

B Online Appendix (not for publication)

This Online Appendix provides some additional results referenced in the paper.

B.1 Proof of Corollary 1

The proof is the same as in a single-sector case, but done with slightly different notation.

B.1.1 Preliminary

The system is rewritten in the following form by redefining $\delta_i^s = \delta_i^s Y_i$.

$$\begin{aligned} X_{ij}^s &= K_{ij}^s \gamma_i \delta_j^s Y_j \\ Y_i &= \sum_s \sum_j X_{ij}^s \\ B_i^s Y_i &= \sum_j X_{ji}^s \\ Y_i &= B_i \gamma_i^{\frac{\alpha}{1-\beta}} (\delta_i)^{\frac{\beta}{1-\beta}} \\ \delta_i &= \prod_t (\delta_i^t)^{\theta^t}, \end{aligned}$$

where $\sum_s B_i^s = 1$. The new set of α^* and β^* is $\alpha^* = \frac{\alpha}{1-\beta}$ and $\beta^* = \frac{\beta}{1-\beta}$. Actually it turns out that it is easier to show existence and uniqueness of the system with this notation. However we need to show that it suffices to establish existence and uniqueness for α^* and β^* . The following lemma tells that the mapping between these two is one-to-one so that if the system has an property for (α^*, β^*) , then the (original) system has the same property under (α, β) .

Lemma 7. *There is an one-to-one mapping between (α, β) and (α^*, β^*) if $\beta \neq 1$.*

Proof. Fix (α, β) , then (α^*, β^*) is uniquely pinned down. Fix (α^*, β^*) , then there exists a unique β such that

$$\beta = \frac{\beta^*}{1 + \beta^*}$$

Then α^* is uniquely pinned down by $\alpha = (1 - \beta) \alpha^* = \frac{\alpha^*}{1 + \beta^*}$. □

Lemma 8. *Denote the function from (α, β) as f . Then $f(D) = \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$, where $D = \{(\alpha, \beta) \in \mathbb{R}^2; \alpha, \beta \leq 0 \text{ or } \alpha, \beta \geq 1\}$.*

Proof. Take $(\alpha, \beta) \in f(D)$. Namely

$$\alpha^* = \frac{\alpha}{1 - \beta}$$

$$\beta^* = \frac{\beta}{1 - \beta}.$$

Then if α and β are both negative, then, α^* and β^* are both negative. Also

$$\alpha^* - 1 - \beta^* = \frac{\alpha - \beta - 1 + \beta}{1 - \beta} = \frac{\alpha - 1}{1 - \beta} \leq 0,$$

which implies $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$. Suppose α, β are greater than 1. Then both α^* and β^* are negative, and

$$\alpha^* - 1 - \beta^* = \frac{\alpha - 1}{1 - \beta} \leq 0.$$

Again we have $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$.

Fix $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$. Then define (α, β) as follows.

$$\alpha = \frac{\alpha^*}{1 + \beta^*}$$

$$\beta = \frac{\beta^*}{1 + \beta^*}.$$

Then if $1 + \beta^* < 0$, then

$$\alpha = \frac{\alpha^*}{1 + \beta^*} > 1$$

$$\beta = \frac{\beta^*}{1 + \beta^*} > 1.$$

If $1 + \beta^* > 0$, then both are negative. Namely $(\alpha^*, \beta^*) \in f(D)$, which completes the proof. \square

Lemma 9. Denote the function from (α, β) as f . Then $f(D) = \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0\}$, where $D = \{(\alpha, \beta) \in \mathbb{R}^2; \alpha, \beta \leq 0 \text{ or } \alpha \geq 0, \beta \geq 1\}$.

Proof. Take $(\alpha, \beta) \in f(D)$. Namely

$$\alpha^* = \frac{\alpha}{1 - \beta}$$

$$\beta^* = \frac{\beta}{1 - \beta}.$$

Then if α and β are both negative, then, α^* and β^* are both negative, which implies $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0\}$. Suppose $\alpha \geq 0$ and $\beta \geq 1$. Then both α^* and β^* are negative. Again we have $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0, \alpha^* - 1 \leq \beta^*\}$.

Fix $(\alpha^*, \beta^*) \in \{(\alpha^*, \beta^*); \alpha^*, \beta^* \leq 0\}$. Then define (α, β) as follows.

$$\alpha = \frac{\alpha^*}{1 + \beta^*}$$

$$\beta = \frac{\beta^*}{1 + \beta^*}.$$

Then if $1 + \beta^* < 0$, then

$$\alpha = \frac{\alpha^*}{1 + \beta^*} \geq 0$$

$$\beta = \frac{\beta^*}{1 + \beta^*} \geq 1.$$

If $1 + \beta^* > 0$, then both are negative. Namely $(\alpha^*, \beta^*) \in f(D)$, which completes the proof. \square

These lemmas imply that if we can establish existence (uniqueness) on (α^*, β^*) -space, then under the associated (α, β) , we can show that the system has a solution (unique solution). Strictly speaking, we loose uniqueness result when $\beta = 1$.

From now on, for notational convenience, we omit the star “*”. From the previous lemma, it suffices to show that if $(\alpha, \beta) \in f(D)$, the system

$$\begin{aligned} X_{ij}^s &= K_{ij}^s \gamma_i \delta_j^s Y_j \\ Y_i &= \sum_s \sum_j X_{ij}^s \\ B_i^s Y_i &= \sum_j X_{ji}^s \\ Y_i &= B_i \gamma_i^\alpha (\delta_i)^\beta \\ \delta_i &= \prod_t (\delta_i^t)^{\theta^t}. \end{aligned}$$

has an unique solution. Then for $(\alpha, \beta) \in D$, the (original) system has an unique solution.

As we did in a single-sector economy, we can re-define variables as follows.

$$\begin{aligned}
x_i &= B_i \gamma_i^{\alpha-1} (\delta_i)^\beta \\
y_i^s &= (\delta_i^s)^{-1} (= (P_i^s)^{1-\sigma}) \\
z_i &= \prod_s \left((y_i^s)^{\theta^t} \right)^{(\alpha-\beta)} \\
\delta_i &= \prod_t (\delta_i^t)^{\theta^t} = \prod_t (y_i^t)^{-\theta^t} \\
&= \left[\prod_t (y_i^t)^{\theta^t} \right]^{-1} = (z_i)^{-\frac{1}{\alpha-\beta}}.
\end{aligned}$$

Here z_i , loosely speaking, represents the aggregate price index for country i P_i . The power $\alpha - \beta$ is $\frac{1}{1-\sigma}$ for Armington with intermediate case, and $(\delta_i^s)^{-1} = (y_i^s) = (P_i^s)^{1-\sigma}$.

Then we can express $(\gamma_i, \delta_i, \delta_i^s)$ by (x_i, y_i^s, z_i) .

$$\begin{aligned}
\delta_i &= (z_i)^{-\frac{1}{\alpha-\beta}} (= (P_i)^{\sigma-1}) \\
(\gamma_i)^{\alpha-1} &= (B_i)^{-1} x_i (\delta_i)^{-\beta} \\
&= (B_i)^{-1} x_i \left((z_i)^{-\frac{1}{\alpha-\beta}} \right)^{-\beta} \\
\gamma_i &= (B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
\delta_i^s &= (y_i^s)^{-1}.
\end{aligned}$$

Then substituting these $(\gamma_i, \delta_i, \delta_i^s)$ into the equilibrium conditions, we get

$$\begin{aligned}
x_i &= \sum_s \sum_j B_j K_{ij}^s \delta_j^s \gamma_j^\alpha (\delta_j)^\beta \\
&= \sum_s \sum_j B_j K_{ij}^s (y_j^s)^{-1} (B_j)^{-\frac{\alpha}{\alpha-1}} (x_j)^{\frac{\alpha}{\alpha-1}} (z_j)^{\frac{\alpha\beta}{(\alpha-\beta)(\alpha-1)}} (z_j)^{-\frac{\beta}{\alpha-\beta}} \\
&= \sum_s \sum_j K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}} \\
y_i^s &= (\delta_i^s)^{-1} = (B_i^s)^{-1} \sum_j K_{ji}^s \gamma_j \\
&= \sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
z_i &= \prod_s \left((y_i^s)^{\theta^t} \right)^{(\alpha-\beta)}.
\end{aligned}$$

The system is finally written in the following form.

$$\begin{aligned}
x_i &= \sum_s \sum_j K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}} \\
y_i^s &= \sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
z_i &= \prod_s \left((y_i^s)^{\theta^t} \right)^{(\alpha-\beta)}.
\end{aligned}$$

B.1.2 Existence proof

The existence proof consists of two steps. First we consider the following system.

$$\begin{aligned}
x_i &= \frac{\sum_s \sum_j K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}}}{\sum_{i,s,j} K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}}} \\
y_i^s &= \sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
z_i &= \prod_s \left((y_i^s)^{\theta^t} \right)^{(\alpha-\beta)}.
\end{aligned}$$

Then we know that x_i is bounded since we normalize x_i . The following lemma ensures that we can obtain the bounds for y_i^s and z_i under certain conditions.

Lemma 10. *If $\alpha, \beta \leq 0$ and $\alpha - 1 \leq \beta$, then y_i^s and z_i are bounded.*

Proof. First we construct (candidate) bounds, and show that actually they are bounds. Suppose that

$$m_y \leq y_i^s \leq M_y.$$

Suppose that $\alpha, \beta \leq 0$, $\alpha \geq \beta$, and $\alpha - 1 \leq \beta$. Then z_i is bounded as follows

$$(m_y)^{\alpha-\beta} \leq z_i \leq (M_y)^{\alpha-\beta}.$$

Then

$$\begin{aligned}
y_i^s &= \sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\underbrace{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}_{\geq 0}} \\
&\leq \max_{i,s} \underbrace{\left[\sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} \right]}_{\triangleq C_y} (m_x)^{\frac{1}{\alpha-1}} \left((M_y)^{\alpha-\beta} \right)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
&= C_y (m_x)^{\frac{1}{\alpha-1}} (M_y)^{\frac{\beta}{\alpha-1}} \\
y_i^s &\geq \min_{i,s} \underbrace{\left[\sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} \right]}_{\triangleq c_y} (M_x)^{\frac{1}{\alpha-1}} (m_y)^{\frac{\beta}{\alpha-1}} \\
&= c_y (M_x)^{\frac{1}{\alpha-1}} (m_y)^{\frac{\beta}{\alpha-1}}.
\end{aligned}$$

Set m_y and M_y as follows,

$$\begin{aligned}
M_y &= C_y (m_x)^{\frac{1}{\alpha-1}} (M_y)^{\frac{\beta}{\alpha-1}} \\
&= \left[C_y (m_x)^{\frac{1}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-1-\beta}} = \left[C_y (m_x)^{\frac{1}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-1-\beta}} \\
m_y &= c_y (M_x)^{\frac{1}{\alpha-1}} (m_y)^{\frac{\beta}{\alpha-1}} = \left[c_y (M_x)^{\frac{1}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-1-\beta}}.
\end{aligned}$$

It is easy to show $M_y > m_y$ since $\frac{\alpha-1}{\alpha-1-\beta} \geq 0$.

Now suppose that $\alpha \leq \beta$. Then z_i is bounded as follows

$$(M_y)^{\alpha-\beta} \leq z_i \leq (m_y)^{\alpha-\beta}.$$

Then

$$\begin{aligned}
y_i^s &= \sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
&\leq \max_{i,s} \underbrace{\left[\sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} \right]}_{\triangleq C_y} (m_x)^{\frac{1}{\alpha-1}} \left((M_y)^{\alpha-\beta} \right)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
&= C_y (m_x)^{\frac{1}{\alpha-1}} (M_y)^{\frac{\beta}{\alpha-1}} \\
y_i^s &\geq \min_{i,s} \underbrace{\left[\sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{-\frac{1}{\alpha-1}} \right]}_{\triangleq c_y} (M_x)^{\frac{1}{\alpha-1}} (m_y)^{\frac{\beta}{\alpha-1}} \\
&= c_y (M_x)^{\frac{1}{\alpha-1}} (m_y)^{\frac{\beta}{\alpha-1}}.
\end{aligned}$$

Set m_y and M_y in the same as before,

$$\begin{aligned}
M_y &= C_y (m_x)^{\frac{1}{\alpha-1}} (M_y)^{\frac{\beta}{\alpha-1}} \\
&= \left[C_y (m_x)^{\frac{1}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-1-\beta}} = \left[C_y (m_x)^{\frac{1}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-1-\beta}} \\
m_y &= c_y (M_x)^{\frac{1}{\alpha-1}} (m_y)^{\frac{\beta}{\alpha-1}} = \left[c_y (M_x)^{\frac{1}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-1-\beta}}.
\end{aligned}$$

Note that $\frac{\alpha-1}{\alpha-1-\beta} \geq 0$. Therefore $M_y > m_y$. □

Since we bound the variables, existence follows immediately from the Schauder's FPT.

Lemma 11. (Scaling) Suppose that (x_i, y_i^s, z_i) solves

$$\begin{aligned}
x_i &= \frac{\sum_s \sum_j K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}}}{\sum_{i,s,j} K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}}} \\
y_i^s &= \sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\
z_i &= \prod_s \left((y_i^s)^{\theta^s} \right)^{(\alpha-\beta)}.
\end{aligned}$$

Then

$$\sum_{i,s,j} B_j K_{ij}^s (B_j)^{-\frac{\alpha}{\alpha-1}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} = 1.$$

Proof. Define λ_x for notational convenience.

$$\lambda_x = \sum_{i,s,j} K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}}.$$

Multiply $\left[(B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \right]$ for the first equation and take a sum w.r.t. i .

$$\begin{aligned} \lambda_x &= \frac{\sum_i x_i \left[(B_j)^{\frac{1}{1-\alpha}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \right]}{\sum_s \sum_j K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \left[(B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \right]} \\ &= \frac{\sum_i \left[(B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{\alpha}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \right]}{\sum_s \sum_j (K_{ij}^s) \left[(B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \right] (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}}} \end{aligned}$$

Also multiply $\left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (y_i^s)^{-1} (z_i)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}} \right]$ for the second equation, and sum up w.r.t. i, s .

$$\begin{aligned} 1 &= \frac{\sum_{i,s} \left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (y_i^s)^{-1} (z_i)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}} \right] B_i^s y_i^s}{\sum_{i,s} \sum_j K_{ji}^s (B_j)^{-\frac{1}{\alpha-1}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (y_i^s)^{-1} (z_i)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}} \right]} \\ &= \frac{\sum_i \left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (z_i)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}} \right] \underbrace{\sum_s B_i^s}_{=1}}{\sum_{i,s} \sum_j K_{ij}^s (B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \left[(B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}} \right]} \\ &= \frac{\sum_i \left[(B_i)^{\frac{1}{1-\alpha}} (x_i)^{\frac{\alpha}{\alpha-1}} (z_i)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}} \right]}{\sum_{i,s} \sum_j (K_{ij}^s) \left[(B_i)^{-\frac{1}{\alpha-1}} (x_i)^{\frac{1}{\alpha-1}} (z_i)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \right] (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}}} \\ &= \lambda_x. \end{aligned}$$

□

This lemma tells that it suffices to prove existence for

$$\begin{aligned} x_i &= \frac{\sum_s \sum_j K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}}}{\sum_{i,s,j} K_{ij}^s (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{\alpha}{\alpha-1}} (y_j^s)^{-1} (z_j)^{\frac{\beta}{\alpha-\beta} \frac{1}{\alpha-1}}} \\ y_i^s &= \sum_j K_{ji}^s (B_i^s)^{-1} (B_j)^{\frac{1}{1-\alpha}} (x_j)^{\frac{1}{\alpha-1}} (z_j)^{\frac{\beta}{(\alpha-\beta)(\alpha-1)}} \\ z_i &= \prod_s \left((y_i^s)^{\theta^t} \right)^{(\alpha-\beta)}. \end{aligned}$$

B.1.3 Uniqueness proof

Proof. Suppose that there are two solutions. As in a single-sector case, we can take one of the solutions as follows without loss of generality

$$x_i = y_i^s = z_i = 1.$$

Suppose that $\alpha \leq \beta \leq 0$. Then we can bound z_i .

$$(N_y)^{\alpha-\beta} \leq z_i \leq (n_y)^{\alpha-\beta}.$$

Then we can bound the maximums of x_i and y_i^s and the minimums of them as follows. As for x , it is easy to show

$$\left(\frac{N_x}{n_x}\right)^{\frac{1}{1-\alpha}} \leq \left(\frac{N_y}{n_y}\right)^{\frac{\beta}{\alpha-1}+1}.$$

As for y , we get

$$\left(\frac{N_y}{n_y}\right)^{\frac{\beta}{\alpha-1}+1} \leq \left(\frac{N_x}{n_x}\right)^{\frac{1}{\alpha-1}}.$$

Since there are two solutions, one of the inequalities is strict.

$$1 < \left(\frac{N_x}{n_x}\right)^{\frac{1}{1-\alpha}} < \left(\frac{N_x}{n_x}\right)^{\frac{1}{\alpha-1}} < 1,$$

which is a contradiction. Therefore the system has an unique solution.

Suppose that $\alpha - \beta > 0$, and $\alpha, \beta < 0$, and $\alpha - 1 < \beta$. Then we can bound z_i .

$$(n_y)^{\alpha-\beta} \leq z_i \leq (N_y)^{\alpha-\beta}.$$

Then we can bound the maximums of x_i and y_i^s and the minimums of them as follows. As for x , it is easy to show

$$\left(\frac{N_x}{n_x}\right)^{-\frac{1}{\alpha-1}} \leq \left(\frac{N_y}{n_y}\right)^{\frac{\beta}{\alpha-1}+1}$$

Then as for y , we get

$$\left(\frac{N_y}{n_y}\right)^{\frac{\beta}{\alpha-1}+1} \leq \left(\frac{N_x}{n_x}\right)^{\frac{1}{\alpha-1}}.$$

Since there are two solutions, one of the inequalities is strict.

$$1 < \left(\frac{N_x}{n_x}\right)^{-\frac{1}{\alpha-1}} < \left(\frac{N_x}{n_x}\right)^{\frac{1}{\alpha-1}} < 1,$$

which is a contradiction. Therefore the system has an unique solution. \square

B.2 Proof of Corollary 2

Proof. As we did in the general equilibrium trade models, we analyze a transformed system.

$$x_i = \lambda \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \quad (56)$$

$$y_i = \lambda \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}} \quad (57)$$

$$\lambda^c = \sum_i \tilde{B}_i \tilde{x}_i^{\tilde{\alpha}} \tilde{y}_i^{\tilde{\beta}} \quad (58)$$

$$\lambda = \sum_i \left(B_i^{\frac{\alpha}{1-\alpha-\beta}} x_i^{\frac{\alpha(1-\beta)}{\alpha+\beta-1}} y_i^{\frac{\alpha\beta}{\alpha+\beta-1}} \right), \quad (59)$$

where

$$\begin{aligned} \tilde{B}_i &= C_i B_i^{\frac{d+e}{1-\alpha-\beta}}, \\ (\tilde{\alpha}, \tilde{\beta}) &= \left(\frac{e\alpha + d(1-\beta)}{\alpha + \beta - 1}, \frac{e(1-\alpha) + d\beta}{\alpha + \beta - 1} \right). \end{aligned}$$

\square

B.2.1 Existence

Proof. We first prove existence. From Theorem 1, there exists (x_i, y_i) satisfying

$$\begin{aligned} x_i &= \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{\alpha}{\alpha+\beta-1}} y_j^{\frac{1-\alpha}{\alpha+\beta-1}} \\ y_i &= \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} x_j^{\frac{1-\beta}{\alpha+\beta-1}} y_j^{\frac{\beta}{\alpha+\beta-1}}. \end{aligned}$$

Consider $(\tilde{x}_i, \tilde{y}_i) = (tx_i, sy_i)$. Then it is easy to show

$$\begin{aligned} tx_i &= t^{1-\frac{\alpha}{\alpha+\beta-1}} s^{-\frac{1-\alpha}{\alpha+\beta-1}} \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{\alpha}{\alpha+\beta-1}} (sy_j)^{\frac{1-\alpha}{\alpha+\beta-1}} \\ sy_i &= t^{-\frac{1-\beta}{\alpha+\beta-1}} s^{1-\frac{\beta}{\alpha+\beta-1}} \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (tx_j)^{\frac{1-\beta}{\alpha+\beta-1}} (sy_j)^{\frac{\beta}{\alpha+\beta-1}}. \end{aligned}$$

since (x_i, y_i) solves (56) and (57) with $\lambda = 1$.

Set (t, s) as follows.

$$\begin{aligned} t^{1-\frac{\alpha}{\alpha+\beta-1}} s^{-\frac{1-\alpha}{\alpha+\beta-1}} &= \sum_i B_i^{\frac{\alpha}{1-\alpha-\beta}} (tx_i)^{\alpha\frac{1-\beta}{\alpha+\beta-1}} (sy_i)^{\alpha\frac{\beta}{\alpha+\beta-1}} \\ \sum_i \tilde{B}_i (tx_i)^{\tilde{\alpha}} (sy_i)^{\tilde{\beta}} &= t^{c(1-\frac{\alpha}{\alpha+\beta-1})} s^{-c\frac{1-\alpha}{\alpha+\beta-1}}. \end{aligned}$$

This is a system of a log-linear equations. The solution to the system is given by

$$\begin{pmatrix} t \\ s \end{pmatrix} = \exp \left(\begin{pmatrix} \frac{\beta-1-\alpha+\alpha\beta}{\alpha+\beta-1} & \frac{-1+\alpha-\alpha\beta}{\alpha+\beta-1} \\ c \left(1 - \frac{\alpha}{\alpha+\beta-1}\right) - \tilde{\alpha} & -c\frac{1-\alpha}{\alpha+\beta-1} - \tilde{\beta} \end{pmatrix}^{-1} \begin{pmatrix} \log \left(\sum_i B_i^{\frac{\alpha}{1-\alpha-\beta}} (x_i)^{\alpha\frac{1-\beta}{\alpha+\beta-1}} (y_i)^{\alpha\frac{\beta}{\alpha+\beta-1}} \right) \\ \log \left(\sum_i \tilde{B}_i (x_i)^{\tilde{\alpha}} (y_i)^{\tilde{\beta}} \right) \end{pmatrix} \right).$$

Then $(\tilde{x}_i, \tilde{y}_i)$ solves the system (56) to (59). To see this carefully, note that $(\tilde{x}_i, \tilde{y}_i)$ solves the following non-linear equations.

$$\begin{aligned} \tilde{x}_i &= t^{1-\frac{\alpha}{\alpha+\beta-1}} s^{-\frac{1-\alpha}{\alpha+\beta-1}} \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (\tilde{x}_j)^{\frac{\alpha}{\alpha+\beta-1}} (\tilde{y}_j)^{\frac{1-\alpha}{\alpha+\beta-1}} \\ &= t^{-\frac{1-\beta}{\alpha+\beta-1}} s^{-\frac{1-\alpha}{\alpha+\beta-1}} \sum_j K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} (\tilde{x}_j)^{\frac{\alpha}{\alpha+\beta-1}} (\tilde{y}_j)^{\frac{1-\alpha}{\alpha+\beta-1}} \\ \tilde{y}_i &= t^{-\frac{1-\beta}{\alpha+\beta-1}} s^{1-\frac{\beta}{\alpha+\beta-1}} \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (\tilde{x}_j)^{\frac{1-\beta}{\alpha+\beta-1}} (\tilde{y}_j)^{\frac{\beta}{\alpha+\beta-1}} \\ &= t^{-\frac{1-\beta}{\alpha+\beta-1}} s^{-\frac{1-\alpha}{\alpha+\beta-1}} \sum_j K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} (\tilde{x}_j)^{\frac{1-\beta}{\alpha+\beta-1}} (\tilde{y}_j)^{\frac{\beta}{\alpha+\beta-1}}. \end{aligned}$$

From the construction of t and s ,

$$\begin{aligned} t^{1-\frac{\alpha}{\alpha+\beta-1}} s^{-\frac{1-\alpha}{\alpha+\beta-1}} \left(= t^{-\frac{1-\beta}{\alpha+\beta-1}} s^{-\frac{1-\alpha}{\alpha+\beta-1}} \right) &= \sum_i B_i^{\frac{\alpha}{1-\alpha-\beta}} (tx_i)^{\alpha\frac{1-\beta}{\alpha+\beta-1}} (sy_i)^{\alpha\frac{\beta}{\alpha+\beta-1}} \\ &= \lambda. \end{aligned}$$

Therefore (56) and (57) are satisfied. Also the world income is normalized since

$$\begin{aligned} \sum_i \tilde{B}_i (\tilde{x}_i)^{\tilde{\alpha}} (\tilde{y}_i)^{\tilde{\beta}} &= \left[t^{(1-\frac{\alpha}{\alpha+\beta-1})} s^{-\frac{1-\alpha}{\alpha+\beta-1}} \right]^c \\ &= \lambda^c, \end{aligned}$$

which completes the proof of existence. □

B.2.2 Uniqueness

Proof. We now prove uniqueness by contradiction. Suppose that there are two solutions (x_i, y_i, λ) and $(\tilde{x}_i, \tilde{y}_i, \tilde{\lambda})$ for (56) to (59), where λ and $\tilde{\lambda}$ are the associated eigenvalues. Then it is easy to show

$$\begin{aligned} 1 &= \sum_j \left[\frac{\tilde{\lambda} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} \tilde{x}_j^{\frac{\alpha}{\alpha+\beta-1}} \tilde{y}_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\tilde{x}_i} \right] \\ 1 &= \sum_j \left[\frac{\tilde{\lambda} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} \tilde{x}_j^{\frac{1-\beta}{\alpha+\beta-1}} \tilde{y}_j^{\frac{\beta}{\alpha+\beta-1}}}{\tilde{y}_i} \right]. \end{aligned}$$

Since (x_i, y_i, λ) also satisfies the two integral equations, we have

$$\begin{aligned} \frac{x_i}{\tilde{x}_i} &= \frac{\lambda}{\tilde{\lambda}} \sum_j \frac{\tilde{\lambda} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} \tilde{x}_j^{\frac{\alpha}{\alpha+\beta-1}} \tilde{y}_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\tilde{x}_i} \left(\frac{x_j}{\tilde{x}_j} \right)^{\frac{\alpha}{\alpha+\beta-1}} \left(\frac{y_j}{\tilde{y}_j} \right)^{\frac{1-\alpha}{\alpha+\beta-1}} \\ \frac{y_i}{\tilde{y}_i} &= \frac{\lambda}{\tilde{\lambda}} \sum_j \left[\frac{\tilde{\lambda} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} \tilde{x}_j^{\frac{1-\beta}{\alpha+\beta-1}} \tilde{y}_j^{\frac{\beta}{\alpha+\beta-1}}}{\tilde{y}_i} \right] \left(\frac{x_j}{\tilde{x}_j} \right)^{\frac{1-\beta}{\alpha+\beta-1}} \left(\frac{y_j}{\tilde{y}_j} \right)^{\frac{\beta}{\alpha+\beta-1}}. \end{aligned}$$

Redefine the variables:

$$x_i \triangleq \frac{x_i}{\tilde{x}_i}, y_i \triangleq \frac{y_i}{\tilde{y}_i}, \lambda \triangleq \frac{\lambda}{\tilde{\lambda}},$$

and the kernels:

$$\begin{aligned} H_{ij} &\triangleq \frac{\tilde{\lambda} K_{ij} B_j^{\frac{1}{1-\alpha-\beta}} \tilde{x}_j^{\frac{\alpha}{\alpha+\beta-1}} \tilde{y}_j^{\frac{1-\alpha}{\alpha+\beta-1}}}{\tilde{x}_i} \\ F_{ij} &\triangleq \frac{\tilde{\lambda} K_{ji} B_j^{\frac{1}{1-\alpha-\beta}} \tilde{x}_j^{\frac{1-\beta}{\alpha+\beta-1}} \tilde{y}_j^{\frac{\beta}{\alpha+\beta-1}}}{\tilde{y}_i}. \end{aligned}$$

Then we can assume, without loss of generality, that one of the solutions is $(\tilde{x}_i, \tilde{y}_i, \tilde{\lambda}) = (1, 1, 1)$, and the kernels satisfy for all j ,

$$\sum H_{ij} = \sum F_{ij} = 1.$$

The rest of the proof is the same as one in Theorem 1. Hence, part (ii) of Theorem 1 can be extended to include the alternative condition C.4' for economic geography models. \square

B.3 The “Universal Gravity” economic geography framework

In this section, we show how the economic geography model developed in [Allen and Arkolakis \(2014\)](#) can be mapped to the equilibrium C.4'. We refer the reader to that paper to discuss how the [Allen and Arkolakis \(2014\)](#) framework is in turn isomorphic to the economic geography models of [Helpman \(1998\)](#) and [Redding \(2014\)](#) (amongst others).

In [Allen and Arkolakis \(2014\)](#), gravity takes the same form as in the Armington model, so that $K_{ij} = \tau_{ij}^{1-\sigma}$, $\gamma_i = \left(\frac{w_i}{A_i}\right)^{1-\sigma}$ and $\delta_i = w_i L_i P_i^{\sigma-1}$. In addition, the labor supply adjusts so that in equilibrium welfare \bar{W} is equalized across locations: $\bar{W} = \frac{w_i}{P_i} u_i$. The total labor is assumed to be constant, so that $\bar{L} = \sum_{i \in S} L_i$. Finally, both productivities and amenities are allowed to depend on the local population: $A_i = \bar{A}_i L_i^a$ and $u_i = \bar{u}_i L_i^b$, where a and b determine the strength of agglomeration and dispersion forces (and create the isomorphisms to many different economic geography models).

To write the [Allen and Arkolakis \(2014\)](#) model in the universal gravity framework, we use the welfare equalization requirement to substitute for the price index and apply the assumed functional form for productivities and amenities. As a result, we can write the exporter and importer shifters solely as a function of the aggregate welfare, the wage, the population, and exogenous variables:

$$\begin{aligned}\gamma_i &= \bar{A}_i^{\sigma-1} w_i^{1-\sigma} L_i^{a(\sigma-1)} \\ \delta_i &= \bar{W}^{1-\sigma} \bar{u}_i^{\sigma-1} w_i^\sigma L_i^{1+b(\sigma-1)}\end{aligned}$$

We can solve this system of equations for the population and wages as a function of the aggregate welfare \bar{W} , exogenous shifters, and the exporter and importer shifters:

$$\begin{aligned}w_i &= \bar{W}^{\frac{a(\sigma-1)}{(1+b(\sigma-1)+\sigma a)}} \bar{A}_i^{\frac{(1+b(\sigma-1))}{(1+b(\sigma-1)+\sigma a)}} \bar{u}_i^{-\frac{a(\sigma-1)}{(1+b(\sigma-1)+\sigma a)}} \gamma_i^{\frac{(1+b(\sigma-1))}{(1-\sigma)(1+b(\sigma-1)+\sigma a)}} \delta_i^{\frac{a}{(1+b(\sigma-1)+\sigma a)}} \\ L_i &= \bar{W}^{-\frac{1-\sigma}{(1+b(\sigma-1)+\sigma a)}} \bar{A}_i^{-\frac{\sigma}{(1+b(\sigma-1)+\sigma a)}} \bar{u}_i^{\frac{(1-\sigma)}{(1+b(\sigma-1)+\sigma a)}} \gamma_i^{-\frac{\sigma}{(1-\sigma)(1+b(\sigma-1)+\sigma a)}} \delta_i^{\frac{1}{(1+b(\sigma-1)+\sigma a)}}\end{aligned}$$

Since $Y_i = w_i L_i$, we have:

$$Y_i = \frac{1}{\lambda} B_i \gamma_i^\alpha \delta_i^\beta,$$

where $\lambda \equiv \bar{W}^{-\frac{(a+1)(\sigma-1)}{(1+b(\sigma-1)+\sigma a)}}$, $B_i \equiv \bar{A}_i^{\frac{(1+b(\sigma-1))-\sigma}{(1+b(\sigma-1)+\sigma a)}} \bar{u}_i^{\frac{(a+1)(1-\sigma)}{(1+b(\sigma-1)+\sigma a)}}$, $\alpha \equiv \frac{1-b}{1+b(\sigma-1)+\sigma a}$, and $\beta \equiv \frac{a+1}{(1+b(\sigma-1)+\sigma a)}$.

Furthermore, from the aggregate labor market clearing condition, we have:

$$\bar{L} = \sum_i \bar{W}^{-\frac{1-\sigma}{(1+b(\sigma-1)+\sigma a)}} \bar{A}_i^{-\frac{\sigma}{(1+b(\sigma-1)+\sigma a)}} \bar{u}_i^{\frac{(1-\sigma)}{(1+b(\sigma-1)+\sigma a)}} \gamma_i^{-\frac{\sigma}{(1-\sigma)(1+b(\sigma-1)+\sigma a)}} \delta_i^{\frac{1}{(1+b(\sigma-1)+\sigma a)}} \iff$$

$$\lambda^{1+a} = \sum_i C_i \gamma_i^d \delta_i^e,$$

where $C_i \equiv \frac{1}{\bar{L}} \bar{A}_i^{-\frac{\sigma}{(1+b(\sigma-1)+\sigma a)}} \bar{u}_i^{\frac{(1-\sigma)}{(1+b(\sigma-1)+\sigma a)}}$, $d \equiv -\frac{\sigma}{(1-\sigma)(1+b(\sigma-1)+\sigma a)}$, and $e \equiv \frac{1}{(1+b(\sigma-1)+\sigma a)}$. Hence, [Allen and Arkolakis \(2014\)](#) satisfies C.4'.

B.4 Existence and Uniqueness using Gross Substitutes Methodology (a la [Alvarez and Lucas \(2007\)](#))

We will illustrate the application of the gross-substitute property to prove uniqueness equilibrium in an excess demand system. This is a necessary step in the proof of [Alvarez and Lucas \(2007\)](#) but it is not sufficient, as a number of other properties need to be proved for an equation to be an excess demand system, as we discuss below.

Because of the complexity of the system that we analyze we cannot apply the gross-substitutes property directly to equations (4) and (5).

$$B_i \gamma_i^{\alpha-1} \delta_i^\beta = \sum_j K_{ij} \delta_j \quad (60)$$

Combining gravity C.1 with balanced trade C.(3) and condition C.4 yields:

$$B_i \gamma_i^\alpha \delta_i^{\beta-1} = \sum_j K_{ji} \gamma_j \quad (61)$$

In order to find the equation that can be used to prove, we need to eliminate one variable. Use (5) to express δ_i as

$$\delta_i = \left(\frac{\sum_{s \in S} \gamma_s K_{si}}{B_i \gamma_i^\alpha} \right)^{\frac{1}{\beta-1}} \quad (62)$$

into equation (4), we obtain

$$\begin{aligned}
B_i \gamma_i^\alpha \left(\frac{\sum_{s \in S} \gamma_s K_{si}}{B_i \gamma_i^\alpha} \right)^{\frac{\beta}{\beta-1}} &= \sum_{j \in S} \gamma_i \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^\alpha} \right)^{\frac{1}{\beta-1}} K_{ij} \iff \\
B_i^{\frac{1}{1-\beta}} \gamma_i^{\frac{\alpha}{1-\beta}-1} \left(\sum_{s \in S} \gamma_s K_{si} \right)^{\frac{\beta}{\beta-1}} &= \sum_{j \in S} \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^\alpha} \right)^{\frac{1}{\beta-1}} K_{ij}
\end{aligned} \tag{63}$$

We define the corresponding excess demand function might be

$$Z_i(\gamma) = \frac{1}{\gamma_i} \left[B_i^{\frac{1}{\beta-1}} \gamma_i^{\frac{\alpha+\beta-1}{1-\beta}} \left(\sum_{s \in S} \gamma_s K_{si} \right)^{\frac{\beta}{\beta-1}} - \sum_{j' \in S} \left(\frac{\sum_{s \in S} \gamma_s K_{sj'}}{B_{j'} \gamma_{j'}^\alpha} \right)^{\frac{1}{\beta-1}} K_{ij'} \right]$$

This system written as such needs to satisfy 5 properties to be an excess demand system and the gross substitute property to establish existence and uniqueness (see Propositions 17.B.2, 17.C.1 and 17.F.3 of [Mas-Colell, Whinston, and Green \(1995\)](#)). The six conditions are:

1. $Z(\gamma)$ is continuous for $\gamma \in (\Delta(R_+^N))^o$
2. $Z(\gamma)$ is homogenous of degree zero.
3. $Z(\gamma) \cdot \gamma = 0$ (Walras' Law).
4. There exists a $k > 0$ such that $Z_j(\gamma) > -k$ for all j .
5. If there exists a sequence $w^m \rightarrow w^0$, where $w^0 \neq 0$ and $w_i^0 = 0$ for some i , then it must be that:

$$\max_j \{Z_j(w^m)\} \rightarrow \infty \tag{64}$$

and the gross-substitute property:

6. Gross substitutes property: $\frac{\partial Z(w_j)}{\partial w_k} > 0$ for all $j \neq k$.

Properties 1-3 are trivial by the way we define the system. Properties 4 and 5 are challenging and may require an analysis case-by-case which restrict further the set of parameters that uniqueness applies. We thus only discuss the region where gross-substitutes applies. To consider this system as an excess demand system and apply the tools originally developed in [Alvarez and Lucas \(2007\)](#), we need to differentiate the expression above. We only use the

bracketed term without loss of generality. We have:

$$\begin{aligned}
\frac{\partial Z_i(\gamma)}{\partial \gamma_j} &= \frac{\beta}{\beta-1} K_{ji} B_i^{\frac{1}{1-\beta}} \gamma_i^{\frac{\alpha+\beta-1}{1-\beta}} \left(\sum_{s \in S} \gamma_s K_{si} \right)^{\frac{1}{\beta-1}} - \frac{1}{\beta-1} \sum_{j' \in S, j' \neq j} \left[K_{ij'} \left(\frac{\sum_{s \in S} \gamma_s K_{sj'}}{B_{j'} \gamma_{j'}^\alpha} \right)^{\frac{-\beta+2}{\beta-1}} K_{jj'} \right] - \\
&\quad - \frac{1}{\beta-1} K_{ij} \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^\alpha} \right)^{\frac{-\beta+2}{\beta-1}} \left[\frac{K_{jj} B_j \gamma_j^\alpha - \alpha B_j \gamma_j^{\alpha-1} \sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^\alpha} \right] \\
&= \frac{\beta}{\beta-1} K_{ji} B_i^{\frac{1}{1-\beta}} \gamma_i^{\frac{\alpha+\beta-1}{1-\beta}} \left(\sum_{s \in S} \gamma_s K_{si} \right)^{\frac{1}{\beta-1}} - \frac{1}{\beta-1} \sum_{j' \in S, j' \neq j} \left[K_{ij'} \left(\frac{\sum_{s \in S} \gamma_s K_{sj'}}{B_{j'} \gamma_{j'}^\alpha} \right)^{\frac{-\beta+2}{\beta-1}} K_{jj'} \right] - \\
&\quad - \frac{1}{\beta-1} K_{ij} \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^\alpha} \right)^{\frac{-\beta+2}{\beta-1}} \left[\frac{\gamma_j K_{jj} - \alpha \sum_{s \in S} \gamma_s K_{sj'}}{\gamma_j} \right]
\end{aligned}$$

Let $\beta < 0$ and $\alpha < 0$ then the expression is positive and the gross-substitute property holds. Similar results can be easily established for $\beta = 0$, $\alpha < 0$ and $\beta < 0$, $\alpha = 0$. The same cannot be, in generally, established if $\beta > 1$ or $\alpha > 1$ since the expression cannot be signed in that case, and in particular we have found parametric specifications where the gross-substitutes property may fail.²⁸ Thus, the region that uniqueness applies with this approach is $\alpha \leq 0, \beta \leq 0$.

B.5 Comparative Statics when $\beta = 0$

Let us consider a particularly interesting special case, $\beta = 0$. We have in this case that the equilibrium is characterized by

$$\begin{aligned}
B_i \gamma_i^{\alpha-1} &= \sum_{j \in S} \left(\frac{\sum_{s \in S} \gamma_s K_{sj}}{B_j \gamma_j^\alpha} \right)^{-1} K_{ij} \implies \\
\gamma_i^{\alpha-1} &= \sum_{j \in S} \left(\frac{B_j \gamma_j^\alpha}{\sum_{s \in S} \gamma_s K_{sj}} \right) B_i K_{ij},
\end{aligned}$$

which is the standard single-equation gravity model that we find in papers such as [Anderson \(1979\)](#); [Eaton and Kortum \(2002\)](#); [Chaney \(2008\)](#). We can rewrite this system re-written using C.4 as

$$Y_i = \sum_{j \in S} \left(\frac{\gamma_i K_{ij}}{\sum_{s \in S} \gamma_s K_{sj}} \right) Y_j$$

²⁸In particular, we analyzed the Armington case with intermediate inputs as in Section 2. We can show that this model for $\sigma = 3$ and $\gamma = 1/4$ corresponds to the case $\alpha, \beta > 1$ but the gross-substitute condition does not obtain in the case of many symmetric regions with symmetric trade costs or even two regions with no trade costs.

In this last equation the technique developed by [Dekle, Eaton, and Kortum \(2008\)](#) can be applied (see details in [Arkolakis, Costinot, and Rodríguez-Clare \(2012\)](#)) so that computing the changes in γ_i require only knowledge of changes in K_{ij} and initial trade and output levels across all the models that can be captured by this formulation.

Notice that given equation 62 and the above equation we have for $\beta = 0$ that we can express the exporter shifter as a function of the importer shifter and parameters

$$\gamma_i = \left(\frac{\sum_j K_{ij} \delta_j}{B_i} \right)^{\frac{1}{\alpha-1}}. \quad (65)$$

B.6 Proof of Auxiliary Lemma 5

We prove the two auxiliary lemmas used in the proof of Proposition 3

Proof. The proof of Lemma 5 is quite straightforward. The summation of each row implies that $A\vec{e} = a\vec{e}$ and $B\vec{e} = b\vec{e}$, where \vec{e} is the identity vector. Thus $AB\vec{e} = Ab\vec{e} = ab\vec{e}$, and thus ab is the eigenvalue. Thus, $\sum_j c_{ij} = ab$ where c_{ij} is the element of AB . \square

B.7 Comparing the Fixed Effects Estimator and General Equilibrium Estimator

In this sub-section, we compare the standard fixed effects estimator to the general equilibrium estimator developed in Section 5.2.

Using the “hat” notation from Section 4.2.2 and applying the gravity structure C.1 yields the following gravity equation in differences:

$$\hat{X}_{ij} = \hat{K}_{ij} \hat{\gamma}_i \hat{\delta}_j. \quad (66)$$

Suppose that the (log) change in bilateral trade frictions can be written as a linear function of a vector of observables, i.e. $\ln \hat{K}_{ij} = \hat{T}'_{ij} \mu$, and than an econometrician observes trade flows with measurement error. Then taking logs of equation 66 yields:

$$\ln \hat{X}_{ij}^o = \hat{T}'_{ij} \mu + \ln \hat{\gamma}_i + \ln \hat{\delta}_j + \varepsilon_{ij}, \quad (67)$$

where \hat{X}_{ij}^o are the observed ratio of trade flows between i and j in period 1 to period 0, \hat{T}_{ij} is an $S \times 1$ vector of observables and \hat{T}'_{ij} denotes its transpose, μ is an $S \times 1$ vector of parameters, and ε_{ij} is the measurement error. The goal of the econometrician is to estimate μ , i.e. the effect of the various observables on bilateral trade frictions.

The fixed effects estimator

To provide a point of comparison for our estimator, it is helpful to first describe what has become the standard method of estimating μ , which we refer to as the “fixed effects estimator.” The fixed effects estimator estimates μ using equation (67) by including a full set of origin and destination fixed effects in an ordinary least squares regression framework.²⁹ Formally, the fixed effects estimator μ^* is the one that minimizes the squared error between observed (hatted) trade flows and the gravity regression, conditional on the optimal set of fixed effects:

$$\mu_{FE}^* \equiv \arg \min_{\mu \in \mathbb{R}^S} \left(\min_{\hat{\gamma}_i, \hat{\delta}_i \in \mathbb{R}^N} \sum_i \sum_j \left(\ln \hat{X}_{ij}^o - \hat{T}'_{ij} \mu - \ln \hat{\gamma}_i - \ln \hat{\delta}_j \right)^2 \right).$$

By taking first order conditions, it is straightforward to derive an analytical solution for μ^* :

$$\mu_{FE}^* = \left(\sum_i \sum_j \hat{T}_{ij} \hat{T}'_{ij} \right)^{-1} \sum_i \sum_j \hat{T}_{ij} \left(\ln \hat{X}_{ij}^o - \ln \hat{\gamma}_i^* - \ln \hat{\delta}_j^* \right), \quad (68)$$

where the estimated fixed effects are identified up to scale:

$$\ln \hat{\gamma}_i^* - \frac{1}{N} \sum_k \ln \hat{\gamma}_k^* = \frac{1}{N} \sum_j \left(\left(\ln \hat{X}_{ij} - \hat{T}'_{ij} \mu^* \right) - \frac{1}{N} \sum_k \left(\ln \hat{X}_{kj} - \hat{T}'_{kj} \mu^* \right) \right)$$

and:

$$\ln \hat{\delta}_j^* - \frac{1}{N} \sum_k \ln \hat{\delta}_k^* = \frac{1}{N} \sum_i \left(\left(\ln \hat{X}_{ij} - \hat{T}'_{ij} \mu^* \right) - \frac{1}{N} \sum_k \left(\ln \hat{X}_{ik} - \hat{T}'_{ik} \mu^* \right) \right).$$

We should emphasize that there are a number of attractive properties of the fixed effects estimator, most notably that it is easy to implement, and, as long as the measurement error is uncorrelated with the observables or fixed effects, it is a consistent and unbiased estimator of μ .

²⁹The fixed effects estimator is discussed in detail in the review articles of [Baldwin and Taglioni \(2006\)](#) and [Head and Mayer \(2013\)](#). The latter review credits [Harrigan \(1996\)](#) as the first to use the fixed effects estimator and [Redding and Venables \(2004\)](#) and [Feenstra \(2003a\)](#) for showing that the fixed effects estimator could be used to control for the endogenous “multilateral resistance” terms present in general equilibrium gravity models. Since then, the fixed effects literature has been used extensively in the empirical trade literature.

Comparing the fixed effects and general equilibrium estimators

To assess the relative benefit of the fixed effects and general equilibrium estimators of μ – given a known set of gravity constants α and β – we conduct a set of Monte Carlo simulations. For each simulation, we draw a random set of initial bilateral frictions $\{K_{ij}^0\}$ and a random set of (time-invariant) income shifters $\{B_i\}$. We then randomly assign half of the locations to be “existing members” of a “multilateral trade organization” and ten percent of locations to be “new members” of the same trade organization. Next, we suppose that the observed change in trade frictions arises from the new members joining the trade organization; in particular, we assume $\hat{K}_{ij} = \hat{T}_{ij}\mu$, where \hat{T}_{ij} is an indicator variable equal to one if either the origin or destination is a new member of the organization and its trading partner is either an existing or new member. For a given set of gravity constants, we calculate the equilibrium in both periods.³⁰ We then add idiosyncratic measurement error to the trade flows in both periods and implement the two estimators based on these “observed” trade flows.³¹ We calculate the coefficient of variation of the root mean squared deviation “CV(RMSD)” for both estimators over five hundred simulations to assess their relative efficiency.³² We repeat this procedure for varying numbers of countries, size of measurement error, and magnitudes of the effect of the trade agreement (i.e. μ).

The top panel of Table 2 presents the results. For the sake of readability, we highlight the most efficient estimator under a particular set of simulation parameters in bold. As is evident, which estimator is more efficient depends on the particular set of simulation parameters. When there are a few number of locations, the general equilibrium estimator is more efficient than the fixed effects estimator; this is because fixed effects estimator requires estimating $2N$ nuisance parameters, which reduces the degrees of freedom available for estimating μ . In contrast, with many locations and a large effect size, the fixed effects estimator outperforms the general equilibrium estimator; this is because the first order approximation (28) is less accurate the larger the effect size.

While the general equilibrium estimator often outperforms the fixed effect estimator, its true advantage arises from the ability to exploit the general equilibrium structure of the gravity model to overcome the common econometric concern of omitted variable bias. For example, whether or not a country signs a trade agreement is likely correlated with

³⁰We choose $\alpha = -\frac{2}{3}$ and $\beta = -\frac{1}{3}$; see below.

³¹The results are very similar if we instead add an error term to \hat{K}_{ij} , i.e. $\hat{K}_{ij} = \hat{T}_{ij}'\mu + \varepsilon_{ij}$.

³²The CV(RMSD) is defined as $\frac{\left(\frac{1}{M} \sum_{m=1}^M (\mu^{true} - \mu_m^*)^2\right)^{\frac{1}{2}}}{\mu^{true}}$, where μ^{true} is the true value of μ , and μ_m^* is the estimated value for simulation $m \in \{1, \dots, M\}$, i.e. the CV(RSMD) reports the ratio of the standard deviation of an estimator to the true parameter value. Like the root mean squared error, the CV(RMSD) is a statistic that combines both the accuracy and precision of an estimator; unlike the root mean squared error, its value is not dependent on the size of μ^{true} .

unobservable variables (e.g. expectations about future trade flows) that are also correlated with observed trade flows. Such omitted variables will result in biased estimates in a typical gravity equation. However, because the identification in the general equilibrium estimator relies on the effect of particular bilateral observables have on all bilateral trade flows, one can estimate μ using only trade flows between locations that did not choose to sign a particular trade agreement. That is, the decision of country i to join a trade agreement will have a general equilibrium effect on trade flows between countries j and k . This general equilibrium effect can be used to infer the effect of a trade agreement without the need to directly consider how the trade flows of country i change.

To illustrate the power of this method, the bottom panel of Table 2 shows the efficiency of the estimators when we include an omitted variable in the error term that increases the observed period 1 trade flows³³ by 5 percent only between countries in which one entered the trade agreement.³⁴ As is evident, the omitted variable biases both the fixed effects estimator and the baseline general equilibrium estimator upward by an amount equal to the size of the omitted variable. However, when we use the general equilibrium estimator but exclude observations of trade between countries in which one entered the trade agreement (the “GE - switchers” column), the effect of omitted variable on the efficiency of the estimator is small. As a result, the general equilibrium estimator substantially outperforms the fixed effects estimator as long as there are a sufficiently large number of countries to allow for the indirect identification of the effect of the trade agreement.

³³We choose to add the omitted variables to period 1 trade flows rather than period 0 trade flows in order to introduce bias into the elasticity calculations used for the general equilibrium estimator.

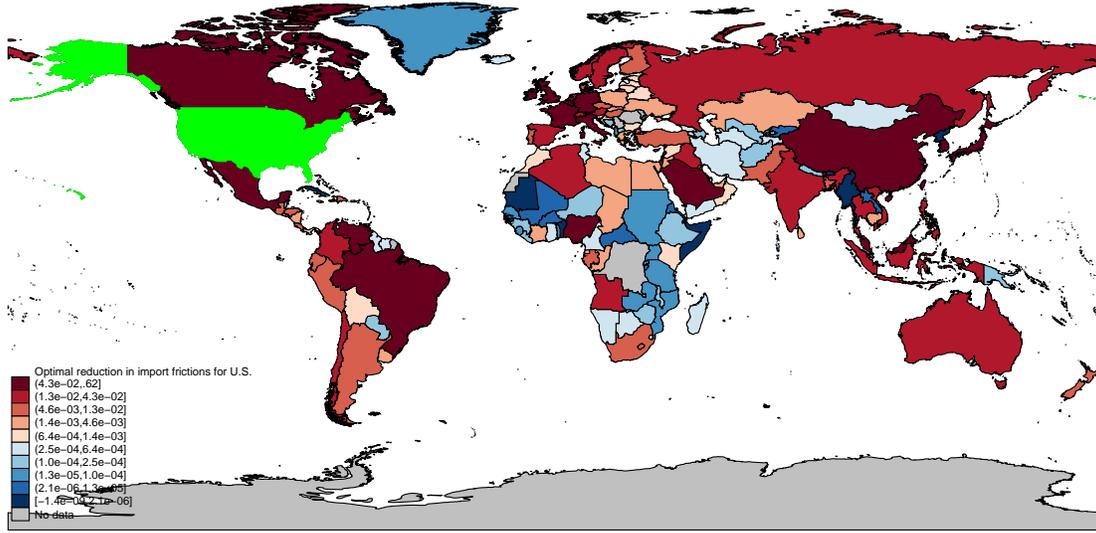
³⁴Because we interpret the error term as measurement error, this procedure should be interpreted as capturing the possibility that countries who sign trade agreements have *observed* trade flows that are on average 5 percent higher than their actual trade flows, while their actual trade flows are affected only by the trade agreement. If we replace the measurement error with an endogeneous error term in the change in bilateral trade frictions (i.e. $\hat{K}_{ij} = \hat{T}'_{ij}\gamma + \varepsilon_{ij}$ where $E[\hat{T}'_{ij}\varepsilon_{ij}] \neq 0$), the general equilibrium estimator excluding the “switchers” still outperforms the fixed effects estimator, although the differences in efficiency are less stark since the endogeneous error term in this case also has a general equilibrium effect on trade flows between all other locations.

Table 2: Monte Carlo results

Measurement error	No omitted variable bias								
	10 countries			20 countries			50 countries		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Size of the effect	FE	GE	GE - no switchers	FE	GE	GE - no switchers	FE	GE	GE - no switchers
0.05	0.1259	0.1044	0.8973	0.0322	0.0276	0.1112	0.0050	0.0051	0.0235
0.1	0.0610	0.0554	0.3508	0.0163	0.0147	0.0603	0.0026	0.0064	0.0283
0.2	0.0307	0.0279	0.1453	0.0079	0.0124	0.0603	0.0013	0.0108	0.0539
0.05	0.2486	0.2058	1.1630	0.0636	0.0548	0.2279	0.0104	0.0095	0.0392
0.1	0.1222	0.1045	0.6837	0.0318	0.0276	0.1050	0.0051	0.0072	0.0327
0.2	0.0614	0.0548	0.2739	0.0153	0.0161	0.0757	0.0026	0.0107	0.0546
0.05	0.6359	0.5353	3.5137	0.1655	0.1373	0.5757	0.0251	0.0207	0.0930
0.1	0.3140	0.2625	1.5472	0.0824	0.0695	0.2950	0.0143	0.0131	0.0519
0.2	0.1594	0.1269	1.2624	0.0401	0.0356	0.1429	0.0062	0.0120	0.0589
	Omitted variable bias								
0.05	1.0142	1.1612	1.1900	1.0025	1.2222	0.1206	1.0000	1.2655	0.0606
0.1	0.5034	0.5828	0.2965	0.4988	0.6124	0.0904	0.5000	0.6371	0.0714
0.2	0.2515	0.2937	0.1941	0.2494	0.3116	0.0965	0.2500	0.3248	0.0959
0.05	1.0541	1.1929	1.1386	1.0042	1.2272	0.2579	1.0006	1.2675	0.0670
0.1	0.5219	0.5953	0.6438	0.5020	0.6153	0.1205	0.4996	0.6376	0.0725
0.2	0.2593	0.2976	0.4048	0.2508	0.3111	0.1087	0.2500	0.3251	0.0964
0.05	1.1800	1.2649	2.5031	1.0198	1.2353	0.5967	1.0021	1.2672	0.1024
0.1	0.6120	0.6554	1.5613	0.5077	0.6190	0.2919	0.4999	0.6363	0.0848
0.2	0.2918	0.3171	0.7293	0.2545	0.3156	0.1625	0.2497	0.3248	0.0994

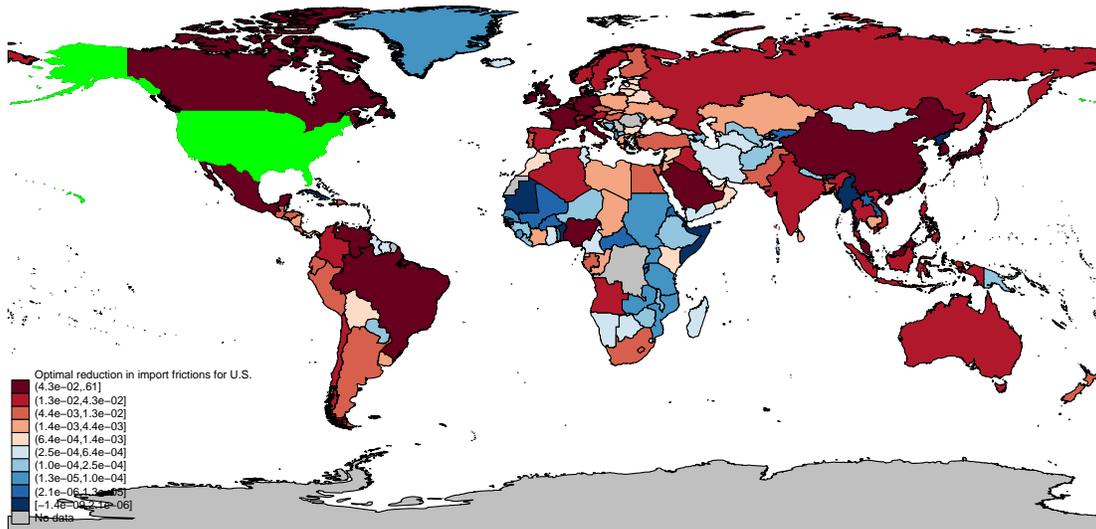
Notes: This table shows the coefficient of variation of the root mean squared deviation of three estimators of the effect of a multilateral trade agreement using a Monte Carlo procedure. Values in bold indicate which estimator is most precise for a given number of countries, size of measurement error, and size of effect. (We use the coefficient of variation of the RMSE because unlike the RMSE itself, the CV(RMSE) is invariant to the effect size being estimated.) The coefficient of variation of the root mean squared deviation is calculated from 500 simulations of the model, where in each simulation, the set of initial bilateral frictions, the exogenous income shifters B_i , and both the set of countries already in the multilateral trade agreement and the new members are randomly generated. We assign half of the countries to be existing members of the trade agreement and ten percent of countries to be new members. In the top panel, the generated measurement error is independent of whether or not a country is a new member of the multilateral trade agreement. In the bottom panel, the generated measurement error, in addition to an i.i.d. term, includes an omitted variable that increases observed trade flows by 0.05 log points when the trade is between a country who joined the trade agreement and either another new or existing trade agreement member. The size of the effect is the percentage reduction in bilateral frictions (i.e. an increase in K_{ij}); the size of the measurement error is the standard deviation of the measurement error relative to an average bilateral trade flow. The FE estimator is the standard ordinary least squares estimator of the change in trade flows on the change in trade agreement membership with both origin and destination country fixed effects; the GE estimator is the estimator introduced in the text which directly accounts for the general equilibrium effects through the network structure of trade; the GE - no switchers estimator is the GE estimator only identified off of the change in trade flows between countries who did not change their trade agreement membership. To calculate the equilibrium and elasticities, we assume gravity constants $\alpha = -2/3$, $\beta = -1/3$ which in an intermediate goods trade model correspond to a trade elasticity of 4 and a labor share in the production function of 1/2.

Figure 9: Optimal unilateral trade friction reduction for the U.S. (EK gravity constants)



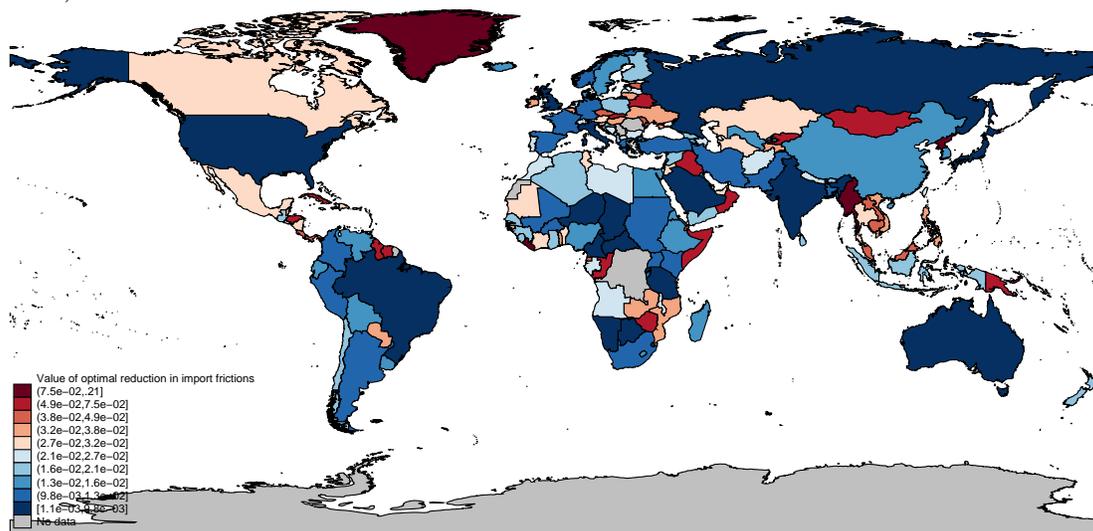
Notes: This figure shows the set of unilateral reductions in import trade frictions for the United States subject to the norm of the total reductions remaining constant that maximizes the welfare of the United States. Countries are sorted by deciles; red indicates a greater reduction in trade frictions while blue indicates a smaller reduction in trade frictions. The gravity constants used are those in [Eaton and Kortum \(2002\)](#).

Figure 10: Optimal unilateral trade friction reduction for the U.S. (AL gravity constants)



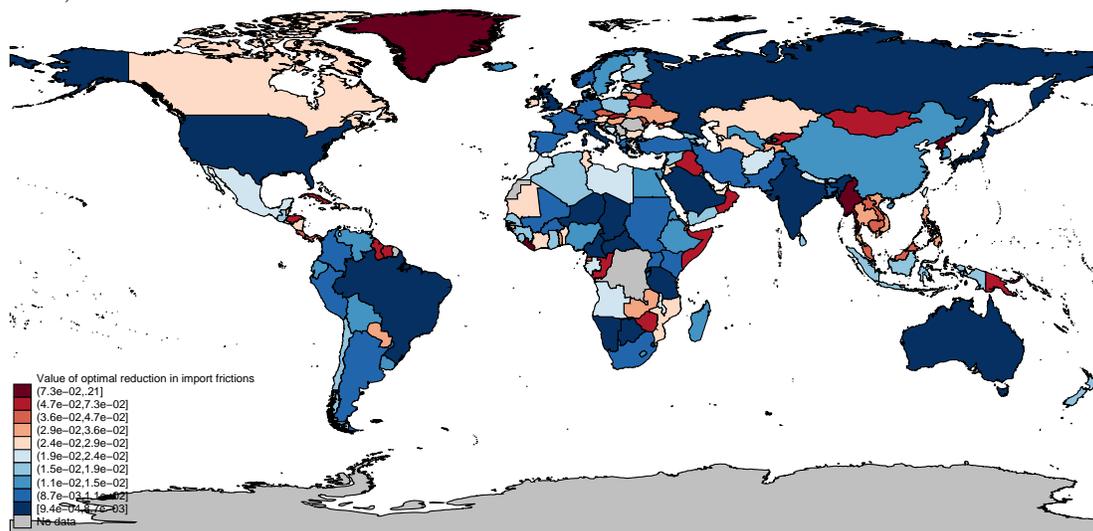
Notes: This figure shows the set of unilateral reductions in import trade frictions for the United States subject to the norm of the total reductions remaining constant that maximizes the welfare of the United States. Countries are sorted by deciles; red indicates a greater reduction in trade frictions while blue indicates a smaller reduction in trade frictions. The gravity constants used are those in [Alvarez and Lucas \(2007\)](#).

Figure 11: Potential welfare gains from unilateral trade friction reductions (EK gravity constants)



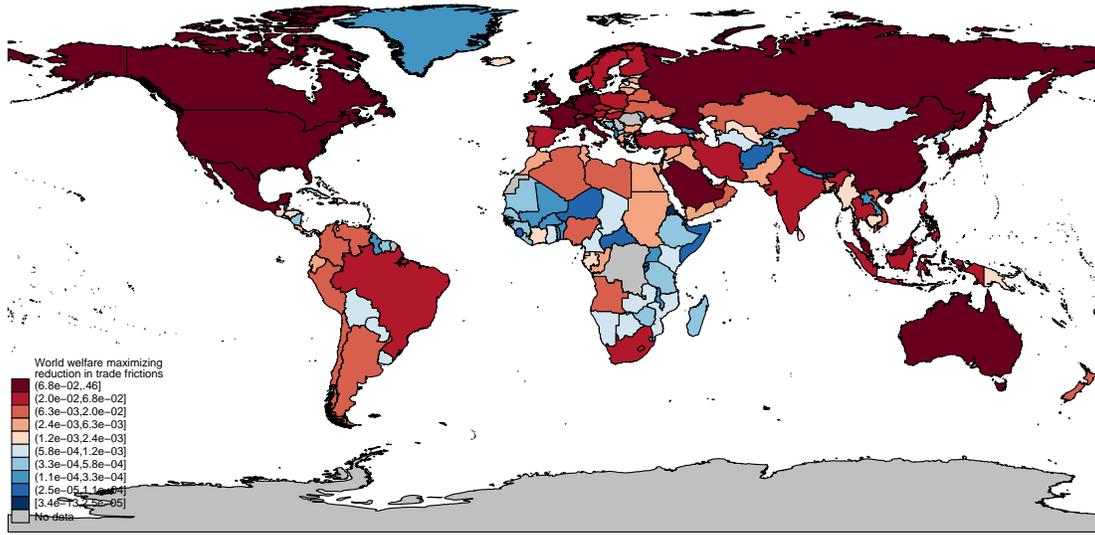
Notes: This figure shows the elasticity of each country's welfare to increasing the amount of unilateral trade friction reductions in the optimal way (i.e. the Lagrange multiplier of equation (25)). Countries are sorted by deciles; red indicates a greater potential welfare gain while blue indicates a smaller potential welfare gain. The gravity constants used are those in [Eaton and Kortum \(2002\)](#).

Figure 12: Potential welfare gains from unilateral trade friction reductions (AL gravity constants)



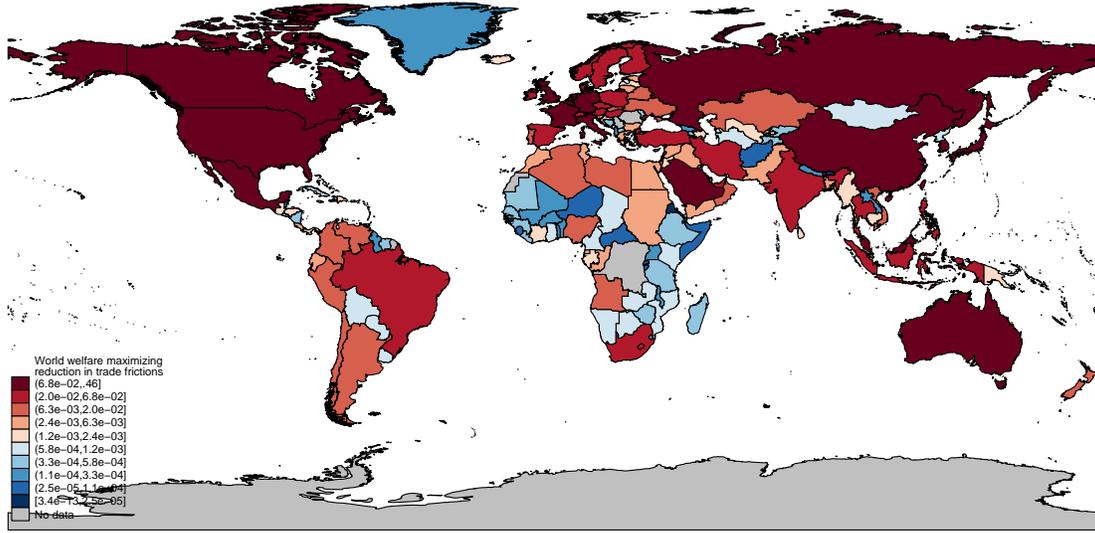
Notes: This figure shows the elasticity of each country's welfare to increasing the amount of unilateral trade friction reductions in the optimal way (i.e. the Lagrange multiplier of equation (25)). Countries are sorted by deciles; red indicates a greater potential welfare gain while blue indicates a smaller potential welfare gain. The gravity constants used are those in [Alvarez and Lucas \(2007\)](#).

Figure 13: World optimal multilateral trade friction reduction (EK gravity constants)



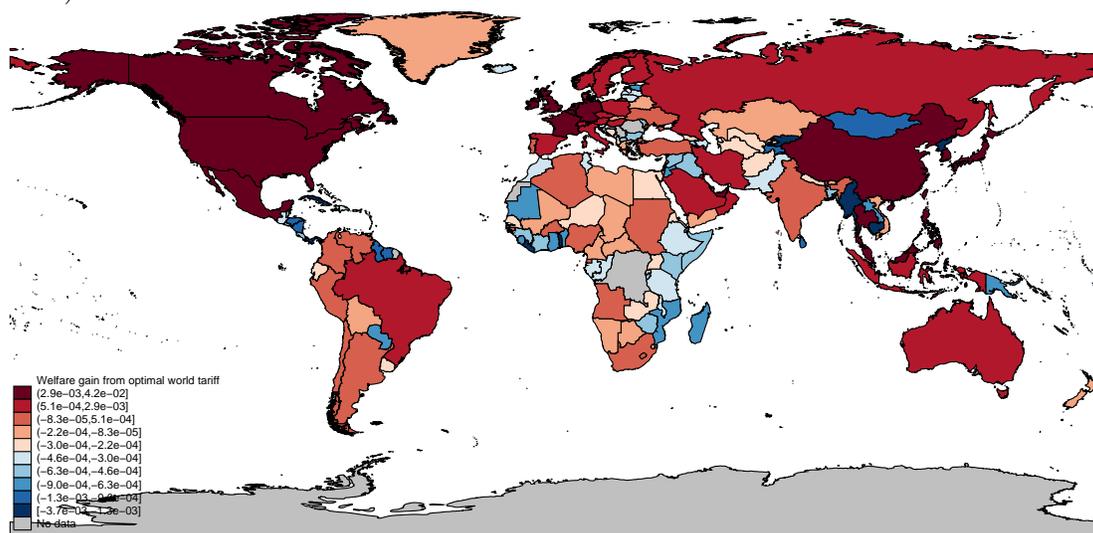
Notes: This figure shows the set of country reductions in trade frictions (subject to the total reduction of bilateral frictions being constant) that maximizes the world welfare (where the country Pareto weights are those imposed by the competitive equilibrium). Countries are sorted by deciles; red indicates a greater reduction in trade frictions while blue indicates a smaller reduction (or even increase) in trade frictions. The gravity constants used are those in [Eaton and Kortum \(2002\)](#).

Figure 14: World optimal multilateral trade friction reduction (AL gravity constants)



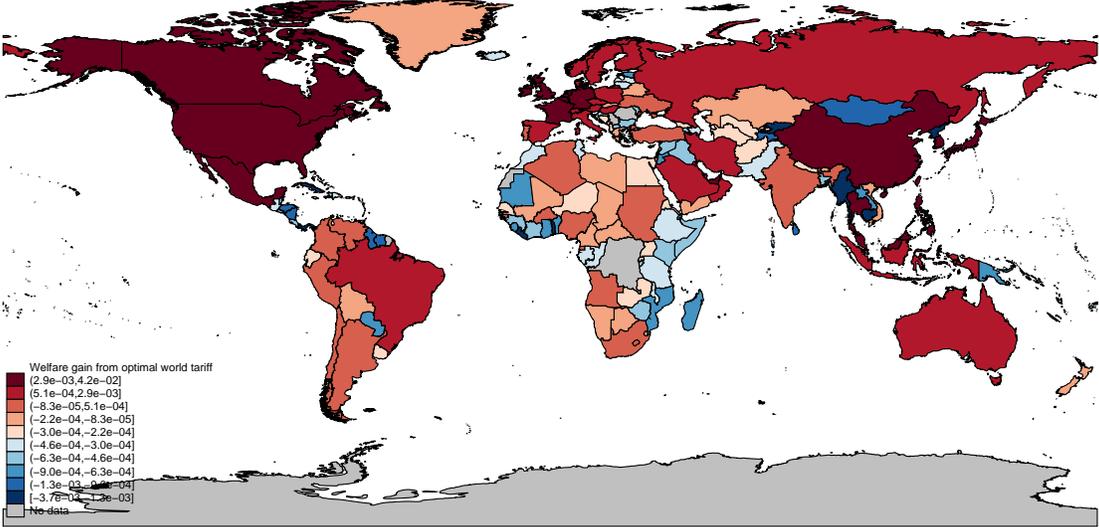
Notes: This figure shows the set of country reductions in trade frictions (subject to the total reduction of bilateral frictions being constant) that maximizes the world welfare (where the country Pareto weights are those imposed by the competitive equilibrium). Countries are sorted by deciles; red indicates a greater reduction in trade frictions while blue indicates a smaller reduction (or even increase) in trade frictions. The gravity constants used are those in [Alvarez and Lucas \(2007\)](#).

Figure 15: Welfare gains from world optimal multilateral trade friction reduction (EK gravity constants)



Notes: This figure shows distribution of welfare gains from an optimal multilateral trade friction. In particular, we report the welfare gain each country would achieve if all countries in the world were to alter their trade frictions in order to maximize world welfare (where the country Pareto weights are those imposed by the competitive equilibrium). Countries are sorted by deciles; red indicates a greater increase in welfare while blue indicates a smaller increase in welfare. The gravity constants used are those in [Eaton and Kortum \(2002\)](#).

Figure 16: Welfare gains from world optimal multilateral trade friction reduction (AL gravity constants)



Notes: This figure shows distribution of welfare gains from an optimal multilateral trade friction. In particular, we report the welfare gain each country would achieve if all countries in the world were to alter their trade frictions in order to maximize world welfare (where the country Pareto weights are those imposed by the competitive equilibrium). Countries are sorted by deciles; red indicates a greater increase in welfare while blue indicates a smaller increase in welfare. The gravity constants used are those in [Alvarez and Lucas \(2007\)](#).