Online Appendix of
Structural Change with Long-Run Income and Price Effects

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A Equilibrium Dynamics and Asymptotics in Continuous Time

This section characterizes the dynamic and asymptotic properties of a general version of the economy introduced in Section 2. Our main result, stated in Proposition 2, shows that the economy converges to a path of constant real consumption growth under milder conditions than the ones analyzed in Section 2. It also provides a sharp characterization of the constant growth path as a function of sectoral income elasticities, sectoral capital shares and productivity growth of different sectors. Moreover, we show that these results hold for a wider class of models. In particular, we introduce a generalization of the baseline nonhomthetic CES aggregator that allows us to establish a precise connection to empirical measures of the growth rate of real aggregate consumption as defined in the Penn World Table 8.0 (Feenstra et al., 2013).

Notation Henceforth, we denote by η_f^x the elasticity of a (potentially multivariate) function f with respect to its argument x, i.e., $\eta_f^x \equiv \partial \log f / \partial \log x$. When function f is defined over a single variable x, we simply refer to the elasticity function as η_f. Moreover, we use bold face notation to indicate a collection of sectoral variables. For instance, $\boldsymbol{p}(t)$ stands for the set of sectoral prices at time t, that is, $\{p_i(t)\}_{i=1}^I$.

A.1 Household Preferences and Demand

Consider a unit mass of households with identical preferences over a stream of real consumption per capita $[c(t)]_{t=0}^{\infty}$, defined as

$$U(0) \equiv \int_0^{\infty} e^{-(\rho-n)t} u(c(t))\, dt, \tag{A.1}$$

where $u(\cdot)$ is the instantaneous utility function, $\rho > 0$ is the discount rate, and $n \geq 0$ denotes population growth. We make the standard assumption that $n < \rho$, and that the instantaneous utility function u is asymptotically isoelastic, that is, $\lim_{c\to\infty} \eta_u \equiv cu'/u = 1 - \theta$ with $\theta > 0$. Households inelastically supply labor, $L(t) \equiv L(0) e^{nt}$.

Per capita real consumption $c(t)$ aggregates cosumption of a bundle $\boldsymbol{c}(t) = \{c_i(t)\}_{i=1}^I$ of goods according to generalized nonhomothetic CES preferences defined implicitly through

$$\sum_{i=1}^{I} \left[g(c(t))^{-\varepsilon_i} c_i(t) \right]^{\frac{\sigma-1}{\sigma}} = 1, \tag{A.2}$$

where each consumption good is characterized by an income elasticity parameter $\varepsilon_i > 0$ and $\sigma \in (0,1)$ denotes the elasticity of substitution. Function $g(\cdot)$ is a positive-valued and

monotonic function and will be characterized in what follows.[1]

When consumers face prices $\boldsymbol{p} \equiv \{p_i\}_{i=1}^{I}$, the expenditure function corresponding to the preferences above is be given by (see derivations in the main Appendix A)

$$e\left(c;\boldsymbol{p}\right) \equiv \left(\sum_{i=1}^{I}\left[g\left(c\right)^{\varepsilon_i} p_i\right]^{1-\sigma}\right)^{\frac{1}{1-\sigma}}. \tag{A.3}$$

Correspondingly, the consumption expenditure share of good i is given by

$$\omega_i\left(c;\boldsymbol{p}\right) \equiv \frac{p_i c_i}{e\left(c;\boldsymbol{p}\right)} = \left(\frac{g\left(c\right)^{\varepsilon_i} p_i}{e\left(c;\boldsymbol{p}\right)}\right)^{1-\sigma}. \tag{A.4}$$

For any variable that varies across sectors, e.g., income elasticity parameters ε_i, we can define expenditure-weighted averages. For instance, we define the economy-wide average income elasticity paramter as

$$\overline{\varepsilon\left(c;\boldsymbol{p}\right)} \equiv \sum_{i=1}^{I}\omega_i\left(c;\boldsymbol{p}\right)\varepsilon_i. \tag{A.5}$$

In the remainder of the paper, we will drop the dependence of expenditure-weighted average functions on the prices whenever the corresponding prices are clear from the context.

The real consumption aggregator defined by equation (A.2) characterizes the consumer's instantaneous utility function up to a monotonic transformation implied by $g(\cdot)$. In order for our concept of real aggregate consumption $c(t)$ to correspond to most empirical measures of real income, the choice of $g(\cdot)$ should express consumer utility in terms of equivalent expenditure in constant prices. Accordingly, we choose here the function $g\left(\cdot\right)$ in such a way that aggregate consumption per capita c is expressed in terms of constant prices $\boldsymbol{q}$, i.e., in real terms. We define $g(\cdot)$ implicitly through:

$$c \equiv \left(\sum_{i=1}^{I}\left[g\left(c\right)^{\varepsilon_i} q_i\right]^{1-\sigma}\right)^{\frac{1}{1-\sigma}}, \tag{A.6}$$

that is, we let $g \equiv e^{-1}\left(\cdot;\boldsymbol{q}\right)$. With this definition, Equation (A.2) defines $c(t)$ as an aggregator of a bundle $\{c_i\left(t\right)\}_{i=1}^{I}$ expressed in terms of the cost of an optimal bundle when consumers face given constant prices $\boldsymbol{q}$.

The next proposition, characterizes the solution to the household problem.

[1] To find the basic isoelastic nonhomothetic CES aggregator presented in Section 2, one can simply replace $g(c) \to c$ and $\varepsilon_i \to \frac{\epsilon_i - \sigma}{1-\sigma}$. We focus our attention to the empirically relevant case (at least, for three sectors) where the elasticity of substitution is not greater than unity to avoid a taxonomical analysis, but these preferences are well defined for $\sigma > 1$ (see main Appendix A) and, as it will become clear in the analysis that follows, the theoretical results extend to $\sigma > 1$.

Proposition 1. *Consider the household's problem of maximizing* (A.1) *where the aggregator is defined by Equations* (A.2) *and* (A.6)*, subject to the flow budget constraint*

$$\dot{a}(t) = w(t) + [r(t) - n]\, a(t) - e(c(t); \boldsymbol{p}(t)), \tag{A.7}$$

and the No-Ponzi condition

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t \left(r\left(t'\right) - n\right) dt'\right) \geq 0, \tag{A.8}$$

for some path of wage $w(\cdot)$, interest rate $r(\cdot)$, and sectoral prices $\boldsymbol{p}(\cdot)$. Then, the Euler Equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \overline{\frac{\dot{p}(t)}{p(t)}} - (1-\sigma) Cov(\frac{\varepsilon_i}{\overline{\varepsilon(c(t))}}, \frac{\dot{p}_i(t)}{p_i(t)})}{\eta^c_{e_c}(c(t); t) - \eta_{u'}(c(t))}, \tag{A.9}$$

along with the transversality condition

$$\lim_{t\to\infty} a(t)\, e^{-(\rho-n)t} \frac{c(t)}{e(c(t); \boldsymbol{p}(t))} \frac{\overline{\varepsilon(c; \boldsymbol{q})}}{\overline{\varepsilon(c; \boldsymbol{p}(t))}} = 0, \tag{A.10}$$

characterize necessary conditions for any paths of consumption and assets to be the solution to the household problem. In the Euler Equation above $\overline{\dot{p}/p}$ and $\overline{\varepsilon}$ are expenditure-weighted sectoral averages at price $\boldsymbol{p}(t)$, and the elasticity of marginal expenditure is given by

$$\begin{aligned} \eta^c_{e_c}(c(t); t) \;=\; & \left(\frac{\overline{\varepsilon(c; \boldsymbol{p}(t))}}{\overline{\varepsilon(c; \boldsymbol{q})}} - 1\right)\left[1 + (1-\sigma)\left(\frac{Var(\varepsilon; c, \boldsymbol{p}(t))}{\overline{\varepsilon(c; \boldsymbol{p}(t))}^2}\right)\right], \\ & + (1-\sigma)\left[\frac{Var(\varepsilon; c, \boldsymbol{p}(t))}{\overline{\varepsilon(c; \boldsymbol{p}(t))}^2} - \frac{Var(\varepsilon; c, \boldsymbol{q})}{\overline{\varepsilon(c; \boldsymbol{q})}^2}\right]. \end{aligned} \tag{A.11}$$

Furthermore, let $\Delta \equiv \varepsilon_{max}/\varepsilon_{min}$ be the ratio of the largest to the smallest sectoral elasticity parameters correpsonding to the household preference's aggregator in Equation (A.2)*, and suppose the elasticity of intertemporal substition is bounded above by $1/\underline{\theta}$, that is, $\eta_{u'}(c) < -\underline{\theta}$ for all c. If the following inequality is satisfied*

$$\underline{\theta} > (\Delta - 1)\left[\frac{1}{\Delta} + \frac{1-\sigma}{4}(\Delta - 1)\right], \tag{A.12}$$

then the household problem has a unique solution, fully characterized by the Euler equation and the transversality conditions above.

The proposition above establishes that under mild conditions on the concavity of instantaneous utility function $u(\cdot)$, the household problem has a unique optimum that can be found by solving an Euler equation. First, note that if all income elasticity parameters are the same,

then the real consumption elasticity of the marginal expenditure in Equation (A.11) becomes zero and the Euler equation reduces to its familiar form of standard homothetic preferences with heterogenous technological growth growth rates across consumption sectors (see, e.g., Ngai and Pissarides, 2007).

When income elasticity parameters are heterogenous across sectors, we have two distinct modifications to the standard Euler equation. To unpack these two different modifications, note that $\left(\eta^{c}_{e_c} - \eta_{u'}\right) \frac{\dot{c}}{c}$ is the rate of decline in the marginal utility of consumer spending (expenditure) at constant prices, which in turn depends on the concavity of the instantaneous utility function and the expenditure function. Accordingly, the numerator of equation (A.9) describes the growth rate of the marginal utility of consumer spending, while the denominator expresses the ratio between growth rates of consumer spending and real aggreate consumption in terms of base prices $\boldsymbol{q}$.

First, due to nonhomotheticity, the expenditure function is a nonlinear function of real consumption. Therefore, the denominator in the right hand side of the Euler equation (A.9) includes an adjustment term that reflects the convexity of the expenditure function. The adjustment effectively increases the concavity of the instantaneous utility function by the degree of the convexity of the expenditure function. Equation (A.11) shows that this nonhomotheticity adjustment depends on the mean and the variance of the income elasticity parameters with distributions implied by expenditure shares, under current and base prices. The larger the mean and the variance of the income elasticity parameters under current prices relative to base prices, the larger is this adjustment. Intuitively, the components of consumer's expenditure corresponding to goods with higher income elasticity have higher convexity in real aggregate consumption. When the income of a typical consumer grows, she spends a larger share of her income on more income elastic goods. As a result, the expenditure function as a whole becomes a more convex function of real aggregate consumption.

The last term on the numerator of Euler equation (A.9) accounts for the interaction of income elasticity and growth rates of sectoral prices. If the rates of growth in sectoral prices are positively correlated with income elasticity parameters, when (real) income grows consumers have to shift a larger share of their expenditure toward *more expensive* goods. This effectively reduces the growth rate of their real consumption.

We emphasize that equations (A.9), (A.10), (A.11), and (A.12) are all invariant to common scaling of sectoral elasticity parameters ε_i's. Therefore, our choice of cardinality for function $g(\cdot)$ in equation (A.6) highlights (and pins down) the one degree of freedom that we face in our choice of sectoral elasticity parameters.

A.2 Production

The production side of our model includes sectoral heterogeneity in rates of technological progress, analyzed first by Ngai and Pissarides (2007), as well as sectoral heterogeneity in factor intensities, studied first in a two-sector setting by Acemoglu and Guerrieri (2008). We show that our growth model remains fully tractable when we incoporate both these supply side channels.

Capital is accumulated using investment goods produced by sector $i = 0$,

$$\dot{K}(t) = Y_0(t) - \delta K(t). \tag{A.13}$$

Labor and capital are combined by producers of consumption good sectors $i \in \{1, \ldots, I\}$ to produce output using a Cobb-Douglas technology

$$Y_i(t) = A_i(t) L_i(t)^{1-\alpha_i} K_i(t)^{\alpha_i}, \qquad \text{for } i \in \{0, \ldots, I\}, \tag{A.14}$$

where the production function in sector i is characterized with sector-specific capital intensity $\alpha_i \in (0, 1)$. The Hicks-neutral technological progress $A_i(t)$ grows exogenously with the (potentially time varying) rate $\gamma_i(t)$ that asymptotically becomes constant, i.e., rate $\lim_{t\to\infty} \dot{A}_i(t)/A_i(t) = \gamma_i > 0$ for all sectors i.

Since we assume competitive labor and capital markets, the marginal revenue product of labor and capital have to equate their respective prices, that is,

$$w(t) = (1-\alpha_i)\frac{p_i(t) Y_i(t)}{L_i(t)}, \tag{A.15}$$

$$R(t) = \alpha_i \frac{p_i(t) Y_i(t)}{K_i(t)}. \tag{A.16}$$

We define sectoral capital-labor ratios as

$$\kappa_i(t) \equiv \frac{K_i(t)}{L_i(t)} = \frac{\alpha_i}{1-\alpha_i}\frac{w(t)}{R(t)}, \qquad \text{for } i \in \{0, \ldots, I\}. \tag{A.17}$$

Equation (A.17) shows that sectoral capital-labor ratios are proportional to each other, and to the relative price of labor to capital.

As before, we normalize the price of the investment sector in each period to unity, $p_0(t) \equiv 1$. We find the prices of sectoral consumption goods by equalizing marginal rev-

enue products of capital from equation (A.16)

$$\begin{aligned} p_i(t) &= \frac{\alpha_0^{\alpha_0}(1-\alpha_0)^{1-\alpha_0}}{\alpha_i^{\alpha_i}(1-\alpha_i)^{1-\alpha_i}} \cdot \frac{A_0(t)}{A_i(t)} \cdot \left(\frac{w(t)}{R(t)}\right)^{\alpha_0-\alpha_i}, \\ &= \left(\frac{\alpha_0}{\alpha_i}\right)^{\alpha_i} \left(\frac{1-\alpha_0}{1-\alpha_i}\right)^{1-\alpha_i} \frac{A_0(t)}{A_i(t)} \kappa_0(t)^{\alpha_0-\alpha_i}, \end{aligned} \tag{A.18}$$

where in the second equality we have substituted for relative price of inputs from equation (A.17), and κ_0 denotes the capital-labor ratio in the investment sector. Equation (A.18) shows that consumption good prices depend only on sectoral TFPs and the capital-labor ratio in the investment sector. As expected, goods produced by sectors with lower TFPs are more expensive. The dependence of prices on capital-labor ratio in the investment sector in equation (A.18) is a proxy for their dependence on the rental price of capital. Goods produced by sectors with higher capital intensities become more expensive as capital-labor ratios rise and the rental price of capital falls.

Equation (A.18) illustrates how both supply-side forces driving structural change appear through sectoral prices. A higher rate of technological progress in sector i (relative to the investment sector) is a force lowering the price in this sector, the mechanism featured in the model of Ngai and Pissarides (2007). As the economy accumulates capital, the capital-labor ratio grows proportionally in all sectors. A higher capital intensity in sector i (relative to the investment sector) is an alternative force lowering the price in this sector, one formalized in the model by Acemoglu and Guerrieri (2008).

A.3 Competitive Equilibrium and Equilibrium Dynamics

Market clearing implies that for all $t \geq 0$,

$$L(t) = \sum_{i=0}^{I} L_i(t) \tag{A.19}$$

$$K(t) = \sum_{i=0}^{I} K_i(t) = a(t)L(t), \tag{A.20}$$

$$p_i(t)Y_i(t) = \omega_i(t)\, e(t)\, L(t), \tag{A.21}$$

where $\omega_i(t) = \omega_i(c(t); \boldsymbol{p}(t))$ and $e(t) = e(c(t); \boldsymbol{p}(t))$ are the expenditure share and expenditure functions as defined in equations (A.3) and (A.4). Equation (A.21) connects the production allocations to the nonhomothetic CES demand system. In particular, this equation characterizes the total sectoral outputs that, together with prices of labor and capital, pin down equilibrium sectoral allocations of labor and capital from equations (A.15) and (A.16). More-

over, the total value of output at time t is given by

$$\begin{aligned} Y(t) &\equiv \sum_{i=0}^{I} p_i(t) Y_i(t), \\ &= Y_0(t) + L(t) e(t). \end{aligned}$$

An equilibrium path in our economy consists of the path $[c(\cdot), a(\cdot), \boldsymbol{\omega}(\cdot), \boldsymbol{Y}(\cdot), \boldsymbol{K}(\cdot), \boldsymbol{L}(\cdot)]_{t=0}^{\infty}$ of per capita real consumption and assets, sectoral shares in consumption expenditure, and sectoral outputs and allocations of capital and employment, along with the path $[r(\cdot), R(\cdot), w(\cdot), \boldsymbol{p}(\cdot)]_{t=0}^{\infty}$ of real interest rate, rental price of capital, wage, and sectoral output prices satisfying equations (A.4), (A.9), (A.10), (A.13), (A.14), (A.19), (A.20), (A.21) and $r(t) = R(t) - \delta$ for all $t \geq 0$.

The next proposition characterizes the asymptotic properties of the competitve equilibrium of our economy.

Proposition 2. *Let constant γ^* be defined as*

$$\gamma^* \equiv \min_{i \in \mathcal{I}/\{0\}} \frac{(1-\alpha_0)\gamma_i + \alpha_i \gamma_0}{(1-\alpha_0)\varepsilon_i/\varepsilon_{max}}, \tag{A.22}$$

where ε_{max} is the maximum among all income elasticity parameters. Assume that condition (A.12) is satisfied (which ensures that the instantaneous utility function defined in equations (A.1), (A.6), and (A.2) is concave in real consumption c), and that

$$\rho > n + (1-\theta)\gamma^*. \tag{A.23}$$

Then, there exists a unique competitive equilibrium path for our economy, along which per capita real consumption asymptotically grows at a constant rate

$$\lim_{t\to\infty} \frac{\dot{c}(t)}{c(t)} = \gamma^*. \tag{A.24}$$

Let $\mathcal{I}^*$ be the set of consumption sectors achieving the minimum in equation (A.22). Asymptotically, the share of sectors belonging to this set in employment and consumption expenditure converges to a time-constant distribution. The employment and consumption expenditure shares of any sector i outside the set $\mathcal{I}^*$ vanishes at a rate

$$\lim_{t\to\infty} \frac{\dot{l}_i(t)}{l_i(t)} = \lim_{t\to\infty} \frac{\dot{\omega}_i(t)}{\omega_i(t)} = -(1-\sigma)\left(\gamma_i + \frac{\alpha_i}{1-\alpha_0}\gamma_0 - \frac{\varepsilon_i}{\varepsilon_{max}}\gamma^*\right). \tag{A.25}$$

Proposition 2 states the key asymptotic properties of our economy. First, as with standard growth models, the equilibrium is unique and the rate of growth of real consumption and real

interest rate coverge to constant values. More notably, the economy asymptotically features realloction of labor across consumption sectors: while some sectors converge to comprising constant shares, others shrink at a constant rate. Crucially, the rate of growth of real consumption and the set of sectors that do not vanish asymptotically are determined through a combination two forces: 1) the supply-side subsitition forces, as captured by the sectoral rates of technical growth and capital intensities in the numerator of equation (A.22), and 2) the demand-side income forces, as captured by the preference elasticity parameter in the denominator of the same equation. This relation generalizes and encompasses the results of both Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008) and unifies them with our account of long-run demand nonhomotheticity.

Here, we offer some basic intuition for the results and leave a detailed proof of the proposition to Appendix A.4. Define the productivity-adjusted capital-labor ratio in the investment sector as

$$\tilde{\kappa}_0(t) \equiv \frac{K_0(t)}{A_0(t)^{\frac{1}{1-\alpha_0}} L_0(t)}. \tag{A.26}$$

This variable has a one-to-one relationship with real interest rate through $r(t) = \alpha_0 \tilde{\kappa}_0(\cdot) t^{\alpha_0 - 1} - \delta$ and, as we will explain below, constitutes the main state variable of the economy. As we expect from a path of asymptotically constant growth, this variable has to converge to a constant. Now, substituting the normalized capital-labor ratio of the investment sector in equation (A.18), we observe that the growth rates of consumption good prices take the form

$$\frac{p_i(t)}{p_i(0)} = \left(\frac{\tilde{\kappa}_0(t)}{\tilde{\kappa}_0(0)}\right)^{\alpha_0 - \alpha_i} e^{-\left(\gamma_i - \frac{1-\alpha_i}{1-\alpha_0}\gamma_0\right)t}. \tag{A.27}$$

Therefore, if the rental price of capital remains constant, price of consumption good i falls at the rate $(\gamma_i - \gamma_0) + \frac{\alpha_i - \alpha_0}{1-\alpha_0}\gamma_0$, where the first terms in the parantheses captures technical growth in sector i and the second term captures the extent to which technical growth in the investment sector differentially impacts growth in sector i through differences in capital intensity.

Since households invest optimally, both investement and household expenditure comprise non-negligible values of the total value as the economy grows. Therefore, total consumption expenditure of households asymptotically grows at the same rate as the rate of growth of the investment sector. Combining these insights and equations (A.27) and (A.4) yields

$$\lim_{t\to\infty} \frac{\dot{\omega}_i(t)}{\omega_i(t)} = (1-\sigma)\left[\eta_g \varepsilon_i \gamma^* - \left(\gamma_i + \frac{\alpha_i}{1-\alpha_0}\gamma_0\right)\right]. \tag{A.28}$$

Since all shares are bounded above by one, the rate of growth of sectoral shares cannot remain asymptotically positive. Therefore, for the equilibrium to be well defined we need to ensure that these rates of growth do not asymptotically exceed 0. This simple rule in fact pins down

the asymptotic rate of growth of real consumption per capita γ^* in equation (A.22), which in general will be different from the rate of growth of per capita consumption expenditure $\frac{\gamma_0}{1-\alpha_0}$.

While Proposition 2 provides a simple account of the asymptotic properties of our model, the dynamics of the equilibrium path in this economy may generally be more complex. However, the model still remains fully tractable and the dynamic equations can be written in terms of a state variable $\tilde{\kappa}_0$, investment-sector capital-labor ratio, and a control variable c, per capita real consumption. Appendix A.5 provides the full derivation of dynamic equations characterizing the competitive equilibrium everywhere along an path with an initial condition $(c(0), \tilde{\kappa}_0(0))$. A system of two linear equations in $\frac{\dot{c}(t)}{c(t)}$ and $\frac{\dot{\tilde{\kappa}}_0(t)}{\tilde{\kappa}_0(t)}$ determines rates of growth of the two variables for $(c(t), \tilde{\kappa}_0(t))$ at time t. Our unified model includes heterogeneity across sectors along three different dimensions, i.e., income elasticity of demand for sectoral outputs, capital intensitiy, and rate of technical growth, the interactions between all these sources of heterogeneity appear in the dynamic equations.

Here, for the sake of brevity, we present the dynamic equations for the special case where $\alpha_i \equiv \alpha$ for all sectors i. This case parallels the workhorse model analyzed by Buera and Kaboski (2009) and Herrendorf et al. (2013) including two competing forces: income non-homotheticity and heterogeneous rates of technological growth. When capital intensities are identical, capital-labor ratios equalize across all sectors in equilibrium and $\tilde{\kappa}_0$ equals the economy-wide capital-labor ratio. Dropping the subscript 0 to reflect this fact, the dynamics of equilibrium paths take the following form:

$$\begin{aligned}
\frac{\dot{c}}{c} &= \frac{\alpha\tilde{\kappa}^{\alpha-1} - (\delta+\rho) + \bar{\gamma}\left(1+(1-\sigma)\,\rho_{\varepsilon,\gamma}\right) - \gamma_0}{-\eta_{u'} - 1 + \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}'} - 1\right)\left(1+(1-\sigma)\,Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)\right) + (1-\sigma)\left(Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right) - Var'\left(\frac{\varepsilon}{\bar{\varepsilon}'}\right)\right)}, \\
\frac{\dot{\tilde{\kappa}}}{\tilde{\kappa}} &= \tilde{\kappa}^{\alpha} - \frac{\tilde{e}}{\tilde{\kappa}} - \left(n+\delta+\frac{\gamma_0}{1-\alpha_0}\right)\tilde{\kappa},
\end{aligned}$$

where $\tilde{\kappa}$ denotes the normalized capital-labor ratio (which is the same across all sectors), $\bar{\cdot}$, $Var(\cdot)$, and $\rho_{\cdot,\cdot}$ indicate mean, variance, and correlation coefficient of sectoral variables with distribution implied by expenditure shares under current prices, while $\bar{\cdot}'$ and $Var'(\cdot)$ denote mean and variance of sectoral variables with distribution implied by expenditure shares under base prices.

A.4 Proofs of Section A

Proof of Proposition 1.

Using an argument similar to the one used for Lemma 1, we can decompose the problem into two intra-temporal and intertemporal problems. To avoid repetition, we focus on the latter, using the definition of the expenditure function in Equation (A.3) as the cost of real consumption $c(t)$ for the representative consumer in terms of the price of investment good at time t.

For a given path of wages $[w(t)]_{t=0}^{\infty}$, rental prices of capital $[r(t)]_{t=0}^{\infty}$, and sectoral good prices $[\boldsymbol{p}(t)]_{t=0}^{\infty}$, the current-value Hamiltonian for the consumer problem (A.1) may be written as:

$$\hat{\mathcal{H}}(t, c(t), a(t), \lambda(t)) \equiv u(c(t)) + \lambda(t)(w(t) + [r(t) - n]a(t) - e(c(t); \boldsymbol{p}(t))).$$

Let us start with the necessary conditions. The FOCs for the Hamiltonian are as follows:

$$\frac{\partial \hat{\mathcal{H}}}{\partial c} = 0 \quad \Rightarrow \quad u'(c) - \lambda \frac{\partial e}{\partial c} = 0, \tag{A.29}$$

$$\frac{\partial \hat{\mathcal{H}}}{\partial a} = (\rho - n)\lambda - \dot{\lambda} \quad \Rightarrow \quad \frac{\dot{\lambda}}{\lambda} = -(r - \rho). \tag{A.30}$$

In addition, we impose that the solution satisfy the transversality condition:

$$\lim_{t \to \infty} e^{-(\rho - n)t} \lambda(t) a(t) = 0. \tag{A.31}$$

Equations (A.29) and (A.30) together with the law of evolution of assets (A.10) and the transversality equation (A.31) characterize paths of per capita real aggregate consumption and asset holdings $[c(\cdot), a(\cdot)]$, and costate $\lambda(\cdot)$ that satisfy necessary conditions for optimality.

Next, we show conditions that ensure the solution above indeed corresponds to the unique solution to the household utility maximization problem. A standard argument (using (A.30) and the No-Ponzi constraint) shows that *for all feasible pairs* $[c(\cdot), a(\cdot)]$, we have that $\lim_{t \to \infty} \exp(-(\rho - n)t)\lambda(t)a(t) \geq 0$. Therefore, we can establish that the pair characterized by Equations (A.29), (A.30), and (A.31) indeed correspond to the optimum if the Hamiltonian is concave in c. Furthermore, since the Hamiltonian is separable in (c, a) and linear a, strict concavity in c implies the uniqueness of the optimum for the household problem. We will come back to the characterization of conditions ensuring the concavity of e at the end of the proof.

From Equation (A.29), we find that

$$\lambda(t) = \frac{u'(c(t))}{\frac{\partial e(c(t); \boldsymbol{p}(t))}{\partial c}} = \frac{u'(c(t))}{\frac{e}{c}\eta_e^c}, \tag{A.32}$$

where $\frac{\partial e}{\partial c}$ is the marginal (dollar) cost of consumption.We can compute the consumption elasticity of expenditure:

$$\begin{aligned}\eta_e^c = \frac{\partial \log e}{\partial \log c} &= \frac{c}{1-\sigma}\frac{\partial}{\partial c}\log\sum_{i=1}^{I}\left(g^{\varepsilon_i}p_i\right)^{1-\sigma}, \\ &= \eta_g \cdot \sum_{i=1}^{I}\varepsilon_i\omega_i,\end{aligned}$$

where the last equation can be explicitly expressed as $\eta_g(c)\cdot\overline{\varepsilon(c;\boldsymbol{p})}$ as a function of real aggregate consumption c and current prices $\boldsymbol{p}$.

To translate the equations above into the Euler format, we need to compute $\dot{\lambda}(t)/\lambda(t)$, which corresponds to the growth in value of income at time t. Using Equation (A.32), we can write the growth in utility value of income as the sum of the contribution of growth of real consumption, decline in the price index, and the decline in the average income elasticity parameter, that is,

$$\frac{\dot{\lambda}}{\lambda} = \eta_{u'}^c\frac{\dot{c}}{c} - \frac{\dot{(e/c)}}{e/c} - \frac{\dot{\eta_e^c}}{\eta_e^c}.$$

First, we simplify the growth of consumption expenditure:

$$\begin{aligned}\frac{\dot{e}}{e} &= \eta_e^c\frac{\dot{c}}{c} + \sum_i\frac{\dot{p}_i}{p_i}\omega_i, \\ &= \eta_g\bar{\varepsilon}\cdot\frac{\dot{c}}{c} + \overline{\frac{\dot{p}_i}{p_i}},\end{aligned}$$

where we have used the fact that $\eta_e^{p_i} = \omega_i$ from Lemma 1 (see below) and have defined $\overline{\frac{\dot{p}_i}{p_i}}$ to be the average rate of growth of prices across sectors, as weighted by their corresponding consumption expenditures. Next, we compute the growth of the real consumption elasticity of expenditure:

$$\begin{aligned}\frac{\dot{\eta_e^c}}{\eta_e^c} &= \frac{\dot{\eta_g}}{\eta_g} + \frac{\dot{\bar{\varepsilon}}}{\bar{\varepsilon}}, \\ &= \eta_{\eta_g}\cdot\frac{\dot{c}}{c} + \eta_{\bar{\varepsilon}}^c\cdot\frac{\dot{c}}{c} + \sum_i\eta_{\bar{\varepsilon}}^{p_i}\left(\frac{\dot{p}_i}{p_i}\right), \\ &= \eta_{\eta_g}\cdot\frac{\dot{c}}{c} + (1-\sigma)\,\eta_g\bar{\varepsilon}\cdot Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right) + (1-\sigma)\sum_i\left(\frac{\epsilon_i}{\bar{\epsilon}}-1\right)\left(\frac{\dot{p}_i}{p_i}\right)\omega_i, \\ &= \eta_{\eta_g}\cdot\frac{\dot{c}}{c} + (1-\sigma)\,\eta_g\bar{\varepsilon}\cdot Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)\cdot\frac{\dot{c}}{c} + (1-\sigma)\,Cov\left(\frac{\varepsilon}{\bar{\varepsilon}},\frac{\dot{p}}{p}\right),\end{aligned}$$

where in the third equality, we have used the results of Lemma 1 substituting for the elasticities of average income elasticity parameter $\bar{\varepsilon}$.

Combining all of the above, we find the Euler equation to be:

$$\frac{\dot{c}}{c} = \frac{r - \rho - \overline{\frac{\dot{p}_i}{p_i}} - (1-\sigma)\, Cov\left(\frac{\varepsilon_i}{\bar{\varepsilon}}, \frac{\dot{p}_i}{p_i}\right)}{-\eta_{u'} - 1 + \eta_{\eta_g} + \eta_g \bar{\varepsilon}\left(1 + (1-\sigma)\, Var\left(\frac{\varepsilon_i}{\bar{\varepsilon}}\right)\right)}. \tag{A.33}$$

The Euler Equation (A.33) is expressed for a general function $g(\cdot)$. Specializing this result to the specific function defined in Equation (A.6), we will now derive the Euler Equation for the real consumption stated in terms of constant prices $\boldsymbol{q}$. Since g is the inverse of the expenditure function at prices $\boldsymbol{q}$, from Equation (A.33), we have:

$$1 = \eta_g(c) \cdot \overline{\varepsilon(c; \boldsymbol{q})},$$

suggesting $\eta_g = \overline{\varepsilon(c; \boldsymbol{q})}^{-1}$. It follows that:

$$\eta_e^c = \frac{\overline{\varepsilon\,(c; \boldsymbol{p})}}{\overline{\varepsilon\,(c; \boldsymbol{q})}},$$

which is positive if we have $\varepsilon_i > 0$ for all sectors. This ensures that the function e is monotonically increasing and one-to-one. This elasticity, which may in general be different from zero, characterizes the way income effects shifts expenditure across sectors with different prices. In particular, whenever the average elasticity parameter $\bar{\varepsilon}$ is higher than the one at base prices, marginal cost of increasing real consumption exceeds the current aggregate price index.

Similarly, substituting for $\boldsymbol{q}$ in Equation (A.38) from Lemma 1 below, we find:

$$\eta_{\eta_g} = -\frac{\partial \log \overline{\varepsilon(c; \boldsymbol{q})}}{\partial \log c} = -(1-\sigma)\left(\frac{\overline{\varepsilon^2(c; \boldsymbol{q})}}{\left(\overline{\varepsilon(c; \boldsymbol{q})}\right)^2} - 1\right),$$

To ensure the sufficiency of the FOCs and the uniqueness of the solution, we need to find conditions under which the Hamiltonian in strictly concave. The second order condition for c is

$$u'' - \lambda \frac{\partial^2 e}{\partial c^2} = \frac{u}{c^2}\eta_u \left(\eta_{u'} - \eta_{\partial e/\partial c}\right),$$

where we have substituted for λ from Equation (A.32). This expression has to always be negative. We need to only focus on the term within the parentheses. The first term on the right hand side, by assumption, is always greater than $\underline{\theta}$. To compute the second term, note

that

$$\begin{aligned}\eta^c_{e_c} \equiv \eta^c_{\partial e/\partial c} &= \frac{c}{e \cdot \eta^c_e / c} \frac{\partial^2 e}{\partial c^2}, \\ &= \eta^c_e + \eta^c_{\eta^c_e} - 1, \end{aligned} \tag{A.34}$$

and $\eta^c_{\eta^c_e}$ is given by Equation (A.12) from Section A.

Finally, we can find η_{η_g} in Equation (A.12) by using the definition of function $g(\cdot)$ in Equation (A.6). Since $e(c; \boldsymbol{q}) = c$, we have that $\eta^c_{\eta^c_e} = 0$ at prices $\boldsymbol{q}$ and therefore:

$$0 = \eta_g \overline{\varepsilon_i} \left[\eta_{\eta_g} + (1-\sigma) \, Var \left(\frac{\varepsilon_i}{\overline{\varepsilon_i(c; \boldsymbol{q})}} \right) \right], \tag{A.35}$$

which implies $\eta_{\eta_g} = -(1-\sigma) \, Var \left(\frac{\varepsilon_i}{\overline{\varepsilon_i(c;\boldsymbol{q})}} \right)$.

Combining all the results above, we find the second order condition to require:

$$\theta + \frac{\overline{\varepsilon(c; \boldsymbol{p})}}{\overline{\varepsilon(c; \boldsymbol{q})}} - 1 + (1-\sigma) \left[\frac{\overline{\varepsilon(c; \boldsymbol{p})}}{\overline{\varepsilon(c; \boldsymbol{q})}} \left(\frac{Var(\varepsilon; c, \boldsymbol{p})}{\overline{\varepsilon(c; \boldsymbol{p})}^2} \right) - \frac{Var(\varepsilon; c, \boldsymbol{q})}{\overline{\varepsilon(c; \boldsymbol{q})}^2} \right] > 0,$$

for all c. Remembering that the variance of any distribution on $\{\varepsilon_i\}_{i=1}^I$ is bounded above by $\frac{1}{4}(\varepsilon_{max} - \varepsilon_{min})^2$, it immediately follows that condition (A.12) is a sufficient condition for the FOC to always characterize an optimal solution.

□

Lemma 1. *The real consumption and sectoral price elasticities of the expenditure function are given by:*

$$\eta^c_e = \eta_g \overline{\varepsilon}, \tag{A.36}$$

$$\eta^{p_i}_e = \omega_i. \tag{A.37}$$

Furthermore, define the average $\overline{x}$ of some sectoral parameters x_i across the consumption sector weighted by expenditure shares:

$$\overline{x} \equiv \sum_{i=1}^{I} x_i \omega_i.$$

The elasticities of this function are given by:

$$\eta^c_{\overline{x}} = \frac{1-\sigma}{\overline{x}} \eta_g Cov(\varepsilon, x), \tag{A.38}$$

$$\eta^{p_i}_{\overline{x}} = (1-\sigma) \left(\frac{x_i}{\overline{x}} - 1 \right) \omega_i. \tag{A.39}$$

Proof. We can compute the consumption elasticity of expenditure:

$$\begin{aligned}\eta_e^c = \frac{\partial \log e}{\partial \log c} &= \frac{c}{1-\sigma}\frac{\partial}{\partial c}\log\sum_{i=1}^{I}\left(g^{\varepsilon_i}p_i\right)^{1-\sigma}, \\ &= \eta_g \cdot \sum_{i=1}^{I}\varepsilon_i\omega_i, \\ &= \eta_g\overline{\varepsilon}.\end{aligned}$$

where the last equation can be explicitly expressed as $\eta_g(c)\cdot\overline{\epsilon(c;\boldsymbol{p})}$ as a function of real aggregate consumption c. Similarly, we can compute the price elasticity of the expenditure function:

$$\begin{aligned}\eta_e^{p_i} = \frac{\partial \log e}{\partial \log p_i} &= \frac{p_i}{1-\sigma}\frac{\partial}{\partial p_i}\log\sum_{j=1}^{I}\left(g^{\varepsilon_j}p_j\right)^{1-\sigma}, \\ &= \left(\frac{g^{\varepsilon_i}p_i}{e}\right)^{1-\sigma}, \\ &= \omega_i.\end{aligned}$$

Next, we use the expressions above to compute the elasticities of the sectoral shares in consumption expenditure. From Equation (A.4) we have:

$$\begin{aligned}\eta_{\omega_i}^c &= (1-\sigma)\left(\eta_g\varepsilon_i - \eta_e^c\right) = (1-\sigma)\eta_g\left(\varepsilon_i - \overline{\varepsilon}\right), \\ \eta_{\omega_i}^{p_j} &= (1-\sigma)\left(\delta_{ij} - \eta_e^{p_j}\right) = (1-\sigma)\left(\delta_{ij} - \omega_j\right),\end{aligned}$$

where δ_{ij} stands for the kronecker delta function.

Finally, we use the elasticities of the expenditures shares to compute elasticities of a general function $\overline{x}$. We find:

$$\begin{aligned}\eta_{\overline{x}}^c &= \frac{1}{\overline{x}}\sum_j x_j\omega_j\eta_{\omega_j}^c, \\ &= \frac{1-\sigma}{\overline{x}}\eta_g\sum_j x_j\omega_j\left(\varepsilon_j - \overline{\varepsilon}\right), \\ &= \frac{1-\sigma}{\overline{x}}\eta_g Cov\left(\varepsilon_i, x_i\right) \\ \eta_{\overline{x}}^{p_i} &= \frac{1}{\overline{x}}\sum_j x_j\omega_j\eta_{\omega_j}^{p_i}, \\ &= \frac{1-\sigma}{\overline{x}}\sum_j x_j\omega_j\left(\delta_{ij} - \omega_i\right), \\ &= (1-\sigma)\left(\frac{x_i}{\overline{x}} - 1\right)\omega_i.\end{aligned}$$

For instance, for $x_i \equiv \varepsilon_i$, we find that:

$$\eta_{\bar{\varepsilon}}^{c} = (1-\sigma)\,\eta_g Cov\left(\frac{\varepsilon_i}{\bar{\varepsilon}}, \varepsilon_i\right) = (1-\sigma)\,\eta_g \bar{\varepsilon} Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right). \tag{A.40}$$

□

Proof of Proposition 2

Our strategy for the proof of this proposition is as follows. To establish the existence and uniqueness of the competitive equilibrium, we invoke the second Welfare Theorem. We formulate the social planner's problem, whose potential solutions have to correspond to different competitive equilibria in our economy. We solve the social planner's problem and show that it has a unique solution, and further establish a direct correspondence between this solution and the competitive equilibrium, which thus has to also be unique.

Let $\hat{u}(c_1, \cdots, c_I) \equiv u(c)$, where c is defined through Equation (A.2), denote the instantaneous utility of the representative household over a bundle of $\boldsymbol{c} = (c_i)_{i=1}^{I}$ per capita consumption of I different goods. The social planner's problem can be stated as the following maximization problem:

$$\max_{\{k_i(\cdot), l_i(\cdot)\}_{i=0}^{I}} \int_0^{\infty} e^{-(\rho-n)t} \hat{u}\left(c_1(t), \cdots, c_I(t)\right),$$

where

$$c_i = A_i k_i^{\alpha_i} l_i^{1-\alpha_i}, \qquad 1 \leq i \leq I, \tag{A.41}$$

$$\dot{k} = A_0 k_0^{\alpha_0} l_0^{1-\alpha_0} - (\delta + n)\,k, \tag{A.42}$$

subject to constraints:

$$\sum_{i=0}^{I} l_i = 1, \tag{A.43}$$

$$\sum_{i=0}^{I} k_i = k, \tag{A.44}$$

and $k(0) = K(0)/L(0)$ and $k(t) \geq 0$ for all t.

The corresponding present value Hamiltonian is given by:

$$\mathcal{H} = \hat{u} + \mu\left(A_0 L k_0^{\alpha_0} l_0^{1-\alpha_0} - (\delta + n)\,k\right),$$

where we substitute for per capita consumption, i.e., Equations (A.41) and (A.44), in the expression for $\hat{u}$ making the latter a function of vectors of sectoral per capita stocks of capital

and employment shares $(\boldsymbol{k},\boldsymbol{l})$.[2]

We can show that the function $\mathcal{M}(k)=\max_{(\boldsymbol{k},\boldsymbol{l})}\hat{\mathcal{H}}$ under the constraints suggested by Equations (A.43) and (A.44) is a strictly concave function of k, if $\hat{u}$ is a strictly concave function of its arguments. To see why, let us define functions $\mathcal{F}(\boldsymbol{k},\boldsymbol{l})\equiv\hat{u}(\boldsymbol{k},\boldsymbol{l})+\mu A_0 L k_0^{\alpha_0} l_0^{1-\alpha_0}$ and $\hat{\mathcal{F}}(\boldsymbol{k})=\max_{\boldsymbol{l}}\mathcal{F}(\boldsymbol{k},\boldsymbol{l})$. First, it is straightforward to see that $\mathcal{F}$ is jointly strictly concave in $(\boldsymbol{k},\boldsymbol{l})$ for $\mu\geq 0$, which implies that $\hat{\mathcal{F}}$ is a strictly concave function of vector $\boldsymbol{k}$. The strict concavity of the latter then implies that $\mathcal{M}(k)=\max_{\sum_i k_i=k}\hat{\mathcal{F}}(\boldsymbol{k})$ is a strictly concave function of k.

Now let us find a candidate solution for the social planner's problem that satisfies the following conditions:

$$-\frac{\partial\mathcal{H}}{\partial k} = (\delta+n)\mu-\varsigma=\dot{\mu}-(\rho-n)\mu, \tag{A.45}$$

$$\frac{\partial\mathcal{H}}{\partial k_i} = \alpha_i\frac{\partial\hat{u}}{\partial c_i}A_i\left(\frac{k_i}{l_i}\right)^{\alpha_i-1}=\varsigma, \qquad 1\leq i\leq I, \tag{A.46}$$

$$\frac{\partial\mathcal{H}}{\partial l_i} = (1-\alpha_i)\frac{\partial\hat{u}}{\partial c_i}A_i\left(\frac{k_i}{l_i}\right)^{\alpha_i}=\xi, \qquad 1\leq i\leq I, \tag{A.47}$$

$$\frac{\partial\mathcal{H}}{\partial k_0} = \alpha_0\mu A_0\left(\frac{k_0}{l_0}\right)^{\alpha_0-1}=\varsigma, \tag{A.48}$$

$$\frac{\partial\mathcal{H}}{\partial l_0} = (1-\alpha_0)\mu A_0\left(\frac{k_0}{l_0}\right)^{\alpha_0}=\xi, \tag{A.49}$$

and the transversality condition

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)k(t)=0, \tag{A.50}$$

where ξ and ς are Lagrange multipliers corresponding to the two constraints Equations (A.41) and (A.44), respectively.

From Equations (A.45) and (A.48) we find:

$$\mu(t)=\mu(0)\exp\left(-\int_0^t\left(\alpha_0\tilde{\kappa}_0\left(t'\right)^{\alpha_0-1}-(\rho+\delta)\right)dt'\right), \tag{A.51}$$

where we have used the productivity adjusted definition of capital-labor ratio:

$$\tilde{\kappa}_0(t)\equiv A_0(t)^{-1/(1-\alpha_0)}\frac{k_0(t)}{l_0(t)}. \tag{A.52}$$

Similarly, we define the economy-wide aggregate capital-labor ratio $\tilde{k}(t)\equiv A_0(t)^{-1/(1-\alpha_0)}k(t)/l(t)$

[2]Let us remember that according to our vector notation $\boldsymbol{k}\equiv(k_0,\cdots,k_I)$ denotes the vector of sectoral per capita stocks of capital, which is distinct from k, the economy-wide total per capita stock of capital.

and rewrite the transversality condition as:

$$\lim_{t\to\infty} \tilde{k}(t)\,\mu(0)\exp\left(-\int_0^t \left(\alpha_0\tilde{\kappa}_0\left(t'\right)^{\alpha_0-1} - \left(\delta+n+\frac{\gamma_0}{1-\alpha_0}\right)\right)dt'\right) = 0, \tag{A.53}$$

where we have used the fact, which we will show later, that asymptotically $k(t) \to \tilde{k}^* e^{\frac{\gamma_0}{1-\alpha_0}t}$. Note that with this transformation Equation (A.42) can be written as:

$$\dot{\tilde{k}}(t) = l_0\tilde{\kappa}_0(t)^{\alpha_0} - \left(n+\delta+\frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}(t). \tag{A.54}$$

Henceforth, we use the notation that $\tilde{x}(t)$ denotes variable $x(t)$ adjusted by productivity in the investment sector, i.e., $\tilde{x}(t) \equiv A_0(t)^{-1/(1-\alpha_0)}\,x(t)$. Dividing (A.47) by (A.46), we find:

$$\frac{1-\alpha_i}{\alpha_i}\tilde{\kappa}_i(t) = \frac{\tilde{\xi}(t)}{\varsigma(t)}, \tag{A.55}$$

suggesting that capital-labor ratios in all sectors are proportional to each other. This relation echoes Equation (A.17), suggesting, as we show below, that ξ and ς correspond to the wage and rental price of capital in a competitive equilibrium.

Starting from any initial value $\mu(0)$, Equation (A.51) along with conditions (A.46) to (A.49) define a unique path for the allocations. The argument follows from the strict concavity of function $\mathcal{F}$ defined earlier, and the intuition is as follows. Equation (A.51) determines $\mu(t)$ and therefore function $\mathcal{F}$ at time t. The optimal allocation at this point it time is simply given by maximizing function $\mathcal{F}$ under constraints (A.43) and (A.43), which has to be unique.

Note that any candidate path that satisfies the conditions above has to further satisfy the following two asymptotic conditions:

$$\lim_{t\to\infty} \tilde{\kappa}_0(t) = \tilde{\kappa}_0^* > 0, \tag{A.56}$$

$$\lim_{t\to\infty} l_0(t) = l_0^*, \qquad 0 < l_0^* < 1, \tag{A.57}$$

implying that the asymptotic capital-labor ratio and employment in the investment sector are constant and interior. If, on the contrary, we asymptotically have $\tilde{\kappa}_0 \to 0$, we can show that $\xi \to 0$ and Equation (A.47) has to be violated.[3] If $\tilde{\kappa}_0 \to \infty$, Equation (A.54) implies that the tranversality condition A.53 has to be violated. Therefore, condition (A.56), has to hold. Now from Equation (A.55) we learn that all sectoral capital-labor ratios asymptote to nonzero constants and therefore $\lim_{t\to\infty}\tilde{k}(t) = \tilde{k}^* = \sum_{i=0}^{I} l_i^*\tilde{\kappa}_i^* > 0$. Hence, from Equation (A.54)

[3] Assume $\tilde{\kappa}_0$ converges to zero exponentially at a constant rate. From Equation (A.51), μ has to converge to zero at the rate of $-\infty$, which implies the same has to be the case for ξ. From Equation (A.47), we need to have that $\partial\hat{u}/\partial c_i$ converges to zero at the rate of $-\infty$, which can hold only if c_i's all grow at an ever increasing rate. This contradicts the initial assumption that capital-labor ratios converge to zero.

we know that $l_0^* > 0$, since otherwise $\tilde{k} \to 0$. Finally, assuming $l_0^* = 1$ would suggest that $\tilde{k}^* = \tilde{\kappa}_0^*$ and $(\tilde{\kappa}_0^*)^{\alpha_0 - 1} = n + \delta + \frac{\gamma_0}{1-\alpha_0}$. Substituting this in Equation (A.53) would violate the transversality condition. Therefore, any candidate path satisfying the conditions above will asymptotically be an interior candidate solution, in the sense that labor-capital ratios grow at the same rate as the rate of technological progress in the investment sector and there is an interior split of employment between the investment and the consumption sector. This implies, as we will see shortly, that the per capita consumption expenditure in the corresponding competitive equilibrium also grows at the same rate as the rate of growth of technology in the investment sector.

Next, we show that the growth of real consumption per capita along any candidate path satisfying the conditions above has to be asymptotically constant. To see this, note that combining Equations (A.41) and (A.55), we find that $c_i = A_0^{\frac{\alpha_i}{1-\alpha_0}} A_i l_i \tilde{\kappa}_0^{\alpha_i}$ for $i \in \{1, \cdots, I\}$. Equation (A.56) then implies:

$$\lim_{t \to \infty} \frac{\dot{c}_i(t)}{c_i(t)} = \gamma_i + \frac{\alpha_i}{1-\alpha_0}\gamma_0 + \hat{\gamma}_i^C, \tag{A.58}$$

where we defined the asymptotic rate of growth of the share of employment in sector i as:[4]

$$\hat{\gamma}_i^C \equiv \lim_{t \to \infty} \frac{\dot{l}_i(t)}{l_i(t)} \leq 0. \tag{A.59}$$

Now, from Equation (A.2), we have that for all $t \geq 0$:

$$\sum_{i=1}^{I} \nu_i \left(\gamma_i + \frac{\alpha_i}{1-\alpha_0}\gamma_0 + g_i - \eta_g \varepsilon_i \frac{\dot{c}}{c} \right) = 0,$$

where we have defined the effective share of sector i in consumption as $\nu_i \equiv \left(g(c)^{-\varepsilon_i} c_i \right)^{(\sigma-1)/\sigma}$.[5] The rate of growth of per capita real consumption is then given by

$$\frac{\dot{c}}{c} = \frac{1}{\eta_g \overline{\varepsilon}} \left(\overline{\gamma} + \frac{\overline{\alpha}}{1-\alpha_0}\gamma_0 + \overline{\hat{\gamma}^C} \right), \tag{A.60}$$

where averages are taken with respect to the distribution implied by $\{\nu_i\}_{i=1}^I$. Equation (A.47) suggests that, up to a constant, consumption share ν_i's are the same as employment shares

[4] Note that Equation (A.57) establishes the total employment share in the consumption sector asymptotically converges to a constant $l_C \to 1 - l_0^*$. However, within the aggregate consumption sector, some sectors could continue to shrink asymptotically which can result in Equation (A.59) taking nonzero values.

[5] As a reminder, share ν_i equals the share of sector i in consumption expenditure $\omega_i \equiv (g(c)^{\varepsilon_i} p_i/e)^{1-\sigma}$ for the corresponding prices in the competitive equilibrium. We introduce a different variable here solely to respect the conceptual distinction between the formulation of the social planner's problem and the competitive equilibrium where the prices are implicit in the former (and the current formulation).

l_i's. To see this, rewrite Equation (A.47) as:

$$\begin{aligned}\xi &= (1-\alpha_i)\, u'(c)\, \frac{\partial c}{\partial c_i}\frac{c_i}{l_i}, \\ &= (1-\alpha_i)\, u'(c)\, \frac{c}{\eta_g \bar{\varepsilon}}\frac{\nu_i}{l_i},\end{aligned}$$

where we have used $\eta_c^{c_i} = \nu_i/\eta_g\bar{\varepsilon}$ from Equation (A.7).

Define set $\mathcal{I}^* \subset \{1, \cdots, I\}$ as the set of consumption sectors with nonzero asymptotic employment shares, i.e., $\mathcal{I}^* = \left\{i | \hat{\gamma}_i^C = 0\right\}$. Consider some sector $i \notin \mathcal{I}^*$, for which $\hat{\gamma}_i^C < 0$ implying that the asymptotic employment share of this sector is zero, i.e., $\lim_{t\to\infty} l_i = 0$. Since consumption shares ν_i's have to grow proportionally to employment shares as we showed above, for any such sector i we have $\lim_{t\to\infty} \nu_i = 0$. It then follows that $\lim_{t\to\infty} \overline{\hat{\gamma}^C} = \sum_{i\in\mathcal{I}^*} \hat{\gamma}_i^C = 0$, and taking the limit of expression (A.60), we find the asymptotic rate of growth of per capital consumption as:

$$\gamma^* \equiv \lim_{t\to\infty} \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta_g \bar{\varepsilon}^*}\left(\bar{\gamma}^* + \frac{\bar{\alpha}^*}{1-\alpha_0}\gamma_0\right),$$

where averages are taken with respect to the distribution implied by $\{\nu_i^*\}_{i=1}^I$ the limit of distribution $\{\nu_i\}_{i=1}^I$, with support $\mathcal{I}^*$.

In order to characterize the set $\mathcal{I}^*$, we need to compute the asymptotic rate of growth of ν_i for each sector i. Substituting Equation (A.55) in Equation (A.47), we find:

$$u'(c)\frac{\partial c}{\partial c_i} = \frac{1}{A_i}\left(\frac{\varsigma}{\alpha_i}\right)^{\alpha_i}\left(\frac{\xi}{1-\alpha_i}\right)^{1-\alpha_i}, \tag{A.61}$$

where the right hand side captures the social cost of producing good i in marginal utility terms at time t. From Equations (A.48) and (A.49), we have:

$$\varsigma = \mu\alpha_0\tilde{\kappa}_0^{\alpha_0-1}, \tag{A.62}$$

$$\xi = \mu A_0^{\frac{1}{1-\alpha_0}}(1-\alpha_0)\,\tilde{\kappa}_0^{\alpha_0}. \tag{A.63}$$

Substituting in the expression above we find:

$$\frac{u'(c)}{\mu}\frac{c}{c_i}\frac{\nu_i}{\eta_g\bar{\varepsilon}} = \left(\frac{\alpha_0}{\alpha_i}\right)^{\alpha_i}\left(\frac{1-\alpha_0}{1-\alpha_i}\right)^{1-\alpha_i}\frac{A_0^{\frac{1-\alpha_i}{1-\alpha_0}}}{A_i}\tilde{\kappa}_0^{\alpha_0-\alpha_i}, \tag{A.64}$$

where we have again used $\eta_c^{c_i} = \nu_i/\eta_g\bar{\varepsilon}$.

We proceed by establishing that the asymptotic rate of growth of the term $\frac{u'(c)}{\mu}\frac{c}{\eta_g\bar{\varepsilon}}$ is $\frac{\gamma_0}{1-\alpha_0}$.

To see this, let us rewrite Equation (A.47) as:

$$\frac{cu'(c)}{\eta_g \bar{\varepsilon}} \nu_i = \frac{l_i}{1-\alpha_i} \xi,$$

which we can then sum over $i \in \{1, \cdots, I\}$ to find:

$$\frac{cu'(c)}{\eta_g \bar{\varepsilon}} = \xi \sum_{i=1}^{I} \frac{l_i}{1-\alpha_i}.$$

Since $\sum_{i=1}^{I} l_i$ converges to a nonzero value $1 - l_0^*$, the rate of growth of the expression on the left hand side is the same as ξ, which we know, from Equation (A.63), grows at the same rate as $\mu A_0^{1/(1-\alpha_0)}$. We use this fact and the fact that $\frac{\nu_i}{c_i} = v_i^{1/(1-\sigma)} g(c)^{\varepsilon_i}$ to conclude from Equation (A.64) that:

$$\lim_{t\to\infty} \frac{\dot{v}_i}{v_i} = (1-\sigma)\left(\eta_g \varepsilon_i \gamma^* - \gamma_i - \frac{\alpha_i}{1-\alpha_0}\gamma_0\right). \tag{A.65}$$

Crucially, Equation (A.65) implies that for any sector $i^* \in \mathcal{I}^*$, whose employment share does not asymptotically vanish, we should have:

$$\gamma^* = \frac{1}{\eta_g \varepsilon_{i^*}}\left(\gamma_{i^*} + \frac{\alpha_{i^*}}{1-\alpha_0}\gamma_0\right).$$

Now, consider the set $\hat{\mathcal{I}}$, defined as follows:

$$\hat{\mathcal{I}} \equiv \text{argmin}_{i \in \mathcal{I}/\{0\}} \frac{(1-\alpha_0)\gamma_i + \alpha_i \gamma_0}{(1-\alpha_0)\varepsilon_i}.$$

Consider the rate of growth of employment share in a consumption sector $\hat{i} \in \hat{\mathcal{I}}$:

$$\begin{aligned}
\lim_{t\to\infty} \frac{\dot{v}_{\hat{i}}}{v_{\hat{i}}} &= (1-\sigma)\left(\eta_g \varepsilon_{\hat{i}} \gamma^* - \gamma_{\hat{i}} - \frac{\alpha_{\hat{i}}}{1-\alpha_0}\gamma_0\right), \\
&= (1-\sigma)\varepsilon_{\hat{i}}\left[\frac{1}{\varepsilon_{i^*}}\left(\gamma_{i^*} + \frac{\alpha_{i^*}}{1-\alpha_0}\gamma_0\right) - \frac{1}{\varepsilon_{\hat{i}}}\left(\gamma_{\hat{i}} + \frac{\alpha_{\hat{i}}}{1-\alpha_0}\gamma_0\right)\right].
\end{aligned}$$

The expression shows that $\hat{\mathcal{I}} = \mathcal{I}^*$. If $\hat{i} \notin \mathcal{I}^*$, this expression has to be strictly negative, violating the assumption that $\hat{i} \in \hat{\mathcal{I}}$. If $i^* \notin \hat{\mathcal{I}}$, will have to be negative violating the assumption that $i^* \in \mathcal{I}^*$. It then follows that:

$$\gamma^* = \min_{i \in \mathcal{I}/\{0\}} \frac{(1-\alpha_0)\gamma_i + \alpha_i \gamma_0}{\eta_g (1-\alpha_0)\varepsilon_i}.$$

Finally, we need to check that the transversality condition is indeed satisfied. Since we

know that the asymptotic rate of growth of $\frac{cu'(c)}{\mu\eta_g\bar{\varepsilon}}$ is $\frac{\gamma_0}{1-\alpha_0}$, we find that:

$$\lim_{t\to\infty}\frac{\dot{\mu}(t)}{\mu(t)}=(1-\theta)\gamma^*-\frac{\gamma_0}{1-\alpha_0},$$

where we have used the fact that $\lim_{t\to\infty}\bar{\varepsilon}(t)=\bar{\varepsilon}^*$ and $\lim_{t\to\infty}\eta_g(c(t))=\eta_g$ are both constants. Therefore, in order to satisfy condition A.53, we need to ensure that

$$\rho>n+(1-\theta)\gamma^*.$$

From Equations (A.45) and (A.48), $\mu(t)\geq 0$ for all t. Therefore, from strict concavity of $\mathcal{F}$ it follows that these equations together give a unique path of $[\boldsymbol{k}(\cdot),\boldsymbol{l}(\cdot),k(\cdot),\mu(\cdot)]_{t=0}^{\infty}$. Due to the strict concavity of $\mathcal{M}$, we conclude that the resulting path corresponds to the unique solution to the social planner's problem (see Theorem 7.14 in Acemoglu, 2008). This completes the proof of the proposition.

For completeness, we state the correspondence between the variables above and the variables characterizing the competitive equilibrium. Comparing expressions (A.62) and (A.63) with expressions (A.15) and (A.16) in the derivations of the competitive equilibrium, we find that $\varsigma\equiv\mu R$ and $\xi\equiv\mu w$. Substituting in the expression (A.61) yields:

$$\frac{u'(c)}{\mu}\frac{\partial c}{\partial c_i}=\frac{1}{A_i}\left(\frac{R}{\alpha_i}\right)^{\alpha_i}\left(\frac{w}{1-\alpha_i}\right)^{1-\alpha_i},$$

where the right hand side corresponds to the expression for unit price of goods in sector i in Equation (A.18). We use $\eta_c^{c_i}=\nu_i/\eta_g\bar{\varepsilon}$ to rewrite this expression as

$$\frac{cu'(c)}{\mu}\frac{\nu_i}{\eta_g\bar{\varepsilon}}=p_ic_i,$$

which, once we sum over $i\in\{1,\cdots,I\}$, implies

$$\frac{cu'(c)}{\mu\eta_g\bar{\varepsilon}}\equiv e.$$

In light of this connection, the key step of the proof above establishing the rate of growth of the expression on the left hand side has a stright forward interpretation: per capita consumption expenditure grows at the same rate as the output of the investment sector.

□

A.5 Equilibrium Dynamic Equations

In this section, we characterize the dynamics of this economy along the equilibrium path. Define investment sector productivity-adjusted aggregate and investment variables:

$$\tilde{k}\left(t\right)\equiv\frac{K\left(t\right)}{A_{0}^{\frac{1}{1-\alpha_{0}}}\left(t\right)L\left(t\right)},\qquad\tilde{y}_{0}\left(t\right)\equiv\frac{Y_{0}\left(t\right)}{A_{0}^{\frac{1}{1-\alpha_{0}}}\left(t\right)L\left(t\right)}.\qquad\tilde{e}\left(t\right)\equiv\frac{e\left(t\right)}{A_{0}^{\frac{1}{1-\alpha_{0}}}\left(t\right)},\qquad\tilde{\kappa}_{0}\left(t\right)\equiv\frac{\kappa_{0}\left(t\right)}{A_{0}^{\frac{1}{1-\alpha_{0}}}\left(t\right)},\tag{A.66}$$

and denote the share of labor employed in the investment sector be $l_0(t) \equiv L_0(t)/L$. With this normalization, the rental price of capitaly and the capital-labor ratio in the investment sector have the following one-to-one relationship $R\left(t\right)=\alpha_{0}\tilde{\kappa}_{0}(t)^{\alpha_{0}-1}$ and we can use them interchangably to characterize the path of the aggregate economy.

Per capita consumption and productivity-adjusted capital-labor ratio in the investment sector $(c(t),\tilde{\kappa}_0(t))$ fully characterize the state of the economy at time t. First, note that productivity-adjusted per capita consumption expenditure $\tilde{e}(t)$ is a function of these two variables and time:

$$\tilde{e}\left(t\right)=\tilde{e}\left(c\left(t\right),\tilde{\kappa}_{0}\left(t\right),t\right)=\left(\sum_{i=1}^{I}\left[\varphi_{i}(t)g\left(c\left(t\right)\right)^{\varepsilon_{i}}\tilde{\kappa}_{0}\left(t\right)^{\alpha_{0}-\alpha_{i}}\right]^{1-\sigma}\right)^{\frac{1}{1-\sigma}},\tag{A.67}$$

where we have substituted from Equations (A.18) and (A.66) in the definition of Equation (A.3) and have defined a (exogenously given) time-dependent function

$$\varphi_{i}(t)\equiv\left(\frac{\alpha_{0}}{\alpha_{i}}\right)^{\alpha_{i}}\left(\frac{1-\alpha_{0}}{1-\alpha_{i}}\right)^{1-\alpha_{i}}A_{0}\left(t\right)^{-\frac{\alpha_{i}}{1-\alpha_{0}}}A_{i}\left(t\right)^{-1}.$$

The direct dependence on time is due to the impact of time-varying sectoral technologies on prices. Similarly, we can write the share of sector i in consumption expenditure as a function of $(c(t),\tilde{\kappa}_0(t))$ and time as $\omega_{i}\left(t\right)=\omega_{i}\left(c\left(t\right),\tilde{\kappa}_{0}\left(t\right),t\right)$. Averages of the income elasticity parameter and capital share in the consumption sector of the economy $\bar{\varepsilon}$ and $\bar{\alpha}$ then also become functions of the two state variables (and time). We emphasize in the special case where capital intensities are identical across all consumption sectors, the expenditure function in Equation (A.3) becomes independent of capital-labor ratios and solely depends on real consumption per capita c and time.

In addition to the expenditure and sectoral shares, the duplet $(c(t),\tilde{\kappa}_0(t))$ also pins down total investment and total per-capita stock of capital at time t along any equilibrium path. To see that, we first compute the employment share of the investment sector, dropping the

dependence on time to simplify notation:

$$\begin{aligned} l_0 \equiv \frac{L_0}{L} &= \frac{1}{1+\frac{L_C}{L_0}}, \\ &= \frac{(1-\alpha_0)\,\tilde{y}_0}{(1-\alpha_0)\,\tilde{y}_0 + (1-\bar{\alpha})\,\tilde{e}}, \end{aligned} \tag{A.68}$$

where in the second line we have used the equality $\frac{L_C(t)}{L_0(t)} = \frac{1-\bar{\alpha}}{1-\alpha_0}\frac{e}{\tilde{y}_0}$. Combining Equation (A.68) with $\tilde{y}_0 = l_0\tilde{\kappa}_0^{\alpha_0}$, we can write both the normalized output and the employment share of the investment sector as:

$$\tilde{y}_0 = \tilde{\kappa}_0^{\alpha_0} - \frac{1-\bar{\alpha}}{1-\alpha_0}\tilde{e}, \tag{A.69}$$

$$l_0 = 1 - \frac{1-\bar{\alpha}}{1-\alpha_0}\frac{\tilde{e}}{\tilde{\kappa}_0^{\alpha_0}}. \tag{A.70}$$

Therefore, since average capital share $\bar{\alpha}$ and normalized expenditure $\tilde{e}$ are both functions of $(c(t), \tilde{\kappa}_0(t))$ and time, so are the employment share and normalized output of the investment sector. Finally, the following lemma establishes that the total per-capita stock of capital is also a function of the same pair of variables.

Lemma 2. *Along any equilibrium path and for all times t, the aggregate productivity-adjusted per-capita stock of capital $\tilde{k}$ and the productivity-adjusted capital-labor ratio in the investment sector $\tilde{\kappa}_0$ satisfy the following equation:*

$$\tilde{k} = \tilde{\kappa}_0\left[1 + \frac{\tilde{e}}{\tilde{\kappa}_0^{\alpha_0}}\left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0\,(1-\alpha_0)}\right)\right], \tag{A.71}$$

where $\tilde{e}$ and $\bar{\alpha}$ are functions of $(c, \tilde{\kappa}_0)$ and time as defined by Equation (A.67). Moreover, for any level of per capita real consumption $c > 0$ at time t, Equation (A.71) defines a monotonically increasing and one-to-one mapping between $\tilde{k}$ and $\tilde{\kappa}_0$.

Proof. Along any equilibrium path, the output of all consumption goods are strictly positive and therefore $\kappa_i > 0$ for all $i \geq 1$. From Equation (A.16), we know that $R \geq \alpha_0 A_0 \kappa_0^{\alpha_0 - 1}$ and therefore along any equilibrium path $\kappa_0 > 0$. Hence, the allocations of labor and capital to all sectors are always interior.

Aggregate capital to labor ratio in the economy may be written as:

$$\begin{aligned} k = \frac{K}{L} &= \frac{L_0}{L}\kappa_0 + \frac{L_C}{L}\kappa_C, \\ &= l_0\kappa_0 + (1-l_0)\,\kappa_0\frac{\kappa_C}{\kappa_0}, \\ &= \kappa_0\left[l_0 + (1-l_0)\,\frac{\bar{\alpha}/(1-\bar{\alpha})}{\alpha_0/(1-\alpha_0)}\right], \\ &= \kappa_0\left[1 + \frac{\tilde{e}}{\tilde{\kappa}_0^{\alpha_0}}\left(\frac{\bar{\alpha}}{\alpha_0} - \frac{1-\bar{\alpha}}{1-\alpha_0}\right)\right], \end{aligned} \tag{A.72}$$

where in the second equality, we have defined L_C and κ_C as the total employment and captial-labor ratio in the consumption sector, and in the third equality, we have used the expressions for capital-to-labor ratios in Equations (A.17) as well as

$$\kappa_C = \frac{\overline{\alpha}}{1-\overline{\alpha}}\frac{w}{R}.$$

In the last equality, we have used the expression for the share of employment in the investment sector from Equation (A.70). Adjusting both sides of Equation (A.72) with respect to the productivity in the investment sector yields the desired result.

We will now show that the function defined by Equation (A.72) is one-to-one and monotonically increasing, mapping values of $\tilde{k}$ to $\tilde{\kappa}_0$ everywhere along any equilibrium path. To show this, it is sufficient to establish that the derivative of this function with respect to $\tilde{\kappa}_0$ is everywhere strictly positive.

$$\begin{aligned} \frac{\partial\tilde{k}}{\partial\tilde{\kappa}_0} &= 1 + \left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0(1-\alpha_0)}\right)\frac{\partial}{\partial\tilde{\kappa}_0}\left(\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\right) + \left(\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\right)\frac{\partial}{\partial\tilde{\kappa}_0}\left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0(1-\alpha_0)}\right), \\ &= 1 + \left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0(1-\alpha_0)}\right)\tilde{\kappa}_0^{-\alpha_0}\tilde{e}\frac{\partial\log\left(\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\right)}{\partial\log\tilde{\kappa}_0} + \left(\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\right)\frac{\bar{\alpha}}{\alpha_0(1-\alpha_0)\tilde{\kappa}_0}\frac{\partial\log\bar{\alpha}}{\partial\log\tilde{\kappa_0}}, \\ &= 1 + \left(\frac{\tilde{\kappa}_0^{-\alpha_0}\tilde{e}}{\alpha_0(1-\alpha_0)}\right)\left[(\bar{\alpha}-\alpha_0)\left(1-\alpha_0-\sum_i\eta_{\tilde{e}}^{p_i}\eta_{p_i}^{\tilde{\kappa}_0}\right) + \overline{\alpha}\sum_i\eta_{\overline{\alpha}}^{p_i}\eta_{p_i}^{\tilde{\kappa}_0}\right], \\ &= 1 + \left(\frac{\tilde{\kappa}_0^{-\alpha_0}\tilde{e}}{\alpha_0(1-\alpha_0)}\right)\left[(\bar{\alpha}-\alpha_0)(1-\alpha_0-(\alpha_0-\overline{\alpha})) + \overline{\alpha}(1-\sigma)\sum_i\omega_i\left(\frac{\alpha_i}{\overline{\alpha}}-1\right)(\alpha_0-\alpha_i)\right], \\ &= 1 + \left(\frac{\tilde{\kappa}_0^{-\alpha_0}\tilde{e}}{\alpha_0(1-\alpha_0)}\right)\left[(\bar{\alpha}-\alpha_0)(1-\overline{\alpha}) - (1-\sigma)\,Var(\alpha)\right], \end{aligned}$$

where in the third equality we have invoked the results of Lemma 1. Recalling the expression for the employment share of the investment sector l_0 from Equation A.70, we can now rewrite

this as follows:

$$\begin{aligned}\frac{\partial \tilde{k}}{\partial \tilde{\kappa}_0} &= 1+(1-l_0)\left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0}-\frac{(1-\sigma)\,Var\,(\alpha)}{\alpha_0\,(1-\bar{\alpha})}\right), \\ &= l_0+\frac{1-l_0}{\alpha_0\,(1-\bar{\alpha})}\left[\bar{\alpha}\,(1-\bar{\alpha})-(1-\sigma)\,Var\,(\alpha)\right].\end{aligned}$$

Finally, note that the expression within the square bracket on the right hand side is always positive. This is because:

$$\begin{aligned}\bar{\alpha}\,(1-\bar{\alpha})-(1-\sigma)\,Var\,(\alpha) &\geq \bar{\alpha}\,(1-\bar{\alpha})-Var\,(\alpha)\,, \\ &= \bar{\alpha}-\overline{\alpha^2}>0.\end{aligned}$$

where the inequality in the second line follows from the fact that for all sectors i, we have $0<\alpha_i<1$. This completes the proof that the mapping of $\tilde{k}$ to $\tilde{\kappa}_0$ is monotonic and one-to-one. □

Equation (A.71) shows that whenever average capital share of the consumption sector exceeds that of the investment sector, the economy-wide capital-labor ratio k is greater than the capital-labor ratio in the investment sector κ_0. Furthermore, the lemma ensures us that the relationship that the two ratios is one-to-one. This point is critical since it allows us to use the investment sector capital-labor ratio $\tilde{\kappa}_0$ as the state variable fully characterizing the path of capital accumulation in the economy.

The next proposition characterizes the dynamics of competitive equilibria in our economy in terms of the two state variables $(c,\tilde{\kappa_0})$. Before introducing the dynamic equations, let us introduce a function a function χ of the state variables:

$$\chi\,(c,\tilde{\kappa}_0;t)\equiv\frac{\tilde{\kappa}_0^{-\alpha_0}\tilde{e}\,(c,\tilde{\kappa}_0;t)}{\alpha_0\,(1-\alpha_0)}. \tag{A.73}$$

This function has the property that $\chi=\frac{l_C}{\alpha_0(1-\bar{\alpha})}=\frac{\tilde{k}/\tilde{\kappa}_0-1}{\bar{\alpha}-\alpha_0}$ and will greatly simplify the exposition of the forthcoming lemma.

Proposition 3. *The following system of two equations characterizes the dynamics of state*

variables $(c, \tilde{\kappa}_0)$ *in any competitive equilibria of our economy:*

$$\begin{aligned}\left[-\eta_{u'} - 1 + \eta_{\eta_g} + \eta_g \bar{\varepsilon}\left(1 + (1-\sigma)\, Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)\right)\right]\frac{\dot{c}}{c} & \\ + \left[\alpha_0 - \overline{\alpha}\left(1 + (1-\sigma)\,\rho_{\varepsilon,\alpha}\right)\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} &= \alpha_0 \tilde{\kappa}_0^{\alpha_0 - 1} - (\delta + \rho) + \bar{\gamma}\left(1 + (1-\sigma)\,\rho_{\varepsilon,\gamma}\right) \\ &\quad - \frac{1 - \overline{\alpha}\left(1 + (1-\sigma)\,\rho_{\varepsilon,\alpha}\right)}{1 - \alpha_0}\gamma_0,\end{aligned} \tag{A.74}$$

$$\begin{aligned}\eta_g \bar{\varepsilon}\left[\overline{\alpha}\left(1 + (1-\sigma)\,\rho_{\varepsilon,\alpha}\right) - \alpha_0\right]\chi\frac{\dot{c}}{c} & \\ + \left[1 + (1-\overline{\alpha})\left(\overline{\alpha} - \alpha_0\right)\chi - (1-\sigma)\, Var\left(\alpha\right)\chi\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} &= \left(1 - \alpha_0\left(1 - \overline{\alpha}\right)\chi\right)\tilde{\kappa}_0^{\alpha_0 - 1} \\ &\quad - \left(1 + \left(\overline{\alpha} - \alpha_0\right)\chi\right)\left(n + \delta + \frac{\gamma_0}{1 - \alpha_0}\right) \\ &\quad + \left[\overline{\alpha}\left(\overline{\alpha} - \alpha_0\right) + (1-\sigma)\, Var\left(\alpha\right)\right]\chi\gamma_0 \\ &\quad + \left[\overline{\alpha}\left(1 + (1-\sigma)\,\rho_{\alpha,\gamma}\right) - \alpha_0\right]\chi\overline{\gamma},\end{aligned} \tag{A.75}$$

where $\overline{x}$ *and* $Var(x)$ *denote the average and variance of a sector-specific set of parameters* $\boldsymbol{x}$ *and* $\rho_{x,x'}$ *denotes the correlation coeffcient between this parameter and another set of parameters* $\boldsymbol{x}'$*, all according to the distribution implied by sectoral expenditures shares at time* t*, and* χ *is defined by Equation* (A.73)*.*

If the condition (A.12) *is satisfied (the instantaneous utility function defined in Equations* (A.1)*,* (A.6)*, and* (A.2) *is concave in real consumption* c*), then the system above uniquely determines* $(\dot{c}/c, \dot{\tilde{\kappa}}_0/\tilde{\kappa}_0)$ *at time* t*.*

Proof. First, let us express the Euler Equation (A.33) in terms of the variables $(c, \tilde{\kappa}_0, t)$ by substituting for the growth of sectoral prices based on the production side of our economy. From Equation (A.18) we can write sectoral prices as

$$p_i = \tilde{\varphi}_i A_i^{-1} A_0^{\frac{1-\alpha_i}{1-\alpha_0}} \tilde{\kappa}_0^{\alpha_0 - \alpha_i},$$

where φ_i is a constant sector-specific parameter. This implies that the rate of growth of sectoral prices is given by:

$$\frac{\dot{p}_i}{p_i} = \frac{1 - \alpha_i}{1 - \alpha_0}\gamma_0 - \gamma_i + \frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\left(\alpha_0 - \alpha_i\right).$$

This allows us to compute the average of growth rates of sectoral prices and their covariance

with income elasticity parameters under the distribution implied by expenditure shares:

$$\begin{aligned}\overline{\left(\frac{\dot{p}_i}{p_i}\right)} &= \frac{1-\overline{\alpha}}{1-\alpha_0}\gamma_0 - \overline{\gamma} + (\alpha_0 - \overline{\alpha})\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}, \\ Cov\left(\varepsilon_i, \frac{\dot{p}_i}{p_i}\right) &= -Cov\left(\varepsilon_i, \gamma_i\right) - \left(\frac{\gamma_0}{1-\alpha_0} + \frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\right) Cov\left(\varepsilon_i, \alpha_i\right).\end{aligned}$$

Substituting this relation and the fact that $r = R - \delta = \alpha_0 \tilde{\kappa}_0^{\alpha_0 - 1} - \delta$ in the Euler equation, yields:

$$\begin{aligned}\left[-\eta_{u'}^{c} - 1 + \eta_{\eta_g} + \eta_g \bar{\varepsilon}\left(1 + (1-\sigma)\, Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)\right)\right]\frac{\dot{c}}{c} & \\ + \left[\alpha_0 - \overline{\alpha}\left(1 + (1-\sigma)\,\rho_{\varepsilon,\alpha}\right)\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} &= \alpha_0 \tilde{\kappa}_0^{\alpha_0 - 1} - (\delta + \rho) + \bar{\gamma}\left(1 + (1-\sigma)\,\rho_{\varepsilon,\gamma}\right) \\ &\quad - \frac{1 - \overline{\alpha}\left(1 + (1-\sigma)\,\rho_{\varepsilon,\alpha}\right)}{1-\alpha_0}\gamma_0,\end{aligned}$$

where $\rho_{\varepsilon,\gamma}$ and $\rho_{\varepsilon,\alpha}$ denote correlation coefficients between the income elasticity parameters and the technological rates of growth and capital shares at the sectoral levels, both under the distributions implied by expenditure shares.

The equation governing the evolution of aggregate capital stock can be written as follows

$$\begin{aligned}\dot{\tilde{k}} &= \left(\frac{\dot{K}}{K} - n - \frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}, \\ &= \frac{Y_0 \tilde{k}}{K} - \left(n + \delta + \frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}, \\ &= \tilde{y}_0 - \left(n + \delta + \frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}, \\ &= \tilde{\kappa}_0^{\alpha_0}\left(1 - \alpha_0\left(1 - \overline{\alpha}\right)\chi\right) - \left(n + \delta + \frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}, \qquad \text{(A.76)}\end{aligned}$$

where we have used $\dot{K} = Y_0 - \delta K$ in the second equality and Equations (A.69) and (A.73) in the fourth equality. Next, we need to transform this equation into one described in terms of the per-capita stock of capital in the investment sector $\tilde{\kappa}_0$. This will complete the characterization of the dynamics of the pair $(c, \tilde{\kappa}_0)$.

Lemma 2 established that along any equilibrium path a one-to-one mapping exists that relates a level of (productivity-adjusted) per capita stock of capital $\tilde{k}$ to a corresponding level of (productivity-adjusted) capital-per-worker $\tilde{\kappa}_0$ in the investment sector. Taking the

derivative of this function (from Equation (A.71)) yields:

$$\begin{aligned}\dot{\tilde{k}} &= \dot{\tilde{\kappa}}_0 + (1-\alpha_0)\,\tilde{\kappa}_0^{-\alpha_0}\tilde{e}\frac{\overline{\alpha}-\alpha_0}{\alpha_0\,(1-\alpha_0)}\dot{\tilde{\kappa}}_0 + \tilde{\kappa}_0^{1-\alpha_0}\frac{\overline{\alpha}-\alpha_0}{\alpha_0\,(1-\alpha_0)}\dot{\tilde{e}} + \tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\frac{1}{\alpha_0\,(1-\alpha_0)}\dot{\overline{\alpha}},\\ &= \left[\tilde{\kappa}_0 + (1-\alpha_0)\,\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\frac{\overline{\alpha}-\alpha_0}{\alpha_0\,(1-\alpha_0)}\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} + \frac{\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}}{\alpha_0\,(1-\alpha_0)}\left[(\overline{\alpha}-\alpha_0)\,\frac{\dot{\tilde{e}}}{\tilde{e}} + \overline{\alpha}\frac{\dot{\overline{\alpha}}}{\overline{\alpha}}\right]. \quad \text{(A.77)}\end{aligned}$$

Therefore, we need to compute the growth rates of expenditure $\dot{\tilde{e}}/\tilde{e}$ and average (consumption-sector) capital intensities $\dot{\overline{\alpha}}/\overline{\alpha}$ in terms of the growth rates of real consumption $\dot{c}/c$ and investment-sector capital-to-labor ratio $\dot{\tilde{\kappa}}_0/\tilde{\kappa}_0$.

Now, we can use the expressions for the elasticity of function $\overline{\alpha}$ with respect to real consumption and price, from Lemma 1, to find:

$$\begin{aligned}\frac{\dot{\overline{\alpha}}}{\overline{\alpha}} &= \eta_{\overline{\alpha}}^{c}\cdot\frac{\dot{c}}{c} + \sum_i \eta_{\overline{\alpha}}^{p_i}\frac{\dot{p}_i}{p_i},\\ &= \frac{1-\sigma}{\overline{\alpha}}\eta_g Cov\left(\varepsilon,\alpha\right)\cdot\frac{\dot{c}}{c} + (1-\sigma)\sum_i\left(\frac{\alpha_i}{\overline{\alpha}}-1\right)\omega_i\left[\frac{1-\alpha_i}{1-\alpha_0}\frac{\dot{A}_0}{A_0} - \frac{\dot{A}_i}{A_i} + (\alpha_0-\alpha_i)\,\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\right],\\ &= (1-\sigma)\,\eta_g\overline{\varepsilon}\rho_{\varepsilon,\alpha}\cdot\frac{\dot{c}}{c} - \frac{1-\sigma}{\overline{\alpha}}\left[Cov\left(\alpha,\gamma\right) + \left(\frac{\gamma_0}{1-\alpha_0}+\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\right)Var\left(\alpha\right)\right].\end{aligned}$$

Similarly, we can write the growth rate of consumption expenditure as:

$$\begin{aligned}\frac{\dot{\tilde{e}}}{\tilde{e}} &= \eta_e^c\frac{\dot{c}}{c} + \sum_i \eta_e^{p_i}\frac{\dot{p}_i}{p_i} - \frac{1}{1-\alpha_0}\frac{\dot{A}_0}{A_0},\\ &= \eta_g\overline{\varepsilon}\frac{\dot{c}}{c} + \sum_i \omega_i\left[\frac{1-\alpha_i}{1-\alpha_0}\frac{\dot{A}_0}{A_0} - \frac{\dot{A}_i}{A_i} + (\alpha_0-\alpha_i)\,\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\right] - \frac{1}{1-\alpha_0}\frac{\dot{A}_0}{A_0},\\ &= \eta_g\overline{\varepsilon}\frac{\dot{c}}{c} - \overline{\gamma} + (\alpha_0-\overline{\alpha})\,\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} - \frac{\overline{\alpha}\gamma_0}{1-\alpha_0}.\end{aligned}$$

We can write Equation (A.77) as:

$$\begin{aligned}\frac{\dot{\tilde{k}}}{\tilde{\kappa}_0} &= \left[1+(1-\alpha_0)\,(\overline{\alpha}-\alpha_0)\,\chi\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} + \chi\left[(\overline{\alpha}-\alpha_0)\,\frac{\dot{\tilde{e}}}{\tilde{e}} + \overline{\alpha}\frac{\dot{\overline{\alpha}}}{\overline{\alpha}}\right],\\ &= \left[1+(1-\overline{\alpha})\,(\overline{\alpha}-\alpha_0)\,\chi - (1-\sigma)\,Var\left(\alpha\right)\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} + \eta_g\overline{\varepsilon}\left[\overline{\alpha}\left(1+(1-\sigma)\,\rho_{\varepsilon,\alpha}\right)-\alpha_0\right]\chi\frac{\dot{c}}{c}\\ &\qquad - \left[\overline{\alpha}\left(1+(1-\sigma)\,\rho_{\alpha,\gamma}\right)-\alpha_0\right]\chi\overline{\gamma} - \left[\overline{\alpha}\left(\overline{\alpha}-\alpha_0\right)+(1-\sigma)\,Var\left(\alpha\right)\right]\chi\gamma_0.\end{aligned}$$

Finally, substituting this expression into Equation (A.76) give us:

$$\begin{aligned}
&\left[1+(1-\overline{\alpha})(\overline{\alpha}-\alpha_0)\chi-(1-\sigma)Var(\alpha)\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\\
&\qquad+\eta_g\bar{\varepsilon}\left[\overline{\alpha}\left(1+(1-\sigma)\rho_{\varepsilon,\alpha}\right)-\alpha_0\right]\chi\frac{\dot{c}}{c} \;=\; \left(1-\alpha_0(1-\overline{\alpha})\chi\right)\tilde{\kappa}_0^{\alpha_0-1}\\
&\qquad\qquad-(1+(\overline{\alpha}-\alpha_0)\chi)\left(\rho+\delta+\frac{\gamma_0}{1-\alpha_0}\right)\\
&\qquad\qquad+\left[\overline{\alpha}(\overline{\alpha}-\alpha_0)+(1-\sigma)Var(\alpha)\right]\chi\gamma_0\\
&\qquad\qquad+\left[\overline{\alpha}\left(1+(1-\sigma)\rho_{\alpha,\gamma}\right)-\alpha_0\right]\chi\overline{\gamma}.
\end{aligned}$$

where we have used the fact that $l_0 = 1 - l_C = 1 - \alpha_0(1-\overline{\alpha})\chi$ and $\tilde{k}/\tilde{\kappa}_0 = 1 + (\overline{\alpha}-\alpha_0)\chi$.

To ensure that the system of Equations above indeed has a solution, we need to establish that the determinant of the following matrix is nonzero:

$$M = \begin{bmatrix} -\eta^c_{u'} - 1 + \eta_{\eta_g} + \eta_g\bar{\varepsilon}\left(1+(1-\sigma)Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)\right) & \alpha_0 - \overline{\alpha}\left(1+(1-\sigma)\rho_{\varepsilon,\alpha}\right) \\ \eta_g\bar{\varepsilon}\left[\overline{\alpha}\left(1+(1-\sigma)\rho_{\varepsilon,\alpha}\right)-\alpha_0\right]\chi & 1+(1-\overline{\alpha})(\overline{\alpha}-\alpha_0)\chi-(1-\sigma)Var(\alpha)\chi \end{bmatrix}.$$

A necessary condition for the Euler equation to have a unique solution is that the expression $M_{1,1}$ is strictly positive. Since $M_{1,2}\cdot M_{2,1} \leq 0$ and $M_{2,2} = \frac{\partial k}{\partial \kappa_0} > 0$ (from Lemma 2), this is sufficient to ensure that the determinant of matrix M is indeed positive.

□

Starting from initial level of per capita stock of capital $k(0)$, Equation (A.71) and a choice of $c(0)$, give the corresponding allocation of capital to the investment sector κ_0. For all $t \geq 0$, Equations (A.74) and (A.75) describe the dynamics of the economy in terms of $(c, \tilde{\kappa}_0)$.

The two state variables $(c, \tilde{\kappa}_0)$ at time t are sufficient to fully specify the economy. As we discussed, both $\tilde{e}$ and $\bar{\alpha}$ are functions of c and $\tilde{\kappa}_0$, with the dependence on the latter going through the dependence of functions (A.4) on prices, as specified by Equation (A.18). Knowing the two state variables at any given time t, capital and labor employed in each consumption good sector i may be written as:

$$\begin{aligned}
K_i &= \left(\frac{\alpha_i}{\alpha_0}\omega_i(c,\tilde{\kappa}_0)\cdot\frac{\tilde{e}(c,\tilde{\kappa}_0)}{\tilde{\kappa}_0^{\alpha_0}}\right)\cdot LA_0^{\frac{1}{1-\alpha_0}},\\
L_i &= \left(\frac{1-\alpha_i}{1-\alpha_0}\omega_i(c,\tilde{\kappa}_0)\cdot\frac{\tilde{e}(c,\tilde{\kappa}_0)}{\tilde{\kappa}_0^{\alpha_0}}\right)\cdot L,
\end{aligned}$$

which completes the characterization of the economy.

B Aggregate Demand with Heterogeneous Households

Thus far, our analysis of aggregate data abstracted from heterogeneity. The growth model presented in Section 2 operated under the assumption that each country is populated by a mass of identical households. As a result, we estimate our model using the macro data based on the average level of consumption in each country, which, under the assumption of identical households, coincides with the consumption of every household in the country.

In this section, we show how to extend the empirical analysis when analyzing aggregate data that comes from the aggregation of households with heterogeneous levels of expenditure. The nonhomothetic CES does not belong to the Gorman or the PIGL class and, in contrast to these demand systems, total household expenditures are deflated by household-specific price indices that depend, not only on prices and model parameters, but also on the level of the household consumption expenditures. Below, we show that under empirically plausible assumptions, we can still approximate aggregate sectoral demand for the nonhomothetic CES preferences as a function of the first and second order moments of the distribution of consumption expenditures. In particular, up the first order approximation in the standard deviation of the distribution of log-consumption-expenditure, the aggregate and the household-level expressions coincide.[6] We further provide the corrections to these expressions up to the second order in the standard deviation of the distribution of log-consumption-expenditure (i.e., up the second order in the variance of the distribution).[7]

Consider a country c populated by a set of households, each with identical intra-period nonhomothetic CES preferences defined in equation (1). Assume that E^h, the total expenditure of household h, is distributed in a given country c according to a distribution $E^h \sim F_t^c$ at time t. Henceforth, we will drop the household superscript h to simplify the notation throughout this section. Define the country-wide average total household expenditure $E_t^c = \int_0^\infty E^h \, dF_t^c(E^h)$ at time t. Henceforth, we will drop the country and household superscripts c and h to simplify the notation throughout this section, denoting household-level variables without superscripts and the country-level variables with a bar. For instance, we let $\overline{E}_t \equiv E_t^c$. Similarly, we define the associated nonhomothetic CES real consumption index $\overline{C}_t$ as the solution to the utility maximization problem of nonhomothetic CES (1) for a household with total expenditure $\overline{E}_t$ facing sectoral prices $\boldsymbol{p}_t$, i.e., we let $\overline{C}_t \equiv C\left(\boldsymbol{p}_t, \overline{E}_t\right)$.

We now derive an expression for the country-wide average household expenditure on goods from sector i, $\overline{E}_{it} = \int_0^\infty \omega_{it}(\boldsymbol{p}_t, E) \cdot E \cdot dF_t(E)$, where $\omega_{it}(\boldsymbol{p}_t, E)$ denotes the expenditure share of good i for a consumer facing prices $\boldsymbol{p}_t$ and spending total amount E.

[6] Battistin et al. (2009) show evidence for the log-normality of the distribution of consumption expenditures. They estimate the standard deviation of the distribution of log-normal to be around 0.5 for the period 1996-2000 based on CEX data from the U.S.

[7] Further results evaluating aggregate measures of welfare are available upon request.

Lemma 3. *Assume that that the logarithm of consumption expenditure is distributed according to a log-normal distribution with variance κ_t^2, that is, $\log E \sim \log \mathcal{N}\left(\log \overline{E}_t, \kappa_t^2\right)$.*[8] *If the standard deviation of the distribution κ_t is small, the average expenditure in good i, $\overline{E}_{it}$, is given by*

$$\overline{E}_{it} = \Omega_i \left(\frac{p_{it}}{\overline{E}_t}\right)^{1-\sigma} \overline{E}_t \, \overline{C}_t^{\epsilon_i} \left(1 + \frac{1}{2}(1-\sigma)^2 \frac{\epsilon_i}{\bar{\epsilon}_t}\left(\frac{\epsilon_i}{\bar{\epsilon}_t} - Var\left(\frac{\epsilon_i}{\bar{\epsilon}_t}\right)\right)\kappa_t^2 + o\left(\kappa_t^2\right)\right), \tag{B.1}$$

where $\overline{C}_t$ and $\bar{\epsilon}_t$ denote the nonhomothetic CES index of real income and the average income elasticity parameter, respectively, both evaluated at the average total expenditure $\overline{E}_t$ and current prices $\boldsymbol{p}_t$.

Therefore, to the first order of approximation in the standard deviation κ_t, we can write the following expression for the relative expenditure shares of aggregate goods:

$$\log \frac{\overline{E}_{it}}{\overline{E}_{jt}} = \log \frac{\Omega_i}{\Omega_j} + (1-\sigma)\log\left(\frac{p_{it}}{p_{jt}}\right) + (\epsilon_i - \epsilon_j)\log \overline{C}_t + o\left(\kappa_t\right),$$

which corresponds to the empirical equations we used in Section 3.[9]

B.1 Proof of Lemma 3

This section derives the results presented above. Since our empirical strategy focuses on the intra-period allocation problem, we discuss the analysis at a given point in time and omit explicit time subscripts.

Let us construct the average expenditure in sector i aggregating individual household expenditures as defined by nonhomothetic CES (1):

$$\begin{aligned} \overline{E}_i &= \Omega_i p_i^{1-\sigma} \int_0^\infty E^\sigma \, C\left(\boldsymbol{p}, E\right)^{\epsilon_i} dF\left(E\right), & \text{(B.2)} \\ &= \Omega_i p_i^{1-\sigma} \int_0^\infty \exp\left(\sigma \log E + \epsilon_i \log C\left(\boldsymbol{p}, E\right)\right) dF\left(E\right), & \text{(B.3)} \end{aligned}$$

where $C\left(\boldsymbol{p}, E\right)$ is the indirect utility function implied by nonhomothetic CES preferences. We next approximate the value of $\log C$ around $\log \overline{C}$ (i.e., the $\overline{C}$ defined by the nonhomothetic CES for the average level of expenditure $\overline{E}$).

[8] Note that the assumption of log-normality is not necessary. Our results continue to hold for any symmetric distribution of the logarithm of consumption expenditure with a small standard deviation around its mean. In fact, the derivations in the proof of the Lemma contain the general expression for an arbitrary distribution.

[9] We can derive an estimation strategy that solely relies on the aggregate-level consumption data to estimate the underlying parameters of demand based on equation (B.1) if we observe the first and the second moments of the distribution of consumption expenditure at any point in time. The method relies on an iterative strategy along the lines of the one we have used in combination with equation (37).

First, note that

$$\begin{aligned}
\frac{\partial \log C}{\partial \log E} &= \frac{1-\sigma}{\bar{\epsilon}} && \text{(B.4)}\\
\frac{\partial^2 \log C}{\left(\partial \log E\right)^2} &= \frac{\partial}{\partial \log E}\frac{1-\sigma}{\bar{\epsilon}} && \text{(B.5)}\\
&= (1-\sigma)\frac{\partial}{\partial \log E}\left(\sum_{i\in\mathcal{I}} \epsilon_i \Omega_i p_i^{1-\sigma} \exp\left((\sigma-1)\log E + \epsilon_i \log C\right)\right)^{-1},\\
&= -\frac{(1-\sigma)}{\left(\sum_{i\in\mathcal{I}} \omega_i \epsilon_i\right)^2}\sum_{i\in\mathcal{I}} \epsilon_i \Omega_i p_i^{1-\sigma} E^{\sigma-1} C^{\epsilon_i}\left((\sigma-1)+(1-\sigma)\frac{\epsilon_i}{\bar{\epsilon}}\right),\\
&= -\frac{(1-\sigma)^2}{\left(\bar{\epsilon}\right)^2}\left(\frac{\overline{\epsilon_i^2}}{\bar{\epsilon}} - \bar{\epsilon}\right)\\
&= -\frac{(1-\sigma)^2}{\bar{\epsilon}} Var\left(\frac{\epsilon_i}{\bar{\epsilon}}\right). && \text{(B.6)}
\end{aligned}$$

Now, using a Taylor expansion up to the second order around $\log \overline{E}$ we find

$$\begin{aligned}
C\left(\boldsymbol{p}, E\right)^{\epsilon_i} &= C\left(p, \overline{E}\right)^{\epsilon_i} + \cdot\frac{\partial C^{\epsilon_i}}{\partial \log E}\bigg|_{E=\overline{E}} \cdot \log\left(\frac{E}{\overline{E}}\right) + \frac{1}{2}\frac{\partial^2 C^{\epsilon_i}}{\left(\partial \log E\right)^2}\bigg|_{E=\overline{E}} \cdot \log\left(\frac{E}{\overline{E}}\right)^2 + o\left(\log\left(\frac{E}{\overline{E}}\right)^2\right),\\
&= C\left(p, \overline{E}\right)\cdot\left[1 + \epsilon_i \frac{\partial \log C}{\partial \log E}\bigg|_{E=\overline{E}} \cdot \log\left(\frac{E}{\overline{E}}\right)\right.\\
&\qquad \left. + \frac{1}{2}\left(\epsilon_i \frac{\partial^2 \log C}{\left(\partial \log E\right)^2}\bigg|_{E=\overline{E}} + \left(\epsilon_i \frac{\partial \log C}{\partial \log E}\bigg|_{E=\overline{E}}\right)^2\right)\cdot \log\left(\frac{E}{\overline{E}}\right)^2 + o\left(\log\left(\frac{E}{\overline{E}}\right)^2\right)\right],\\
&= C\left(p, \overline{E}\right)^{\epsilon_i}\cdot\left[1 + (1-\sigma)\frac{\epsilon_i}{\bar{\epsilon}}\cdot\log\left(\frac{E}{\overline{E}}\right)\right.\\
&\qquad \left. + \frac{1}{2}(1-\sigma)^2\left(\left(\frac{\epsilon_i}{\bar{\epsilon}}\right)^2 - \frac{\epsilon_i}{\bar{\epsilon}} Var\left(\frac{\epsilon_i}{\bar{\epsilon}}\right)\right)\cdot \log\left(\frac{E}{\overline{E}}\right)^2 + o\left(\log\left(\frac{E}{\overline{E}}\right)^2\right)\right].
\end{aligned}$$

Let us use the change of variable $x \equiv \log\left(E/\overline{E}\right)$ to simplify the expressions, and note that $x \sim \mathcal{N}\left(0, \kappa^2\right)$. We can now write

$$\begin{aligned}
\overline{E}_i &= \Omega_i p_i^{1-\sigma}\overline{E}^{\sigma}\overline{C}^{\epsilon_i}\int\left(1 + (1-\sigma)\frac{\epsilon_i}{\bar{\epsilon}}\cdot x + \frac{1}{2}(1-\sigma)^2\left(\left(\frac{\epsilon_i}{\bar{\epsilon}}\right)^2 - \frac{\epsilon_i}{\bar{\epsilon}} Var\left(\frac{\epsilon_i}{\bar{\epsilon}}\right)\right)\cdot x^2 + o\left(x^2\right)\right) f\left(x\right) dx,\\
&= \Omega_i p_i^{1-\sigma}\overline{E}^{\sigma}\overline{C}^{\epsilon_i}\left(1 + \frac{1}{2}(1-\sigma)^2\frac{\epsilon_i}{\bar{\epsilon}}\left(\frac{\epsilon_i}{\bar{\epsilon}} - Var\left(\frac{\epsilon_i}{\bar{\epsilon}}\right)\right)\kappa^2 + o\left(\kappa^2\right)\right).
\end{aligned}$$

Finally, we find the following expressions for the *relative* aggregate demand:

$$\begin{aligned}
\log\frac{\overline{\omega}_i}{\overline{\omega}_j} &= \log\frac{\Omega_i}{\Omega_j} + (1-\sigma)\log\left(\frac{p_i}{p_j}\right) + (\epsilon_i - \epsilon_j)\log\overline{C} + \log\left(1 + \frac{1}{2}(1-\sigma)^2\frac{\epsilon_i}{\overline{\epsilon}}\left(\frac{\epsilon_i}{\overline{\epsilon}} - Var\left(\frac{\epsilon_i}{\overline{\epsilon}}\right)\right)\kappa^2 + o\left(\kappa^2\right)\right), \\
&\approx \log\frac{\Omega_i}{\Omega_j} + (1-\sigma)\log\left(\frac{p_i}{p_j}\right) + (\epsilon_i - \epsilon_j)\log\overline{C} + \frac{1}{2}(1-\sigma)^2\left[\left(\frac{\epsilon_i}{\overline{\epsilon}}\right)^2 - \left(\frac{\epsilon_j}{\overline{\epsilon}}\right)^2 - \left(\frac{\epsilon_i}{\overline{\epsilon}} - \frac{\epsilon_j}{\overline{\epsilon}}\right)Var\left(\frac{\epsilon_i}{\overline{\epsilon}}\right)\right]\kappa^2, \\
&= \log\frac{\Omega_i}{\Omega_j} + (1-\sigma)\log\left(\frac{p_i}{p_j}\right) + (\epsilon_i - \epsilon_j)\log\overline{C} + \frac{1}{2}(1-\sigma)^2\kappa^2\left(\frac{\epsilon_i}{\overline{\epsilon}} - \frac{\epsilon_j}{\overline{\epsilon}}\right)\left(\frac{\epsilon_i + \epsilon_j}{\overline{\epsilon}} - Var\left(\frac{\epsilon_i}{\overline{\epsilon}}\right)\right).
\end{aligned}$$

C Details On the Calibration Exercise

In this section, we present a dynamic programming formulation for the characterization of the transitional dynamics of the economy, which parallels Equations (26) and (27). We use this formulation to solve for the dynamics of the economy for the calibration exercise of Section 6 under the assumption that $\alpha_i = \alpha$ for all is.

We can write the Bellman equation

$$V_t(k) = \max_C \frac{C^{1-\theta}}{1-\theta} + \beta V_{t+1}(k_+(k, C; t)), \tag{C.1}$$

where the next-period per-capital stock of capital k_+ is given by[10]

$$k_+(k, C; t) = \frac{1}{1+n}\left[k(1-\delta) + A_{0t}k_t^\alpha - E(C; t)\right]. \tag{C.2}$$

Once again, we use the normalization and anticipating the policy function that grows at rate $1+\gamma^*$, we define $V_t(k_t) = (1+\gamma^*)^{(1-\theta)t}\tilde{V}_t\left(\tilde{k}\right)$ where $\tilde{k}$ is the normalized per-capita stock of capital at time t (that is, it corresponds to $\tilde{k}_t$ at time t). We find

$$\tilde{V}_t\left(\tilde{k}\right) = \max_{\tilde{C}} \frac{\tilde{C}^{1-\theta}}{1-\theta} + \beta(1+\gamma^*)^{1-\theta}\tilde{V}_{t+1}\left(\tilde{k}_+\left(\tilde{k}, \tilde{C}; t\right)\right), \tag{C.3}$$

where the normalized next-period per-capital stock of capital $\tilde{k}_+$ is given by

$$\tilde{k}_+\left(\tilde{k}, \tilde{C}; t\right) = \frac{(1+\gamma_0)^{-\frac{1}{1-\alpha}}}{1+n}\left(\tilde{k}^\alpha - \tilde{E}\left(\tilde{C}; t\right) + \tilde{k}(1-\delta)\right). \tag{C.4}$$

Function $\tilde{E}$ is given by

$$\tilde{E}\left(\tilde{C}, t\right) \equiv \left[\sum_{i=1}^{I} \tilde{\Omega}_i \tilde{C}^{\epsilon_i}(1+\xi_i)^t\right]^{\frac{1}{1-\sigma}}, \tag{C.5}$$

where $\tilde{\Omega}_i \equiv \Omega_i(A_{o,o}/A_{i,0})^{1-\sigma}$.

To solve the problem numerically for a given set of model parameters, we start at T large enough such that for $t \geq T$, we have

$$\tilde{E}\left(\tilde{C}, T\right) \approx \left(\sum_{i\in\mathcal{I}^*} \tilde{\Omega}_{i^*}\tilde{C}^{\epsilon_{i^*}}\right)^{\frac{1}{1-\sigma}}.$$

With this approximation, the problem becomes a standard stationary problem and a simple

[10] Throughout we assume the rate of population growth is n. In the main paper, this rate is assumed to be zero.

iterative method allows us to find $V_T(\cdot)$ as the solution to the following fixed point problem:

$$V_T\left(\tilde{k}\right) = \left(\mathcal{L}\tilde{V}_T\right)\left(\tilde{k}\right) \equiv \max_{\tilde{C}} \frac{\tilde{C}^{1-\theta}}{1-\theta} + \beta\left(1+\gamma^*\right)^{1-\theta}\, \tilde{V}_t\left(\tilde{k}^{\alpha} - \tilde{E}\left(\tilde{C};\, T\right) + \tilde{k}\left(1-\delta\right)\right).$$

We then start with the resulting solution $\tilde{V}_T(\cdot)$ and iteratively apply the following operator to solve for the policy function at all $t \leq T$:

$$V_{t-1}\left(\tilde{k}\right) = \left(\mathcal{L}_t\tilde{V}_t\right)\left(\tilde{k}\right) \equiv \max_{\tilde{C}} \frac{\tilde{C}^{1-\theta}}{1-\theta} + \beta\left(1+\gamma^*\right)^{1-\theta}\, \tilde{V}_t\left(\tilde{k}_+\left(\tilde{k},\, \tilde{C};\, t\right)\right). \tag{C.6}$$

D Data Description

In this section we describe the data construction for the CEX (final expenditure and value added formulations), NSS and macro datasets.

D.1 CEX

As discussed in the main text, we use U.S. household quarterly consumption data for the period 1999-2010 from the Consumption Expenditure Survey (CEX). The baseline exercises that estimate demand specified over final-good consumption are constructed aggregating the data as cleaned by Aguiar and Bils (2015) and available online at the AER website[11] and BLS CPI-U data. The aggregation over Aguiar and Bils categories is done to maximize the match with Herrendorf et al. (2013), which use table 2.3.5 NIPA PCE.[12] Agriculture is composed by food at home consumption. Manufacturing is composed by vehicle purchases and leasing, all other transportation, alcoholic beverages, adult and children clothing, shoes and other apparel, furniture and fixtures, appliances, entertainment equipment, personal care items, tobacco and other smoking. Services are composed of housing, utilities, health expenditures (including insurance), food away from home, entertainment fees, domestic services, education, cash contributions (not for alimony or support).

We combine the CEX expenditure data with disaggregated regional quarterly price series from the BLS's urban CPI (CPI-U). These data starts in 1999. This is the reason for starting our sample in 1999.[13] These data divides the U.S. in four broad regions. We do the following matching from price series to expenditure consumption category: For housing, we use the shelter price index. For utilities, we use the utilities and fuels price index. For food at home, we use food at home. For other transportation we use the transport price index. For health, we use health. For equipment we use durables. For food away from home, we use food away from home. For adult and children clothing we use apparel. For entertainment and entertainment fees we use the recreation price index. For vehicles we use the vehicles price index. For education we use the education and communication price index (since the education only price index starts later in the sample). For cash we use the overall service price index (we have tried using the overall price index or just multiplying it by 1, obtaining similar results). For furniture and fixtures we use furnishing operations. For shoes and other apparel we use the apparel price index. For domestic services we use professional services. For tobacco we use the price index 'nondurables less food, beverages, and apparel.' For alcoholic beverages we use use alcohol price index. For personal care items we use the durables price

[11] The current link is https://www.aeaweb.org/articles?id=10.1257/aer.20120599

[12] It is available from https://www.bea.gov/

[13] If instead of using regional CPI's, we used the national urban CPI, we could start our sample in 1980–when the CEX data starts. We find similar estimates using national urban CPI, but in this case we cannot use the instrumentation strategy discussed in Section 3.3 of the main text.

index.

We match the three-month average price of each consumption category to each household according to the region they live in and the time of the interview (we do the three month average because this is the window of time that corresponds to an interview). Following Hobijn and Lagakos (2005) and Hobijn et al. (2009), we construct the price for each sector faced by a household by taking the expenditure-weighted average of the price of each of the expenditure categories belonging to the sector. More precisely, suppose that for sector i we have information on the price of subcomponents $k \in \{1, \ldots, K\}$, (e.g., for services we have price indexes for housing, health, education, etc.). We construct the price of sector i faced by household h at time t, p_{it}^h, as $\log p_{it}^h = \sum_{k=1}^{K} \omega_{kit}^h \log p_{kit}$ where ω_{kit}^h denotes the expenditure share in component k of sector i by household h and $\log p_{kit}$ denotes the region-specific price index. The fact that we use the Stone price index is motivated by the fact that if there are scale differences in the construction of price indexes, they will be separated out into the intercept when we run our regression.[14]

Armed with this household-sector specific price indexes, we construct the overall household specific price index using a Tornqvist price index that takes as base the first quarter in which the household appears. Also, we compute the ratio of household prices that enter the baseline equation from these prices. Following Aguiar and Bils (2015) we focus on urban households with a present household head aged between 25 and 64. We also only consider households that report expenditure in the 4 interviews and for which total income is not coded incomplete. Finally, we deflate the agriculture, manufacturing and services with the household-specific price indexes and total expenditure with the Tornqvist price index. We trim the top and bottom deciles of sectoral expenditure and total expenditure (such that for some households we do not have 4 observation after the trimming). This deals with top-coding and with misreporting at the bottom of the distribution.[15]

To construct the value-added dataset we start from the CEX microdata. We use the publicly available MBTI and FMLI data sets. As in the previous discussion, our data construction procedure is based on Aguiar and Bils (2015). Accordingly, we use the same procedure to clean the data as indicated above. For example, we drop households for which we do not have observations for the four quarters. We drop expenditures corresponding to gifts, and financial expenditure categories that are not covered in all 4 rounds. These correspond to UCC codes 006001 (Total amount owed to creditors corresponding to the 2nd interview), 006002 (total amount owed to creditors, corresponds to the 5th interview) and 710110 (finance charges,

[14] As a robustness check we have re-estimated our baseline regressions re-normalizing all price indices to 1 in 1999, obtaining similar results.

[15] Restricting the regressions to households with 4 complete reports does not change the estimates significantly. In practice, the trimming that is most relevant for the estimation results is the one corresponding to the bottom observations of manufacturing. Since we are estimating everything relative to manufacturing, having values close to zero in manufacturing expenditure generates more noisy measures.

excluding mortgage and vehicles, which corresponds to the 5th interview).

We then follow Buera et al. (2015) and map each individual UCC expenditure code to a PCE line of table 2.4.5. We start with the correspondence in Buera et al. provide for 2012. We manually match the missing categories that appear going back in time. We also match prices to each PCE line as described above. We sum over all household expenditures within a PCE for each interview round. We use the I/O weights from Buera et al. (2015) to map PCE lines to NIPA input lines. For nominal expenditure we simply divide each PCE line across input lines according to the I/O weights. For prices, we compute the input price according to the Stone index where we use the I/O weights as weights for the index. That is, we take the a weighted average of the log of PCE prices with weights the I/O weights to obtain the log price of NIPA input. Note that we find a household-specific price for each input category. We then aggregate across NIPA input lines following Herrendorf et al. (2013) to obtain the aggregate expenditure across agriculture, manufacturing and services. Codes and crosswalks are available upon request.

D.2 NSS

We use data from rounds 64, 66 and 68 of the India National Sample Survey (NSS), which span the years 2007 to 2012. The NSS is a representative survey of household expenditure that covers almost entirely India and is used, among other things, to estimate national expenditures. The survey covers the whole of the Indian Union except (i) interior villages of Nagaland situated beyond five kilometres of the bus route and (ii) villages in Andaman and Nicobar Islands. It collects repeated cross-sections of expenditures incurred by households in goods and services. The survey contains information on both direct and indirect expenditures. The indirect expenditures include the imputed value of home-produced goods, in kind remunerations and social transfers from the government. We use the data cleaning procedure described in Breza and Kinnan (2016). We construct total expenditure in agriculture, manufactures and services following the same classification as for the CEX data. Agricultural expenses are comprised by raw products such as meat, milk, eggs, cereals, fruit and vegetables, pulses, oil, etc. Manufacturing is comprised of clothing, tobacco, personal care products, durable goods, tobacco and fuel. Services is composed of electricity, medical, rent, entertainment, and education.

D.3 Macro Dataset

Our aggregate data comes from two sources. The sectoral data comes from Groningen's 10-Sector Database (Vries et al., 2014). The 10-Sector Database provides a long-run internationally comparable dataset on sectoral measures for 10 countries in Asia, 9 in Europe, 9 in Latin America, 10 in Africa and the United States. The variables covered in the data set

are annual series of production value added (nominal and real) and employment for 10 broad sectors starting in 1947. Not all time series start in 1947, see figures F.1, F.2 and F.3 in the online appendix where the time series for each country are depicted. Chained indexing has been used for all countries to compute real sectoral value added. However there is heterogeneity in the specific implementation country by country given the different data sources. As described in Vries et al. (2014), for most developing countries a fixed-based Laspeyres index is used and its base is usually updated every 5 to 10 years. For richer countries, Fisher indices are typically used. As our identification comes from the within country time series, this cross-country heterogeneity should not bias our estimation. In our baseline exercise we aggregate the ten sectors into agriculture, manufacturing and services. The ten sectors are: agriculture, mining, manufacturing, construction, public utilities, retail and wholesale trade, transport and communication, finance and business services, other market services and government services. We lump into manufacturing mining, manufacturing and construction, while the rest are classified as services (except agriculture).[16]

[16]More information on the data set is available online at http://www.rug.nl/research/ggdc/data/10-sector-database.

E Additional Results and Robustness Checks on Micro Data

Table E.1: First-Stage Type Regression for CEX Data

	$\log C_t^h$	$\log\left(\frac{p_{a,t}^h}{p_{m,t}^h}\right)$	$\log\left(\frac{p_{s,t}^h}{p_{m,t}^h}\right)$
Constant	7.282	-0.066	-0.067
	(0.089)	(0.027)	(0.038)
Second Quintile	0.015	0.002	0.004
	(0.008)	(0.003)	(0.004)
Third Quintile	0.087	0.003	0.000
	(0.011)	(0.003)	(0.005)
Fourth Quintile	0.169	0.004	-0.009
	(0.014)	(0.004)	(0.006)
Fifth Quintile	0.232	-0.001	-0.018
	(0.017)	(0.005)	(0.007)
Net Annual Income	0.135	0.005	0.014
	(0.010)	(0.003)	(0.004)
$\log\left(\frac{p_{a,t,-r}}{p_{m,t,-r}}\right)$	0.232	1.096	0.318
	(0.087)	(0.032)	(0.043)
$\log\left(\frac{p_{s,t,-r}}{p_{m,t,-r}}\right)$	0.398	-0.150	0.670
	(0.077)	(0.027)	(0.036)
Household Controls	Y	Y	Y
Expenditure Re-Weighted	N	N	N
Region FE	Y	Y	Y
Year × quarter FE	N	N	N
Observations	60925	60925	60925

Standard errors clustered at the household level shown in parentheses. Note that the first quintile is omitted since it is subsumed in the constant of the regression. Nominal expenditure is deflated by a household-specific Stone price index.

F Additional Results and Robustness Checks on Macro Data

Table F.1: Residual Relative Log-Expenditure Shares on Residual Aggregate Consumption

Dep. Var.:	$\log\left(\frac{\text{Agriculture}}{\text{Manufact.}}\right)$		$\log\left(\frac{\text{Services}}{\text{Manufact.}}\right)$	
Residual Log-Expenditure	(1)	(2)	(3)	(4)
Residual Log Aggregate Income	-0.34	-0.42	0.27	0.23
	(0.11)	(0.12)	(0.08)	(0.07)
Residual Log Agg. Income below Median Income		0.19		0.09
		(0.12)		(0.13)
R^2	0.27	0.29	0.20	0.21
Observations	513	513	513	513

Note: Standard errors clustered by country. Residual Aggregate Income is constructed by taking the residuals of the following OLS regresssion: $\log Y_t^c = \alpha \log p_{at}^c + \beta \log p_{mt}^c + \gamma \log p_{st}^c + \xi^c + \nu_t^c$ where superscript c denotes country, and subscript t, time. p_{at}^c denotes price of agriculture in country c at time t. Likewise p_{mt}^c and p_{st}^c denote the prices of manufacturing and services, respectively. ξ^c denotes a country fixed effect and ν_t^c the error term. Residual log-expenditures are constructed in an analogous manner.

Table F.2: Sectoral Value-Added as Dependent Variable, Cross-country Regressions, $\epsilon_m = 1$

	World	
	(1)	(2)
σ	0.19	0.16
	(0.03)	(0.03)
$\epsilon_a - 1$	-0.51	-0.42
	(0.03)	(0.03)
$\epsilon_s - 1$	0.17	0.19
	(0.01)	(0.01)
Country × Sector FE	Y	Y
Trade Controls	N	Y
Observations	1626	1626

Note: Robust standard errors in parenthesis.

Table F.3: Services as a Reference Sector, $\epsilon_m = 1$

	Household CEX			Aggregate Data		
	(1)	(2)	(3)	World	OECD	Non-OECD
σ	0.38	0.38	0.35	0.52	0.38	0.57
	(0.04)	(0.04)	(0.05)	(0.03)	(0.05)	(0.03)
$\epsilon_a - 1$	-1.00	-1.00	-1.00	-0.99	-1.00	-0.99
	(0.09)	(0.05)	(0.06)	(0.06)	(0.07)	(0.05)
$\epsilon_s - 1$	0.92	0.92	0.91	0.19	0.19	0.14
	(0.06)	(0.05)	(0.06)	(0.03)	(0.04)	(0.03)
Household Controls	Y	Y	Y	-	-	-
Region FE	N	Y	Y	-	-	-
Year $\times$ Quarter FE	N	N	Y	-	-	-
Country-Sector FE	-	-	-	Y	Y	Y

Note: Clustered standard errors at the household level reported in parenthesis for regressions (1) to (3). Robust standard errors in parenthesis for regressions with aggregate data. Household regressions are computed using the CEX-replicate weights. We constrain $\epsilon_a \geq 0$ in the estimation by taking the exponential transformation of it in the estimation (standard errors are computed through the delta method).

Table F.4: Iterative Method, Cross-Country Estimates, $\epsilon_m = 1$

	World		OECD		Non-OECD	
	(1)	(2)	(3)	(4)	(5)	(6)
σ	0.86	0.75	0.92	0.81	0.76	0.87
	(0.14)	(0.15)	(0.17)	(0.11)	(0.14)	(0.14)
$\epsilon_a - 1$	-1.00	-0.99	-1.00	-0.93	-0.98	-0.95
	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.01)
$\varepsilon_s - 1$	0.21	0.30	0.16	0.21	0.41	0.17
	(0.14)	(0.19)	(0.13)	(0.08)	(0.22)	(0.09)
$c \cdot sm$ FE	Y	Y	Y	Y	Y	Y
Trade Controls	N	Y	N	Y	N	Y
Observations	1626	492	1134	1626	492	1134

Notes: Standard errors clustered at the country level.

Table F.5: First-Step Estimates, Cross-Country Estimates

	World		OECD		Non-OECD	
	(1)	(2)	(3)	(4)	(5)	(6)
σ	0.89	0.87	0.75	0.76	0.94	0.91
	(0.10)	(0.09)	(0.15)	(0.14)	(0.13)	(0.12)
$\tilde{\epsilon}_a - \tilde{\epsilon}_m$	-1.07	-1.06	-1.02	-0.91	-1.09	-1.08
	(0.07)	(0.07)	(0.17)	(0.19)	(0.08)	(0.08)
$\tilde{\epsilon}_s - \tilde{\epsilon}_m$	0.28	0.29	0.39	0.48	0.19	0.21
	(0.09)	(0.09)	(0.17)	(0.13)	(0.10)	(0.10)
Country × Sector FE	Y	Y	Y	Y	Y	Y
Trade Controls	N	Y	N	Y	N	Y
Observations	1626	1626	492	492	1134	1134

Notes: Standard errors clustered at the country level.

Table F.6: Growth Rates of Relative Prices in the Country Panel

$$\log\left(\frac{p^c_{i,t}}{p^c_{m,t}}\right) = \alpha^c_{im} + \beta_i \cdot \text{Year} + \varepsilon^c_{im,t}, \qquad i = \{s, a\}$$

	$\log\left(\frac{p^c_a}{p^c_m}\right)$	$\log\left(\frac{p^c_s}{p^c_m}\right)$
Year	-0.59	0.13
	(0.05)	(0.04)
Country-Sector FE	Yes	Yes
R^2	0.49	0.41
Observations	1680	1680

Note: Year has been re-scaled to Year/100.

Figure F.1: Baseline Country Fit, common $\{\sigma, \epsilon_a, \epsilon_s\}$

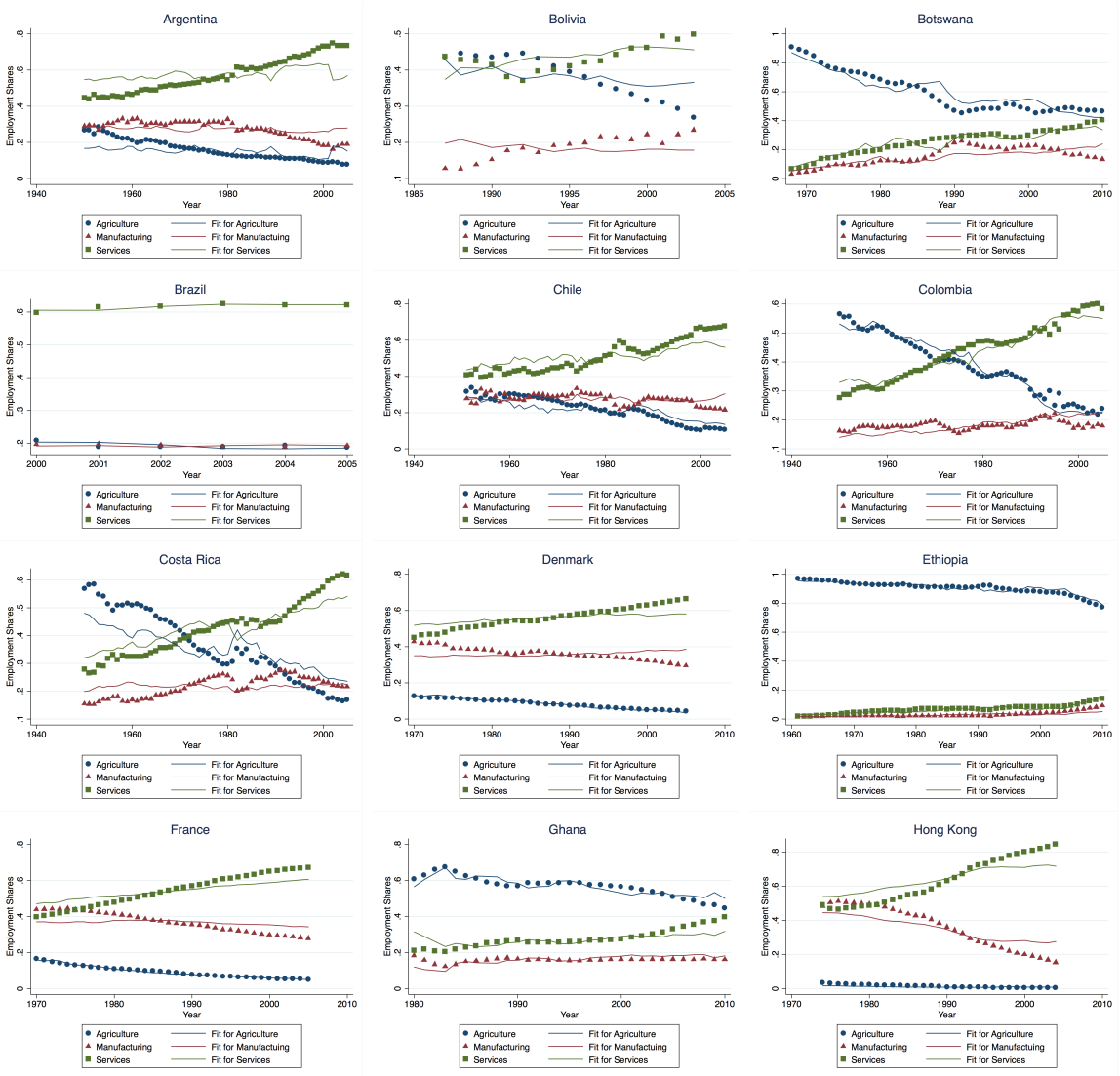


Figure F.2: Baseline Country Fit, common $\{\sigma, \epsilon_a, \epsilon_s\}$ (continued)

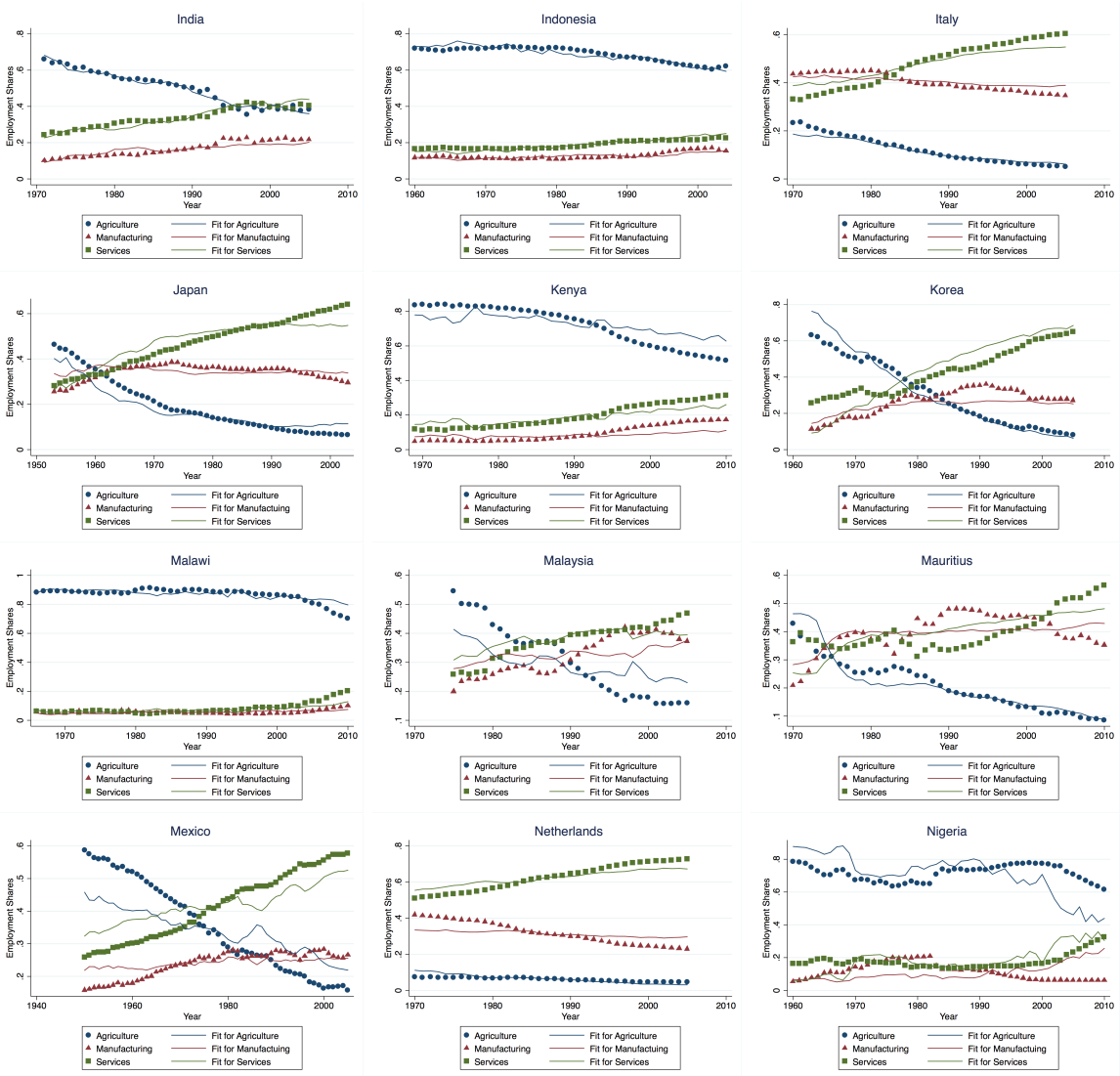


Figure F.3: Baseline Country Fit, common $\{\sigma, \epsilon_a, \epsilon_s\}$ (continued)

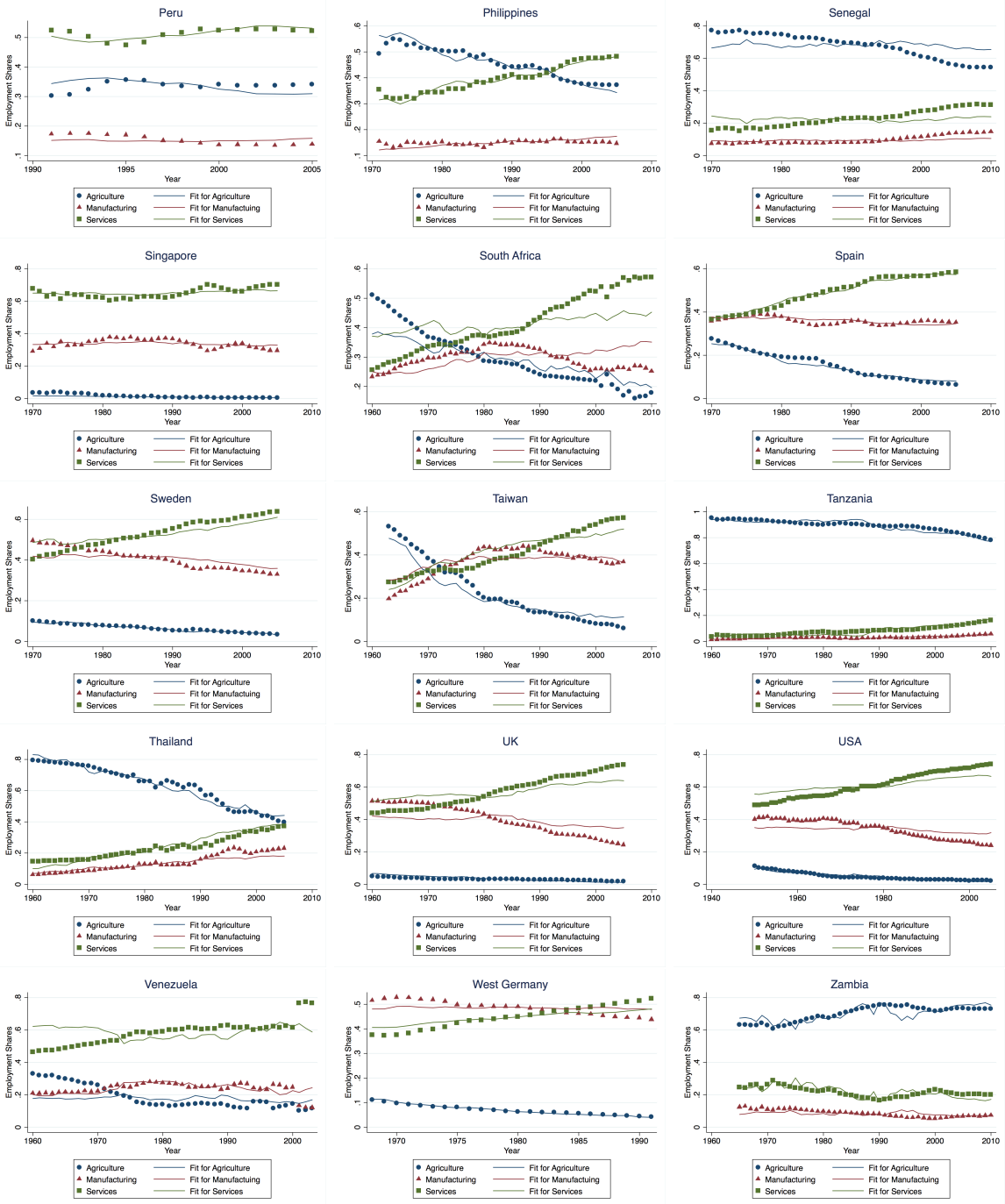


Figure F.4: Country-Specific Fit, $\{\sigma^c, \epsilon_a^c, \epsilon_s^c\}$

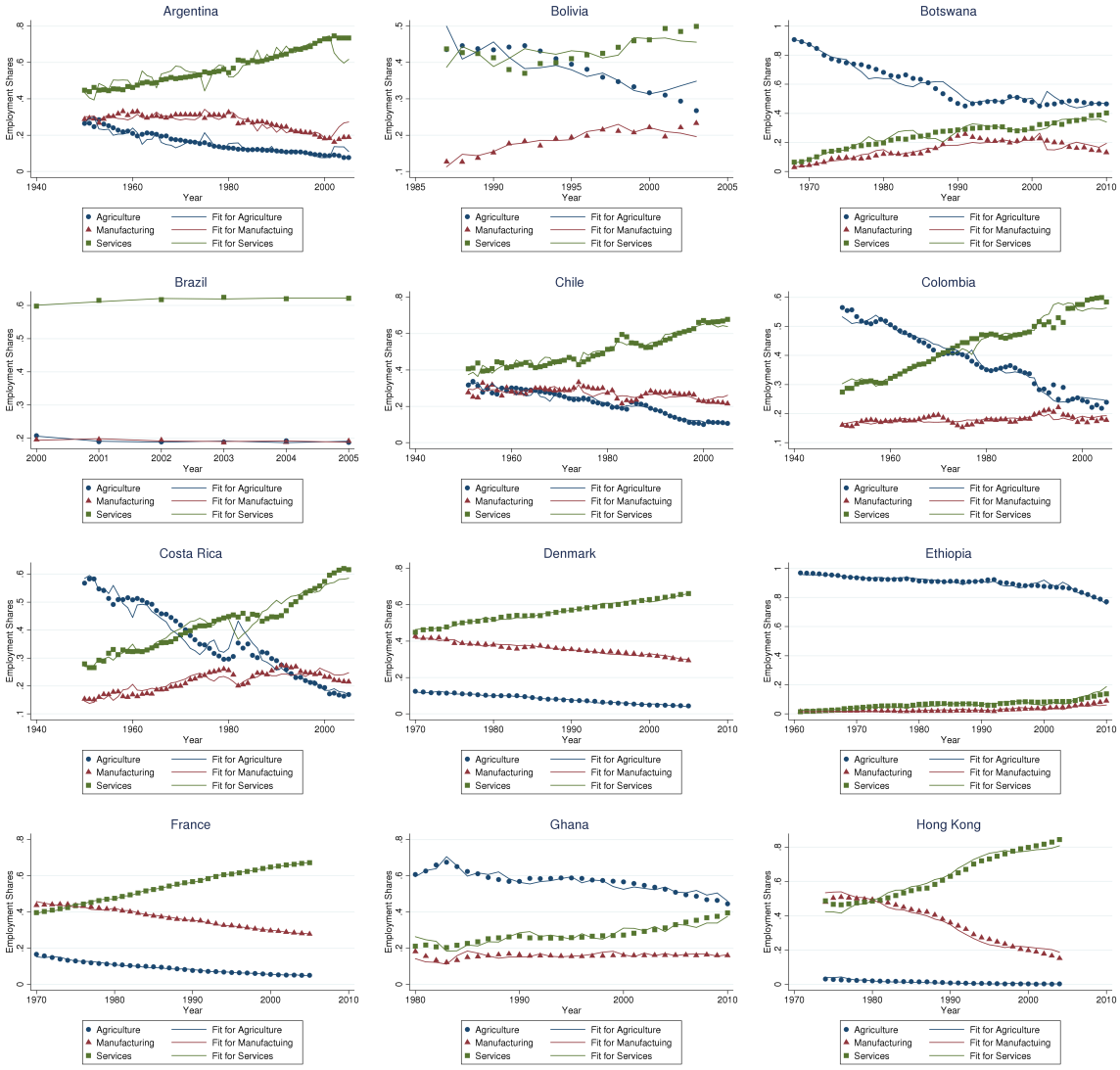


Figure F.5: Country-Specific Fit, $\{\sigma^c, \epsilon_a^c, \epsilon_s^c\}$ (continued)

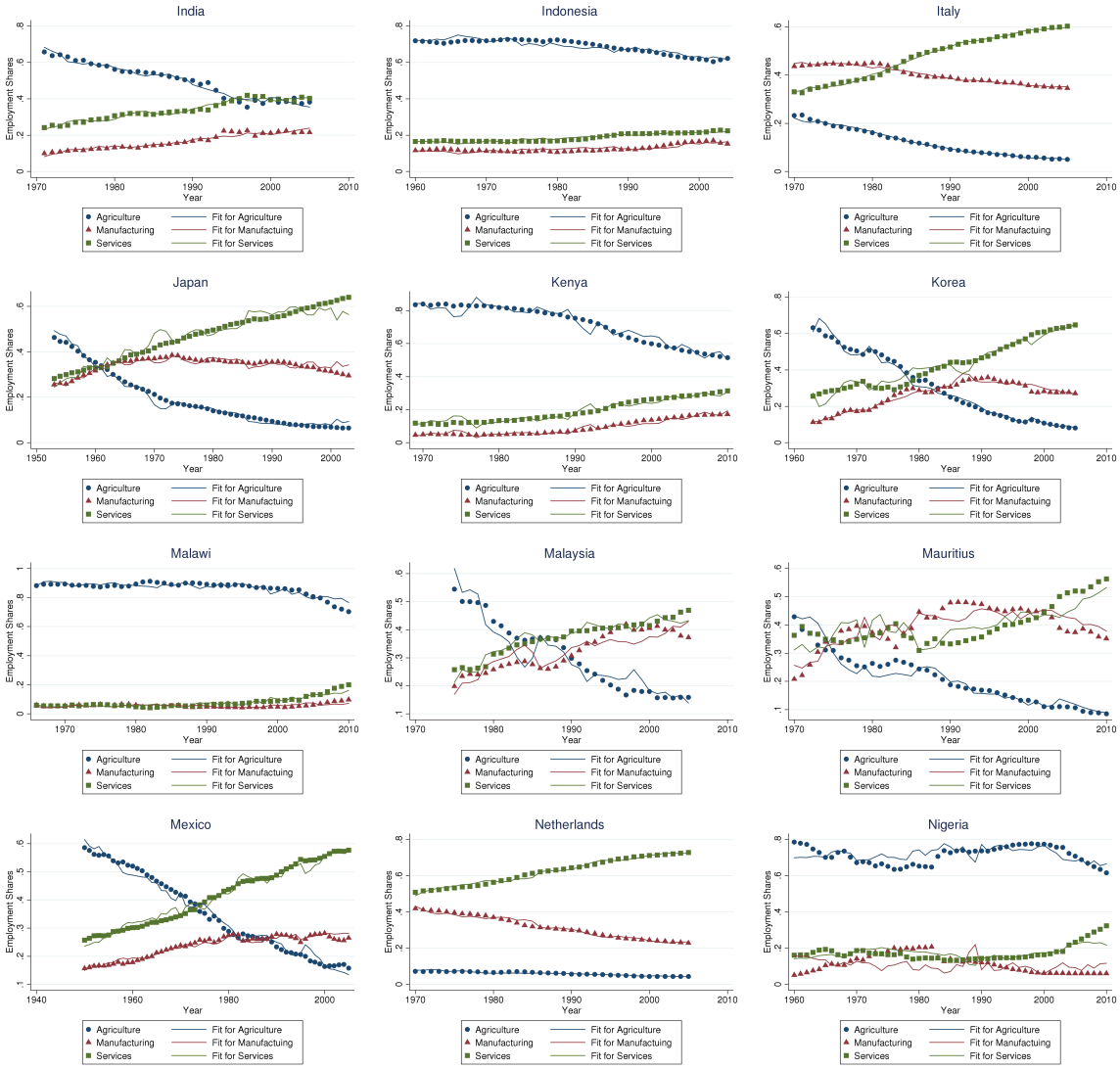
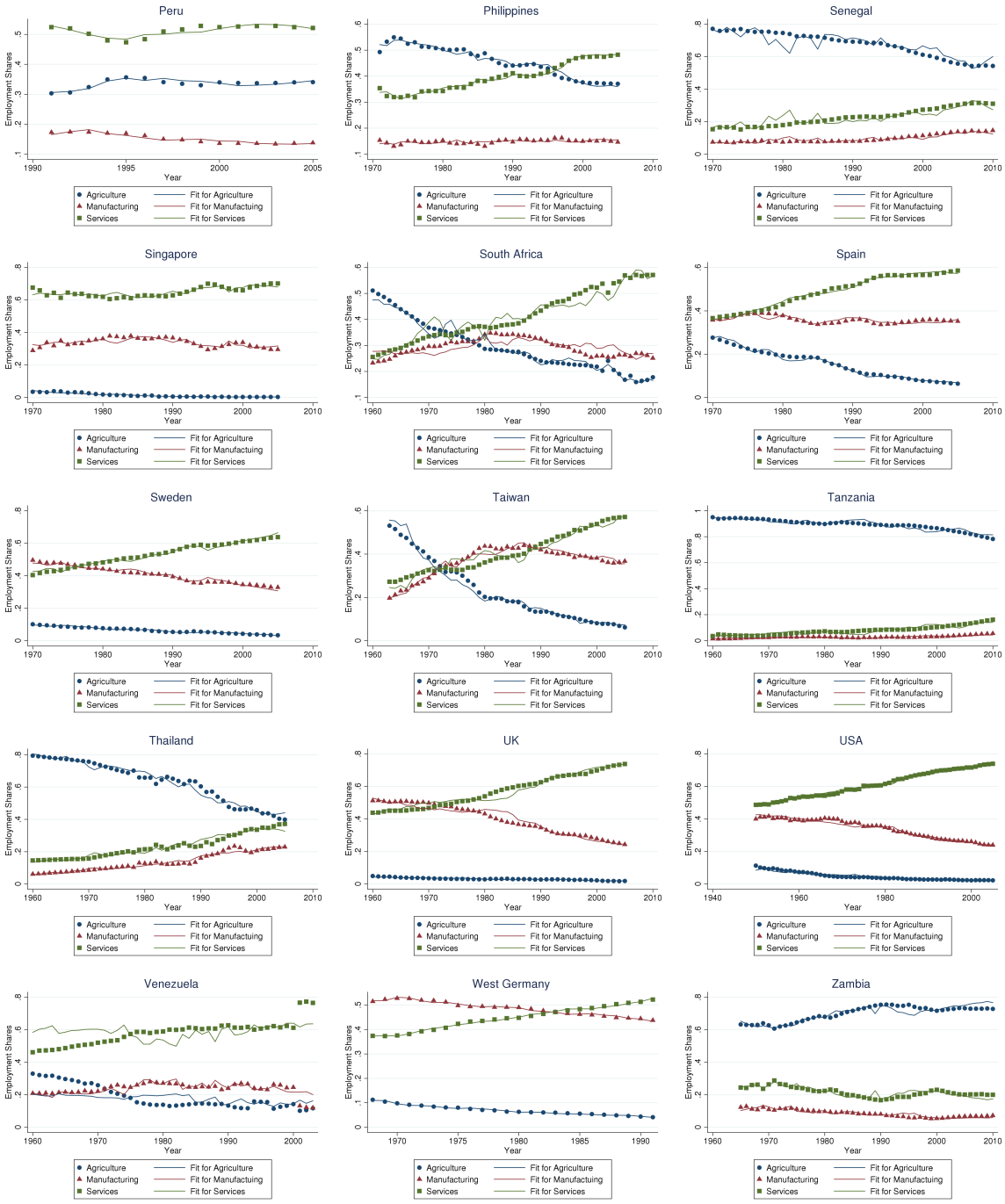


Figure F.6: Country-Specific Fit, $\{\sigma^c, \epsilon_a^c, \epsilon_s^c\}$ (continued)



G Stone-Geary and PIGL Properties and Estimation

G.1 Comparison of Nonhomothetic CES to Generalized Stone-Geary and PIGL Preferences

For comparison, now consider generalized Stone-Geary preferences that have been widely used in previous work on structural change (see, e.g., Kongsamut et al., 2001):

$$C = \left(\sum_{i=1}^{I} \left(C_i - \underline{C}_i \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{G.1}$$

where $\underline{C}_i$ are the usual coordinate shifters.[17] The expenditure elasticity of demand for good i is given by:

$$\eta_{C_i}^E = 1 - \frac{\underline{C}_i}{C_i}, \tag{G.2}$$

which is different from 1 as long as the shifter $\underline{C}_i \neq 0$. However, note that due to constancy of $\underline{C}_i$, this elasticity converges to unity at the same rate as the rate of growth of C_i. Therefore, nonhomotheticity is a short-run feature of Stone-Geary preferences: as the income grows Stone-Geary preferences asymptote to homothetic CES preferences.

Another important feature of the nonhomothetic CES preferences is the fact that elasticity of substitution σ_{ij} between all goods i and j remains constant σ and remains independent of expenditure (income) elasticities. In contrast, for Stone-Geary preferences we find:

$$\sigma_{ij} = \sigma \cdot \frac{E}{E - \sum_k p_k \underline{C}_k} \cdot \left(1 - \frac{\underline{C}_i}{C_i} \right) \cdot \left(1 - \frac{\underline{C}_j}{C_j} \right), \tag{G.3}$$

$$\eta_{C_i}^E = \frac{E}{E - \sum_k p_k \underline{C}_k} \cdot \left(1 - \frac{\underline{C}_i}{C_i} \right), \tag{G.4}$$

where η_i is the nominal income elasticity of demand in sector i (Hanoch, 1975). It then follows that the elasticities of substitution between goods i and j always satisfies the following equality:

$$\sigma_{ij} = \sigma \eta_{C_i}^E \eta_{C_j}^E, \tag{G.5}$$

creating a direct linkage between elasticities of substitution and expenditure for different sectors. As expected, when E goes to infinity we find that $\sigma_{ij} \to \sigma$ and $\eta_{C_i}^E \to 1$ for all sectors.

An alternative specification for nonhomothetic preferences in the structural change literature, recently used by Boppart (2014a), is the Price Independent Generalized Linear (PIGL) preferences. The canonical definition for these preferences involves a two-good system. In

[17] In particular, standard 3-sector models of structural transformation generally assume $\underline{C}_a > 0$, $\underline{C}_s < 0$ and $\underline{C}_m = 0$.

general, no closed-form representation for the utility function exists, but the indirect utility/expenditure function relationship can be specified as:

$$C + \frac{\vartheta}{\varrho}\left[\left(\frac{p_1}{p_2}\right)^{\varrho} - 1\right] = \frac{1}{\xi}\left[\left(\frac{E}{p_2}\right)^{\xi} - 1\right], \tag{G.6}$$

where $\boldsymbol{p} = (p_1, p_2)$ is the pair of good prices, C is the aggregator (utility) and E is expenditure, $0 \leq \xi \leq \varrho < 1$, and $\vartheta > 0$.[18] For these preferences, the expenditure elasticity of demand for good i is constant and less than unity: $\eta_{C_1}^{E} = 1 - \xi < 1$. Therefore, like nonhomothetic CES preferences and unlike Stone-Geary, PIGL preferences also feature nonhomotheticity at all levels of income. In contrast to nonhomothetic CES, however, there is no generalization of PIGL preferences to more than two good demand systems that preserves the independence of income elasticities across different goods.[19]

Needless to say, since PIGL preferences are outside the CES family, the elasticity of substitution varies with income and prices. As Boppart (2014a) shows, the elasticity of substitution between goods 1 and 2 are given by

$$\sigma = 1 - \varrho - (\varrho - \xi)\frac{\vartheta\left(\frac{p_1}{p_2}\right)^{\varrho}}{\left(\frac{E}{p_2}\right)^{\xi} - \vartheta\left(\frac{p_1}{p_2}\right)^{\varrho}}. \tag{G.7}$$

As a result, when PIGL preferences are embedded in a growth model, along an equilibrium path that involves growing income, the elasticity of substitution will be monotonically increasing and converges toward $1 - \varrho$. Therefore, the choice of PIGL preferences involves specific assumptions about the dynamics of substitution elasticities in a two-good model.

G.2 Estimation

In this section, we develop in more detail the comparison in the cross-country fit of our model to alternative specifications where we replace the nonhomothetic CES with *(i)* Stone-Geary, probably the most popular non-homothetic aggregator, and *(ii)* PIGL preferences as they have been proposed by Boppart (2014a).

[18] PIGL preferences are *not* additive in the sense of Hanoch (1975).

[19] As a reminder, from Engel aggregation we know that we can have up to $I-1$ independent income elasticities in a demand system involving I goods. This is why we have one degree of freedom in specifying the I income elasticity parameters $\bar{\epsilon}$ in nonhomothetic CES preferences defined in Section A of the appendix in the main text.

G.2.1 Stone-Geary Estimation

We start considering a generalized Stone-Geary formulation, as used for example in Herrendorf et al. (2014). These preferences define the intra-period consumption aggregator as

$$C_t^c = \left[\Omega_a^c \left(C_{a,t}^c + \bar{c}_a\right)^{\frac{\sigma-1}{\sigma}} + \Omega_m^c \left(C_{m,t}^c\right)^{\frac{\sigma-1}{\sigma}} + \Omega_c^c \left(C_{s,t}^c + \bar{c}_s\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{G.8}$$

where C_t^c denotes aggregate consumption of country c at time t, $\Omega_i^c > 0$ are constant preference parameters that are country specific$C_{i,t}^c$ denotes consumption in sector $i = \{a, m, s\}$, $\bar{c}_a$ and $\bar{c}_s$ are constants that govern the non-homotheticity of these preferences, and σ is a parameter that tends to the price elasticity of substitution as $C_{i,t}^c \gg \max\{\bar{c}_a, \bar{c}_s\}$. Since these preferences are not implicitly additive, the price and income elasticities are not independent. As we have previously shown, the elasticity of substitution between i and j is $\sigma_{ij} = \sigma\eta_i\eta_j$, where η's denote income elasticities. We follow the estimation procedure described in Herrendorf et al. (2013). As with nonhomothetic CES preferences, we estimate three parameters that are common across countries $\{\sigma, \bar{c}_a, \bar{c}_s\}$ that govern the price and income elasticities, and $\{\Omega_i^c\}_{i\in\mathcal{I}, c\in C}$ which are country specific parameters.

We follow the estimation procedure described in Herrendorf et al. (2013). The identification of the parameters in the utility function is based on the estimation of the first order conditions of the intra-period problem to estimate the parameters of the aggregator (D.1),

$$\frac{L_{it}^c}{L_t^c} = \frac{\Omega_i^c p_{it}^{c\,1-\sigma}}{\sum_{i\in\{a,m,s\}} \Omega_i^c p_{it}^{c\,1-\sigma}} \left(1 + \frac{\sum_{i\in\{a,s\}} p_{it}^c \bar{c}_i}{\sum_{i\in\{a,m,s\}} p_{it}^c C_{it}^c}\right) - \frac{p_{it}^c \bar{c}_i}{\sum_{i\in\{a,m,s\}} p_{it}^c C_{it}^c} \tag{G.9}$$

with the understanding that $\bar{c}_m = 0$. We perform the estimation with and without sectoral trade controls for net exports. As with nonhomothetic CES preferences, we estimate three parameters that are common across countries $\{\sigma, \bar{c}_a, \bar{c}_s\}$ that govern the price and income elasticities, and $\{\Omega_i^c\}_{i\in\mathcal{I}, c\in C}$ which are country specific parameters.[20]

Table G.1 in reports the estimates. As expected, we find that the three sectors are gross complements $\sigma = 0.23$ and that income effects play a role, $\bar{c}_a < 0$ and $\bar{c}_s > 0$.[21] To have a better grasp of the magnitude of the income effects, we compute the values of $-\frac{p_{a,t}^c \bar{c}_a}{\sum_{i\in\{a,m,s\}} p_{it}^c C_{it}^c}$ and $\frac{p_{s,t}^c \bar{c}_s}{\sum_{i\in\{a,m,s\}} p_{it}^c C_{it}^c}$. For the U.S., they are never higher (in absolute terms) than .1%, which suggests that non-homtheticities are insignificant when compared to aggregate consumption. The highest values of the non-homotheticities in the sample are 37% and 18% for services.

Figure G.1 shows the fit of the model for six countries using the common parameters

[20] Note that, in fact, for (G.9) to be true, we have to impose that $\alpha_i = \alpha$ for all i. Otherwise we need to add the correction factor implied by (20), $L_{jt}\frac{1-\alpha_i}{1-\alpha_j}\frac{\omega_{it}}{\omega_{jt}}$. If we add this additional factor, the model fit becomes worse. So, we use (G.9) as baseline

[21] We report results with and without controls for net sectorial exports in agriculture and manufacturing in each regression. The estimated parameters are similar .

Table G.1: Stone-Geary Estimation Results

Dep. Var.: Emp. Shares	(1)	(2)
σ	0.23	0.28
	(0.00)	(0.00)
$\bar{c}_a$	-2.53	-2.01
	(0.02)	(0.00)
$\bar{c}_s$	0.99	1.33
	(0.07)	(0.00)
Sectoral Trade Control	N	Y
Measures of Non-homotheticity		
Entire Sample		
$\max \frac{p_s \bar{c}_s}{\sum_{i\in\{a,m,s\}} p_i C_i}$	0.14	0.18
$\min \frac{p_s \bar{c}_s}{\sum_{i\in\{a,m,s\}} p_i C_i}$	0.00	0.00
$\max \frac{p_a \bar{c}_a}{\sum_{i\in\{a,m,s\}} p_i C_i}$	-0.00	-0.00
$\min \frac{p_a \bar{c}_a}{\sum_{i\in\{a,m,s\}} p_i C_i}$	-0.37	-0.29
United States		
$\max \frac{p_s^{U.S.} \bar{c}_s}{\sum_{i\in\{a,m,s\}} p_i^{U.S.} C_i^{U.S.}}$	0.0001	0.0002
$\min \frac{p_a^{U.S.} \bar{c}_a}{\sum_{i\in\{a,m,s\}} p_i^{U.S.} C_i^{U.S.}}$	-0.0009	-0.0007

Note: Robust standard errors shown in parenthesis. All regressions have 1626 observations.

$\{\sigma, \bar{c}_a, \bar{c}_s\}$ and the country-specific preference shifters $\{\Omega_i^c\}_{i\in\mathcal{I}, c\in C}$.[22] We see that the overall fit is worse than with non-homothetic CES preferences. This finding is confirmed for the full sample.

In Table G.2 we compare the within R^2 measures generated with Stone-Geary preferences and nonhomothetic CES. Note that in the case that we only include only country-sector fixed taste parameters both demand systems collapse to the same one and the best prediction for sectoral shares are the country-sector sample average. The R^2 for Stone-Geary is 0.14 while it is 0.19 for nonhomothetic CES (structural estimates). Thus, the within R^2 compares the sum of squared errors made by the full demand system with the sum of squared errors based on predicting sectoral shares by the country sample average.[23] The reason for the

[22] Figures H.1 and H.2 shows the fit for all countries in our sample.

[23] Formally, the within R^2 is defined as $R^2 = 1 - \frac{1}{I}\sum_{i=1}^{I}\left(\sum_{j=1}^{N}(y_{ij}^c - \hat{y}_{ij}^c)^2 / \sum_{j=1}^{N}(y_{ij}^c - \bar{y}_i^c)^2\right)$ where N denotes the total number of observations per sector, I, the number of sectors, y_{ij}^c, observed employment shares in sector i and country c, $\hat{y}_{ij}^c$, predicted employment shares, $\bar{y}_i^c$ the sample average of y_{ij} for country c in sector i, and $i \in \mathcal{I} = \{a, m, s\}$.

Figure G.1: Fit for Stone-Geary with common preference parameters $\{\sigma, \bar{c}_a, \bar{c}_s\}$ for six countries

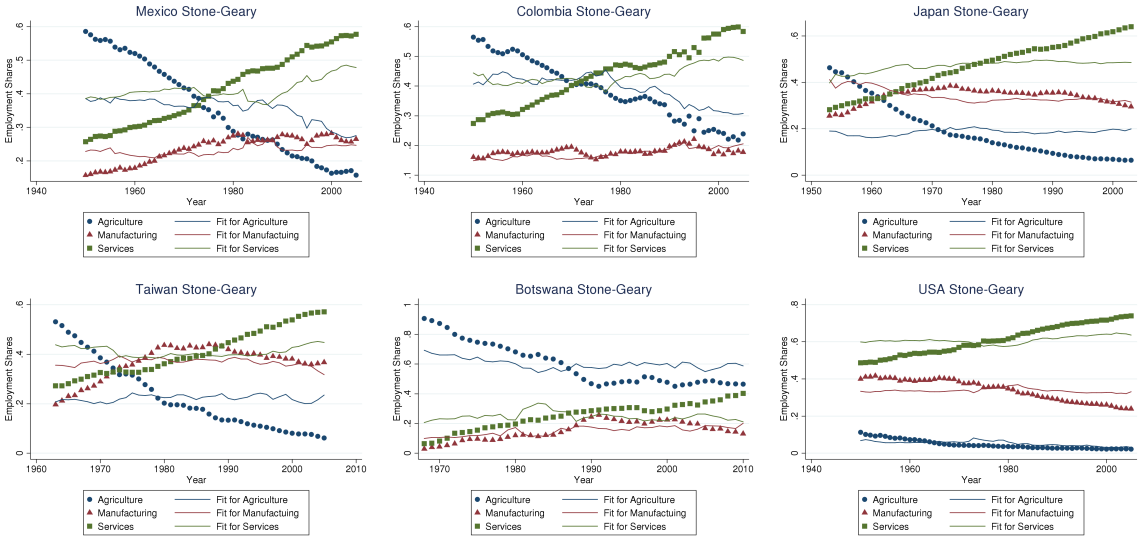


Table G.2: Within R^2 measures for Stone-Geary, PIGL and nonhomothetic CES

Model:	Within-R^2
Stone-Geary	0.14
PIGL	0.13
Nonhom.-CES	0.29

The within R^2 constructed from the structural estimates of each demand system without trade controls. The R^2 for nonhomotethic CES correspond to the structural estimates from column (1) in Table 3.

worsening of the fit is that, with Stone-Geary preferences, income effects are very low for rich countries, as for high levels of income the subsistence levels responsible for introducing the non-homotheticity $\{\sigma, \bar{c}_a, \bar{c}_s\}$ are negligible. Thus, only variation in prices and trade shares are left to explain the variation in employment shares.[24]

G.2.2 PIGL Estimation

We next study the cross-country fit of PIGL preferences as specified in Boppart (2014a). The proposed within-period indirect utility V of a household with total expenditure E^c in country

[24] An equivalent intuition is that the subsistence constants $\{\bar{c}_a, \bar{c}_s\}$ are not stable over time (see e.g., Dennis and Iscan, 2009).

c is

$$V = \frac{1}{\varepsilon}\left(\frac{E^c}{p^c_{s,t}}\right)^{\varepsilon} - \frac{\Omega^c_s}{\gamma}\cdot\frac{\left(\Omega^c_a\cdot\left(p^c_{a,t}\right)^{1-\sigma}+\Omega^c_m\cdot\left(p^c_{m,t}\right)^{1-\sigma}\right)^{\frac{\gamma}{1-\sigma}}}{\left(p^c_{s,t}\right)^{\gamma}} - \frac{1}{\varepsilon} + \frac{\Omega^c_s}{\gamma}, \tag{G.10}$$

with $0 \leq \varepsilon \leq \gamma < 1$ and $\Omega^c_i > 0$ for $i \in \{a, m, s\}$. This preference structure features a homothetic CES aggregator between agriculture and manufacturing with elasticity σ and a nonhomothetic aggregator between services and the agriculture-manufacturing composite. The baseline model in Boppart contains only two sectors. Here we follow the extension proposed in Appendix B.3.3 (Boppart, 2014b) to account for three sectors such that there can be a hump-shape in manufacturing, where we have generalized it to allow for constant taste parameters to be heterogeneous across countries and not symmetric between agricultural and manufacturing goods.[25]

The parameter ε governs the non-homotheticity of preferences between services and the composite of agricultural and manufacturing goods. If $\varepsilon > 0$, the expenditure elasticity is larger than one for services and less than one for agricultural and manufacturing goods (and identical for both). The price elasticity of substitution between services and the agriculture-manufacturing composite never exceeds one, it is increasing with the level of income and it asymptotes to $1-\gamma$.

We use the demand implied by these preferences to estimate the demand parameters. The sectoral demands expressed in expenditure shares per capita are also given in Boppart (2014b),

$$\begin{aligned}
\omega^c_{i,t} &= \Omega^c_s\phi^c(E^c)^{-\varepsilon}(p^c_{s,t})^{\varepsilon}(p^c_{i,t})^{1-\sigma}\frac{\left(\Omega^c_a\cdot\left(p^c_{a,t}\right)^{1-\sigma}+\Omega^c_m\cdot\left(p^c_{m,t}\right)^{1-\sigma}\right)^{\frac{\gamma+\sigma-1}{1-\sigma}}}{\left(p^c_{s,t}\right)^{\gamma}}, \qquad i \in \{a, m\} \\
\omega^c_{i,t} &= 1 - \Omega^c_s\phi^c(E^c)^{-\varepsilon}(p^c_{s,t})^{\epsilon}\frac{\left(\Omega^c_a\cdot\left(p^c_{a,t}\right)^{1-\sigma}+\Omega^c_m\cdot\left(p^c_{m,t}\right)^{1-\sigma}\right)^{\frac{\gamma}{1-\sigma}}}{\left(p^c_{s,t}\right)^{\gamma}},
\end{aligned} \tag{G.11}$$

where E^c_t denotes expenditure per capita and ϕ^c is a constant term capturing the distribution of expenditures in the population (the theory implies that it is constant) and it is effectively subsumed in Ω^c_s in the estimation (note that in the demand equations it always appears the product of the two).[26] To estimate the PIGL demand system, we use the same strategy we have used for nonhomothetic CES and Stone-Geary and use the market clearing condition (20) in the main text that implies that labor shares are proportional to expenditure shares.

[25] We have also experimented with another proposed extension such that the expenditure share in the manufacturing sector constant (Appendix B.3.2) obtaining a worse fit.

[26] In our theory we abstract away from income heterogeneity and effectively assume a representative consumer with expenditures equal to the average expenditure per capita.

Since this version of the model is not estimated in Boppart (2014a), we have experimented with different formulations of the estimating equations to maximize the model fit. We have found that the fit is maximized when we estimate the ratio of log-employment shares in agriculture to manufacturing and the employment share in services,[27]

$$\log\left(\frac{L^c_{a,t}}{L^c_{m,t}}\right) = (1-\phi)\frac{\Omega^c_a}{\Omega^c_m}\log\left(\frac{p^c_{a,t}}{p^c_{m,t}}\right), \tag{G.12}$$

$$\frac{L^c_{s,t}}{L^c_t} = 1-\Omega^c_s\phi^c(E^c)^{-\varepsilon}(p^c_{s,t})^{\varepsilon}\frac{\left(\Omega^c_a\cdot\left(p^c_{a,t}\right)^{1-\sigma}+\Omega^c_m\cdot\left(p^c_{m,t}\right)^{1-\sigma}\right)^{\frac{\gamma}{1-\sigma}}}{\left(p^c_{s,t}\right)^{\gamma}}. \tag{G.13}$$

Finally, note that as with nonhomothetic CES and Stone-Geary, we estimate three elasticities that are common across countries $\{\varepsilon,\gamma,\sigma\}$ and we allow for country-specific constant taste shifters $\{\Omega^c_i\}_{i\in\mathcal{I},c\in C}$.

Table G.3 reports the estimation results. The estimate for σ is less than one, suggesting gross complementarity between agriculture and manufacturing. We also find that nonhomotheticities are present. We note that the point estimates suggest that $\epsilon > \gamma$, however we cannot reject the hypothesis that they are identical and the estimates are still consistent with the theory. We take column (1) as our baseline estimate since is the estimation with the best fit. Column (2) reports the estimates we obtain when imposing that $\epsilon \leq \gamma$. Figure G.2 shows the fit for the same sample of countries as in the examples of Figure G.1. The fit appears to be comparable, as confirmed by Table G.2.

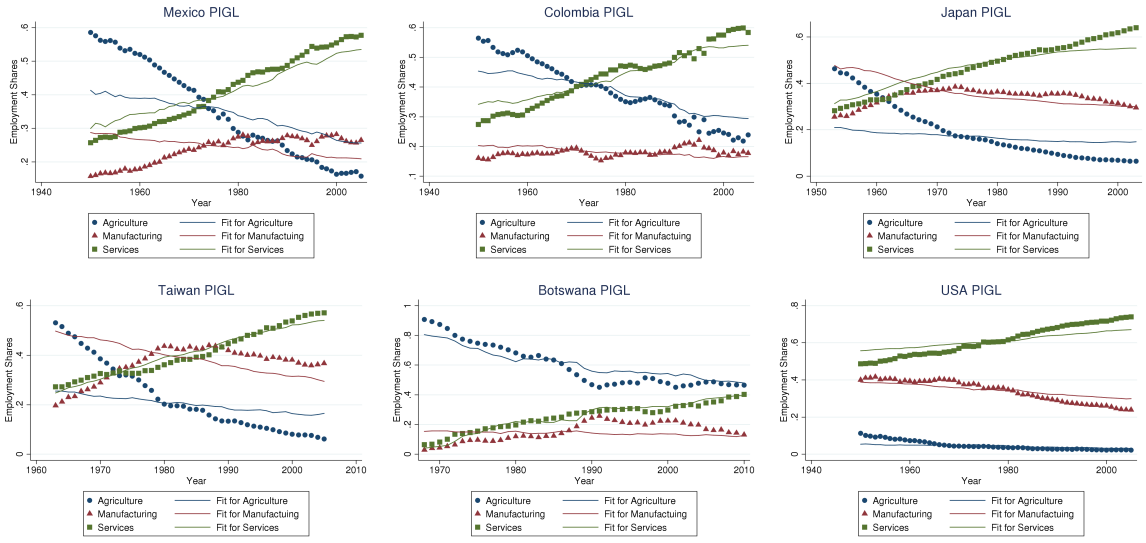
[27] Note that in practice, (G.13) is equivalent too estimate the reciprocal compounded good of agriculture and services which is log-linear–since, $1-\omega^c_{s,t}$ is log-linear. Note also that, as in the Stone-Geary estimation, we are making the same assumption that $\alpha_i = \alpha$. If we allowed for heterogeneous labor shares in production, (20), then we should include the term $L_{jt}\frac{1-\alpha_i}{1-\alpha_j}\frac{\omega_{it}}{\omega_{jt}}$ in the estimation for services. If we add this additional factor, the model fit becomes substantially worse. So, we use (G.9) as baseline. We have also experimented in estimating $\log\left(\frac{L^c_{s,t}}{L^c_{m,t}}\right)$ but we obtain $\epsilon \gg \gamma$. Moreover, the taste parameters Ω^c_s seem hard to identify (and very sensitive to initial guesses of the parameters). We have also tried estimating the model in levels, or with shares for agriculture and manufacturing (not ratios) with and without trade controls obtaining a worse fit in all cases and $\epsilon > \gamma$ in some instances.

Table G.3: PIGL Estimation Results

	(1)	(2)	(3)
σ	0.71	0.72	0.77
	(0.05)	(0.05)	(0.05)
ϵ	0.18	0.28	0.34
	(0.04)	(0.02)	(0.02)
γ	0.16	0.28	0.34
	(0.05)	(0.02)	(0.02)
Sectoral Trade Controls	N	N	Y
Constraint $\varepsilon < \gamma$	N	Y	Y

Note: Standard errors clustered by country. All regressions have 1626 observations.

Figure G.2: Fit for PIGL with common preference parameters $\{\sigma, \bar{c}_a, \bar{c}_s\}$ for six countries



H Additional Figures and Tables

Figure H.1: Country Fit Using Stone-Geary Preferences

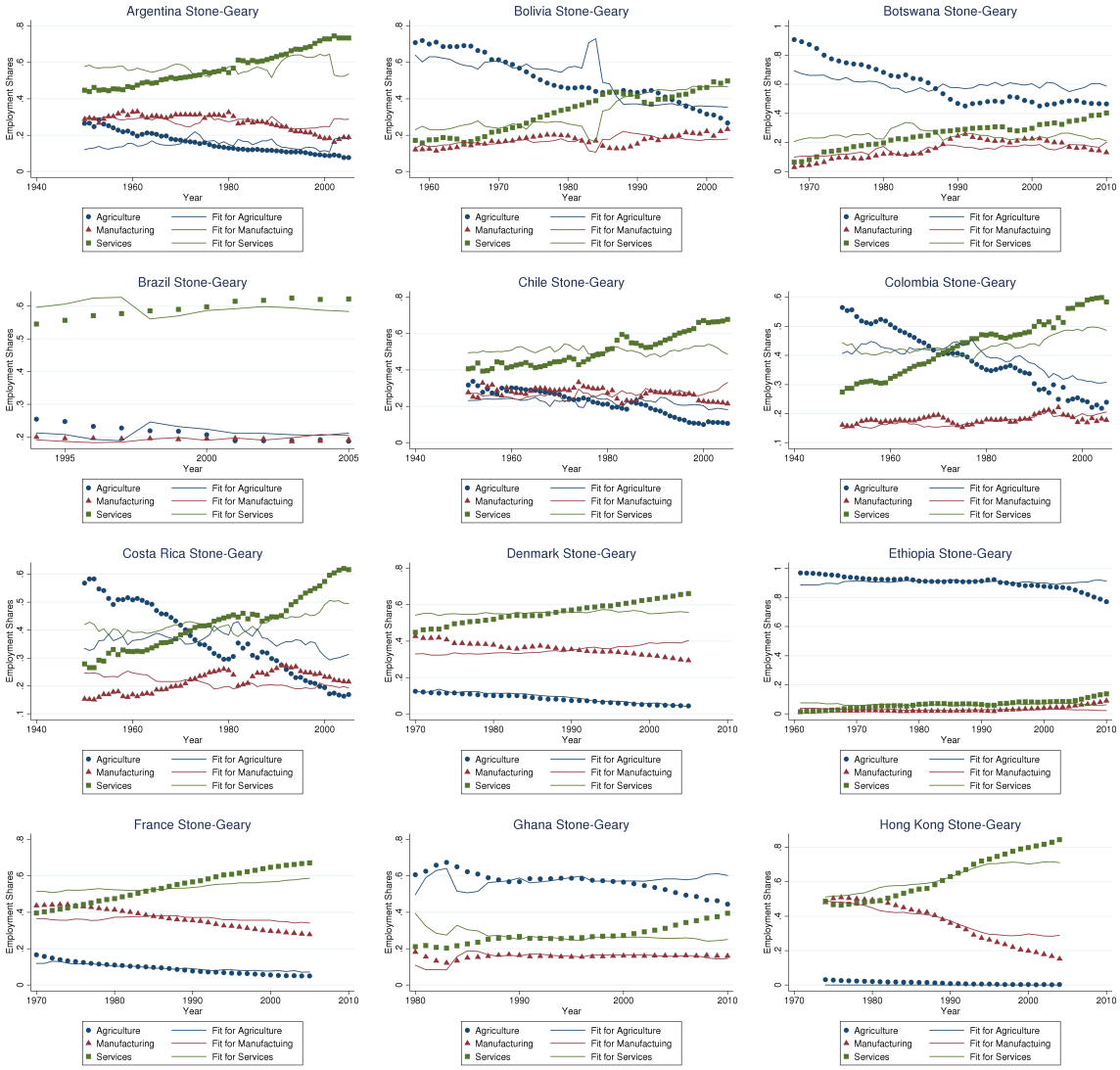


Figure H.2: Country Fit Using Stone-Geary Preferences (continued)

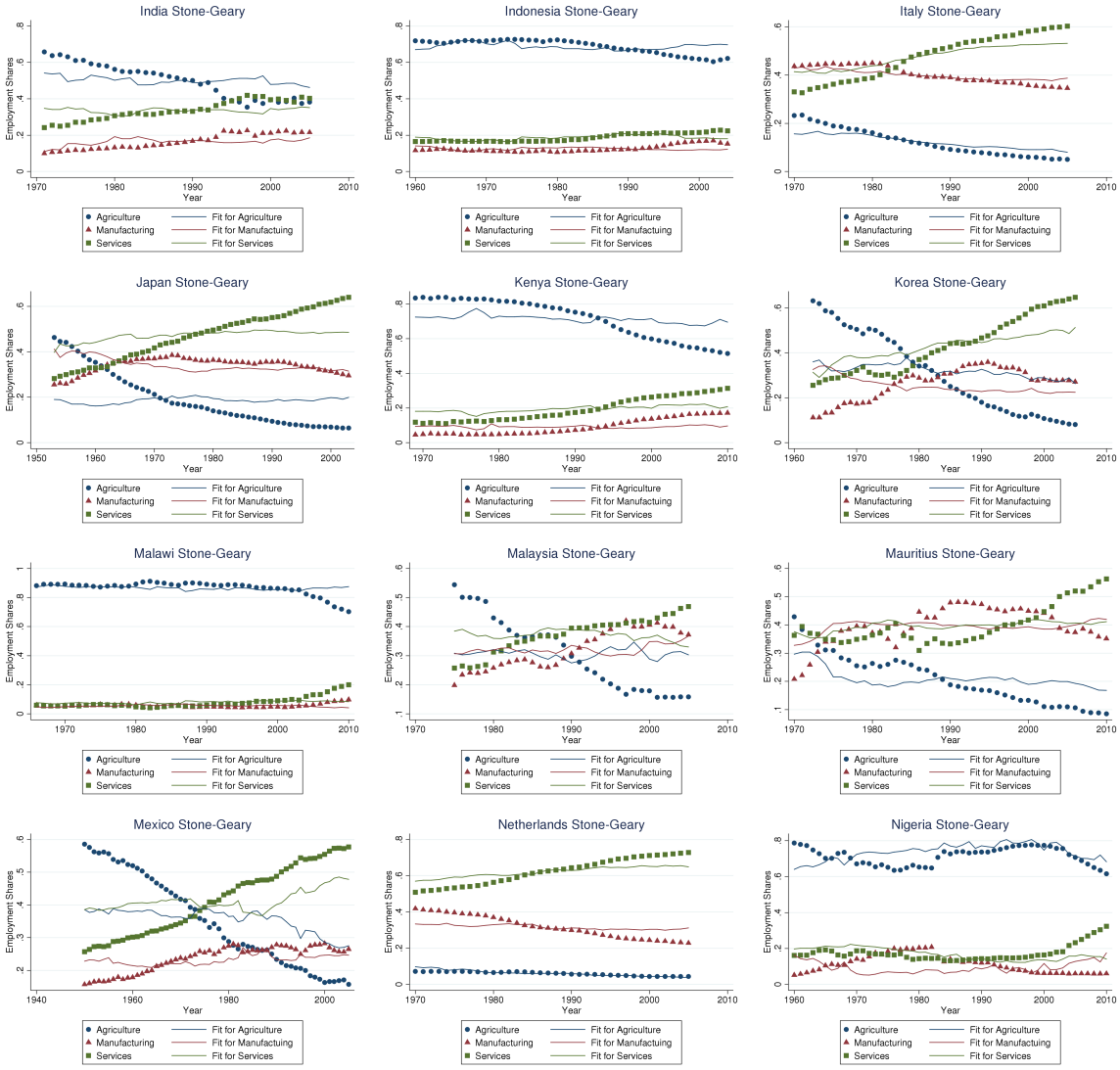


Figure H.3: Country Fit Using Stone-Geary Preferences (continued)

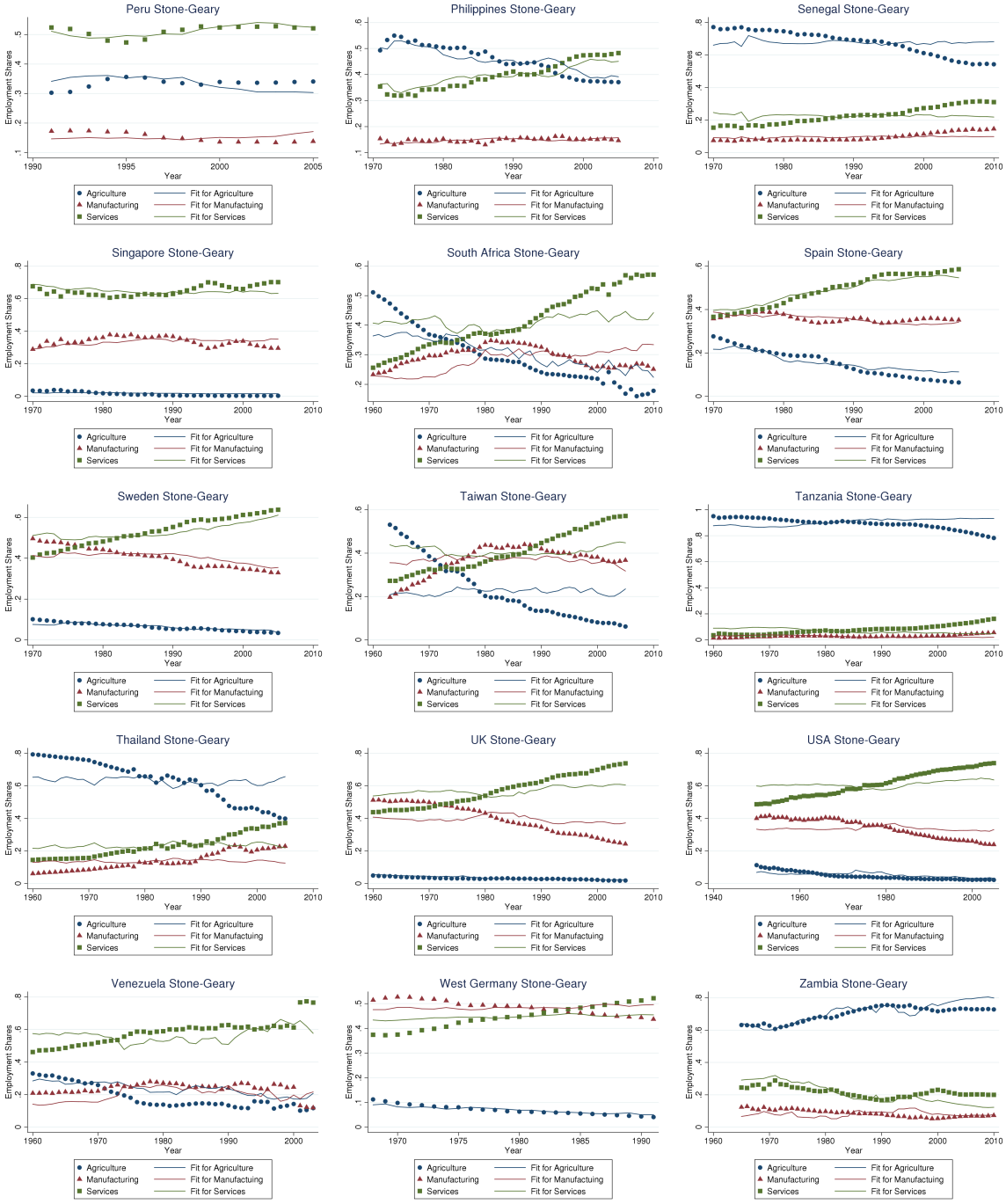


Figure H.4: Country Fit Using PIGL Preferences

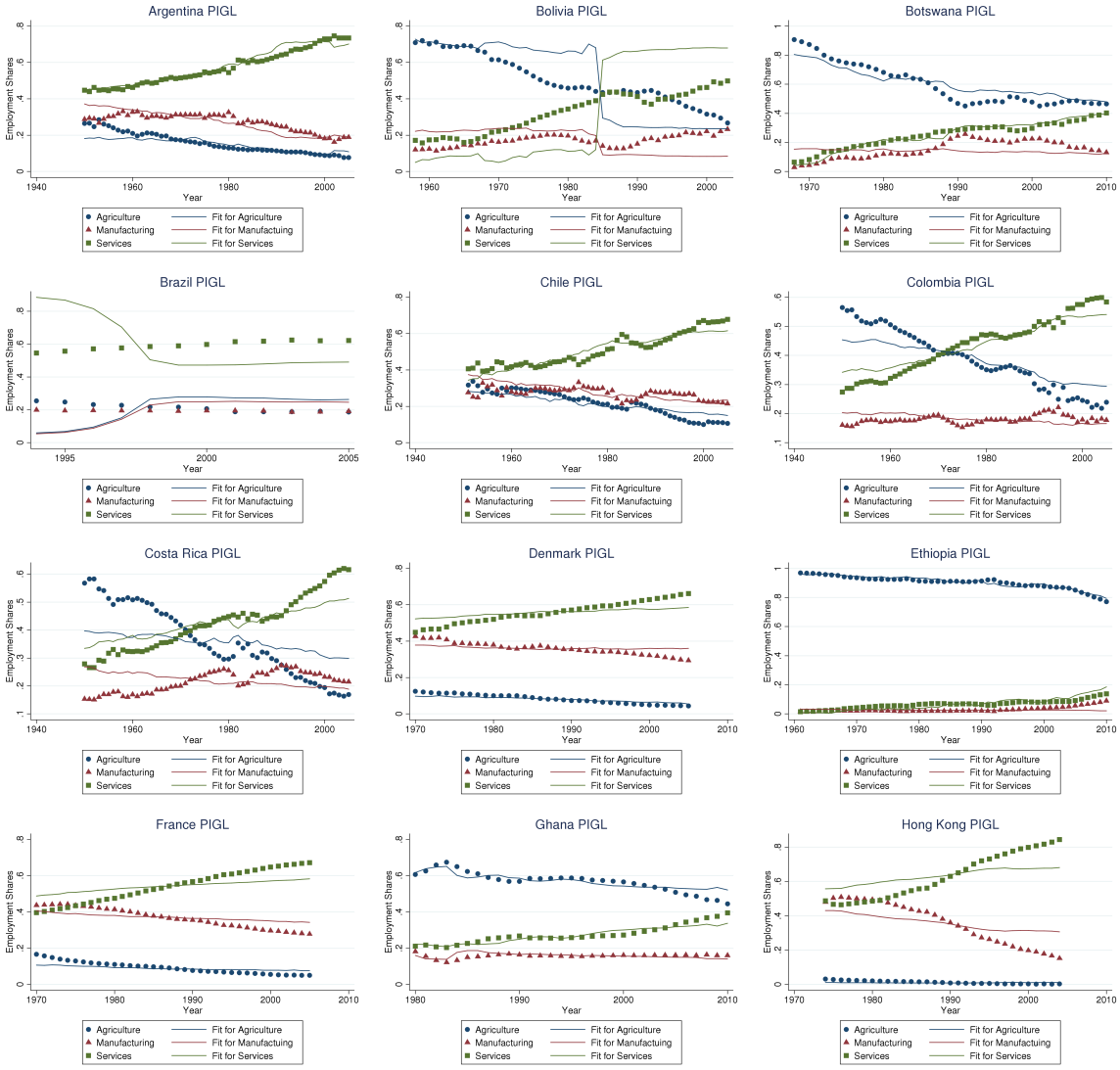


Figure H.5: Country Fit Using PIGL Preferences (continued)

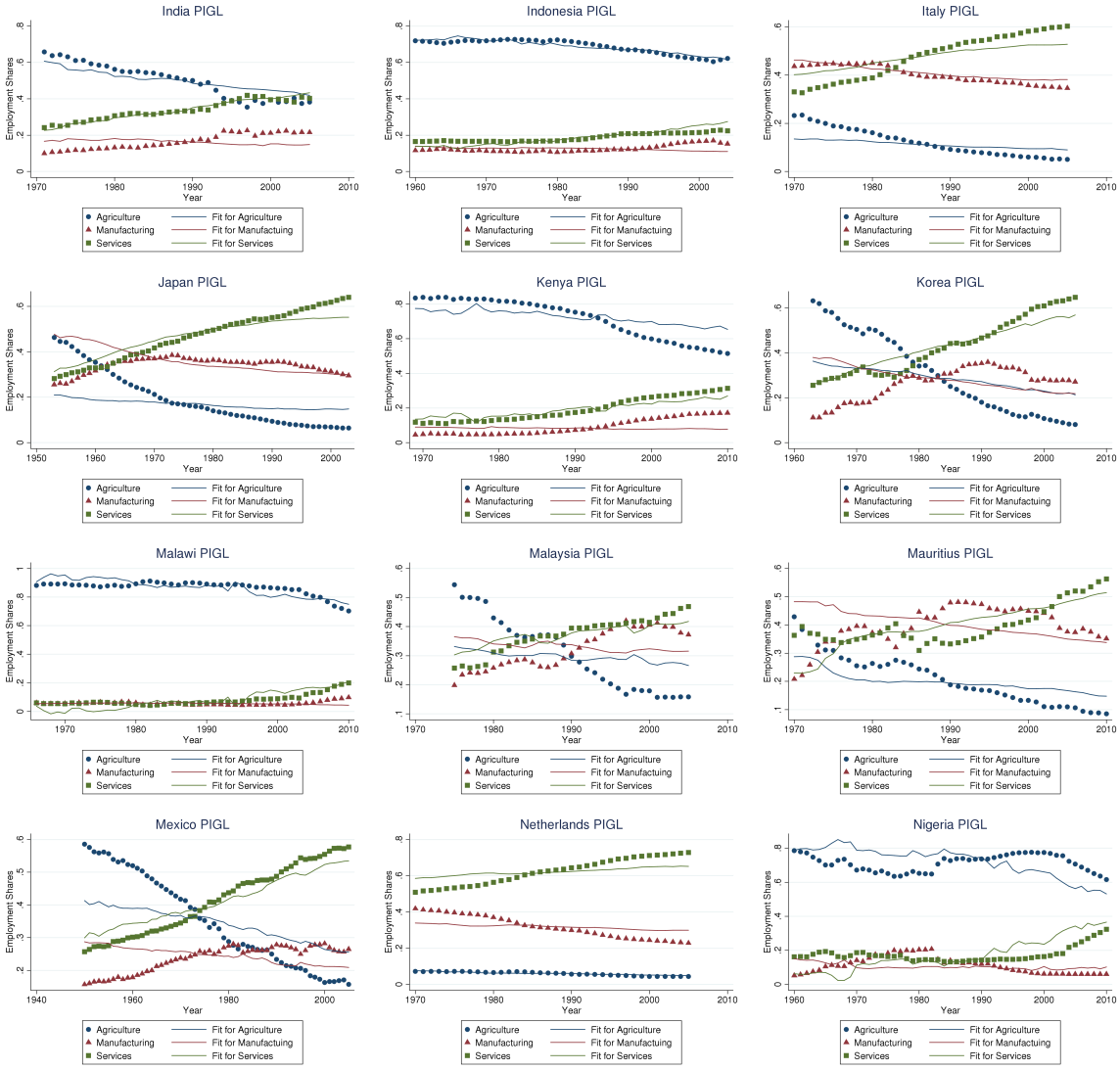
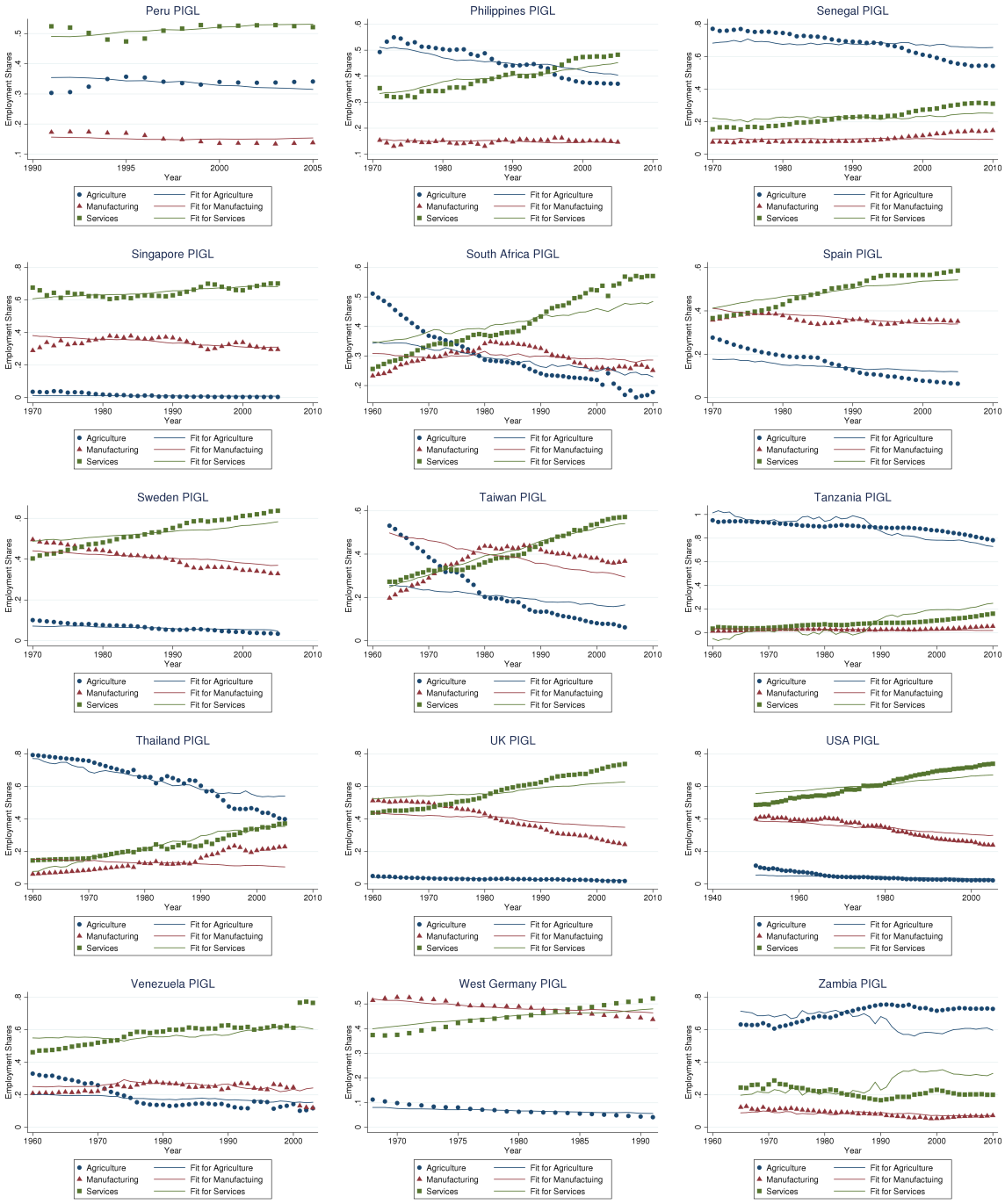


Figure H.6: Country Fit Using PIGL Preferences (continued)



I Monte-Carlo Simulations

This section reports the results of different estimation methods on a synthetic panel of cross-country data. Our synthetic data mimics the broad patterns and sample size of our cross-country data. We generate a panel of 20 countries with data for 42 years for each country. We generate the data such that countries feature around 16-fold differences in expenditures per capita between the 10th and 90th percentile of the distribution. Price trends are, on average, the ones that we observe in the data.

We simulate our data such that each country has a stochastic trend on each of the prices and in the real consumption index. We use the cross-country sample average growth rates in agriculture, manufacturing and services which are -2.2%, -1.7% and -1.6%, respectively. For the real consumption index we calibrate it to grow at 2%. We create the stochastic trends by drawing random variables for each country and time, and imposing a growth rate equal to the average plus the random draw. We draw from a i.i.d normal distribution with zero mean and a standard deviation of 1%. We also impose differences in the initial levels of real consumption, which span three-fold differences in initial levels.

We then simulate mismeasurement in the observed prices, total expenditure and expenditure shares. We do so by drawing from an i.i.d. normal distribution with mean zero and standard deviation x. We multiply each variable (prices, expenditure and expenditure shares) by the exponent of the draw. For the measures of expenditure shares that are above one, we substitute them to 1. Note that the shocks to expenditure shares and top-coding of expenditure shares generates inconsistencies in the data, because expenditure shares do not add up to one. We report simulations on the estimated data for different values of the shock x.

We present estimation results for the different econometric approaches discussed in the text. We report the baseline results corresponding to Equation (30) in Section 3, the non-linear estimation discussed in Section 5.1.3, Equation (36), and the iterative procedure discussed in Section 5.1.4. We also report the estimates of the log-linear approach estimates from Equation (33) discussed in Section 5.1.1, and the first step of the iterative approach discussed in Section 5.1.4, Equation (38). We show that they indeed identify $I-2$ parameters of the demand system.

Finally, we also show the results for two more specifications. The first additional specification corresponds to the hypothetical case were we could observe the exact consumption index, $\log C_t^n$, without *any* measurement error. In the notation of the paper, we run

$$\log\left(\frac{\omega_{it}^n}{\omega_{mt}^n}\right) = (1-\sigma)\log\left(\frac{p_{it}^n}{p_{mt}^n}\right) + (\epsilon_i - \epsilon_m)\log\log C_t^n + \zeta_i^n + \nu_{it}^n. \tag{I.1}$$

Of course, this exercise is not of empirical relevance but it serves the purpose of benchmarking the role of mismeasurement in prices and expenditure shares, as any discrepancy between the

true estimates and what we obtain in this estimation is due to mismeasurement in these two variables.

The second additional specification we estimate is an alternative hybrid, two-step estimation method that is not discussed in the main text. This estimation procedure uses the first step of the iterative procedure to identify $I-2$ elasticity parameters and σ. Then, it estimates the reminder degree of freedom in the income elasticity parameters that makes it consistent with a particular cardinalization of utility using GMM. We show that this estimation procedure yields very similar estimates to the baseline results (but it is faster). [28]

Table I.1 reports the estimates for a low-level of mismeasurement, $x = 0.1\%$. That is, each observable variable is multiplied by a factor that is on average one, but the log of the shock has has a standard error of 0.1%. All estimations have country fixed effects. The first column reports the results of estimating the system of Equations (I.1) where $\log C$ is estimated without an error. Columns (2) to (4) present the exact estimating methods reported in the text. Columns (5) and (6) present rank methods that only identify relative measures of income elasticity parameters. Column (7) reports the hybrid method discussed above. We see that in this case, all exact estimation methods retrieve the true elasticity values (columns 1 to 4 and 7). As implied by Lemma 4, specifications in columns (5) and (6) retrieve the true ratio of estimates. However, the estimated $\tilde{\epsilon}_i - \tilde{\epsilon}_m$ correspond to an alternative cardinalization of utility/normalization of ϵ_m (which turns out to be $\tilde{\epsilon}_m \simeq 0.5$). Note also that in the last row we report the computation time. The methods that rely on linear estimation perform significantly faster, on the order of seconds, while the GMM estimations take significantly longer. Our baseline specification takes 2.7 minutes, for 4.5 minutes of the non-linear specification. The alternative hybrid specification takes slightly less than our baseline, around 2.5 minutes.

Table I.2 reports our estimates for an intermediate level of mismeasurement, $x = 1\%$. Overall, all estimation methods perform well. In fact, all estimation methods retrieve the true parameter values up to the second decimal value, except for the iterative method in column (4). The iterative method obtains median estimates that differ from the true parameter values at the second decimal digit by 0.01 for the price elasticity and the income elasticity parameter for agriculture, suggesting that this method is more sensitive to mismeasurement.

Table I.3 reports our estimates for a large level of mismeasurement, $x = 5\%$. This simulation implies a significant amount of inconsistency in our estimated data. To have a sense, in a

[28] Its main advantage comes when estimating a larger number of sectors than our baseline exercise, because it identifies $I-2$ income elasticity parameters through linear regressions without constraints which are faster to estimate than our baseline estimation. More specifically, in the first step we run the fist step of the iterative procedure, Equation (38) in Section 5.1.4 (we could also run the log-linear specification instead). Then, we identify the reminder degree of freedom α by using the moment condition implied by

$$\log\left(\frac{\omega_{it}^n}{\omega_{mt}^n}\right) = (1-\hat{\sigma})\log\left(\frac{p_{it}^n}{p_{mt}^n}\right) + \alpha(1-\hat{\sigma})\hat{\epsilon}_i \log\left(E_t^n p_{mt}^n\right) + (\alpha\hat{\epsilon}_i + 1)\log\omega_{mt}^n + +\zeta_i^n + \nu_{it}^n, \tag{I.2}$$

where $\hat{\sigma}$ and $\hat{\epsilon}_i$ denote the estimates from the first stage.

typical synthetic dataset, 10% of the observations have expenditure shares adding up to less than 0.95 and 10% of the observations add to more than 1.05. Likewise, log-relative prices have a standard deviation in the mismeasurement shock of 10% (since we are summing two normal errors with standard deviation of 5% each). The first column reports the results of estimating the system of Equations (I.1) where $\log C$ is estimated without an error. We see that the mismeasurement in prices translates in downward bias of $1 - \sigma$, which implies an upward bias on σ. The median estimate is 0.52 (vs. a true value of 0.50). Moreover, the 95% confidence interval does not contain the true value. In contrast, income elasticity parameters are perfectly estimated (since we have used the true measure of $\log C$).

Our baseline estimation method in column (2) appears to be the exact method that performs best, and in fact performs similarly well to the one where we use the exact $\log C$, despite the fact that we use mismeasured nominal expenditures. The price elasticity is 0.51 and the 95% confidence interval contains the true parameter value. For income elasticity parameters they are -0.92 and 0.24 and the true parameter values, -0.90 and 0.25, are contained in the 95% confidence interval. Comparatively, the non-linear method and the iterative method based on the second-order approximation, perform somewhat worse. For example, the non-linear method method estimates a price elasticity of 0.52, and relative income elasticities of -0.93 and 0.26. The rank methods appear to be very robust in identifying the ratio of true parameters –if anything, they do better than the exact methods.

Table I.1: Small Mismeasurement ($x = 0.1\%$) Monte-Carlo Simulation.
True Parameters: $\sigma = 0.5$, $\epsilon_a - 1 = -0.9$, and $\epsilon_m - 1 = 0.25$

		Exact Methods			Rank Methods		
	Exact $\ln C$ (1)	Baseline (2)	Non-Linear (3)	Iterative (4)	Linear (5)	Tornqvist (6)	Hybrid (7)
σ	0.50	0.50	0.50	0.50		0.50	0.50
	[0.50,0.50]	[0.50,0.50]	[0.50,0.50]	[0.50,0.50]		[0.50,0.50]	[0.50,0.50]
$\epsilon_a - 1$	-0.90	-0.90	-0.90	-0.90	-	-	-0.90
	[-0.90,-0.90]	[-0.90,-0.90]	[-0.90,-0.90]	[-0.90,-0.89]			[-0.90,-0.90]
$\epsilon_s - 1$	0.25	0.25	0.25	0.25	-	-	0.25
	[0.25,0.25]	[0.25,0.25]	[0.25,0.25]	[0.25,0.25]			[0.25,0.25]
	-	-	-	-	-0.43	-0.44	-
					[-0.44,-0.43]	[-0.44,-0.43]	
$\tilde{\epsilon}_s - \tilde{\epsilon}_m$	-	-	-	-	0.12	0.12	-
					[0.12,0.12]	[0.12,0.12]	
Ratio	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28
	[-0.28,-0.28]	[-0.28,-0.28]	[-0.28,-0.28]	[-0.28,-0.28]	[-0.28,-0.28]	[-0.28,-0.28]	[-0.28,-0.28]
Time	0.02	2.47	4.03	0.06	0.01	0.02	2.29

Notes: Median estimated parameter across 100 samples reported as main estimate. 95% confidence interval reported in square brackets. These correspond to the 2.5 and 97.5 percentiles of the estimated parameters over 100 random samples. Time is in minutes. The initial guesses for the Baseline and Non-linear Methods are $\sigma^0 = 0.75$, $\epsilon_a^0 = 0.5$ and $\epsilon_s^0 = 1.75$.

Table I.2: Intermediate Mismeasurement ($x = 1\%$) Monte-Carlo Simulation.
$\sigma = 0.5$, $\epsilon_a - 1 = -0.9$ and $\epsilon_m - 1 = 0.25$

		Exact Methods			Rank Methods		
	Exact $\ln C$ (1)	Baseline (2)	Non-Linear (3)	Iterative (4)	Linear (5)	Tornqvist (6)	Hybrid (7)
σ	0.50	0.50	0.50	0.51		0.50	0.50
	[0.50,0.50]	[0.50,0.50]	[0.50,0.50]	[0.51,0.51]		[0.50,0.51]	[0.50,0.51]
$\epsilon_a - 1$	-0.90	-0.90	-0.90	-0.91	-	-	-0.91
	[-0.90,-0.90]	[-0.91,-0.90]	[-0.91,-0.90]	[-0.92,-0.91]			[-0.92,-0.90]
$\epsilon_s - 1$	0.25	0.25	0.25	0.25	-	-	0.25
	[0.25,0.25]	[0.25,0.25]	[0.25,0.25]	[0.25,0.26]			[0.25,0.26]
	-	-	-	-	-0.43	-0.44	-
					[-0.44,-0.43]	[-0.44,-0.43]	
$\tilde{\epsilon}_s - \tilde{\epsilon}_m$	-	-	-	-	0.12	0.12	-
					[0.12,0.12]	[0.12,0.12]	
Ratio	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28	-0.28
	[-0.28,-0.27]	[-0.28,-0.27]	[-0.28,-0.27]	[-0.28,-0.27]	[-0.28,-0.27]	[-0.28,-0.27]	[-0.28,-0.27]
Time	0.02	2.44	4.01	0.06	0.01	0.02	2.33

Notes: Median estimated parameter across 100 samples reported as main estimate. 95% confidence interval reported in square brackets. These correspond to the 2.5 and 97.5 percentiles of the estimated parameters over 100 random samples. Time is in minutes. The initial guesses for the Baseline and Non-linear Methods are $\sigma^0 = 0.75$, $\epsilon_a^0 = 0.5$ and $\epsilon_s^0 = 1.75$.

Table I.3: Intermediate Mismeasurement ($x = 5\%$) Monte-Carlo Simulation.
$\sigma = 0.5$, $\epsilon_a - 1 = -0.9$ and $\epsilon_m - 1 = 0.25$

		Exact Methods			Rank Methods		
	Exact $\ln C$ (1)	Baseline (2)	Non-Linear (3)	Iterative (4)	Linear (5)	Tornqvist (6)	Hybrid (7)
σ	0.52	0.51	0.52	0.53		0.53	0.53
	[0.51,0.53]	[0.50,0.52]	[0.49,0.54]	[0.52,0.54]		[0.51,0.54]	[0.51,0.54]
$\epsilon_a - 1$	-0.90	-0.92	-0.93	-0.95	-	-	-0.95
	[-0.92,-0.89]	[-0.95,-0.89]	[-1.16,-0.90]	[-0.98,-0.91]			[-0.98,-0.91]
$\epsilon_s - 1$	0.25	0.24	0.26	0.26	-	-	0.26
	[0.24,0.27]	[0.22,0.27]	[0.24,0.34]	[0.25,0.28]			[0.25,0.29]
	-	-	-	-	-0.43	-0.44	-
					[-0.44,-0.42]	[-0.45,-0.43]	
$\tilde{\epsilon}_s - \tilde{\epsilon}_m$	-	-	-	-	0.12	0.12	-
					[0.11,0.13]	[0.11,0.13]	
Ratio	-0.28	-0.26	-0.28	-0.28	-0.28	-0.28	-0.28
	[-0.30,-0.26]	[-0.29,-0.24]	[-0.31,-0.25]	[-0.30,-0.26]	[-0.31,-0.26]	[-0.30,-0.26]	[-0.30,-0.26]
Time	0.01	0.62	1.13	0.02	0.00	0.01	0.58

Notes: Median estimated parameter across 100 samples reported as main estimate. 95% confidence interval reported in square brackets. These correspond to the 2.5 and 97.5 percentiles of the estimated parameters over 100 random samples. Time is in minutes. The initial guesses for the Baseline and Non-linear Methods are $\sigma^0 = 0.75$, $\epsilon_a^0 = 0.5$ and $\epsilon_s^0 = 1.75$.

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