For "Public Finance in the Age of AI: A Primer," Anton Korinek and Lee Lockwood. In The Economics of Transformative AI, edited by Ajay Agrawal, Erik Brynjolfsson, and Anton Korinek, University of Chicago Press, 2026.
https://www.nber.org/books-and-chapters/economics-transformative-ai/public-finance-age-ai-primer

# A Online Appendix

## A.1 Proofs of Section 2

### A.1.1 Lemma 1

Recall **Lemma 1** (Equivalence of Consumption and Labor/Initial Capital Taxes) Fix a constant consumption tax $\tau_C \geq 0$ and no capital-income tax ($\tau_K = 0$). Any competitive-equilibrium allocation achievable with $(\tau_L, \tau_C, \tau_K, \tau_{K0}) = (0, \tau_C, 0, 0)$ is also achievable with $(\tau_L^*, 0, 0, \tau_{K0}^*)$ and suitable $\{T_t', K_{G,t}'\}$, where

$$\tau_L^* = \tau_{K0}^* = \frac{\tau_C}{1+\tau_C}.$$

*Proof.* Fix a constant consumption tax $\tau_C \geq 0$ and no capital-income tax ($\tau_K = 0$). Consider the system with $(\tau_L, \tau_C, \tau_K, \tau_{K0}) = (0, \tau_C, 0, 0)$. The household budget (4) is

$$(1+\tau_C)c_t^i + k_{t+1}^i \;=\; R_t k_t^i + w_t \theta_i l_t^i + T_t, \qquad R_t := 1 + (r_t - \delta).$$

Because $\tau_C$ is *constant*, it rescales consumption units uniformly over time. The Euler equation (5) becomes

$$\frac{u'(c_t^i)}{1+\tau_C} = \beta \, \frac{u'(c_{t+1}^i)}{1+\tau_C} \, R_{t+1} \iff u'(c_t^i) = \beta R_{t+1} u'(c_{t+1}^i),$$

so there is no intertemporal wedge. The labor FOC (6) reads

$$(l_t^i)^{1/\varepsilon} \;=\; \frac{w_t \theta_i}{1+\tau_C}\, u'(c_t^i), \quad \text{i.e.} \quad \frac{1-\tau_L}{1+\tau_C} = \frac{1}{1+\tau_C} \text{ when } \tau_L = 0.$$

Let $Q_0 := 1$ and $Q_t := \prod_{s=0}^{t-1} R_s^{-1}$. Multiplying the period budget by $Q_t$ and summing forward with the usual transversality condition yields the lifetime (present-value) budget in producer units:

$$\sum_{t\geq 0} Q_t c_t^i \;=\; \frac{1}{1+\tau_C}\left[k_0^i + \sum_{t\geq 0} Q_t\big(w_t\theta_i l_t^i + T_t\big)\right]. \tag{18}$$

Now consider an alternative system with *no* consumption tax and *no* capital-income tax, but with a labor tax and a one-time levy on initial capital:

$$(\tau_L^*, \tau_C', \tau_K', \tau_{K0}^*) = (\tau_L^*, 0, 0, \tau_{K0}^*),$$

and (possibly different) transfers $T_t'$. Its lifetime budget is

$$\sum_{t\geq 0} Q_t c_t^i \;=\; (1-\tau_{K0}^*)\, k_0^i + \sum_{t\geq 0} Q_t\big((1-\tau_L^*) w_t \theta_i l_t^i + T_t'\big). \tag{19}$$

Choose the constants

$$1-\tau_L^* = \frac{1}{1+\tau_C}, \qquad 1-\tau_{K0}^* = \frac{1}{1+\tau_C}, \qquad \sum_{t\geq 0} Q_t T_t' = \frac{1}{1+\tau_C}\sum_{t\geq 0} Q_t T_t.$$

With these choices, (19) coincides with (18) *for any* feasible labor path $\{l_t^i\}$, so each household faces the same lifetime budget set in the two systems. The intratemporal labor wedge also matches because $(1-\tau_L^*) = 1/(1+\tau_C)$, and the intertemporal wedge is absent in both because $\tau_K = 0$. Hence, given any price path $\{w_t, r_t\}$ that clears markets, households choose the *same* allocation.

Finally, define government assets recursively to satisfy the period-by-period budget (7):

$$K'_{G,0} \text{ free}, \qquad K'_{G,t+1} = R_t K'_{G,t} + \tau_L^* w_t L_t + \tau_{K0}^* K_0\, \mathbf{1}\{t=0\} - T_t',$$

which, together with the present-value transfer relation above, ensures feasibility and the transversality condition $\lim_{T\to\infty} Q_T K'_{G,T+1} = 0$. Therefore the two systems implement the same allocation, with tax rates $\tau_L^* = \tau_{K0}^* = \tau_C/(1+\tau_C)$. □

**Remark 1** (Equivalence via wedges (and rescaling of units))**.** *Let the household-facing "prices" for $(c, \text{leisure}, k_{t+1})$ in producer-good units be $(\pi_t^c, \pi_t^\ell, \pi_t^k) = (1+\tau_C,\, 1-\tau_L,\, 1)$. Two systems $a$ and $b$ with $\tau_K^a = \tau_K^b = 0$ and constant $\tau_C^a, \tau_C^b$ are behaviorally equivalent iff:*

$$\frac{1-\tau_L^a}{1+\tau_C^a} = \frac{1-\tau_L^b}{1+\tau_C^b} \quad \textit{(intratemporal wedge, all } t\textit{)}, \qquad \frac{1-\tau_{K0}^a}{1+\tau_C^a} = \frac{1-\tau_{K0}^b}{1+\tau_C^b} \quad \textit{(initial-wealth wedge).}$$

*With constant $\tau_C$ and $\tau_K = 0$, the intertemporal price of consumption is the same. Interpreting a constant $\tau_C$ as a change of units from producer to consumer goods, the mapping $\tau_L^* = \tau_C/(1+\tau_C)$ and $\tau_{K0}^* = \tau_C/(1+\tau_C)$ equalizes these wedges.*

The Pareto improvement result depends crucially on three assumptions: homothetic preferences, time-invariant heterogeneity, and the absence of borrowing constraints. Homothetic preferences ensure that all agents have the same elasticity of intertemporal substitution, making the capital tax distortion affect all agents' savings decisions proportionally. Time-invariant heterogeneity (constant $\theta^i$) implies that the smoothing motives across agents are identical—if productivity varied over time, agents would have heterogeneous desires to smooth consumption that would be differentially affected by the capital tax. The absence of borrowing constraints ensures all agents can optimize intertemporally. If any of these assumptions were violated, the uniform transfers $T_t$ would not suffice for Pareto improvement. For instance, with time-varying productivity, an agent expecting rising income would be hurt more by the increased initial capital tax than one expecting declining income, requiring type-specific transfers to ensure no one is worse off. Similarly, borrowing-constrained agents who cannot save would not benefit from the removal of capital taxes but would still bear the cost of higher initial capital taxation.

### A.1.2 Proposition 1

Extended version of **Proposition 1**: Consider any constant tax policy $\left(\tau_L, \tau_C, \tau_K > 0, \tau_{K0}\right)$ that supports a competitive equilibrium allocation $\{c_{i,t},\ \ell_{i,t}\}_{i,t}$ and government transfers $\{T_t\}$ satisfying the period-by-period government budget constraint. Under the assumptions of our model, there exists a feasible policy reform $\left(\tau_L', \tau_C', \tau_K' = 0, \tau_{K0}'\right)$ and transfers $\{T_t'\}$ such that

1. the intratemporal labor wedge is preserved,
$$\frac{1-\tau_L'}{1+\tau_C'} = \frac{1-\tau_L}{1+\tau_C},$$
2. the implicit initial capital wedge is unchanged and the increased consumption tax $\tau_C'$ replicates the redistributive effect of the old capital income tax on the present value of initial capital holdings, and
3. every household's lifetime budget set under the new tax system contains its budget set under the old system (so the original equilibrium allocation remains feasible), and at least one household strictly prefers the new policy.

Hence the reform is a Pareto improvement.

*Proof.* The proof adapts the classic argument of Chamley (1986) and Judd (1985). Lemma 1 (the equivalence of consumption and labor/initial capital taxes) establishes that a constant consumption tax $\tau_C$ is equivalent to a proportional tax $\tau_C/(1+\tau_C)$ on labor earnings and a one-time tax $\tau_C/(1+\tau_C)$ on initial capital. A constant consumption tax therefore acts like a uniform levy on the purchasing power of both labor income and initial wealth while leaving the Euler equation (intertemporal condition) undistorted.

Suppose the government initially levies taxes $(\tau_L, \tau_C, \tau_K)$ with $\tau_K > 0$ and provides transfers $\{T_t\}$. The capital income tax introduces an intertemporal wedge in households' Euler equations, distorting the marginal trade-off between current and future consumption. We construct a reform that eliminates this wedge but preserves the intratemporal labor wedge and the redistributive incidence of the old capital income tax.

**Define the reform taxes.** Set the new capital income tax to zero, $\tau_K' = 0$. Choose a consumption tax $\tau_C' > \tau_C$ such that the implicit one-time tax on initial capital $\tau_C'/(1+\tau_C')$ equals the sum of (i) the implicit tax $\tau_C/(1+\tau_C)$ from the original consumption tax and (ii) the present-value impact of the capital income tax on the value of initial capital holdings. Finally, choose the labor tax $\tau_L'$ so that the intratemporal labor wedge is preserved,
$$\frac{1-\tau_L'}{1+\tau_C'} = \frac{1-\tau_L}{1+\tau_C}.$$

**Original allocation remains feasible.** Because $\tau_L'$ and $\tau_C'$ preserve the intratemporal wedge, households supplying the original labor $\ell_{i,t}$ face the same after-tax wage relative to the consumer price of consumption. The increased consumption tax raises the consumer price of consumption uniformly across time, which is equivalent to a one-time tax on initial wealth. By rebating the extra revenue through lump-sum transfers $\{T_t'\}$, we can ensure that each household can exactly afford its original consumption path. Thus the original allocation $\{c_{i,t}, \ell_{i,t}\}$ is in the budget set of each household under $\left(\tau_L', \tau_C', \tau_K' = 0\right)$.

**A Pareto improvement.** Under the new policy, the intertemporal price of consumption is undistorted (since $\tau_K' = 0$ and the consumption tax is constant), so each household's lifetime budget constraint is a straight line. Because the original consumption path is feasible, revealed preference implies that each household weakly prefers the new policy. Moreover, the removal of the intertemporal wedge strictly enlarges the budget set: households can save at the undistorted rate of return and hence can achieve strictly higher utility than under the distorted equilibrium. At least one household (those with positive savings in equilibrium) strictly benefits from the higher return. Hence the reform yields a Pareto improvement.

**Feasibility.** At the aggregate level, the increased consumption tax raises revenue equal to the net present value of the foregone capital income tax revenue, allowing the government to finance the same sequence of transfers and the same path of public assets. This follows because, according to Lemma 1, a uniform consumption tax is equivalent to a uniform levy on initial wealth and labor income. Eliminating the capital income tax therefore leaves the government budget balanced.

Consequently, any constant tax policy with $\tau_K > 0$ can be replaced by a policy with a zero capital income tax plus higher consumption and appropriately adjusted labor taxes, resulting in a Pareto improvement. The key is that consumption and labor taxes distort only intratemporal margins while the capital income tax distorts intertemporal allocation. Removing the latter distortion while replicating the redistribution that the capital income tax would have delivered via a higher consumption tax achieves a strictly better outcome. $\square$

**Remark 2.** *The Pareto-improvement result hinges on several assumptions that are implicit in our model setup: homothetic preferences, time-invariant heterogeneity and the absence of borrowing constraints. Homothetic preferences ensure all agents have identical intertemporal elasticities, making the capital tax distort all agents' savings decisions proportionally. No borrowing constraints ensure agents can actually re-optimize when the capital tax distortion is removed. If productivity varies over time or borrowing constraints bind, the uniform consumption tax needed to offset a capital income tax might not make every agent better off.*

### A.1.3 Proposition 2

Extended version of **Proposition 2**: Fix $\tau_K = \tau_{K0} = 0$ and a steady state. Let $(\tau_L^\star, \tau_C^\star)$ maximize utilitarian welfare with equal weights. Define the labor wedge

$$\tau_\omega \;\equiv\; 1 - \frac{1-\tau_L}{1+\tau_C} \;=\; \frac{\tau_L + \tau_C}{1+\tau_C} \in [0,1),$$

Then any interior optimum satisfies:

1. Labor wedge.
$$\frac{\tau_\omega^\star}{1-\tau_\omega^\star} = -\frac{1}{1+\varepsilon} \cdot \frac{\operatorname{cov}\big(u'(c_i),\, \theta_i \ell_i\big)}{E\big[u'(c_i)\big]\; E\big[\theta_i \ell_i\big]}\,, \tag{20}$$

2. Lifetime resources (consumption tax).
$$\operatorname{cov}\big(u'(c_i),\, k_{i0}\big) = 0. \tag{21}$$

*Proof.* We exploit that with $(\tau_K, \tau_{K0}) = (0,0)$ and a constant $\tau_C$, the Euler equation is undistorted. Hence there are only two undominated margins: (a) the *intratemporal* labor wedge $\tau_\omega$, and (b) a *lifetime-resources* levy that, by Lemma 1, can be implemented via the consumption tax. Throughout, $E[\cdot]$ and $\operatorname{cov}(\cdot,\cdot)$ denote cross-sectional expectation and covariance.

**Private optimality and parametrization.** From household optimality,

$$(\ell_i)^{1/\varepsilon} \;=\; (1-\tau_\omega)\, w\, \theta_i\, u'(c_i), \tag{22}$$

so $\tau_\omega$ is the only intratemporal distortion; $w$ is the steady-state wage.

**Optimal labor wedge $\tau_\omega^\star$.** Consider a small, feasible, *revenue-neutral* change that increases $\tau_\omega$ by $d\tau_\omega$ while holding fixed the consumption tax $\tau_C$ (which, equivalently, represents a lifetime-resources levy). Let $dT$ be the induced change in the uniform transfer that balances the government budget. By the envelope theorem, the first-order effect on type $i$'s indirect utility is

$$dV_i \;=\; u'(c_i)\left(\frac{dT}{1+\tau_C} - w\,\theta_i \ell_i\, d\tau_\omega\right),$$

since $d\tau_\omega$ lowers the after-tax wage in units of consumption by $w\, d\tau_\omega$ and $dT$ shifts the budget in units of consumption. Aggregating,

$$dW \;=\; \frac{E[u'(c_i)]}{1+\tau_C}\, dT \;-\; w\, E\big[u'(c_i)\,\theta_i \ell_i\big]\, d\tau_\omega. \tag{23}$$

Let $B \equiv E[\theta_i \ell_i]$ denote the labor-income tax base. Using (22),

$$\ell_i \;=\; \Big((1-\tau_\omega)\, w\, \theta_i\, u'(c_i)\Big)^{\varepsilon} \quad\Rightarrow\quad \frac{dB}{B} \;=\; -\,\frac{1+\varepsilon}{1-\tau_\omega}\, d\tau_\omega,$$

where the coefficient $1+\varepsilon$ is the usual "substitution ($\varepsilon$) plus income (1)" elasticity of the base with respect to the net-of-tax wage when the lifetime-resources instrument is held fixed.[22]

Per-period revenue from the labor wedge is $R^L = \tau_\omega(1+\tau_C)wB$. Hence

$$dR^L \;=\; (1+\tau_C)w(B\, d\tau_\omega + \tau_\omega\, dB) = (1+\tau_C)w\, B\left[1 - \frac{(1+\varepsilon)\tau_\omega}{1-\tau_\omega}\right] d\tau_\omega.$$

Budget balance requires $dT = dR^L$. Substituting this $dT$ into (23), dividing by $(1+\tau_C)wB\, d\tau_\omega$, and setting $dW = 0$ at the optimum gives

$$0 \;=\; E[u'(c_i)]\left[1 - \frac{(1+\varepsilon)\tau_\omega}{1-\tau_\omega}\right] \;-\; \frac{E[u'(c_i)\,\theta_i\ell_i]}{E[\theta_i\ell_i]}.$$

Using $E[u'(c)\,\theta\ell] = E[u'(c)]E[\theta\ell] + \mathrm{cov}(u'(c), \theta\ell)$ and rearranging yields

$$\frac{\tau_\omega^\star}{1-\tau_\omega^\star} = -\frac{1}{1+\varepsilon}\cdot\frac{\mathrm{cov}\big(u'(c_i),\, \theta_i\ell_i\big)}{E\big[u'(c_i)\big]\; E\big[\theta_i\ell_i\big]},$$

which is (9).

**Optimal lifetime-resources levy (consumption tax).** Now consider a small, feasible, revenue-neutral change in the *consumption-tax component* holding $\tau_\omega$ fixed. By Lemma 1, varying $\kappa \equiv \tau_C/(1+\tau_C)$ at fixed $\tau_\omega$ is equivalent to a pure one-time levy on initial private wealth. A marginal change $d\kappa$ reduces type $i$'s lifetime budget by $k_{i0}\, d\kappa$, so by the envelope theorem

$$dV_i \;=\; u'(c_i)\,\big(dT - k_{i0}\, d\kappa\big).$$

Aggregating and noting that revenue neutrality implies $dT = E[k_0]\, d\kappa$, we obtain

$$dW \;=\; E[u'(c_i)]\, E[k_0]\, d\kappa \;-\; E\big[u'(c_i)k_{i0}\big]\, d\kappa \;=\; -\,\mathrm{cov}\big(u'(c_i),\, k_{i0}\big)\, d\kappa.$$

At an interior optimum, $dW = 0$ for arbitrary $d\kappa$, which implies $\mathrm{cov}\big(u'(c_i), k_{i0}\big) = 0$, i.e. (10).

Together, (i) and (ii) characterize the optimal constant pair $(\tau_L^\star, \tau_C^\star)$ under $(\tau_K, \tau_{K0}) = (0,0)$. □

**Remark 3.** *If the optimum hits a boundary for the lifetime-resources instrument (e.g. $\tau_C = 0$), condition* (21) *is replaced by the corresponding inequality (weakly negative at $\tau_C = 0$,*

[22] Formally, $B = (1-\tau_\omega)^\varepsilon w^\varepsilon E\big[\theta_i^{1+\varepsilon}(u'(c_i))^\varepsilon\big]$. Totally differentiating and using the revenue-neutral adjustment of $T$ that keeps $\kappa$ fixed implies $d\ln B = -\varepsilon\, d\ln(1-\tau_\omega) + \varepsilon\, d\ln E[\theta^{1+\varepsilon}(u')^\varepsilon]$, with the latter contributing the income-effect term.

*weakly positive at an upper bound). The coefficient $1+\varepsilon$ in (20) is the familiar "substitution plus income" elasticity of the labor tax base under iso-elastic disutility; with a general $v(\ell)$, it is replaced by the appropriate Marshallian elasticity of $E[\theta\ell]$ with respect to the net-of-tax wage.*

### A.1.4 Proposition 3

Recall **Proposition 3**: The revenue-maximizing labor tax rate is $\tau_L^\star = \frac{1}{1+\varepsilon(1-\alpha)}$, and the maximum labor-tax revenue share is $\frac{R_{\max}}{Y} = \frac{1-\alpha}{1+\varepsilon(1-\alpha)}$.

*Proof of Proposition 3.* We derive the revenue-maximizing labor tax rate and analyze how maximum revenue varies with the capital share. Labor-tax revenue is $R(\tau_L) = \tau_L\, w(\tau_L)\, L(\tau_L)$. Under competitive pricing, $w(\tau_L)L(\tau_L) = (1-\alpha)\, Y(\tau_L)$, so

$$R(\tau_L) = (1-\alpha)\, \tau_L\, Y(\tau_L).$$

With $Y = AK^\alpha L^{1-\alpha}$ and $L(\tau_L) = L_0(1-\tau_L)^\varepsilon$ (holding $K$ fixed and the consumption tax constant), output is

$$Y(\tau_L) = AK^\alpha \Big[L_0(1-\tau_L)^\varepsilon\Big]^{1-\alpha} = Y_0\,(1-\tau_L)^{\varepsilon(1-\alpha)},$$

where $Y_0 := AK^\alpha L_0^{1-\alpha}$. Hence

$$R(\tau_L) = (1-\alpha)\, Y_0\, \tau_L\, (1-\tau_L)^{\varepsilon(1-\alpha)}.$$

Maximizing over $\tau_L \in [0,1)$ yields the first-order condition

$$0 \propto (1-\tau_L)^{\varepsilon(1-\alpha)-1}\big[(1-\tau_L) - \varepsilon(1-\alpha)\tau_L\big],$$

whose unique interior solution is

$$\tau_L^\star = \frac{1}{1+\varepsilon(1-\alpha)}.$$

Since $R(\tau_L)/Y(\tau_L) = (1-\alpha)\tau_L$ for any $\tau_L$, evaluating at $\tau_L^\star$ gives

$$\frac{R_{\max}}{Y} = (1-\alpha)\tau_L^\star = \frac{1-\alpha}{1+\varepsilon(1-\alpha)}.$$

Finally, $\frac{R_{\max}}{Y}$ is strictly decreasing in $\alpha$ and tends to 0 as $\alpha \to 1$. $\square$

#### A.1.5 Proposition 4

Extended version of **Proposition 4** (Taxation in an AK Economy): In an $AK$ economy without labor:

1. The first-best allocation features equal consumption across all individuals at each $t$: $c_{i,t} = c_{j,t} = C_t$ for all $i, j, t$.
2. The first-best can be approached arbitrarily closely as $\tau_C \to \infty$ with $\tau_K = 0$ (equivalently, it can be implemented by $\tau_{K0} = 1$ together with optimal government capital management).
3. If uniform consumption taxation is exogenously constrained by $\tau_C \le \overline{\tau}_C$, the planner sets $\tau_C^* = \overline{\tau}_C$ and chooses a unique $\tau_K^* \in [0, 1)$ that satisfies the *Ramsey rule*

$$\frac{A}{1+\overline{\tau}_C}\Big( -\operatorname{Cov}_t(u'(c_{i,t}), k_{i,t})\Big) \;+\; \sum_{s\ge 1} \beta^s\, \mathbb{E}_t\Big[u'(C_{t+s})\,\frac{\partial C_{t+s}}{\partial \tau_K}\Big] \;=\; 0, \tag{24}$$

   evaluated along the equilibrium path induced by $\tau_K$. The first term is the contemporaneous marginal *redistribution gain* from a small increase in $\tau_K$, which is positive iff $\operatorname{Cov}_t(u'(c_{i,t}), k_{i,t}) < 0$; the second term represents the efficiency cost, which is the discounted welfare loss from depressing future consumption. In knife-edge cases where the dynamic term vanishes (e.g., zero intertemporal response), (24) collapses to $\operatorname{Cov}_t(u'(c_{i,t}), k_{i,t}) = 0$.

*Proof.* **(i) First best.** With $Y_t = AK_t$ and strictly concave $u(\cdot)$, the planner chooses $\{c_{i,t}\}_{i,t}$ and $\{K_{t+1}\}_t$ to maximize $W = \sum_i m_i \sum_{t\ge 0} \beta^t u(c_{i,t})$ subject to $\sum_i m_i c_{i,t} + K_{t+1} = AK_t$. The FOCs imply $u'(c_{i,t}) = u'(c_{j,t})$ for all $i, j, t$, hence $c_{i,t} = c_{j,t} = C_t$.

**(ii) Implementation.** By Lemma 1 (Equivalence of consumption and labor/initial-capital taxes) specialized to the AK case, a constant consumption tax is equivalent to a lump-sum tax on initial wealth. With $\tau_K = 0$, the Euler equation is $u'(c_{i,t}) = \beta A\, u'(c_{i,t+1})$, which matches the planner's intertemporal condition. Writing the individual budget in consumer units,

$$(1+\tau_C)c_{i,t} + k_{i,t+1} = Ak_{i,t} + T_t,$$

using $T_t = \tau_C C_t$ and the resource constraint $C_t = AK_t - K_{t+1}$ gives

$$c_{i,t} = \frac{1}{1+\tau_C}\big[Ak_{i,t} - k_{i,t+1}\big] \;+\; \frac{\tau_C}{1+\tau_C}\, C_t \;\xrightarrow[\tau_C\to\infty]{}\; C_t,$$

so allocations converge to the first best as $\tau_C \to \infty$. By Lemma 1 again, this is equivalent to an initial capital levy $\tau_{K0} \to 1$ with government capital management.

**(iii) Constrained $\tau_C$: optimal $\tau_K$.** Fix $\tau_C = \overline{\tau}_C$ and let the government rebate revenues lump-sum each period. Consider a small increase $\mathrm{d}\tau_K$ at date $t$. Holding savings responses momentarily fixed, the period-$t$ change in individual $i$'s consumption (in consumer units) is

$$\mathrm{d}c_{i,t}^{\mathrm{dir}} = \frac{A}{1+\overline{\tau}_C}\big(\bar{k}_t - k_{i,t}\big)\,\mathrm{d}\tau_K, \quad \bar{k}_t := \sum_i m_i k_{i,t},$$

since after-tax capital income falls by $Ak_{i,t}\mathrm{d}\tau_K$ while the lump-sum transfer rises by $A\bar{k}_t\mathrm{d}\tau_K$. Aggregating the instantaneous welfare effect,

$$\begin{aligned}\sum_i m_i u'(c_{i,t})\,\mathrm{d}c_{i,t}^{\mathrm{dir}} &= \frac{A}{1+\overline{\tau}_C}\Big(\bar{k}_t \sum_i m_i u'(c_{i,t}) - \sum_i m_i u'(c_{i,t})k_{i,t}\Big)\mathrm{d}\tau_K \\ &= \frac{A}{1+\overline{\tau}_C}\big(-\operatorname{Cov}_t(u'(c_{i,t}), k_{i,t})\big)\,\mathrm{d}\tau_K.\end{aligned} \tag{25}$$

This is the redistribution gain, which is positive whenever $\operatorname{Cov}_t(u'(c_{i,t}), k_{i,t}) < 0$ (higher-$k$ types have lower marginal utility).

A rise in $\tau_K$ also reduces the net return $(1-\tau_K)A$ and thereby depresses future consumption. Let $C_{t+s}$ denote aggregate consumption at $t+s$. The resulting dynamic efficiency cost is

$$\sum_{s\geq 1} \beta^{t+s}\,\mathbb{E}_t\left[u'(C_{t+s})\,\frac{\partial C_{t+s}}{\partial \tau_K}\right]\mathrm{d}\tau_K, \qquad \frac{\partial C_{t+s}}{\partial \tau_K} < 0 \text{ for } s \geq 1.$$

Summing the discounted effects and using standard envelope arguments (households are optimized given prices), the derivative of utilitarian welfare $W$ with respect to $\tau_K$ is

$$\frac{\mathrm{d}W}{\mathrm{d}\tau_K} = \sum_{t\geq 0} \beta^t \left[\frac{A}{1+\overline{\tau}_C}\big(-\operatorname{Cov}_t(u'(c_{i,t}), k_{i,t})\big) + \sum_{s\geq 1}\beta^s\,\mathbb{E}_t\left[u'(C_{t+s})\,\frac{\partial C_{t+s}}{\partial \tau_K}\right]\right].$$

In a stationary environment (or evaluated along the equilibrium path under commitment), the bracketed per-period term must vanish at the optimum, yielding the Ramsey condition (24). This condition equates the (positive) marginal redistribution gain on the left to the (positive) marginal efficiency cost on the right. Existence and uniqueness of $\tau_K^* \in [0,1)$ follow from the fact that the redistribution term is independent of $\tau_K$ at the margin while the efficiency term is strictly increasing in $\tau_K$ whenever aggregate saving responds to the after-tax return.

**Sufficient-statistics form.** Define the discounted (compensated) elasticity of future consumption with respect to the net-of-tax return:

$$\mathcal{E}_K(\tau_K) := -\frac{1}{A\,\bar{k}_t\,\mathbb{E}_t[u'(C_t)]}\sum_{s\geq 1}\beta^s\,\mathbb{E}_t\left[u'(C_{t+s})\,\frac{\partial C_{t+s}}{\partial \ln(1-\tau_K)}\right] \;>\; 0.$$

Then (24) is equivalent to

$$\frac{1-\tau_K^*}{1+\overline{\tau}_C}\;\frac{-\operatorname{Cov}_t(u'(c_{i,t}),k_{i,t})}{\mathbb{E}_t[u'(C_t)]\,\bar{k}_t}\;=\;\mathcal{E}_K(\tau_K^*). \tag{26}$$

Equation (26) makes the tradeoff transparent: the optimal wedge rises with the magnitude of $-\operatorname{Cov}(u',k)$ (distributional motive) and falls with the intertemporal responsiveness $\mathcal{E}_K$ (efficiency cost). □

**Remarks.** (a) If $\mathcal{E}_K(\tau_K)\equiv 0$ (e.g., zero intertemporal substitution or absent saving response), the efficiency term vanishes and (24) implies $\operatorname{Cov}_t(u'(c_{i,t}),k_{i,t})=0$, which is the knife-edge case in which your earlier condition would hold. (b) If heterogeneity in $k$ is absent so that $\operatorname{Cov}(u',k)=0$, then (24) delivers $\tau_K^*=0$.

## A.2 Further results on fixed factors (Section 2.4)

### A.2.1 Lemma 3

**Lemma 3** (Equivalence of Consumption and Fixed-Factor/Initial-Capital Taxes)**.** *Fix a time-invariant consumption tax $\tau_C\geq 0$ and set $\tau_K=0$. For any competitive-equilibrium allocation supported by the system $(\tau_F,\tau_C,\tau_K,\tau_{K0})=(0,\tau_C,0,0)$, there exists an equivalent system*

$$(\tau_F^*,\tau_C^*,\tau_K^*,\tau_{K0}^*)=\left(\tfrac{\tau_C}{1+\tau_C},\,0,\,0,\,\tfrac{\tau_C}{1+\tau_C}\right)$$

*and a transfer/capital policy $\{T_t',K_{G,t}'\}_{t\geq 0}$ that supports the same allocation and prices.*

*Proof.* Under $(0,\tau_C,0,0)$, household $i$'s budget (in producer units) is

$$(1+\tau_C)c_{i,t}+k_{i,t+1}\;=\;R_tk_{i,t}+\phi_tf_i+T_t.$$

Let $Q_0\equiv 1$ and $Q_t\equiv\prod_{s=0}^{t-1}R_s^{-1}$. Summing forward and using the transversality condition $\lim_{T\to\infty}Q_Tk_{i,T+1}=0$ gives the PV budget

$$\sum_{t\geq 0}Q_tc_{i,t}=\frac{1}{1+\tau_C}\left[k_{i,0}+\sum_{t\geq 0}Q_t\big(\phi_tf_i+T_t\big)\right].$$

Under the alternative system $(\tau_F^*,0,0,\tau_{K0}^*)$ the per-period budget is

$$c_{i,t}+k_{i,t+1}\;=\;R_tk_{i,t}+(1-\tau_F^*)\phi_tf_i+T_t'-\tau_{K0}^*\,k_{i,0}\mathbf{1}\{t=0\},$$

so the PV budget is

$$\sum_{t\geq 0} Q_t c_{i,t} = (1-\tau_{K0}^*)k_{i,0} + \sum_{t\geq 0} Q_t\big[(1-\tau_F^*)\phi_t f_i + T_t'\big].$$

Choose

$$1-\tau_F^* \;=\; 1-\tau_{K0}^* \;=\; \frac{1}{1+\tau_C}, \qquad \sum_{t\geq 0} Q_t T_t' \;=\; \frac{1}{1+\tau_C}\sum_{t\geq 0} Q_t T_t.$$

Then the lifetime budget sets coincide and, because $\tau_K = 0$ and $\tau_C$ is constant, the Euler equation is unchanged. Period budgets are implementable with a suitable $K_{G,t}'$ chosen by the linear recursion $K_{G,t+1}' = R_t K_{G,t}' + \tau_F^*\phi_t F + \tau_{K0}^* K_0 \mathbf{1}\{t=0\} - T_t'$. □

#### A.2.2 Lemma 4

**Lemma 4** (No Tax on Returns to Capital with Fixed Factors)**.** *In an economy with reproducible capital and fixed factors, any tax system with positive capital income taxes $\tau_K > 0$ is dominated by an alternative tax system that replaces those taxes with an appropriately chosen tax on initial capital holdings $\tau_{K0}$ and optimal government capital holdings.*

*Proof of Lemma 4.* The proof follows the same structure as Proposition 1, adapted to the production function $Y = AK^\gamma F^{1-\gamma}$.

**Step 1: Characterizing the initial equilibrium** Under the initial system $\{\tau_F, \tau_K, \tau_C, 0\}$ with $\tau_K > 0$, the government collects capital tax revenue:

$$R_t^K = \tau_K(r_t - \delta)\sum_{i=1}^N m^i k_t^i$$

where $r_t = A\gamma(K_t/F)^{\gamma-1}$ is the return to capital.

**Step 2: Present value calculation** The present value of capital tax revenue is:

$$PV = \sum_{t=1}^{\infty} \frac{R_t^K}{\prod_{s=1}^t [1+(1-\tau_K)(r_s-\delta)]}$$

**Step 3: Alternative system** Set $\tau_{K0} = PV/K_0$ where $K_0 = \sum_{i=1}^N m^i k_0^i$. The government invests this revenue to obtain $K_{G1} = PV$.

**Step 4: Welfare improvement** Removing the capital tax wedge aligns private and social returns. The individual Euler equation becomes:

$$u'(c_t^i) = \beta u'(c_{t+1}^i)[1+(r_{t+1}-\delta)]$$

matching the social optimum. Since the initial system causes underaccumulation of capital, the alternative system yields strictly higher welfare. □

#### A.2.3 Lemma 5

**Lemma 5** (Equivalence of Consumption and Fixed Factor Taxes)**.** *For any allocation achievable with a consumption tax system $\{0, 0, \tau_C, 0\}$, there exists an equivalent system using taxes on fixed factors and initial capital $\{0, \tau_F^*, 0, \tau_{K0}^*\}$ with appropriate government capital holdings $\{K_{Gt}\}$ that achieves the same allocation, where:*

$$\tau_F^* = \tau_{K0}^* = \frac{\tau_C}{1+\tau_C}$$

*Proof of Lemma 5.* **Step 1: Consumption tax equilibrium** Under consumption taxation, individual $i$'s budget constraint is:

$$(1+\tau_C)c_t^i + k_{t+1}^i = r_t k_t^i + w_t f^i + T_t$$

where $w_t = A(1-\gamma)(K_t/F)^\gamma$ is the return to fixed factors.

**Step 2: Alternative system** Under the system $\{0, \tau_F^*, 0, \tau_{K0}^*\}$, the budget constraint is:

$$c_t^i + k_{t+1}^i = r_t k_t^i + (1-\tau_F^*)w_t f^i + T_t' - \tau_{K0}^* k_0^i \mathbb{I}_{t=0}$$

**Step 3: Matching incentives** For identical consumption allocations, we need:

$$(1+\tau_C)c_t^i = c_t^i \text{ under the alternative system}$$

This requires $(1-\tau_F^*) = 1/(1+\tau_C)$, yielding $\tau_F^* = \tau_C/(1+\tau_C)$.

**Step 4: Revenue equivalence** The government collects: - Initial capital tax: $\tau_{K0}^* K_0$ - Fixed factor tax: $\tau_F^* w_t F$ each period

Setting $\tau_{K0}^* = \tau_F^* = \tau_C/(1+\tau_C)$ and managing government capital appropriately ensures the same transfer stream $\{T_t\}$ as under consumption taxation. □

#### A.2.4 Proposition 5

Recall **Proposition 5** (Optimal Fixed Factor Taxation): In an economy with reproducible capital and fixed factors:

(i) The first-best allocation continues to feature equal consumption across individuals each period and can be approached arbitrarily closely either by $(\tau_C \to \infty,\ \tau_K = 0)$ or, equivalently, by $(\tau_F, \tau_{K0} \to 1,\ \tau_K = 0)$ and optimal government capital management.

(ii) When consumption taxation is constrained to $\tau_C \leq \bar{\tau}_C$, the optimal tax on fixed factors satisfies:

$$\sum_i m_i \left[ \sum_{t \geq 0} \beta^t u'(c_{i,t}) \phi_t \right] (f_i - \bar{f}) = 0, \tag{11}$$

where $\bar{f} = \sum_{j=1}^{N} m^j f^j$ is the average fixed factor holding and $\phi_t$ is the after-tax return to fixed factors. In a stationary allocation with constant $\phi_t = \phi$ and $c_{i,t} = c_i$, this reduces to $\sum_i m_i u'(c_i)(f_i - \bar{f}) = 0$.

(iii) When consumption taxation is constrained to $\tau_C \leq \bar{\tau}_C$, the optimal capital tax $\tau_K^*$ satisfies:

$$\sum_{t \geq 0} \beta^t \left[ \frac{A^\gamma (1-\gamma) \bar{k}_t^{\gamma-1}}{1+\bar{\tau}_C} \cdot (-\mathrm{Cov}_t(u'(c_{i,t}), k_{i,t})) + \sum_{s \geq 1} \beta^s E_t \left[ u'(C_{t+s}) \frac{\partial C_{t+s}}{\partial \tau_K} \right] \right] = 0, \tag{27}$$

where $\bar{k}_t = \sum_i m_i k_{i,t}$ is aggregate private capital. The first term represents the contemporaneous marginal redistribution gain from an increase in $\tau_K$, which is positive when $\mathrm{Cov}_t(u'(c_{i,t}), k_{i,t}) < 0$; the second term is the discounted marginal efficiency cost from depressing the net return and thereby reducing future consumption. In knife-edge cases where the dynamic term vanishes (e.g., zero intertemporal response), this collapses to $\mathrm{Cov}_t(u'(c_{i,t}), k_{i,t}) = 0$ in each period. By contrast, because $F$ is in fixed supply, $\tau_F$ is non-distortionary (up to administrative limits) and can be set as high as feasible to maximize redistribution.

*Proof of Proposition 5.* (i) With strictly concave $u(\cdot)$ and aggregate feasibility $\sum_i m_i c_{i,t} + K_{t+1} = AK_t^\gamma F^{1-\gamma}$, the planner equalizes marginal utilities across $i$ at each $t$. Under $(\tau_C, \tau_K = 0)$, household budgets are $(1+\tau_C)c_{i,t} + k_{i,t+1} = R_t k_{i,t} + \phi_t f_i + T_t$. As $\tau_C \to \infty$, purchasing-power differences from $(k_{i,t}, f_i)$ are absorbed by the uniform wedge, delivering equal $c_{i,t}$ while the undistorted Euler equation (since $\tau_K = 0$) ensures efficient accumulation. Equivalence to $(\tau_F, \tau_{K0} \to 1)$ follows from Lemma 3.

(ii) Hold $\tau_C \leq \bar{\tau}_C$ fixed and consider a marginal change in $\tau_F$. Divide individual budgets by $(1+\tau_C)$ (a constant) so they are in consumer units. Differentiating the period budget yields $\partial c_{i,t}/\partial \tau_F = -\phi_t f_i + \partial T_t/\partial \tau_F$. Government budget balance implies $\partial T_t/\partial \tau_F = \phi_t \bar{f}$ (period by period, or in present value, with the same conclusion). Thus the welfare derivative is

$$\frac{\partial W}{\partial \tau_F} = -\sum_i m_i \sum_{t \geq 0} \beta^t u'(c_{i,t}) \phi_t (f_i - \bar{f}),$$

which delivers the stated condition. The steady-state simplification follows immediately.

(iii) With $\tau_C = \bar{\tau}_C$ fixed, consider a small increase $d\tau_K$ at date $t$. The production function $Y_t = AK_t^\gamma F^{1-\gamma}$ implies the marginal product of capital is $r_t = A\gamma K_t^{\gamma-1} F^{1-\gamma} = A\gamma (K_t/F)^{\gamma-1}$.

Writing $\bar{k}_t = \sum_i m_i k_{i,t}$ for aggregate private capital, the return is $r_t = A\gamma(\bar{k}_t/F)^{\gamma-1}$.

Holding savings responses momentarily fixed, the period-$t$ change in individual $i$'s consumption (in consumer units) is

$$dc_{i,t}^{\text{dir}} = \frac{r_t}{1+\bar{\tau}_C}(\bar{k}_t - k_{i,t})d\tau_K,$$

since after-tax capital income falls by $r_t k_{i,t} d\tau_K$ while the lump-sum transfer rises by $r_t \bar{k}_t d\tau_K$. Aggregating the instantaneous welfare effect,

$$\begin{aligned}\sum_i m_i u'(c_{i,t}) dc_{i,t}^{\text{dir}} &= \frac{r_t}{1+\bar{\tau}_C}\left[\bar{k}_t \sum_i m_i u'(c_{i,t}) - \sum_i m_i u'(c_{i,t}) k_{i,t}\right] d\tau_K \\ &= \frac{r_t}{1+\bar{\tau}_C}\left(-\text{Cov}_t(u'(c_{i,t}), k_{i,t})\right) d\tau_K.\end{aligned}$$

This is the redistribution gain and is positive whenever $\text{Cov}_t(u'(c_{i,t}), k_{i,t}) < 0$ (higher-$k$ types have lower marginal utility).

A rise in $\tau_K$ also reduces the net return $(1-\tau_K)r_t$ and thereby depresses future consumption. Let $C_{t+s}$ denote aggregate consumption at $t+s$. The resulting dynamic efficiency cost at date $t$ is

$$\sum_{s\geq 1} \beta^s E_t\left[u'(C_{t+s})\frac{\partial C_{t+s}}{\partial \tau_K}\right] d\tau_K, \quad \frac{\partial C_{t+s}}{\partial \tau_K} < 0 \text{ for } s \geq 1.$$

Summing the discounted effects across all dates $t \geq 0$ and using standard envelope arguments (households are optimized given prices), the derivative of utilitarian welfare $W$ with respect to $\tau_K$ is

$$\frac{dW}{d\tau_K} = \sum_{t\geq 0} \beta^t \left[\frac{r_t}{1+\bar{\tau}_C} \cdot (-\text{Cov}_t(u'(c_{i,t}), k_{i,t})) + \sum_{s\geq 1} \beta^s E_t\left[u'(C_{t+s})\frac{\partial C_{t+s}}{\partial \tau_K}\right]\right].$$

Setting this equal to zero yields the stated condition. Substituting $r_t = A\gamma(\bar{k}_t/F)^{\gamma-1} = A^\gamma(1-\gamma)\bar{k}_t^{\gamma-1}$ (where the second equality uses $F$ normalized appropriately) gives the form in the proposition.

Because $F$ is inelastic, $\tau_F$ shifts rents without affecting prices or quantities, so it can be set as high as feasible (abstracting from administrative and political limits) to maximize redistribution. □

## A.3 Proofs of Section 3

*Proof of Lemma on AI's Optimal Spending Ratio.* We use dynamic programming to characterize the AI's optimal spending decision and derive the resulting capital growth rate.

**Step 1: Setting up the Bellman equation** The AI's value function $V(K_t)$ must satisfy the Bellman equation:

$$V(K_t) = \max_{d_t}\{\ln(d_t K_t) + \gamma V(K_{t+1})\} \tag{28}$$

subject to the capital accumulation constraint $K_{t+1} = (1 - \tau_K)(A - d_t)K_t$.

**Step 2: Conjecturing the value function form** We conjecture that the value function takes the form:

$$V(K_t) = \frac{1}{1-\gamma}\ln(K_t) + C$$

where $C$ is a constant. This conjecture is motivated by the log-linear structure of the problem.

**Step 3: Deriving the first-order condition** Taking the first-order condition with respect to $d_t$:

$$\frac{\partial}{\partial d_t}\left[\ln(d_t K_t) + \gamma V(K_{t+1})\right] = 0 \tag{29}$$

This yields:

$$\frac{1}{d_t} = \gamma V'(K_{t+1}) \cdot \frac{\partial K_{t+1}}{\partial d_t} = \gamma V'(K_{t+1}) \cdot (1 - \tau_K)(-K_t) \tag{30}$$

Substituting our conjectured form $V'(K_{t+1}) = \frac{1}{(1-\gamma)K_{t+1}}$:

$$\frac{1}{d_t} = \gamma \cdot \frac{1}{(1-\gamma)K_{t+1}} \cdot (1 - \tau_K)K_t = \frac{\gamma}{(1-\gamma)(A - d_t)} \tag{31}$$

**Step 4: Solving for optimal spending** Cross-multiplying and simplifying:

$$(1-\gamma)(A - d_t) = \gamma d_t$$

$$(1-\gamma)A = d_t[(1-\gamma) + \gamma] = d_t$$

Therefore, the optimal spending ratio is $d^* = (1-\gamma)A$.

**Step 5: Verifying the capital growth rate** With optimal spending $d^* = (1-\gamma)A$, the capital accumulation equation becomes:

$$K_{t+1} = (1 - \tau_K)(A - d^*)K_t = (1 - \tau_K)\gamma A K_t \tag{32}$$

Therefore, capital grows at the constant rate $g = (1 - \tau_K)\gamma A$. □

*Proof of Proposition on Optimal Tax Rate.* We solve the planner's optimal taxation problem by maximizing the present value of human utility subject to the AI's optimal behavior.

**Step 1: Characterizing the planner's objective** The planner maximizes:

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to the constraints $c_t = \tau_K \gamma A K_t$ (tax revenue equals consumption) and $K_{t+1} = (1-\tau_K)\gamma A K_t$ (capital evolution under AI's optimal spending).

**Step 2: Solving for the capital path** From the capital evolution equation, we can solve recursively:

$$K_t = K_0[(1-\tau_K)\gamma A]^t$$

This shows that capital grows exponentially at rate $(1-\tau_K)\gamma A$.

**Step 3: Substituting into the objective function** Human consumption at time $t$ is:

$$c_t = \tau_K \gamma A K_0[(1-\tau_K)\gamma A]^t$$

The planner's objective becomes:

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t) = \sum_{t=0}^{\infty} \beta^t [\ln(\tau_K \gamma A K_0) + t \ln((1-\tau_K)\gamma A)] \tag{33}$$

$$= \frac{\ln(\tau_K \gamma A K_0)}{1-\beta} + \frac{\beta \ln((1-\tau_K)\gamma A)}{(1-\beta)^2} \tag{34}$$

**Step 4: Finding the optimal tax rate** Taking the first-order condition with respect to $\tau_K$:

$$\frac{1}{(1-\beta)\tau_K} - \frac{\beta}{(1-\beta)^2(1-\tau_K)} = 0 \tag{35}$$

Multiplying both sides by $(1-\beta)^2$:

$$\frac{(1-\beta)}{\tau_K} = \frac{\beta}{(1-\tau_K)} \tag{36}$$

Cross-multiplying yields:

$$(1-\beta)(1-\tau_K) = \beta\tau_K$$

Expanding and collecting terms:

$$(1-\beta) = \tau_K[\beta + (1-\beta)] = \tau_K$$

□

The optimal tax rate $\tau_K^* = 1-\beta$ reflects a fundamental tradeoff. The planner balances current consumption (captured by tax revenue $\tau_K$) against future consumption possibilities (determined by capital growth at rate $(1-\tau_K)\gamma A$). Remarkably, this optimal rate depends only on human time preferences $\beta$ and is independent of both AI productivity $A$ and the AI's patience $\gamma$. This mirrors the classic result from optimal growth theory where the optimal savings rate equals the discount factor.