \frac{1}{2}\mathcal{R}(\phi)$. Then (75) holds and Corollary 6 implies monotonicity of $p \mapsto \Pi(\phi, p)$ and boundary selection at the joint optimum. □

Interpretation. Boundary optimality is guaranteed for sufficiently high ρ; at moderate ρ (e.g., Cobb-Douglas), it holds whenever the inequality (75) (condition on primitives) is satisfied;

at very low ρ (Leontief), income feedback can overturn monotonicity unless primitives make $\mathcal{R}(\phi)$ large.

Comparative statics intuition (all else equal). Boundary optimality is more likely with large s_B/w_S (strong cost-share transfer), low m_B, high σ (low μ), and *smaller* ϕ relative to F (since $\phi/(F+\phi)$ increases in ϕ and raises the denominator of $\mathcal{R}(\phi)$). It is less likely with large α *and* high markups (both amplify $\mathcal{C}_Y$), or with especially strong income feedback (large $\bar{\epsilon}_Y^X$).

3.4.2 Mixed partial inside the frontier (CES): sign and primitives

We show that raising ϕ reduces the marginal profitability of p_X under primitive conditions, so the instruments are local substitutes.

In this subsection, we work in the feasible interior.

Notation. Let $\epsilon_Y^X \equiv -\frac{1}{Y}\frac{dY}{dp_X} \geq 0$, and write (cf. Lemma 4)

$$\frac{d\Pi}{dp_X} \;=\; Y\Big[-\,\epsilon_Y^X\,\mathcal{C}_Y(\phi,p_X)+\mathcal{C}_M(p_X)\Big], \tag{76}$$

$$\mathcal{C}_Y(\phi,p_X) := \frac{\phi\,\kappa(\mu)}{F+\phi}+\frac{\alpha}{\mu}\frac{p_X}{m_B}, \quad \mathcal{C}_M(p_X) := \frac{\alpha}{\mu}\left(\frac{1}{m_B}-\frac{p_X}{m_B^2}\frac{dm_B}{dp_X}\right). \tag{77}$$

Lemma 5 (Markup derivative)**.** *With* $\mathcal{C}_M(p_X)=\frac{\alpha}{\mu}\left(\frac{1}{m_B}-\frac{p_X}{m_B^2}\frac{dm_B}{dp_X}\right)$*, define*

$$\tau := \frac{dm_B}{dp_X}\in(0,1), \qquad s := \frac{\partial m_B}{\partial \phi}<0 \quad (\textit{under } \Delta_\phi>0,\ \Lambda_S>0).$$

Then

$$\frac{\partial \mathcal{C}_M}{\partial \phi}=\frac{\alpha}{\mu}\left\{-\frac{s}{m_B^2}-\frac{p_X}{m_B^2}\frac{\partial \tau}{\partial \phi}+\frac{2p_X}{m_B^3}\,s\,\tau\right\},$$

and

$$\frac{\partial \mathcal{C}_M}{\partial \phi} \;\leq\; 0 \quad\Longleftrightarrow\quad \frac{\partial \tau}{\partial \phi} \;\geq\; \frac{-\,s}{p_X\,m_B}\left(1-\frac{2p_X}{m_B}\,\tau\right). \tag{78}$$

Corollary 7 (Primitive sufficient condition for $\partial_\phi \mathcal{C}_M \leq 0$)**.** *Let* $\tau=\frac{dm_B}{dp_X}\in(0,1)$*,* $s=\frac{\partial m_B}{\partial \phi}<0$*, and* $m_B=s_Bw_S+p_X$*. A primitive sufficient condition for* $\frac{\partial \mathcal{C}_M}{\partial \phi}\leq 0$ *is*

$$\boxed{\frac{\partial \tau}{\partial \phi} \;\geq\; 0 \quad \textit{and} \quad 2p_X\,\tau \;\geq\; m_B}.$$

Two facts we use repeatedly:

$$\frac{\partial \mathcal{C}_Y}{\partial \phi} = \frac{\kappa(\mu) F}{(F+\phi)^2} + \frac{\alpha}{\mu} \frac{p_X}{m_B^2} \left(-\frac{\partial m_B}{\partial \phi} \right) \; > \; 0, \tag{79}$$

since $\kappa(\mu), F > 0$ and, by (80), $\frac{\partial m_B}{\partial \phi} < 0$.

Finally, with $dp_X = 0$, $m_B = s_B w_S + p_X$ and Prop. 9 yield:

$$\frac{\partial m_B}{\partial \phi} = s_B \frac{\partial w_S}{\partial \phi} < 0 \qquad \text{(whenever } \Delta_\phi > 0, \; \Lambda_S > 0), \tag{80}$$

i.e. the unit cost in B falls as ϕ rises via the GE effect on w_S.

Monotonicity of the income feedback. We now give primitive sufficient conditions under which ϕ makes the usage–price income contraction weakly larger:

$$\frac{\partial \epsilon_Y^X}{\partial \phi}(\phi, p_X) \; \geq \; 0.$$

Lemma 6 (Monotonicity checks)**.** *In the CES environment with $s_B > 0$, $\sigma > 1$:*

(a) ***Leontief in*** *A **($\rho \to 0$).** With fixed proportions in A, higher ϕ raises $\vartheta(\phi)$ and lowers m_B; the usage–price income feedback is nondecreasing in ϕ: $\frac{\partial \epsilon_Y^X}{\partial \phi} \geq 0$ without additional restrictions.*

(b) ***Cobb–Douglas in*** *A **($\rho = 1$).** A primitive sufficient condition is either $\beta \geq \frac{1}{2}$ or*

$$\frac{s_B^2 \, \alpha}{\mu} \cdot \frac{w_S^2}{m_B^2} \; \geq \; (1-\alpha)\,(1-2\beta)\,\beta. \tag{81}$$

Under either inequality, $\frac{\partial \epsilon_Y^X}{\partial \phi} \geq 0$.

(c) ***Perfect substitutes in*** *A **($\rho \to \infty$).** As $\rho \to \infty$, the usage–price income feedback is $O(1/\rho)$ and nondecreasing in ϕ due to the entry term $\vartheta(\phi)$; for sufficiently large ρ, $\frac{\partial \epsilon_Y^X}{\partial \phi} \geq 0$ holds without further restrictions.*

Proposition 12 (Mixed partial at an interior usage–price optimizer)**.** *Fix $\phi \geq 0$ and suppose $\hat{p}_X \in (0, \infty)$ is an unconstrained interior maximizer of $p \mapsto \Pi(\phi, p)$. Using* (76) *and Lemma 4,*

$$\frac{d\Pi}{dp_X} = Y \big[-\, \epsilon_Y^X \, \mathcal{C}_Y + \mathcal{C}_M \big],$$

and the mixed partial satisfies

$$\frac{\partial^2 \Pi}{\partial p_X \, \partial \phi}(\phi, \hat{p}_X) = Y\Big\{ - \frac{\partial \epsilon_Y^X}{\partial \phi}(\phi, \hat{p}_X)\, \mathcal{C}_Y(\phi, \hat{p}_X) \; - \; \epsilon_Y^X(\phi, \hat{p}_X)\, \frac{\partial \mathcal{C}_Y}{\partial \phi}(\phi, \hat{p}_X) \; + \; \frac{\partial \mathcal{C}_M}{\partial \phi}(\phi, \hat{p}_X) \Big\}. \tag{82}$$

In particular, if

$$\frac{\partial \epsilon_Y^X}{\partial \phi}(\phi, \hat{p}_X) \geq 0 \qquad \textit{and} \qquad \Big(\frac{\partial \tau}{\partial \phi}(\phi, \hat{p}_X) \; \geq \; 0 \;\; \textit{and} \;\; 2 p_X \, \tau(\phi, \hat{p}_X) \; \geq \; m_B \Big), \tag{83}$$

then

$$\frac{\partial^2 \Pi}{\partial p_X \, \partial \phi}(\phi, \hat{p}_X) \; \leq \; 0, \quad \textit{with strict } < 0 \textit{ if at least one inequality in (83) is strict.} \tag{84}$$

Moreover, $\frac{\partial \mathcal{C}_Y}{\partial \phi} > 0$ *always holds by* (79).

Corollary 8 (Primitive special cases by ρ)**.** *Let* $\mu = \sigma/(\sigma - 1) > 1$, $m_B = s_B w_S + p_X$.

(a) ***Leontief in*** A ***(***$\rho \to 0$***).*** *If the condition in Corollary 7 holds (i.e.* $\partial_\phi \tau \geq 0$ *and* $2 p_X \tau \geq m_B$*), then* $\dfrac{\partial^2 \Pi}{\partial p_X \, \partial \phi} < 0$.

(b) ***Cobb–Douglas in*** A ***(***$\rho = 1$***).*** *If either* $\beta \geq \frac{1}{2}$ *or the inequality* (81) *holds, and, in addition, the primitive condition in Corollary 7 holds (i.e.* $\partial_\phi \tau \geq 0$ *and* $2 p_X \, \tau \geq m_B$ *with* $\tau := dm_B/dp_X \in (0, 1)$*), then* $\dfrac{\partial^2 \Pi}{\partial p_X \, \partial \phi} < 0$.

(c) ***Perfect substitutes in*** A ***(***$\rho \to \infty$***).*** *For sufficiently large* ρ*, the usage–price income feedback is* $O(1/\rho)$ *and* $\partial_\phi \epsilon_Y^X \geq 0$ *due to the entry term. If, in addition, the primitive condition in Corollary 7 holds (i.e.* $\partial_\phi \tau \geq 0$ *and* $2 p_X \, \tau \geq m_B$*), then* $\dfrac{\partial^2 \Pi}{\partial p_X \, \partial \phi} < 0$*. Moreover, even if that condition fails, there exists* $\bar{\rho} < \infty$ *(depending on* $(\alpha, \mu, s_B, w_S, m_B, p_X, F, \phi)$*) such that for all* $\rho \geq \bar{\rho}$ *the mixed partial remains negative.*

Interpretation. At an interior p_X optimum, the cross-partial in (82) is the sum of: (i) a *negative* income-feedback term (Lemma 6), (ii) a *negative* demand/entry term since $\partial_\phi \mathcal{C}_Y > 0$ (79), and (iii) a markup term. The primitive condition in Corollary 7 (i.e. $\partial_\phi \tau \geq 0$ and $2 p_X \, \tau \geq m_B$) makes the markup term nonpositive, so raising ϕ reduces the marginal profitability of p_X: the instruments behave as local substitutes.

3.5 Summary: Case of High Sector A Factor Substitutability

For ease of reference, here we summarize the results about the case with high Sector A substitutability between skilled and unskilled labor. When $\rho \to \infty$, the Sector A determinant is proportional $(1-\alpha)\rho\theta_S$, so wage semi-elasticities with respect to usage price fees are, from (44)–(45), $\partial \ln w_S/\partial p_X, \partial \ln w_U/\partial p_X = O(1/\rho)$ and $\partial m_B/\partial p_X \to 1$. With $N \propto Y$ we have $\partial \ln Y/\partial p_X < 0$ (Corollary 3), and within the regime where AI is adopted, $P = P_B^{\alpha}$ implies $\partial \ln P/\partial p_X > 0$ and hence $\partial \ln U/\partial p_X < 0$ (Proposition 8); moreover the real-wage derivatives converge to the same constant:

$$\lim_{\rho\to\infty} \frac{\partial \ln(w_S/P)}{\partial p_X} = \lim_{\rho\to\infty} \frac{\partial \ln(w_U/P)}{\partial p_X} = -\frac{\alpha}{m_B}.$$

For changes in access fees, $\Lambda_S \to 0$ and $\Delta_\phi \to 1 - \vartheta(\phi) > 0$, so $\partial Y/\partial\phi < 0$ and $\partial \ln w_S/\partial\phi < 0$ unconditionally (Proposition 9); the unskilled-wage incidence reduces to $\partial \ln w_U/\partial\phi = -\frac{\theta_U}{(1-\alpha)\theta_S}\,\epsilon_Y^{\phi} + O(1/\rho)$, and since $\epsilon_Y^{\phi} > 0$ for large ρ, also $\partial \ln w_U/\partial\phi < 0$ (Corollary 5). On quantities, the bracket in $\partial Q_B/\partial\phi = \frac{\alpha}{\mu}\,\partial Y/\partial\phi\,[\,\frac{1}{m_B} - \frac{s_B w_S}{m_B^2}\Lambda_S\,]$ converges to $1/m_B > 0$, so with $\partial Y/\partial\phi < 0$ we get $\partial Q_B/\partial\phi < 0$ unconditionally. $N = \kappa(\mu)Y/(F+\phi)$ falls and $q = Q_B/N = (F+\phi)/[(\mu-1)m_B]$ rises. On the adoption frontier, $p'_{X,\max}(\phi) < 0$ holds for any $\sigma > 1$ ((66)), and for sufficiently large ρ the inner usage-price problem is monotone so the joint optimum lies on the frontier, $p_X^{\star} = p_{X,\max}(\phi^{\star})$ (Proposition 11). Finally, inside the frontier the mixed partial is negative for large ρ—the first two terms in (82) dominate and, together with the primitive markup condition of Corollary 7, imply $\partial^2\Pi/(\partial p_X\,\partial\phi) < 0$ (Corollary 8).