

# Internet Appendix

## A. Derivation of the bond price

As stated in the main text, the bond price evolves according to the following ordinary differential equation:

$$r \cdot p(\lambda) = c + \xi(1 - p(\lambda)) + a(\lambda)p'(\lambda) + \lambda(\eta - p(\lambda)), \quad (\text{A.1})$$

where

$$a(\lambda) \equiv \lambda^2 - (\lambda_h + \lambda_l + 2q)\lambda + q(\lambda_h + \lambda_l) + \lambda_h \lambda_l. \quad (\text{A.2})$$

We conjecture that the bond price is a linear function of  $\lambda$ :  $p(\lambda) = m_1 + m_2 \cdot \lambda$ . Substituting this conjecture into equation (A.1), we obtain

$$\begin{aligned} r(m_1 + m_2\lambda) = & c + m_2(\lambda^2 - (\lambda_h + \lambda_l + 2q)\lambda + q(\lambda_h + \lambda_l) + \lambda_h \lambda_l) \\ & + \lambda(\eta - m_1 - m_2\lambda) + \xi(1 - m_1 - m_2\lambda). \end{aligned} \quad (\text{A.3})$$

Note that, on the right hand side of (A.3), the two  $\lambda^2$  terms,  $m_2\lambda^2$  and  $-m_2\lambda^2$ , cancel each other out. Thus, both sides of the equation are linear function of  $\lambda$ . Matching terms gives

$$\begin{aligned} m_1 &= \frac{(c + \xi)(2q + \lambda_h + \lambda_l + r + \xi) + \eta q(\lambda_h + \lambda_l) + \eta \lambda_h \lambda_l}{(r + \xi)(2q + \lambda_h + \lambda_l + r + \xi) + q(\lambda_h + \lambda_l) + \lambda_h \lambda_l} > 0, \\ m_2 &= -\frac{(c + \xi) - (r + \xi)\eta}{(r + \xi)(2q + \lambda_h + \lambda_l + r + \xi) + q(\lambda_h + \lambda_l) + \lambda_h \lambda_l} < 0. \end{aligned} \quad (\text{A.4})$$

## B. Numerical procedure for solving the default extrapolation model

### B.1. Change of variables

For numerical considerations, we first define

$$W(X, \lambda) \equiv K \cdot e\left(\frac{X}{K}, \lambda\right), \quad \text{so } e(v, \lambda) = K^{-1}W(vK, \lambda), \quad (\text{B.1})$$

where  $K$  is a large scaling factor (a constant). We substitute (B.1) into (16) of the main text and obtain

$$r \cdot W(X, \lambda) = \left[ \begin{array}{l} \mu X W_X + \frac{1}{2} \sigma^2 X^2 W_{XX} + a(\lambda) W_\lambda \\ + \delta(1 - \tau) \cdot X - [(c(1 - \tau) + \xi) - \xi p(\lambda)] K \\ + \frac{1}{2\psi K} (p(\lambda) K + W - X W_X)^2 \end{array} \right]. \quad (\text{B.2})$$

The two boundary conditions in (18) of the main text now become

$$W(X_b(\lambda), \lambda) = 0, \quad W_X(X_b(\lambda), \lambda) = 0. \quad (\text{B.3})$$

Our focus is to numerically solve the differential equation (B.2) as well as the default boundary  $X_b(\lambda)$ .

To do so, we first consider the limiting case when  $\lambda = \lambda_m$ . In this case, the partial differential equation in (B.2) becomes an ordinary differential equation

$$r \cdot W = \left[ \begin{array}{l} \mu X W' + \frac{1}{2} \sigma^2 X^2 W'' \\ + \delta(1 - \tau) \cdot X - [(c(1 - \tau) + \xi) - \xi p(\lambda_m)] K \\ + \frac{1}{2\psi K} (p(\lambda_m) K + W - X W')^2 \end{array} \right], \quad (\text{B.4})$$

where  $W(X, \lambda_m)$  can be viewed as a univariate function of  $X$ . As  $X$  goes to infinity, we conjecture that

$$W(X, \lambda_m) \rightarrow \frac{(1 - \tau)\delta}{r - \mu} X + W_2(\lambda_m); \quad (\text{B.5})$$

this conjecture is easy to verify.

Substituting (B.5) back into (B.4), we obtain

$$r \cdot W_2 = \left[ -[(c(1 - \tau) + \xi) - \xi p(\lambda_m)] K + \frac{1}{2\psi K} (p(\lambda_m) K + W_2)^2 \right]. \quad (\text{B.6})$$

The two solutions are

$$W_{2,\pm}(\lambda_m) = -(p(\lambda_m) - r\psi)K \pm K \sqrt{(p(\lambda_m) - r\psi)^2 + (2\psi c(1 - \tau) + 2\xi\psi(1 - p(\lambda_m)) - p^2(\lambda_m))}. \quad (\text{B.7})$$

Note that  $-W_2$  can be interpreted as debt value, so the negative root  $W_{2,-}(\lambda_m)$  is the relevant one.

We now define the market-timing component of equity value as

$$H(X; \lambda_m) \equiv W(X, \lambda_m) - \frac{\delta(1 - \tau)}{r - \mu}X. \quad (\text{B.8})$$

Substituting (B.8) back into (B.4) gives

$$r \cdot H = \left[ \begin{array}{l} \mu X H' + \frac{1}{2}\sigma^2 X^2 H'' \\ -[(c(1 - \tau) + \xi) - \xi p(\lambda_m)]K + \frac{1}{2\psi K}(p(\lambda_m)K + H - X H')^2 \end{array} \right]. \quad (\text{B.9})$$

Further substituting (B.8) into the two boundary conditions in (B.3) gives

$$H(X_b(\lambda_m); \lambda_m) = -\frac{\delta(1 - \tau)}{r - \mu}X_b(\lambda_m), \quad H_X(X_b(\lambda_m); \lambda_m) = -\frac{\delta(1 - \tau)}{r - \mu}. \quad (\text{B.10})$$

These two conditions, together with the following condition

$$\lim_{X \rightarrow \infty} H(X; \lambda_m) = W_2(\lambda_m), \quad (\text{B.11})$$

allows us to solve for  $H$  and  $X_b(\lambda_m)$ , the default boundary evaluated at  $\lambda = \lambda_m$ .

## B.2. Solving $H(X)$

To facilitate subsequent numerical analyses, we make the following change of variables

$$x = \frac{X - \zeta}{X + \zeta} \quad (\text{B.12})$$

and define  $l(x) \equiv H(X(x))$  with  $X = \zeta(1 + x)/(1 - x)$ . Now, equation (B.9) becomes

$$rl = \left[ \begin{array}{l} \frac{1}{2}\mu(1 - x^2)l' + \frac{1}{2}\sigma^2 \left( \zeta \frac{1+x}{1-x} \right)^2 \left( l'' \frac{(1-x)^4}{4\zeta^2} - l' \frac{(1-x)^3}{2\zeta^2} \right) \\ -[(c(1 - \tau) + \xi) - \xi p(\lambda_m)]K + \frac{1}{2\psi K} [p(\lambda_m)K + l - \frac{1}{2}(1 - x^2)l']^2 \end{array} \right]. \quad (\text{B.13})$$

The two boundary conditions in (B.10) become

$$l(x_b) = -\frac{\zeta\delta(1 - \tau)}{r - \mu} \cdot \frac{1 + x_b}{1 - x_b}, \quad l'(x_b) \frac{(1 - x_b)^2}{2\zeta} = -\frac{\delta(1 - \tau)}{r - \mu}, \quad (\text{B.14})$$

where  $x_b \equiv (X_b(\lambda_m) - \zeta)/(X_b(\lambda_m) + \zeta)$  is the transformed value of the default boundary  $X_b(\lambda_m)$ . The limiting condition in (B.11) is

$$l(1) = W_2(\lambda_m). \quad (\text{B.15})$$

Next, we approximate the function  $l$  by

$$\hat{l}(x) = \sum_{r=0}^n b_r T_r(x), \quad (\text{B.16})$$

where  $\{b_r\}_{0 \leq r \leq n}$  represents  $n + 1$  coefficients and  $T_r(x)$  represents the  $r^{\text{th}}$  degree Chebyshev polynomial of the first kind.<sup>35</sup> For a given value of  $x_b$ , equations (B.14) and (B.15) allow us to express  $\{b_0, b_1, b_2\}$  each as a combination of the remaining  $n - 2$  coefficients. Then, equation (B.13) allows us to solve for the remaining coefficients as well as  $x_b$  through minimizing a weighted sum of squared errors.

Define the optimal debt issuance as  $g^*(x) \equiv g^*(v(x), \lambda_m)$ . Then, from (17) of the main text, (B.1), (B.8), and (B.12), we obtain

$$g^*(x) = \xi + \frac{p(\lambda_m)K + l - \frac{1}{2}l'(1 - x^2)}{\psi K}. \quad (\text{B.17})$$

It can be approximated by

$$\hat{g}(x) = \xi + \frac{p(\lambda_m)K + \hat{l} - \frac{1}{2}\hat{l}'(1 - x^2)}{\psi K}. \quad (\text{B.18})$$

We illustrate the numerical procedure by showing an example. The parameter values are:  $\lambda_l = 0.005$ ,  $\lambda_h = 2$ ,  $q = 0.05$ ,  $\eta = 0.5$ ,  $r = 0.04$ ,  $c = 0.04$ ,  $\xi = 1$ ,  $\tau = 0.3$ ,  $\psi = 20$ ,  $\mu = 0.01$ ,  $\delta = 0.04$ , and  $\sigma = 0.25$ . The scaling factor  $K$  is set to 10. The parameter  $\zeta$  for the non-linear transformation in (B.12) is set to 25. The optimal default boundary is  $X_b(\lambda_m) = 4.88$ . We also know from (B.7) that  $W_{2,-}(\lambda_m) = -12.63$ . With this parameterization, Figure B.1 plots  $\hat{l}$  and  $\hat{g} - \xi$  against  $x$ , the transformed value of  $X$ .

### B.3. Solving the two-state variable partial differential equation

We now solve the full model. Recall that the partial differential equation we are solving is

$$r \cdot W(X, \lambda) = \left[ \begin{array}{l} \mu X W_X + \frac{1}{2} \sigma^2 X^2 W_{XX} + a(\lambda) W_\lambda \\ + \delta(1 - \tau) \cdot X - [(c(1 - \tau) + \xi) - \xi p(\lambda)] K \\ + \frac{1}{2\psi K} (p(\lambda)K + W - X W_X)^2 \end{array} \right]. \quad (\text{B.19})$$

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<sup>35</sup>See Mason and Handscomb (2003) for a detailed discussion of the properties of Chebyshev polynomials.

Optimal debt issuance is

$$g^*(X, \lambda) = \xi + \frac{p(\lambda)K + W - XW_X}{\psi K}. \quad (\text{B.20})$$

We define the “residual” equity value as

$$H(X, \lambda) \equiv W(X, \lambda) - \frac{\delta(1-\tau)}{r-\mu}X. \quad (\text{B.21})$$

The evolution of  $H$  is governed by

$$rH = \left[ \begin{array}{l} \mu X H_X + \frac{1}{2}\sigma^2 X^2 H_{XX} + a(\lambda)H_\lambda \\ -[(c(1-\tau) + \xi) - \xi p(\lambda)]K + \frac{1}{2\psi K}(p(\lambda)K + H - XH_X)^2 \end{array} \right]. \quad (\text{B.22})$$

Optimal debt issuance becomes

$$g^*(X, \lambda) \equiv \frac{G}{F} = \xi + \frac{p(\lambda)K + H - XH_X}{\psi K}. \quad (\text{B.23})$$

The boundary conditions are

$$H(X_b(\lambda), \lambda) = -\frac{\delta(1-\tau)}{r-\mu}X_b(\lambda), \quad H_X(X_b(\lambda), \lambda) = -\frac{\delta(1-\tau)}{r-\mu}, \quad (\text{B.24})$$

for  $\lambda_m \leq \lambda \leq \lambda_h$ .

Numerically, we have solved one other boundary condition

$$H(X, \lambda_m) = \hat{l}(x(X)), \quad (\text{B.25})$$

which, as we see below, will be helpful when solving (B.22).

We then have the following changes of variables

$$x = \frac{X - \zeta}{X + \zeta}, \quad z = \xi_1 \lambda + \xi_2, \quad \text{where } \xi_1 = \frac{2}{\lambda_h - \lambda_m}, \quad \xi_2 = -\frac{\lambda_h + \lambda_m}{\lambda_h - \lambda_m}. \quad (\text{B.26})$$

Given (B.26), define  $q(x, z) \equiv H(X(x), \lambda(z))$ . The default boundary is  $k(z) \equiv X_b(\lambda(z))$ . Equation (B.22) becomes

$$rq(x, z) = \left[ \begin{array}{l} \mu \frac{1}{2}(1-x^2)q_x + \frac{1}{2}\sigma^2 \left( \zeta \frac{1+x}{1-x} \right)^2 \left( q_{xx} \frac{(1-x)^4}{4\zeta^2} - q_x \frac{(1-x)^3}{2\zeta^2} \right) \\ + a(\lambda(z))\xi_1 q_z - [(c(1-\tau) + \xi) - \xi p(\lambda(z))]K \\ + \frac{1}{2\psi K} (p(\lambda(z))K + q(x, z) - \frac{1}{2}(1-x^2)q_x)^2 \end{array} \right]. \quad (\text{B.27})$$

The optimal debt issuance from (B.23) becomes

$$g^*(x, z) = \xi + \frac{p(\lambda(z))K + q(x, z) - \frac{1}{2}(1 - x^2)q_x}{\psi K}. \quad (\text{B.28})$$

The two boundary conditions in (B.24) are now written as

$$q\left(\frac{k(z)-\zeta}{k(z)+\zeta}, z\right) = -\frac{\delta(1-\tau)}{r-\mu}k(z), \quad q_x\left(\frac{k(z)-\zeta}{k(z)+\zeta}, z\right) = -\frac{2\delta(1-\tau)\zeta}{(r-\mu)\left(1 - \frac{k(z)-\zeta}{k(z)+\zeta}\right)^2}. \quad (\text{B.29})$$

Given (B.25), we approximate the residual equity value  $q(x, z)$  by the following

$$\hat{q}(x, z) = \hat{l}(x) + (1+z) \sum_{0 \leq i+j \leq m} a(i, j) T_i(z) T_j(x), \quad (\text{B.30})$$

where  $\{a(i, j)\}_{0 \leq i+j \leq m}$  represents  $(m+1)(m+2)/2$  coefficients. Note that

$$\begin{aligned} \hat{l}(1) &= W_{2,-}(\lambda_m) \\ &= -(p(\lambda_m) - r\psi)K \\ &\quad - K \sqrt{(p(\lambda_m) - r\psi)^2 + (2\psi c(1-\tau) + 2\xi\psi(1-p(\lambda_m)) - p^2(\lambda_m))}. \end{aligned} \quad (\text{B.31})$$

We approximate the default boundary  $k$  by

$$\hat{k}(z) = \sum_{r=0}^n d_r T_r(z). \quad (\text{B.32})$$

When numerically solving (B.13), we have obtained a numerical value for  $X_b(\lambda_m)$ . As such,

$$\lim_{z \rightarrow -1} \hat{k}(z) = \sum_{r=0}^n d_r \cdot (-1)^r = X_b(\lambda_m). \quad (\text{B.33})$$

That is, we can write  $d_n$  as a linear function of  $\{d_r\}_{r=0}^{n-1}$ .

Within the boundary—that is, when  $(k(z) - \psi)/(k(z) + \psi)$  and when  $-1 \leq z \leq 1$ —the partial differential equation in (B.27) is satisfied. We consider  $M$  grid points between  $-1$  and  $1$  for the variable  $z$ ; these are the  $M$  zeros of  $T_M(z)$  and we denote them as  $\{z_i\}_{i=1}^M$ . These  $M$  zeros are given by

$$z_i = \cos\left(\frac{(i-0.5)\pi}{M}\right), \quad 1 \leq i \leq M. \quad (\text{B.34})$$

We assume the same  $M$  grid points for the variable  $x$ ; we denote them as  $\{x_j\}_{j=1}^M$ . For each  $z_i$ , the following set of  $x_j$  corresponds to firm fundamentals that are above the default boundary

$$1 \leq j \leq \bar{j}(i) \equiv \left\lfloor \frac{M}{\pi} \arccos\left(\frac{\hat{k}(z_i) - \psi}{\hat{k}(z_i) + \psi}\right) + 0.5 \right\rfloor. \quad (\text{B.35})$$

Finally, we choose coefficients  $\{a(i, j)\}_{0 \leq i+j \leq m}$  and coefficients  $\{d_r\}_{r=0}^{n-1}$  to minimize the following weighted sum of squared errors

$$\begin{aligned}
& \sum_{j=1}^{\bar{j}(i)} \sum_{i=1}^M w(j, i) \left[ \begin{aligned} & \mu \frac{1}{2} (1 - x_j^2) \hat{q}_x + \frac{1}{2} \sigma^2 \left( \zeta \frac{1 + x_j}{1 - x_j} \right)^2 \left( \hat{q}_{xx} \frac{(1 - x_j)^4}{4\zeta^2} - \hat{q}_x \frac{(1 - x_j)^3}{2\zeta^2} \right) \\ & + a(\lambda(z_i)) \xi_1 \hat{q}_z \\ & - [(c(1 - \tau) + \xi) - \xi p(\lambda(z_i))] K + \frac{1}{2\psi K} (p(\lambda(z_i)) K + q(x_j, z_i) - \frac{1}{2} (1 - x_j^2) \hat{q}_x)^2 \\ & - r \hat{q}(x_j, z_i) \end{aligned} \right]^2 \\
& + K_1 \cdot \sum_{i=1}^M \frac{1}{\sqrt{1 - z_i^2}} \left[ \hat{q} \left( \frac{\hat{k}(z_i) - \zeta}{\hat{k}(z_i) + \zeta}, z_i \right) + \frac{\delta(1 - \tau)}{r - \mu} \hat{k}(z_i) \right]^2 \\
& + K_2 \cdot \sum_{i=1}^M \frac{1}{\sqrt{1 - z_i^2}} \left[ \hat{q}_x \left( \frac{\hat{k}(z_i) - \zeta}{\hat{k}(z_i) + \zeta}, z_i \right) + \frac{2\zeta\delta(1 - \tau)}{(r - \mu)(1 - \frac{\hat{k}(z_i) - \zeta}{\hat{k}(z_i) + \zeta})^2} \right]^2, \tag{B.36}
\end{aligned}$$

where  $w(j, i) = [(1 - x_j^2)(1 - z_i^2)]^{-1/2}$  and  $K_1$  and  $K_2$  are large positive coefficients.

In (B.36), the expressions of  $\hat{q}_x$ ,  $\hat{q}_{xx}$ , and  $\hat{q}_z$  are given by

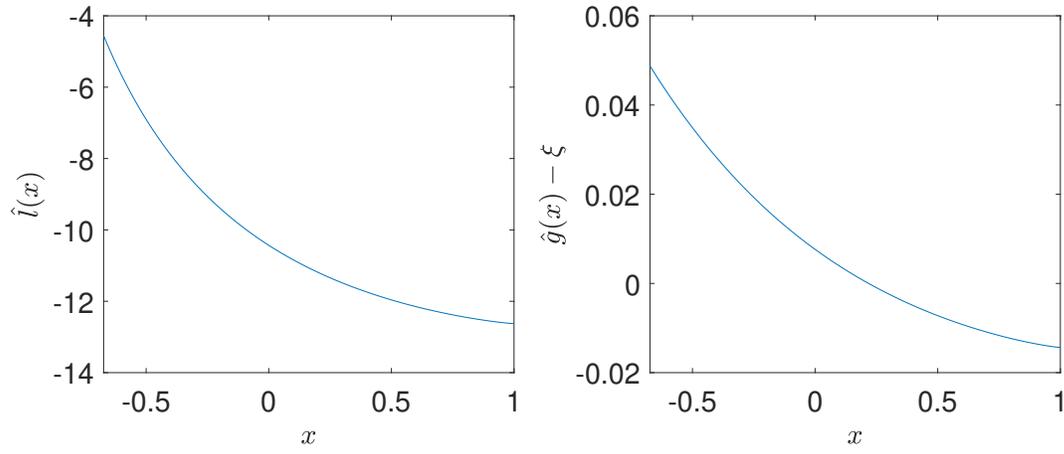
$$\begin{aligned}
\hat{q}_x(x, z) &= \hat{l}_x(x) + (1 + z) \sum_{0 \leq i+j \leq m} a(i, j) T_i(z) T'_j(x), \\
\hat{q}_{xx}(x, z) &= \hat{l}_{xx}(x) + (1 + z) \sum_{0 \leq i+j \leq m} a(i, j) T_i(z) T''_j(x), \tag{B.37}
\end{aligned}$$

and

$$\hat{q}_z(x, z) = \sum_{0 \leq i+j \leq m} a(i, j) T'_i(z) T_j(x) + (1 + z) \sum_{0 \leq i+j \leq m} a(i, j) T'_i(z) T_j(x). \tag{B.38}$$

For the numerical results in the main text, we set  $m = 30$ ,  $n = 30$ ,  $M = 70$ ,  $K_1 = 10^6$ ,  $K_2 = 2,500$ ,  $K = 10$ , and  $\zeta = 25$ . We then apply the Levenberg-Marquardt algorithm. Recall that the parameter values are:  $\lambda_l = 0.005$ ,  $\lambda_h = 2$ ,  $q = 0.05$ ,  $\eta = 0.5$ ,  $r = 0.04$ ,  $c = 0.04$ ,  $\xi = 1$ ,  $\tau = 0.3$ ,  $\psi = 20$ ,  $\mu = 0.01$ ,  $\delta = 0.04$ , and  $\sigma = 0.25$ . For (B.36), we obtain a minimized weighted sum of squared errors at  $8.2 \times 10^{-3}$ .

**Figure B.1.** The market-timing component of equity value  $\hat{l}(x)$  and the optimal net debt issuance  $\hat{g}(x) - \xi$ , each as a function of  $x$ . The parameter values are:  $\lambda_l = 0.005$ ,  $\lambda_h = 2$ ,  $q = 0.05$ ,  $\eta = 0.5$ ,  $r = 0.04$ ,  $c = 0.04$ ,  $\xi = 1$ ,  $\tau = 0.3$ ,  $\psi = 20$ ,  $\mu = 0.01$ ,  $\delta = 0.04$ , and  $\sigma = 0.25$ .



## C. Numerical procedure for solving the rational model

### C.1. Reduction of state variables

Note that the  $V_t$  process in equation (3) exhibits constant stochastic returns to scale and that the adjustment cost in (10) is homogeneous of degree one in  $G_t$  and  $F_t$ . These assumptions imply that the equity value  $E$  is homogeneous of degree one in  $V_t$  and  $F_t$ , that the default boundary  $V_b$  is homogeneous of degree one in  $F_t$ , and that the bond price  $p$  is homogeneous of degree zero in  $V_t$  and  $F_t$ . Without loss of generality, we write

$$E(V, F) = e\left(\frac{V}{F}\right)F, \quad V_b(F) = v_b F, \quad p(V, F) = p\left(\frac{V}{F}\right), \quad (\text{C.1})$$

as in equation (35) of the main text.

Substituting (C.1) back into (29) and (33) gives

$$r \cdot e = \left[ \begin{array}{l} \mu v e' + \frac{1}{2} \sigma^2 v^2 e'' \\ + \delta(1 - \tau) \cdot v - (c(1 - \tau) + \xi) + \xi p + \frac{1}{2\psi} (p + e - v e')^2 \end{array} \right] \quad (\text{C.2})$$

and

$$r \cdot p = c + \xi(1 - p) + (\mu + \xi - g^*) v p' + \frac{1}{2} \sigma^2 v^2 p''. \quad (\text{C.3})$$

In (C.3), the optimal debt issuance is

$$g^*(v) \equiv \frac{G}{F} = \xi + \frac{p(v) + e - v e'}{\psi}. \quad (\text{C.4})$$

The reduced boundary conditions are

$$e(v_b) = 0, \quad e'(v_b) = 0, \quad p(v_b) = \eta, \quad \lim_{v \rightarrow \infty} p(v) = \frac{c + \xi}{r + \xi}. \quad (\text{C.5})$$

As  $v$  goes to infinity, we exclude any bubble component from  $e(v)$ —we set  $e''(v)$  in (C.2) to zero—and we know from (C.5) that  $p(v)$  goes to  $(c + \xi)/(r + \xi)$ . In this case, (C.2) implies

$$\lim_{v \rightarrow \infty} e(v) = A + \frac{\delta(1 - \tau)}{r - \mu} v, \quad (\text{C.6})$$

where

$$A = -(p - \psi r) - \sqrt{(p - \psi r)^2 - 2(\psi(\xi(p - 1) - c(1 - \tau)) + \frac{1}{2} p^2)} \quad (\text{C.7})$$

and  $p = (c + \xi)/(r + \xi)$ . Equation (C.6) serves as another boundary condition.

We further define the “residual” equity value

$$h(v) \equiv e(v) - \frac{\delta(1-\tau)}{r-\mu}v, \quad (\text{C.8})$$

which is the difference between the equity value and the present value of the firm’s unlevered cash flows. From (C.6) we know that

$$\lim_{v \rightarrow \infty} h(v) = A. \quad (\text{C.9})$$

Equations (C.2) to (C.6) characterize a system of two ordinary differential equations, with a single state variable  $v$ . We solve equations (C.2) to (C.6) and the value of  $v_b$  using numerical methods, which we elaborate next.

### C.2. Change of variables

For numerical considerations, we first define

$$W(X) \equiv K \cdot e\left(\frac{X}{K}\right), \quad \text{so } e(v) = K^{-1}W(vK), \quad (\text{C.10})$$

where  $K$  is a large scaling factor (a constant). We substitute (C.10) into (C.2) and obtain

$$r \cdot W = \left[ \begin{array}{l} \mu X W' + \frac{1}{2} \sigma^2 X^2 W'' \\ + \delta(1-\tau) \cdot X - [(c(1-\tau) + \xi) - \xi \cdot p] K \\ + \frac{1}{2\psi K} (p \cdot K + W - X W')^2 \end{array} \right]. \quad (\text{C.11})$$

The two boundary conditions regarding  $e(v)$  in (C.5) now become

$$W(X_b) = 0, \quad W_X(X_b) = 0. \quad (\text{C.12})$$

We then define

$$P(X) \equiv p\left(\frac{X}{K}\right), \quad \text{so } p(v) = P(vK). \quad (\text{C.13})$$

We substitute (C.11) into (C.3) and obtain

$$r \cdot P = c + \xi(1 - P) + (\mu + \xi - g^*)X P' + \frac{1}{2} \sigma^2 X^2 P''. \quad (\text{C.14})$$

The two boundary conditions regarding  $p(v)$  in (C.5) now become

$$P(X_b) = \eta, \quad \lim_{X \rightarrow \infty} P(X) = \frac{c + \xi}{r + \xi}. \quad (\text{C.15})$$

Our focus is to numerically solve the differential equations (C.11) and (C.14) as well as the default boundary  $X_b$ .

To do so, we now define the market-timing component of equity value as

$$H(X) \equiv W(X) - \frac{\delta(1-\tau)}{r-\mu}X. \quad (\text{C.16})$$

Substituting (C.16) back into (C.11) gives

$$r \cdot H = \left[ \begin{array}{l} \mu X H' + \frac{1}{2} \sigma^2 X^2 H'' \\ -[(c(1-\tau) + \xi) - \xi \cdot P]K + \frac{1}{2\psi K} (P \cdot K + H - X H')^2 \end{array} \right]. \quad (\text{C.17})$$

Further substituting (C.16) into the two boundary conditions in (C.12) gives

$$H(X_b) = -\frac{\delta(1-\tau)}{r-\mu}X_b, \quad H'(X_b) = -\frac{\delta(1-\tau)}{r-\mu}. \quad (\text{C.18})$$

Moreover, from (C.6), we know that

$$\lim_{X \rightarrow \infty} H(X) = K \cdot A, \quad (\text{C.19})$$

where coefficient  $A$  is given by (C.7).

### C.3. Solving $H(X)$ and $P(X)$

To facilitate subsequent numerical analyses, we make the following change of variable

$$x = \frac{X - \zeta}{X + \zeta} \quad (\text{C.20})$$

and define  $l(x) \equiv H(X(x))$  and  $f(x) \equiv P(X(x))$  with  $X = \zeta(1+x)/(1-x)$ . Now, equation (C.17) becomes

$$rl = \left[ \begin{array}{l} \frac{1}{2} \mu (1-x^2) l' + \frac{1}{2} \sigma^2 \left( \zeta \frac{1+x}{1-x} \right)^2 \left( l'' \frac{(1-x)^4}{4\zeta^2} - l' \frac{(1-x)^3}{2\zeta^2} \right) \\ -[(c(1-\tau) + \xi) - \xi \cdot f]K + \frac{1}{2\psi K} [f \cdot K + l - \frac{1}{2}(1-x^2)l']^2 \end{array} \right], \quad (\text{C.21})$$

with the following optimal debt issuance

$$g^*(x) = \xi + \frac{f \cdot K + l - \frac{1}{2}(1-x^2)l'}{\psi K}. \quad (\text{C.22})$$

The two boundary conditions in (C.18) become

$$l(x_b) = -\frac{\zeta \delta(1-\tau)}{r-\mu} \frac{1+x_b}{1-x_b}, \quad l'(x_b) \frac{(1-x_b)^2}{2\zeta} = -\frac{\delta(1-\tau)}{r-\mu}. \quad (\text{C.23})$$

The boundary condition in (C.19) becomes

$$l(1) = K \cdot A. \quad (\text{C.24})$$

Equation (C.14) becomes

$$\begin{aligned} r \cdot f = & \quad c + \xi(1 - f) + (\mu + \xi - g^*)\frac{1}{2}(1 - x^2)f' \\ & + \frac{1}{2}\sigma^2 \left( \zeta \frac{1+x}{1-x} \right)^2 \left( f'' \frac{(1-x)^4}{4\zeta^2} - f' \frac{(1-x)^3}{2\zeta^2} \right). \end{aligned} \quad (\text{C.25})$$

The two boundary conditions in (C.15) become

$$f(x_b) = \eta, \quad f(1) = \frac{c + \xi}{r + \xi}. \quad (\text{C.26})$$

We then approximate the two functions  $l$  and  $f$  by

$$\hat{l}(x) = \sum_{r=0}^n b_r T_r(x), \quad \hat{f}(x) = \sum_{r=0}^n c_r T_r(x), \quad (\text{C.27})$$

where  $\{b_r\}_{0 \leq r \leq n}$  and  $\{c_r\}_{0 \leq r \leq n}$  each represents  $n+1$  coefficients, and  $T_r(x)$  represents the  $r^{\text{th}}$  degree Chebyshev polynomial of the first kind. Define  $x_b \equiv (X_b - \zeta)/(X_b + \zeta)$  as the transformed value of the default boundary  $X_b$ . For a given value of  $x_b$ , equations (C.23) and (C.24) allow us to express  $\{b_0, b_1, b_2\}$  each as a combination of the remaining  $n - 2$  coefficients. Similarly, equation (C.26) allows us to express  $\{c_0, c_1\}$  each as a combination of the remaining  $n - 1$  coefficients. We also search for  $x_b$ . Together, we have a total of  $2n - 2$  unknown coefficients.

Note that the optimal debt issuance in (C.22) is approximated by

$$\hat{g}(x) = \xi + \frac{\hat{f} \cdot K + \hat{l} - \frac{1}{2}\hat{l}'(1 - x^2)}{\psi K}. \quad (\text{C.28})$$

Finally, we choose coefficients  $\{b_r\}_{0 \leq r \leq n}$ , coefficients  $\{c_r\}_{0 \leq r \leq n}$ , as well as the default boundary  $x_b$  to minimize the following weighted sum of squared errors

$$\begin{aligned} & \sum_{j=1}^M w(j) \mathbf{1}_{x_j \geq x_b} \left[ \begin{aligned} & \frac{1}{2}\mu(1 - x_j^2)\hat{l}' + \frac{1}{2}\sigma^2 \left( \zeta \frac{1+x_j}{1-x_j} \right)^2 \left( \hat{l}'' \frac{(1-x_j)^4}{4\zeta^2} - \hat{l}' \frac{(1-x_j)^3}{2\zeta^2} \right) \\ & - [(c(1 - \tau) + \xi) - \xi \cdot \hat{f}]K + \frac{1}{2\psi K} \left( \hat{f} \cdot K + \hat{l} - \frac{1}{2}(1 - x_j^2)\hat{l}' \right)^2 - r\hat{l} \end{aligned} \right]^2 \\ & + K_1 \cdot \sum_{j=1}^M w(j) \mathbf{1}_{x_j \geq x_b} \left[ \begin{aligned} & c + \xi(1 - \hat{f}) + (\mu + \xi - g^*)\frac{1}{2}(1 - x_j^2)\hat{f}' \\ & + \frac{1}{2}\sigma^2 \left( \zeta \frac{1+x_j}{1-x_j} \right)^2 \left( \hat{f}'' \frac{(1-x_j)^4}{4\zeta^2} - \hat{f}' \frac{(1-x_j)^3}{2\zeta^2} \right) - r\hat{f} \end{aligned} \right]^2, \end{aligned} \quad (\text{C.29})$$

where  $w(j) = (1 - x_j^2)^{-1/2}$  and  $K_1$  is a positive scaling coefficient. The indicator functions in (C.29) mean that the system of ordinary differential equations only hold in the region where  $v$  is above the default boundary.

For the numerical results in the main text, we set  $n = 30$ ,  $M = 300$ ,  $\zeta = 25$ ,  $K_1 = 4$ ,  $K = 10$ , and  $\zeta = 25$ . We then apply the Levenberg-Marquardt algorithm. The parameter values are:  $\eta = 0.5$ ,  $r = 0.04$ ,  $c = 0.04$ ,  $\xi = 1$ ,  $\tau = 0.3$ ,  $\psi = 20$ ,  $\mu = 0.01$ ,  $\delta = 0.04$ , and  $\sigma = 0.25$ . For (C.29), we obtain a minimized weighted sum of squared errors at  $1.47 \times 10^{-4}$ . The default boundary is solved to be  $v_b = 1.46$ .