Online Appendix

C Proofs and derivations: Section 5

C.1 Theoretical results

Equilibrium price and portfolio allocations The variance-covariance matrix, given by Equation (18) in the text, is positive semi-definite when $\sigma_d^2\sigma_h^2 > (\sigma_{dh})^2$, where σ_{dh} defines the covariance between both random variables, given by $\sigma_{dh} = \rho\sigma_d\sigma_h$. This restriction is implied by the fact that covariance matrices must be positive semi-definite or, equivalently, by the fact that the correlation coefficient is bounded, that is, $\rho \in [-1, 1]$.

Because the joint cross-sectional distribution of beliefs and hedging needs is symmetric, the equilibrium price can be expressed as follows

$$P_1 = \frac{\int_{i\in\mathcal{T}} \left(\mathbb{E}_i[D] - A\left(\mathbb{C}ov[M_{2i}, D] + \mathbb{V}ar[D]\,Q\right)\right) dF(i)}{\int_{i\in\mathcal{T}} dF(i)}$$
$$= \mu_d - \mu_h - A\mathbb{V}ar[D]\,Q, \tag{34}$$

which corresponds to Equation (20) in the text, since $\mu_h = 0$. Note that the equilibrium price P_1 is fully characterized as a function of primitives and is independent of τ. Therefore, we can express equilibrium net trades (when non-zero) as

$$\Delta X_{1i} = \frac{\mathbb{E}_i[D] - A\mathbb{C}ov[M_{2i}, D] - P_1\left(1 + \text{sgn}(\Delta X_{1i})\,\tau\right) - A\mathbb{V}ar[D]\,X_{0i}}{A\mathbb{V}ar[D]}$$
$$= \frac{\left(\mathbb{E}_i[D] - \mathbb{E}_{\mathcal{T}}[\mathbb{E}_i[D]]\right) - \left(A\mathbb{C}ov[M_{2i}, D] - \mathbb{E}_{\mathcal{T}}[A\mathbb{C}ov[M_{2i}, D]]\right) - \text{sgn}(\Delta X_{1i})\,\tau P_1}{A\mathbb{V}ar[D]}$$
$$= \frac{\varepsilon_{di} - \varepsilon_{hi} - \text{sgn}(\Delta X_{1i})\,\tau P_1}{A\mathbb{V}ar[D]}, \tag{35}$$

where P_1 is already determined as a function of primitives in Equation (34) and $\mathbb{E}_{\mathcal{T}}[\mathbb{E}_i[D]] = \frac{\int_{i\in\mathcal{T}} \mathbb{E}_i[D]dF(i)}{\int_{i\in\mathcal{T}} dF(i)}$ and $\mathbb{E}_{\mathcal{T}}[A\mathbb{C}ov[M_{2i}, D]] = \frac{\int_{i\in\mathcal{T}} A\mathbb{C}ov[M_{2i},D]dF(i)}{\int_{i\in\mathcal{T}} dF(i)}$. I refer to ΔX_{1i}^+ and ΔX_{1i}^- as latent net buying/selling positions, since we only look at the positive and negative part. In particular, following the formulation in Equation (5) in the text, we can write the distribution of equilibrium latent net trades in the population as follows

$$\Delta X_{1i}^+ = \frac{\varepsilon_{di} - \varepsilon_{hi} - \tau P_1}{A\mathbb{V}ar[D]} \sim N\left(\frac{-\tau P_1}{A\mathbb{V}ar[D]}, \frac{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}{\left(A\mathbb{V}ar[D]\right)^2}\right)$$
$$\Delta X_{1i}^- = \frac{\varepsilon_{di} - \varepsilon_{hi} + \tau P_1}{A\mathbb{V}ar[D]} \sim N\left(\frac{\tau P_1}{A\mathbb{V}ar[D]}, \frac{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}{\left(A\mathbb{V}ar[D]\right)^2}\right),$$

where P_1 is already determined in Equation (34) and where we use the fact that $\mathbb{C}ov\left[\frac{\varepsilon_{di}}{A\mathbb{V}ar[D]}, \frac{-\varepsilon_{hi}}{A\mathbb{V}ar[D]}\right] = \frac{-\sigma_{dh}}{(A\mathbb{V}ar[D])^2}$.

Note that $\mathbb{V}ar[\varepsilon_{di} - \varepsilon_{hi}] = \sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h$ can take different values depending on the correlation between both motives for trading. Specifically

$$\mathbb{V}ar[\varepsilon_{di} - \varepsilon_{hi}] = \begin{cases} (\sigma_d + \sigma_h)^2, & \text{if } \rho = -1 \\ \sigma_d^2 + \sigma_h^2, & \text{if } \rho = 0 \\ (\sigma_d - \sigma_h)^2, & \text{if } \rho = 1. \end{cases}$$

Therefore, when $\rho \to -1$, the dispersion of equilibrium allocations is maximal. Note also that, when $\rho \to 1$, if $\sigma_d = \sigma_h$, there is no-trade in equilibrium. In terms of (latent) individual turnover, we can express $\frac{\Delta X_{1i}^+}{Q}$ and $\frac{\Delta X_{1i}^-}{Q}$ as follows

$$\frac{\Delta X_{1i}^+}{Q} \sim N\left(\frac{\tau}{\Pi}, 2\pi\left(\Xi(0)\right)^2\right) \quad \text{and} \quad \frac{\Delta X_{1i}^-}{Q} \sim N\left(\frac{-\tau}{\Pi}, 2\pi\left(\Xi(0)\right)^2\right), \tag{36}$$

where $\Pi = \frac{A\mathbb{V}ar[D]Q}{P_1} = \frac{\mu_d}{P_1} - 1$ denotes the risk premium and $\Xi(0)$ denotes laissez-faire aggregate turnover, defined in Equation (49) below.

Trading volume and shares of buyers and sellers Because there is a continuum of investors, a Law of Large Numbers guarantees that the level of trading volume in equilibrium in this model, given by $\mathcal{V}(\tau)$, is deterministic. Exploiting the properties of truncated normal distributions, stated in Section C.3, we can express the two elements that will determine $\mathcal{V}(\tau)$ as follows

$$\begin{aligned}\mathbb{P}[\Delta X_{1i} > 0] &= 1 - \Phi(\alpha_+)\\ \mathbb{E}[\Delta X_{1i} | \Delta X_{1i} > 0] &= \frac{1}{A\mathbb{V}ar[D]}\left(-\tau P_1 + \sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}\lambda(\alpha_+)\right).\end{aligned}$$

Throughout the Appendix, $\phi(\cdot)$ and $\Phi(\cdot)$ respectively denote the p.d.f. and c.d.f. of the standard normal distribution. Note that the share of buyers (and sellers, by symmetry) corresponds to $\mathbb{P}[\Delta X_{1i} > 0] = \int_{i\in\mathcal{B}(\tau)} dF(i)$.

Trading volume expressed in dollars, which follows from the definition of $\mathcal{V}(\tau)$ in Equation (7), can be expressed as a function of the tax rate and primitives as follows

$$\begin{aligned}P_1\mathcal{V}(\tau) &= P_1\int_{i\in\mathcal{B}} \Delta X_{1i} dF(i) = P_1\mathbb{P}[\Delta X_{1i} > 0]\,\mathbb{E}[\Delta X_{1i} | \Delta X_{1i} > 0]\\ &= \frac{P_1}{A\mathbb{V}ar[D]}(1 - \Phi(\alpha_+))\left(-\tau P_1 + \sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}\lambda(\alpha_+)\right)\\ &= \frac{P_1}{A\mathbb{V}ar[D]}\left(-\tau P_1(1 - \Phi(\alpha_+)) + \sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}\phi(\alpha_+)\right),\end{aligned}$$

where $\lambda(\alpha_+) = \frac{\phi(\alpha_+)}{1-\Phi(\alpha_+)}$ corresponds to the inverse Mills ratio, whose properties are described in Section C.3, and α_+ and α are defined as follows

$$\alpha_+ = \max\{\alpha, 0\} \quad \text{where} \quad \alpha = \frac{\tau P_1}{\sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}}. \tag{37}$$

We can use the fact that $\phi'(\alpha_+) = -\alpha_+\phi(\alpha_+)$ and the definition of α_+ to compute the response of trading volume to a tax change, given by

$$\begin{aligned}\frac{d\mathcal{V}}{d\tau} &= \frac{1}{A\mathbb{V}ar[D]}\left(-P_1(1 - \Phi(\alpha_+)) + \tau P_1\phi(\alpha)\frac{d\alpha_+}{d\tau} + \sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}\phi'(\alpha)\frac{d\alpha_+}{d\tau}\right)\\ &= \frac{1}{A\mathbb{V}ar[D]}\left(-P_1(1 - \Phi(\alpha_+)) + \left(\tau P_1 - \alpha\sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}\right)\phi(\alpha)\frac{d\alpha_+}{d\tau}\right)\\ &= -\frac{P_1}{A\mathbb{V}ar[D]}(1 - \Phi(\alpha_+)),\end{aligned}$$

which takes strictly negative values. Note that we can express $\frac{d\log\mathcal{V}}{d\tau} = \frac{d\mathcal{V}}{d\tau}\frac{1}{\mathcal{V}}$ as follows

$$\begin{aligned}\frac{d\mathcal{V}}{d\tau}\frac{1}{\mathcal{V}} &= \frac{-\frac{P_1}{A\mathbb{V}ar[D]}(1 - \Phi(\alpha_+))}{\frac{1}{A\mathbb{V}ar[D]}(1 - \Phi(\alpha_+))\left(-\tau P_1 + \sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}\lambda(\alpha_+)\right)}\\ &= \frac{-1}{-\tau + \frac{\sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}}{P_1}\lambda(\alpha_+)},\end{aligned}$$

which implies that

$$\left.\frac{d\log\mathcal{V}}{d\tau}\right|_{\tau=0} = \frac{-P_1}{\sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}\lambda(0)},$$

where $\lambda(0) = \sqrt{\frac{2}{\pi}}$, as shown in Section C.3.

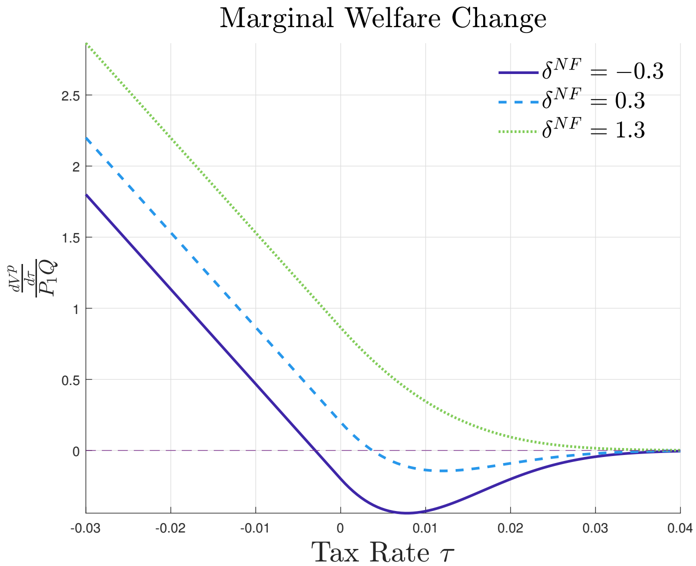


Figure A.1: Marginal welfare change

Note: Figure A.1 shows the normalized aggregate marginal welfare impact of a tax change from the planner's perspective, $\frac{\frac{dV^P}{d\tau}}{P_1 Q}$, defined in Equation (39), for the following values of the share of non-fundamental trading volume δ^{NF}: $\delta^{NF} = \{-0.3, 0.3, 1.3\}$. The optimal tax in each case is respectively given by $\tau^* = -0.30\%$, $\tau^* = 0.37\%$, and $\tau^* = \infty$.

Marginal welfare impact We can express the aggregate marginal welfare change $\frac{dV^p}{d\tau} = \int_{i \in \mathcal{T}(\tau)} \left[-\mathbb{E}_i [D] + \text{sgn}(\Delta X_{1i}) P_1 \tau\right] \frac{dX_{1i}}{d\tau} dF(i)$ as follows

$$\begin{aligned}
\frac{dV^p}{d\tau} &= \frac{P_1}{A \mathbb{V}ar[D]} \left(\int_{i \in \mathcal{B}(\tau)} \varepsilon_{di} dF(i) - \int_{i \in \mathcal{S}(\tau)} \varepsilon_{di} dF(i) - \tau P_1 2 \int_{i \in \mathcal{B}(\tau)} dF(i) \right) \\
&= \frac{P_1}{A \mathbb{V}ar[D]} \left(2 \int_{i \in \mathcal{B}(\tau)} dF(i) \right) \left(\frac{\mathbb{E}_{\mathcal{B}(\tau)} [\mathbb{E}_i [D]] - \mathbb{E}_{\mathcal{S}(\tau)} [\mathbb{E}_i [D]]}{2} - \tau P_1 \right) \\
&= \frac{P_1}{A \mathbb{V}ar[D]} \left(2 \int_{i \in \mathcal{B}(\tau)} dF(i) \right) \left(\frac{\sigma_d (\sigma_d - \rho \sigma_h)}{\sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho \sigma_d \sigma_h}} \lambda(\alpha_+) - \tau P_1 \right) \\
&= \frac{P_1}{A \mathbb{V}ar[D]} \sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho \sigma_d \sigma_h} 2 (1 - \Phi(\alpha_+)) \left(\delta^{NF} \lambda(\alpha_+) - \alpha \right),
\end{aligned}$$

where the definitions of α_+ and α are given in Equation (37) and where we can define δ^{NF}, shown below to correspond to the share of non-fundamental trading volume, as follows

$$\delta^{NF} = \frac{\sigma_d (\sigma_d - \rho \sigma_h)}{\sigma_d^2 + \sigma_h^2 - 2\rho \sigma_d \sigma_h} = \frac{\frac{\sigma_d}{\sigma_h} \left(\frac{\sigma_d}{\sigma_h} - \rho \right)}{1 + \left(\frac{\sigma_d}{\sigma_h} \right)^2 - 2\rho \frac{\sigma_d}{\sigma_h}}. \tag{38}$$

Note that $\mathbb{E}_{\mathcal{B}(\tau)} [\mathbb{E}_i [D]] - \mathbb{E}_{\mathcal{S}(\tau)} [\mathbb{E}_i [D]]$ is explicitly computed in Equation (42) below. Since $\int_{i \in \mathcal{B}(\tau)} dF(i)$ is strictly positive for any value of τ, and α is a positive linear function of τ, it is sufficient to study the properties of $\delta^{NF} \lambda(\alpha_+) - \alpha$ to determine whether there is a uniquely optimal tax.

When normalized by the total value of the risky asset, we can express the marginal welfare impact of a tax

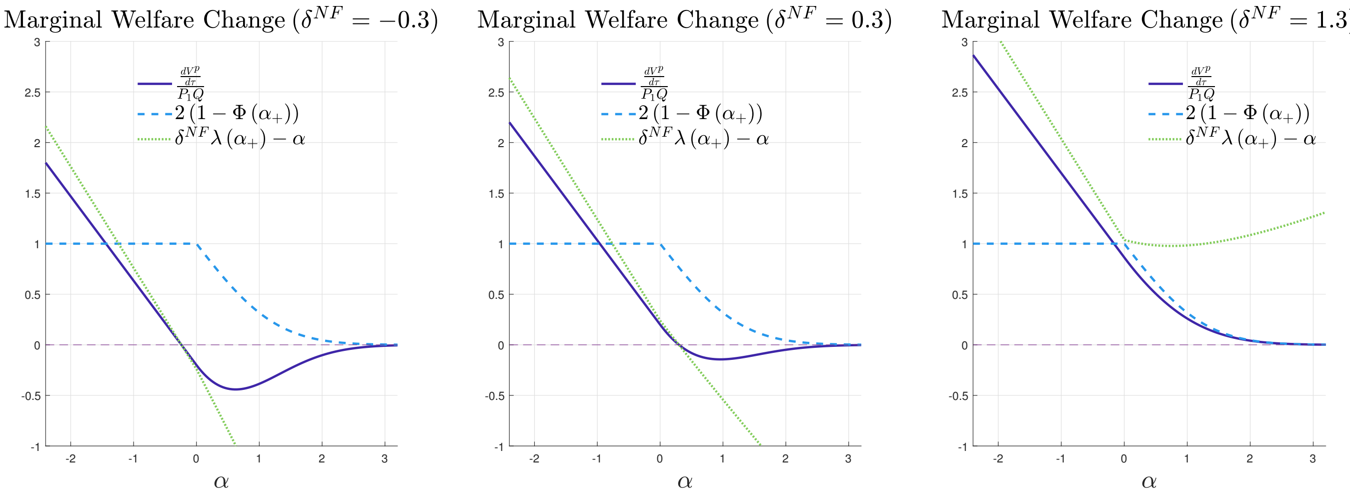


Figure A.2: Marginal welfare change (decomposition)

Note: Figure A.2 shows the normalized aggregate marginal welfare impact of a price change from the planner's perspective, $\frac{\frac{dV^P}{d\tau}}{P_1 Q}$, and its components, as defined in Equation (39), for the following values of the share of non-fundamental trading volume δ^{NF}: $\delta^{NF} = \{-0.3, 0.3, 1.3\}$. The optimal tax in each case is respectively given by $\tau^* = -0.30\%$, $\tau^* = 0.37\%$, and $\tau^* = \infty$.

change as follows

$$\frac{\frac{dV^p}{d\tau}}{P_1 Q} = \underbrace{\frac{P_1}{A\mathbb{V}ar\left[D\right]Q}}_{1/\Pi}\underbrace{\frac{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}{P_1}}_{\sqrt{2\pi}\Xi(0)\Pi} 2\left(1-\Phi\left(\alpha_+\right)\right)\left(\delta^{NF}\lambda\left(\alpha_+\right)-\alpha\right)$$
$$= \sqrt{2\pi}\Xi\left(0\right)2\left(1-\Phi\left(\alpha_+\right)\right)\left(\delta^{NF}\lambda\left(\alpha_+\right)-\alpha\right), \tag{39}$$

where $\Xi\left(0\right)$ denotes laissez-faire turnover, characterized in Equation (49) below. Note that the normalized aggregate welfare impact at zero can be expressed exclusively as a function of Π, $\varepsilon_\tau^{\log \mathcal{V}}\big|_{\tau=0}$, and δ^{NF}, since

$$\left.\frac{\frac{dV^p}{d\tau}}{P_1 Q}\right|_{\tau=0} = \Xi\left(0\right)\sqrt{2\pi}\delta^{NF}\lambda\left(0\right) = \Xi\left(0\right)2\delta^{NF} = \frac{1}{\Pi}\frac{1}{\left|\varepsilon_\tau^{\log \mathcal{V}}\big|_{\tau=0}\right|}\delta^{NF},$$

where we use the fact that $\varepsilon_\tau^{\log \mathcal{V}}\big|_{\tau=0} = -\frac{1}{2}\frac{1}{\Pi\Xi(0)}$.

There are three possible scenarios to consider regarding the sign of the optimal tax, as illustrated in Figure A.1. First, when $\delta^{NF} < 0$, there is a uniquely optimal negative tax (a subsidy), given by $\alpha^* = \delta^{NF}\lambda(0)$. Second, when $0 \leq \delta^{NF} < 1$, there is a uniquely optimal finite and positive tax, which solves $\delta^{NF}\lambda\left(\alpha^*\right) = \alpha^*$. It is easy to verify that the function $\delta^{NF}\lambda\left(\alpha_+\right)-\alpha$ has a single root in that case. Third, when $\delta^{NF} \geq 1$, it is the case that $\frac{dV^p}{d\tau} > 0$, $\forall\tau$, so the optimal tax is $\tau^* = +\infty$. This follows from the fact that $\lambda\left(\alpha\right) > \alpha$, $\forall\alpha$, established in Section C.3. Formally,

$$\text{if} \quad \begin{cases} \delta^{NF} < 0 & \Rightarrow \tau^* < 0 \\ 0 \leq \delta^{NF} < 1 & \Rightarrow 0 \leq \tau^* < \infty \\ \delta^{NF} \geq 1 & \Rightarrow \tau^* = \infty. \end{cases}$$

The second order condition of the planner's problem can be written, for $\tau \neq 0$, as follows

$$\frac{d^2V^p}{d\tau^2} = \frac{P_1}{A\mathbb{V}ar\left[D\right]}\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}2\left(\left(\delta^{NF}\phi'\left(\alpha_+\right)+\alpha\phi\left(\alpha_+\right)\right)\frac{d\alpha_+}{d\tau}-\left(1-\Phi\left(\alpha_+\right)\right)\frac{d\alpha}{d\tau}\right)$$
$$= \frac{P_1}{A\mathbb{V}ar\left[D\right]}\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}2\left(\left(\left(\delta^{NF}-1\right)\phi'\left(\alpha_+\right)\right)\frac{d\alpha_+}{d\tau}-\left(1-\Phi\left(\alpha_+\right)\right)\frac{d\alpha}{d\tau}\right).$$

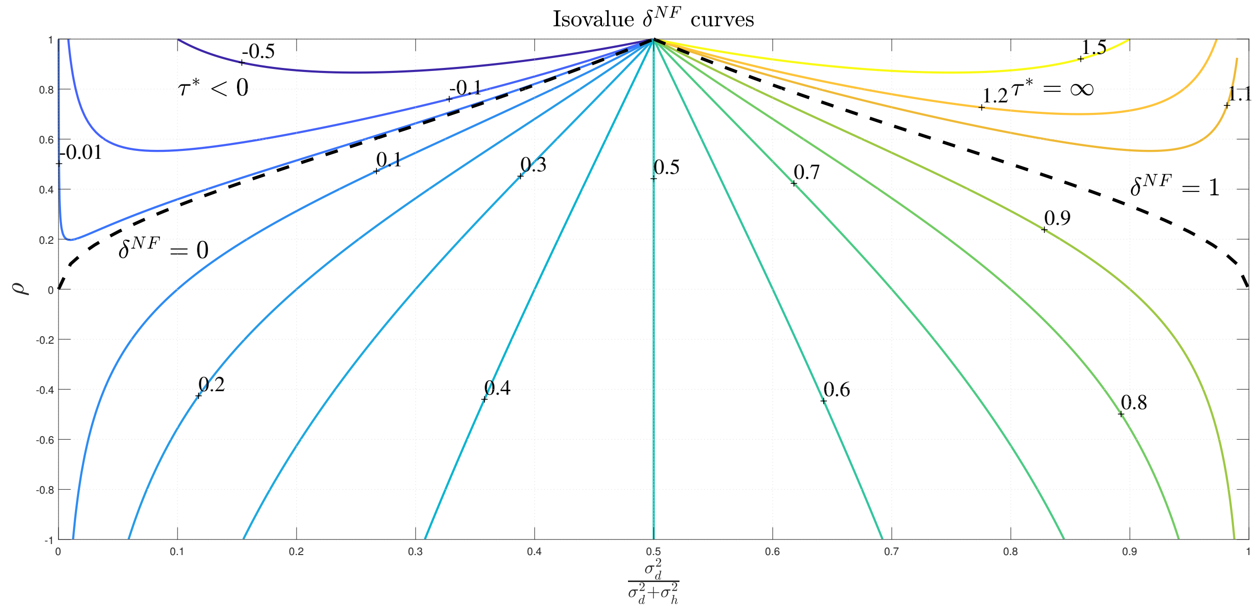


Figure A.3: Isovalue δ^{NF} curves for combinations of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ and ρ.

Note: Figure A.3 shows the different combination of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2} \in [0,1]$ and $\rho \in [-1,1]$ that are associated with the same value of the share of non-fundamental trading volume, δ^{NF}, defined in Equation (19), and consequently with the same optimal tax τ^*. The left black dashed line delimits the area in which δ^{NF} takes negative values (associated with a negative optimal tax). The right black dashed line delimits the area in which δ^{NF} takes values higher than one (associated with an infinite optimal tax). Figure A.4 below shows how δ^{NF} varies as a function of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ for different values of δ.

The first term in the parentheses corresponds to the extensive margin effects described after introducing Equation (32). This term is zero whenever $\alpha < 0$ (equivalently, $\tau < 0$), so the planner's problem is always concave in that region. However, in general, the sign of this term is ambiguous. When $0 \leq \delta^{NF} < 1$, the first term is positive, since $\phi'(\alpha_+) \leq 0$ and $\frac{d\alpha_+}{d\tau} \geq 0$. When $\delta^{NF} > 1$, the first term is negative. The second term is always negative, as already shown after introducing Equation (32). Therefore, in the relevant case in which $0 \leq \delta^{NF} < 1$, the planner's problem is not necessarily concave. However, the characterization of $\frac{dV^p}{d\tau}$ immediately implies that the planner's problem is quasi-concave whenever $\delta^{NF} < 1$ for any set of parameters, so whenever the optimal tax is finite, it is unique.

It is also valuable to understand how different combinations of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ (or equivalently $\frac{\sigma_d}{\sigma_h}$) and ρ map to δ^{NF}. Figure A.3 shows a contour plot with the combinations of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ and ρ that generate the same value of δ^{NF}. The left black dashed line delimits the area in which δ^{NF} takes negative values (associated with a negative optimal tax). The right black dashed line delimits the area in which δ^{NF} takes values higher than one (associated with an infinite optimal tax). Figure A.4 below shows how δ^{NF} varies as a function of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ for different values of δ.

For extreme choices of ρ, the share of non-fundamental trading volume δ^{NF} takes the following values

$$\delta^{NF} = \begin{cases} \frac{\sigma_d^2+\sigma_d\sigma_h}{\sigma_d^2+\sigma_h^2+2\sigma_d\sigma_h}, & \text{if } \rho = -1 \\ \frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}, & \text{if } \rho = 0 \\ \frac{\sigma_d^2-\sigma_d\sigma_h}{\sigma_d^2+\sigma_h^2-2\sigma_d\sigma_h}, & \text{if } \rho = 1. \end{cases}$$

Note that, when $\rho \leq 0$, the optimal tax is finite and non-negative for any value of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$.

Note that from the perspective of a planner who respects investors' beliefs, one can express the aggregate marginal impact of a tax change as follows

$$\frac{\frac{dV^i}{d\tau}}{P_1Q} = -\frac{\tau}{\Pi}\int_{i\in\mathcal{T}(\tau)} dF(i) = -\frac{\tau}{\Pi}2\left(1-\Phi\left(\alpha_+\right)\right).$$

As discussed in the text, social welfare in this case always decreases with the level of the tax.

Finally, note that it is also possible to characterize the individual marginal welfare impact of a tax change for specific investors from the perspective of the planner. Formally, we can write $\frac{dV_i^p}{d\tau}$ as follows

$$\frac{dV_i^p}{d\tau} = \left[\mathbb{E}_p[D] - \mathbb{E}_i[D] + \operatorname{sgn}(\Delta X_{1i}) P_1\tau\right]\frac{dX_{1i}}{d\tau} - \Delta X_{1i}\frac{dP_1}{d\tau} + \frac{d\tilde{T}_{1i}}{d\tau}.$$

After normalizing for the total market capitalization of the risky asset, and since $\frac{dP_1}{d\tau} = 0$, we can further express $\frac{\frac{dV_i^p}{d\tau}}{P_1Q}$ as follows

$$\frac{\frac{dV_i^p}{d\tau}}{P_1Q} = \left[\frac{\mathbb{E}_p[D] - \mu_d + \overbrace{\mu_d - \mathbb{E}_i[D]}^{-\varepsilon_{di}}}{P_1} + \operatorname{sgn}(\Delta X_{1i})\tau\right]\frac{\frac{dX_{1i}}{d\tau}}{Q} + \frac{\frac{d\tilde{T}_{1i}}{d\tau}}{P_1Q}, \tag{40}$$

where $\frac{\frac{dX_{1i}}{d\tau}}{Q} = \frac{-\operatorname{sgn}(\Delta X_{1i})}{\Pi}$ and $\frac{\frac{d\tilde{T}_{1i}}{d\tau}}{P_1Q}$ can take the value of zero if the planner fully rebates the tax liability to each individual investor or can be something different, as described in Section E.4, which characterizes the uniform rebate rule case. After the proof of Proposition 4, I fully describe the necessary informational requirements that characterize the distribution of $\frac{\frac{dV_i^p}{d\tau}}{P_1Q}$. The counterpart of Equation (40) when welfare is computed from the perspective of an investor i is given by

$$\frac{dV_i^i}{d\tau} = \operatorname{sgn}(\Delta X_{1i})\tau\frac{\frac{dX_{1i}}{d\tau}}{Q} + \frac{\frac{d\tilde{T}_{1i}}{d\tau}}{P_1Q}. \tag{41}$$

Optimal tax Because Assumption [S] is satisfied, the optimal tax formula can be written as in Equation (15), that is,

$$\tau^* = \frac{\mathbb{E}_{\mathcal{B}(\tau^*)}\left[\frac{\mathbb{E}_i[D]}{P_1}\right] - \mathbb{E}_{\mathcal{S}(\tau^*)}\left[\frac{\mathbb{E}_i[D]}{P_1}\right]}{2}.$$

The average belief of buyers and sellers corresponds to

$$\begin{aligned}
\mathbb{E}_{\mathcal{B}}\left[\mathbb{E}_i[D]\right] &= \mu_d + \mathbb{E}\left[\varepsilon_{di}\middle|\Delta X_{1i} > 0\right] = \mu_d + \mathbb{E}\left[\varepsilon_{di}\middle|\varepsilon_{di} - \varepsilon_{hi} - \max\{\tau P_1, 0\} > 0\right]\\
\mathbb{E}_{\mathcal{S}}\left[\mathbb{E}_i[D]\right] &= \mu_d + \mathbb{E}\left[\varepsilon_{di}\middle|\Delta X_{1i} < 0\right] = \mu_d + \mathbb{E}\left[\varepsilon_{di}\middle|\varepsilon_{di} - \varepsilon_{hi} + \max\{\tau P_1, 0\} < 0\right].
\end{aligned}$$

Note that to compute those average beliefs, it is useful to write the joint distribution of the following relevant random variables as follows

$$\begin{pmatrix} \varepsilon_{di} \\ \varepsilon_{di} - \varepsilon_{hi} - \max\{\tau P_1, 0\} \\ \varepsilon_{di} - \varepsilon_{hi} + \max\{\tau P_1, 0\} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ -\max\{\tau P_1, 0\} \\ \max\{\tau P_1, 0\} \end{pmatrix}, \begin{pmatrix} \sigma_d^2 & \sigma_d^2 - \rho\sigma_d\sigma_h & \sigma_d^2 - \rho\sigma_d\sigma_h \\ & \sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h & \sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h \\ & & \sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h \end{pmatrix}\right),$$

using the fact that $\mathbb{C}ov\left[\varepsilon_{di}, \varepsilon_{di} - \varepsilon_{hi}\right] = \sigma_d^2 - \rho\sigma_d\sigma_h = \sigma_d\left(\sigma_d - \rho\sigma_h\right)$.

The correlation coefficient between ε_{di} and $\varepsilon_{di} - \varepsilon_{hi} - \max\{\tau P_1, 0\}$, or between ε_{di} and $\varepsilon_{di} - \varepsilon_{hi} + \max\{\tau P_1, 0\}$, is denoted by $\rho^{\mathcal{BS}}$ and given by

$$\rho^{\mathcal{BS}} = \frac{\sigma_d^2 - \rho\sigma_d\sigma_h}{\sigma_d\sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}} = \frac{\sigma_d - \rho\sigma_h}{\sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}}.$$

Exploiting the properties of truncated normal distributions,

$$\begin{aligned}
\mathbb{E}\left[\varepsilon_{di}\middle|\varepsilon_{di} - \varepsilon_{hi} - \max\{\tau P_1, 0\} > 0\right] &= \mathbb{E}\left[\varepsilon_{di}\right] + \rho^{\mathcal{BS}}\sigma_d\lambda\left(\alpha_+\right) = \delta^{NF}\frac{\phi\left(\alpha_+\right)}{1 - \Phi\left(\alpha_+\right)} > 0\\
\mathbb{E}\left[\varepsilon_{di}\middle|\varepsilon_{di} - \varepsilon_{hi} + \max\{\tau P_1, 0\} < 0\right] &= \mathbb{E}\left[\varepsilon_{di}\right] + \rho^{\mathcal{BS}}\sigma_d\lambda_-\left(-\alpha_+\right) = -\delta^{NF}\frac{\phi\left(-\alpha_+\right)}{\Phi\left(-\alpha_+\right)} < 0,
\end{aligned}$$

where $\lambda_-(\cdot)$ is defined in Section C.3, α_+ is defined in Equation (37), and δ^{NF} is defined in Equation (38). When $\delta^{NF} > 0$, the case associated with a positive tax, $\mathbb{E}_{\mathcal{B}}[\mathbb{E}_i[D]]$ is increasing in τ while $\mathbb{E}_{\mathcal{S}}[\mathbb{E}_i[D]]$ is decreasing in τ, that is, the average belief of marginal buyers increases with the tax rate while the average belief of sellers decreases. The opposite occurs when $\delta^{NF} < 0$.

Combining all these results, we can express the numerator of the optimal tax formula in Equation (15) as

$$\mathbb{E}_{\mathcal{B}(\tau)}[\mathbb{E}_i[D]] - \mathbb{E}_{\mathcal{S}(\tau)}[\mathbb{E}_i[D]] = 2\delta^{NF}\lambda(\alpha_+), \tag{42}$$

which follows from the fact that

$$\frac{\phi(\alpha_+)}{1-\Phi(\alpha_+)} + \frac{\phi(-\alpha_+)}{\Phi(-\alpha_+)} = 2\frac{\phi(\alpha_+)}{1-\Phi(\alpha_+)} = 2\lambda(\alpha_+).$$

We can therefore write τ^* as

$$\begin{aligned}\tau^* &= \frac{\mathbb{E}_{\mathcal{B}(\tau^*)}\left[\frac{\mathbb{E}_i[D]}{P_1}\right] - \mathbb{E}_{\mathcal{S}(\tau^*)}\left[\frac{\mathbb{E}_i[D]}{P_1}\right]}{2} \\ &= \frac{\sigma_d(\sigma_d - \rho\sigma_h)}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}\lambda\left(\max\left\{\frac{\tau^* P_1}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}, 0\right\}\right)\frac{1}{P_1}.\end{aligned}$$

We can rearrange this expression to find that

$$\frac{\tau^* P_1}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}} = \frac{\sigma_d(\sigma_d-\rho\sigma_h)}{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}\lambda\left(\max\left\{\frac{\tau^* P_1}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}, 0\right\}\right),$$

which allows us to define $\alpha^* \equiv \frac{\tau^* P_1}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}$, implying that the fixed point that characterizes the optimal tax can be expressed as

$$\alpha^* = \delta^{NF}\lambda(\max\{\alpha^*, 0\}), \tag{43}$$

where δ^{NF} is defined in Equation (38) above. Note that α^* is exclusively a function of $\frac{\sigma_d}{\sigma_h}$ and ρ through δ^{NF}. Once a solution for α^* is found, τ^* simply corresponds to $\tau^* = \alpha^* \frac{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}{P_1}$.

Trading volume implementation Under the new parametric assumption, it is possible to find explicit expressions for the trading volume decomposition. Before doing so, it is useful to compute $\mathbb{E}_{\mathcal{B}}[A\mathbb{C}ov[M_{2i}, D]]$ and $\mathbb{E}_{\mathcal{S}}[A\mathbb{C}ov[M_{2i}, D]]$, given by

$$\begin{aligned}\mathbb{E}_{\mathcal{B}}[A\mathbb{C}ov[M_{2i}, D]] &= \mathbb{E}_{\mathcal{B}}[\varepsilon_{hi}] = \mathbb{E}[\varepsilon_{hi}|\,\Delta X_{1i} > 0] = \mathbb{E}[\varepsilon_{hi}|\,\varepsilon_{di} - \varepsilon_{hi} - \max\{\tau P_1\} > 0] \\ &= \rho^{\mathcal{BS}}\sigma_h\lambda(\alpha_+) = \frac{\sigma_h(\rho\sigma_d - \sigma_h)}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}\frac{\phi(\alpha_+)}{1-\Phi(\alpha_+)} \\ \mathbb{E}_{\mathcal{S}}[A\mathbb{C}ov[M_{2i}, D]] &= \mathbb{E}_{\mathcal{S}}[\varepsilon_{hi}] = \mathbb{E}\left[\varepsilon_{hi}|\,\Delta X_{1i}^- < 0\right] = \mathbb{E}[\varepsilon_{hi}|\,\varepsilon_{di} - \varepsilon_{hi} + \tau P_1 < 0] \\ &= \rho^{\mathcal{BS}}\sigma_h\lambda_-(-\alpha_+) = -\frac{\sigma_h(\rho\sigma_d-\sigma_h)}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}\frac{\phi(-\alpha_+)}{\Phi(-\alpha_+)},\end{aligned}$$

where α_+ is defined in Equation (37). Therefore, it follows that

$$\mathbb{E}_{\mathcal{B}}[A\mathbb{C}ov[M_{2i}, D]] - \mathbb{E}_{\mathcal{S}}[A\mathbb{C}ov[M_{2i}, D]] = 2\frac{\sigma_h(\rho\sigma_d-\sigma_h)}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}\frac{\phi(\alpha_+)}{1-\Phi(\alpha_+)}.$$

Now we can express total trading volume in dollars as follows

$$\begin{aligned}P_1\mathcal{V}(\tau) &= \frac{1}{2}\int_{i\in\mathcal{T}}\left(\left(-\frac{\partial X_{1i}}{\partial\tau}\right)(\mathbb{E}_i[D] - A_i\mathbb{C}ov[M_{2i}, D] - P_1\,\mathrm{sgn}(\Delta X_{1i})\,\tau - A_i\mathbb{V}ar[D]\,X_{0i})\right)dF(i) \\ &= \Theta_F(\tau) + \Theta_{NF}(\tau) - \Theta_\tau(\tau) \\ &= \frac{P_1}{A\mathbb{V}ar[D]}(1-\Phi(\alpha_+))\left[\frac{\sigma_d(\sigma_d-\rho\sigma_h)}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}\lambda(\alpha_+) + \frac{-\sigma_h(\rho\sigma_d-\sigma_h)}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}\lambda(\alpha_+) - \tau P_1\right] \\ &= \frac{P_1}{A\mathbb{V}ar[D]}(1-\Phi(\alpha_+))\left(-\tau P_1 + \sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}\,\lambda(\alpha_+)\right).\end{aligned} \tag{44}$$

The fundamental component of trading volume can be expressed as follows

$$
\begin{aligned}
\Theta_F(\tau) &= \frac{1}{2}\int_{i\in\mathcal{T}}\left(\frac{\operatorname{sgn}(\Delta X_{1i})P_1}{A\mathbb{V}ar[D]}\right)(-A\mathbb{C}ov[M_{2i},D])\,dF(i)\\
&= -\frac{1}{2}\frac{P_1}{A\mathbb{V}ar[D]}\left(\int_{i\in\mathcal{B}}A\mathbb{C}ov[M_{2i},D]\,dF(i)-\int_{i\in\mathcal{S}}A\mathbb{C}ov[M_{2i},D]\,dF(i)\right)\\
&= -\frac{1}{2}\frac{P_1}{A\mathbb{V}ar[D]}\int_{i\in\mathcal{B}}dF(i)\left(\mathbb{E}_{\mathcal{B}}[A\mathbb{C}ov[M_{2i},D]]-\mathbb{E}_{\mathcal{S}}[A\mathbb{C}ov[M_{2i},D]]\right)\\
&= -\frac{1}{2}\frac{P_1}{A\mathbb{V}ar[D]}(1-\Phi(\alpha_+))\left(\mathbb{E}_{\mathcal{B}}[A\mathbb{C}ov[M_{2i},D]]-\mathbb{E}_{\mathcal{S}}[A\mathbb{C}ov[M_{2i},D]]\right)\\
&= -\frac{P_1}{A\mathbb{V}ar[D]}(1-\Phi(\alpha_+))\frac{\sigma_h(\rho\sigma_d-\sigma_h)}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}\frac{\phi(\alpha_+)}{1-\Phi(\alpha_+)}\\
&= \frac{P_1}{A\mathbb{V}ar[D]}\frac{\sigma_h(\sigma_h-\rho\sigma_d)}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}\phi(\alpha_+).
\end{aligned}
\tag{45}
$$

The non-fundamental component of trading volume can be expressed as follows

$$
\begin{aligned}
\Theta_{NF}(\tau) &= \frac{1}{2}\int_{i\in\mathcal{T}}\left(\frac{\operatorname{sgn}(\Delta X_{1i})P_1}{A\mathbb{V}ar[D]}\right)\mathbb{E}_i[D]\,dF(i)\\
&= \frac{1}{2}\frac{P_1}{A\mathbb{V}ar[D]}\int_{i\in\mathcal{B}}dF(i)\left(\mathbb{E}_{\mathcal{B}}[\mathbb{E}_i[D]]-\mathbb{E}_{\mathcal{S}}[\mathbb{E}_i[D]]\right)\\
&= \frac{1}{2}\frac{P_1}{A\mathbb{V}ar[D]}(1-\Phi(\alpha))\left(\mathbb{E}_{\mathcal{B}}[\mathbb{E}_i[D]]-\mathbb{E}_{\mathcal{S}}[\mathbb{E}_i[D]]\right)\\
&= \frac{P_1}{A\mathbb{V}ar[D]}\frac{\sigma_d(\sigma_d-\rho\sigma_h)}{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}\phi(\alpha_+).
\end{aligned}
\tag{46}
$$

The tax component of trading volume can be expressed as follows

$$
\begin{aligned}
\Theta_\tau(\tau) &= \frac{1}{2}\tau P_1\int_{i\in\mathcal{T}}\left(\frac{\operatorname{sgn}(\Delta X_{1i})P_1}{A\mathbb{V}ar[D]}\right)\operatorname{sgn}(\Delta X_{1i})\,dF(i)\\
&= \frac{1}{2}\tau P_1\frac{P_1}{A\mathbb{V}ar[D]}\int_{i\in\mathcal{T}}dF(i)\\
&= \tau P_1\frac{P_1}{A\mathbb{V}ar[D]}\int_{i\in\mathcal{B}}dF(i)\\
&= \tau P_1\frac{P_1}{A\mathbb{V}ar[D]}(1-\Phi(\alpha_+)).
\end{aligned}
\tag{47}
$$

Note that $\Theta_F(\tau)$ and $\Theta_{NF}(\tau)$ can take negative values for extreme values of ρ. That is, if $\sigma_h-\rho\sigma_d<0$ or $\sigma_d-\rho\sigma_h<0$. However, its sum $\Theta_F(\tau)+\Theta_{NF}(\tau)$ is always non-negative. To derive $\Theta_{NF}(\tau)$, we use the expression for $\mathbb{E}_{\mathcal{B}}[\mathbb{E}_i[D]]-\mathbb{E}_{\mathcal{S}}[\mathbb{E}_i[D]]$ given in Equation (42).

Note that the ratio of the non-fundamental to fundamental components of trading volume can be expressed as follows

$$
\frac{\Theta_{NF}(\tau)}{\Theta_F(\tau)}=\frac{\sigma_d(\sigma_d-\rho\sigma_h)}{\sigma_h(\sigma_h-\rho\sigma_d)}=\frac{\sigma_d^2-\rho\sigma_d\sigma_h}{\sigma_h^2-\rho\sigma_d\sigma_h}=\frac{\frac{\sigma_d}{\sigma_h}-\rho}{\frac{\sigma_h}{\sigma_d}-\rho}.
$$

Three facts are worth highlighting. First, this ratio is independent of the tax rate τ. Second, when $\rho=0$, the ratio is exactly $\left(\frac{\sigma_d}{\sigma_h}\right)^2$. Third, when $\sigma_d=\sigma_h$, the ratio is equal to 1. More importantly, it is possible to compute the share of the non-fundamental component of trading to the sum of fundamental and non-fundamental components as follows

$$
\delta^{NF}=\frac{\Theta_{NF}(\tau)}{\Theta_F(\tau)+\Theta_{NF}(\tau)}=\frac{\sigma_d(\sigma_d-\rho\sigma_h)}{\sigma_h(\sigma_h-\rho\sigma_d)+\sigma_d(\sigma_d-\rho\sigma_h)}=\frac{\sigma_d^2-\rho\sigma_d\sigma_h}{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}=\frac{\frac{\sigma_d}{\sigma_h}-\rho}{\frac{\sigma_h}{\sigma_d}-\rho+\frac{\sigma_d}{\sigma_h}-\rho}.
$$

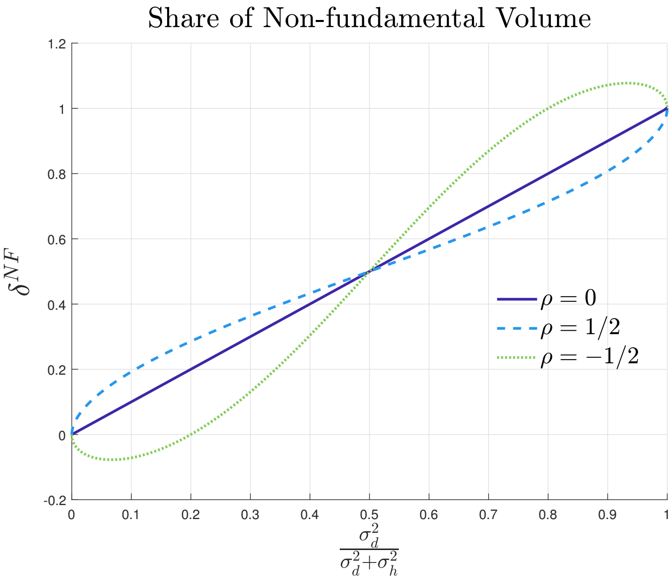


Figure A.4: Share of non-fundamental volume

Note: Figure A.4 shows how the share of non-fundamental trading volume δ^{NF}, defined in Equation (38), varies with the level of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ for different values of the correlation coefficient ρ. Note that δ^{NF} is increasing in $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ (equivalently $\frac{\sigma_d}{\sigma_h}$) whenever $\delta^{NF} \in [0,1]$.

Figure A.4 illustrates how δ^{NF} varies with the level of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ for different values of the correlation coefficient ρ — see also Figure A.3 above.[30] Whenever $\delta^{NF} \in [0,1]$, an increase in $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ increases the value of δ^{NF}, as expected. For a given value of $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$, increasing ρ increases the share of non-fundamental trading volume δ^{NF} when $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2} < 1/2$, but decreases δ^{NF} otherwise. Note that, when $\rho > 0$, δ^{NF} can take negative values or values larger than unity, which makes it harder to interpret as a share.

Finally, note that in terms of turnover, we can express $\Xi(\tau)$, $\Xi_F(\tau)$, $\Xi_{NF}(\tau)$, and $\Xi_\tau(\tau)$ as follows

$$\begin{aligned}
\Xi(\tau) &= \frac{\mathcal{V}(\tau)}{Q} = (1-\Phi(\alpha_+))\left(-\frac{\tau}{\Pi}\right) + \Xi(0)\sqrt{2\pi}\phi(\alpha_+) \qquad (48)\\
\Xi_F(\tau) &= \frac{\Theta_F(\tau)}{P_1 Q} = \Xi(0)\sqrt{2\pi}\delta^{NF}\phi(\alpha_+)\\
\Xi_{NF}(\tau) &= \frac{\Theta_{NF}(\tau)}{P_1 Q} = \Xi(0)\sqrt{2\pi}\left(1-\delta^{NF}\right)\phi(\alpha_+)\\
\Xi_\tau(\tau) &= \frac{\Theta_\tau(\tau)}{P_1 Q} = \frac{\tau}{\Pi}(1-\Phi(\alpha_+)),
\end{aligned}$$

where α_+ is defined in Equation (37), the risk premium is denoted by $\Pi = \frac{A\mathbb{V}ar[D]Q}{P_1}$, and laissez-faire turnover $\Xi(0)$ is given by

$$\Xi(0) = \frac{1}{\sqrt{2\pi}}\frac{1}{\Pi}\frac{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}{P_1}. \qquad (49)$$

Non-quasi-concave planner's objective Figure 4 shows the normalized aggregate welfare impact of a tax change from the planner's perspective in a scenario in which the planner's objective fails to be quasi-concave. The easiest scenario in which symmetry is preserved is one in which there n groups of investors, with proportions π^n, some risk aversion and initial asset holdings, and whose fundamental and non-fundamental trading motives

[30] Note that $\frac{\sigma_d}{\sigma_h}$ and $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2}$ are related through the fact that $\frac{\sigma_d^2}{\sigma_d^2+\sigma_h^2} = \frac{1}{1+\left(\frac{\sigma_d}{\sigma_h}\right)^{-2}}$.

are centered around the same mean as follows

$$\mathbb{E}_i^n [D] \sim \mu_d + \varepsilon_{di}^n$$
$$A\mathbb{C}ov [M_{2i}, D] \sim \mu_h + \varepsilon_{hi}^n,$$

where $\mu_d \geq 0$ and $\mu_h = 0$ are the same across all n groups. The random variables ε_{hi}^n and ε_{di}^n are jointly normally distributed for i investors in group n as follows

$$\begin{pmatrix} \varepsilon_{di}^n \\ \varepsilon_{hi}^n \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (\sigma_d^n)^2 & \rho_n \sigma_d^n \sigma_h^n \\ \rho_n \sigma_d^n \sigma_h^n & (\sigma_h^n)^2 \end{pmatrix}\right).$$

In this case, the equilibrium price still corresponds to Equation (6), and most of the equations derive in this section remain valid for each, including individual and aggregate turnover. In the environment considered in Figure 4, 90% of investors belong to group 1, while the remaining 10% belong to group 2. Group 1 investors have a turnover of 1/4 and a share of non-fundamental trading volume of 0.3. Group 2 investors have a turnover of 1 and a share of non-fundamental trading volume of 0.65. The risk premium is 1.5%.

The aggregate marginal impact of a tax change in this case is given by

$$\frac{\frac{dV^p}{d\tau}}{P_1 Q} = \sum_n \pi^n \frac{\frac{dV_n^p}{d\tau}}{P_1 Q}, \quad \text{where} \quad \frac{\frac{dV_n^p}{d\tau}}{P_1 Q} = \sqrt{2\pi}\Xi_n(0)\, 2\left(1 - \Phi\left(\alpha_+^n\right)\right)\left(\delta_n^{NF} \lambda\left(\alpha_+^n\right) - \alpha^n\right),$$

where α_+^n, α^n, δ_n^{NF}, and $\Xi_n(0)$ are computed as in Equations (37), (38), and (49) for each group. Even though each of the group-specific $\frac{\frac{dV_n^p}{d\tau}}{P_1 Q}$ are quasi-concave, it is well-known that the sum of two quasi-concave functions is not quasi-concave, so the planner's objective $\frac{\frac{dV^p}{d\tau}}{P_1 Q}$ may or may not be quasi-concave.

Proposition 3. (Optimal tax and comparative statics)

Proof. a) It easily follows that, when $\rho \leq 0$ and $\sigma_d > 0$, Equation (43) has a non-negative solution, since $\delta^{NF} > 0$.

b) From Equation (43), for any $\tau^* \geq 0$, the sign of $\frac{d\alpha^*}{d\left(\frac{\sigma_d}{\sigma_h}\right)}$ is determined by the sign of $-\frac{d\delta^{NF}}{d\left(\frac{\sigma_h}{\sigma_d}\right)}$.We can express $\frac{d\delta^{NF}}{d\left(\frac{\sigma_h}{\sigma_d}\right)}$ as follows

$$\frac{d\delta^{NF}}{d\left(\frac{\sigma_h}{\sigma_d}\right)} = \frac{-\rho\left(1 + \left(\frac{\sigma_h}{\sigma_d}\right)^2 - 2\rho\frac{\sigma_h}{\sigma_d}\right) - \left(1 - \rho\frac{\sigma_h}{\sigma_d}\right)2\left(\frac{\sigma_h}{\sigma_d} - \rho\right)}{\left(1 + \left(\frac{\sigma_h}{\sigma_d}\right)^2 - 2\rho\frac{\sigma_h}{\sigma_d}\right)^2} = \frac{\rho\left(\frac{\sigma_h}{\sigma_d}\right)^2 - 2\frac{\sigma_h}{\sigma_d} + \rho}{\left(1 + \left(\frac{\sigma_h}{\sigma_d}\right)^2 - 2\rho\frac{\sigma_h}{\sigma_d}\right)^2}.$$

When $\rho \leq 0$ this expression is everywhere negative implying that $\frac{d\alpha^*}{d\left(\frac{\sigma_d}{\sigma_h}\right)}$ is positive. Note that we can find the following cases, for $\rho \geq 0$:

$$\text{if} \begin{cases} \frac{\sigma_d}{\sigma_h} < \rho, & \alpha^* < 0 \\ \rho \leq \frac{\sigma_d}{\sigma_h} < \frac{1}{\rho}, & \alpha^* \geq 0 \\ \frac{\sigma_d}{\sigma_h} \geq \frac{1}{\rho}, & \alpha^* = \infty. \end{cases}$$

It can be established that $\rho\left(\frac{\sigma_h}{\sigma_d}\right)^2 - 2\frac{\sigma_h}{\sigma_d} + \rho$ is negative in between its two roots when $\rho \in (0, 1]$. Its roots are given by $\frac{1 \pm \sqrt{1-\rho^2}}{\rho}$. It is sufficient to show that $\frac{1 - \sqrt{1-\rho^2}}{\rho} < \rho$ and that $\frac{1}{\rho} < \frac{1+\sqrt{1-\rho^2}}{\rho}$, which are trivially satisfied for any $\rho \in (0, 1]$. This fact is sufficient to show the desired comparative static for the $\rho > 0$ case.

c) When $\delta^{NF} < 0$, there is a unique negative solution to Equation (43) (a subsidy), given by $\alpha^* = \delta^{NF}\lambda(0)$. When $0 \leq \delta^{NF} < 1$, there is a uniquely optimal finite and positive tax, which solves $\delta^{NF}\lambda(\alpha^*) = \alpha^*$. It is easy to verify that Equation (43) has a single solution in that case. Third, when $\delta^{NF} \geq 1$, it is the case that $\frac{dV^p}{d\tau} > 0$,

$\forall\tau$, so the optimal tax is $\tau^* = +\infty$. This follows from the fact that $\lambda(\alpha) > \alpha$, $\forall\alpha$, established in Section C.3. Formally,

$$\text{if} \begin{cases} \delta^{NF} < 0, & \tau^* < 0 \\ 0 \leq \delta^{NF} < 1, & 0 \leq \tau^* < \infty \\ \delta^{NF} \geq 1, & \tau^* = \infty. \end{cases}$$

□

C.2 Quantitative assessment

Proposition 4. (Optimal tax identification/Sufficient statistics)

Proof. a) Equation (43) establishes that α^* exclusively depends on the value of δ^{NF}, itself a function of $\frac{\sigma_d}{\sigma_h}$ and ρ. Therefore, for a given α^*, we can use the definition of α, given in Equation (37), to express τ^* as follows

$$\tau^* = \alpha^* \frac{\sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}}{P_1}.$$

Therefore, for a given α^*, it is sufficient to know $\frac{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}{P_1}$ to find the optimal tax. A possible way of computing that object exploits the definition of volume semi-elasticity to tax changes. First, note that

$$\frac{d\log\mathcal{V}}{d\tau} = \frac{1}{\tau - \frac{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}{P_1}\lambda(\alpha_+)}. \tag{50}$$

Therefore, if $\frac{d\log\mathcal{V}}{d\tau}$ is observed for a given value of τ, Equation (50) can be inverted to recover $\frac{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}{P_1}$. In general, finding $\frac{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}{P_1}$ involves solving for a complicated non-linear equation. However, if one observes $\varepsilon_\tau^{\log\mathcal{V}}\big|_{\tau=0}$, then it is possible to find $\frac{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}{P_1}$ explicitly, since

$$\varepsilon_\tau^{\log\mathcal{V}}\Big|_{\tau=0} = \left.\frac{d\log\mathcal{V}}{d\tau}\right|_{\tau=0} = -\frac{1}{\frac{\sqrt{\sigma_d^2+\sigma_h^2-2\rho\sigma_d\sigma_h}}{P_1}\sqrt{\frac{2}{\pi}}}.$$

In that case,

$$\tau^* = -\frac{\alpha^*}{\varepsilon_\tau^{\log\mathcal{V}}\Big|_{\tau=0}}\sqrt{\frac{\pi}{2}}.$$

This argument is sufficient to prove part a). Note that Equation (49) implies the following relation

$$\frac{\sqrt{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}}{P_1} = \Pi\Xi(0)\sqrt{2\pi},$$

which allows us to write the optimal tax as $\tau^* = \Pi\Xi[0]\sqrt{2\pi}\alpha^*$. Hence, in this model, $\varepsilon_\tau^{\log\mathcal{V}}\big|_{\tau=0} = -\frac{1}{2}\frac{1}{\Pi\Xi(0)}$, so the frequency of trading can be chosen to jointly match $\varepsilon_\tau^{\log\mathcal{V}}\big|_{\tau=0}$, Π, and $\Xi(0)$,

b) The normalized aggregate marginal welfare impact of a tax change, already characterized in Equation (39), corresponds to

$$\frac{\frac{dV^p}{d\tau}}{P_1 Q} = \sqrt{2\pi}\Xi(0)\,2\,(1-\Phi(\alpha_+))\left(\delta^{NF}\lambda(\alpha_+) - \alpha\right).$$

In addition to knowing δ^{NF} and $\frac{d\log\mathcal{V}}{d\tau}$ for some value of τ, to compute $\frac{\frac{dV^p}{d\tau}}{P_1Q}$ it is now necessary to also know either laissez-faire turnover $\Xi[0]$ or the risk premium Π, which concludes the proof. □

Identification of individual marginal welfare gains It is worth making two additional remarks. First, combining Equation (36) with the fact that $\varepsilon_{\tau}^{\log \mathcal{V}}\big|_{\tau=0} = -\frac{1}{2}\frac{1}{\Pi\Xi(0)}$, it follows easily that only information on two of the following three objects: $\varepsilon_{\tau}^{\log \mathcal{V}}\big|_{\tau=0}$, Π, and $\Xi(0)$, is necessary to determine the distribution of investors individual turnover. Second, it is possible to recover the distribution of $\frac{\frac{dV_i^p}{d\tau}}{P_1 Q}$ for individual investors for a given value of $\frac{\mathbb{E}_p[D]-\mu_d}{P_1}$. Note that $\frac{\mu_d}{P_1} = 1 + \Pi$. However, in this case, in addition to $\varepsilon_{\tau}^{\log \mathcal{V}}\big|_{\tau=0}$, and Π or $\Xi(0)$, it is necessary to separately know $\frac{\sigma_d}{\sigma_h}$ and ρ. Formally, starting from Equation (40), we can express $\frac{\frac{dV_i^p}{d\tau}}{P_1 Q}$ as follows

$$\frac{\frac{dV_i^p}{d\tau}}{P_1 Q} = \left[\frac{\mathbb{E}_p[D] - \mu_d - \varepsilon_{di}}{P_1} + \operatorname{sgn}(\Delta X_{1i})\,\tau\right]\frac{-\operatorname{sgn}(\Delta X_{1i})}{\Pi} + \frac{\frac{d\tilde{T}_{1i}}{d\tau}}{P_1 Q}.$$

Hence in this case, one needs to characterize the distributions of $\frac{\varepsilon_{di}}{P_1}$ and $\frac{\varepsilon_{hi}}{P_1}$, given by

$$\frac{\varepsilon_{di}}{P_1} \sim N\left(0, \frac{\sigma_d}{P_1}\right) \quad \text{and} \quad \frac{\varepsilon_{hi}}{P_1} \sim N\left(0, \frac{\sigma_h}{P_1}\right),$$

where

$$\frac{\sigma_d}{P_1} = \frac{\Pi\Xi(0)\sqrt{2\pi}}{\sqrt{1+\left(\frac{\sigma_h}{\sigma_d}\right)^2 - 2\rho\frac{\sigma_h}{\sigma_d}}} = \frac{\frac{1}{\varepsilon_{\tau}^{\log \mathcal{V}}\big|_{\tau=0}}\sqrt{\frac{\pi}{2}}}{\sqrt{1+\left(\frac{\sigma_h}{\sigma_d}\right)^2 - 2\rho\frac{\sigma_h}{\sigma_d}}}$$

$$\frac{\sigma_h}{P_1} = \frac{\Pi\Xi(0)\sqrt{2\pi}}{\sqrt{1+\left(\frac{\sigma_d}{\sigma_h}\right)^2 - 2\rho\frac{\sigma_d}{\sigma_h}}} = \frac{\frac{1}{\varepsilon_{\tau}^{\log \mathcal{V}}\big|_{\tau=0}}\sqrt{\frac{\pi}{2}}}{\sqrt{1+\left(\frac{\sigma_d}{\sigma_h}\right)^2 - 2\rho\frac{\sigma_d}{\sigma_h}}}.$$

The net trading positions for active investors as a function of $\frac{\varepsilon_{di}}{P_1}$ and $\frac{\varepsilon_{hi}}{P_1}$ are given by

$$\frac{\Delta X_{1i}}{Q} = \frac{\varepsilon_{di} - \varepsilon_{hi} - \operatorname{sgn}(\Delta X_{1i})\,\tau P_1}{A\mathbb{V}ar[D]\,Q} = \frac{1}{\Pi}\left(\frac{\varepsilon_{di} - \varepsilon_{hi}}{P_1} - \operatorname{sgn}(\Delta X_{1i})\,\tau\right).$$

Therefore, for given values of $\frac{\varepsilon_{di}}{P_1}$ and $\frac{\varepsilon_{hi}}{P_1}$, as well as Π, it is possible to compute $\frac{\Delta X_{1i}}{Q}$ as well as $\frac{\frac{dV_i^p}{d\tau}}{P_1 Q}$ if $\frac{\frac{d\tilde{T}_{1i}}{d\tau}}{P_1 Q} = 0$. See Section 2 for how to compute $\frac{\frac{d\tilde{T}_{1i}}{d\tau}}{P_1 Q}$ with the same information under a uniform rebate rule.

Estimation of share of non-fundamental trading volume Here, I provide and study the properties of an estimation procedure for δ^{NF} based on information on individual investors' portfolio choices and hedging needs. I assume that the estimation is conducted in an economy with $\tau = 0$, although the approach could be extended to scenarios with positive taxation.

First, note that Equation (35) implies that the following relation must hold for all investors:

$$\underbrace{\Delta X_{1i}}_{\text{observed}} = \underbrace{-\frac{\varepsilon_{hi}}{A\mathbb{V}ar[D]}}_{\text{observed}} + \underbrace{\frac{\varepsilon_{di}}{A\mathbb{V}ar[D]}}_{\text{error term}},$$

which implies that the following variance decomposition must hold

$$TSS \equiv \mathbb{V}ar_F[\Delta X_{1i}] = \frac{\mathbb{V}ar[\varepsilon_{di} - \varepsilon_{hi}]}{(A\mathbb{V}ar[D])^2} = \frac{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}{(A\mathbb{V}ar[D])^2},$$

where TSS denotes the total sum of squares.

Next, under the assumption that an econometrician observes ΔX_{1i} and ε_{hi} (or equivalently, $\mathbb{C}ov[M_{2i}, D]$, which would only change the analysis by including a constant), but cannot observe investors' beliefs (ε_{di}, or equivalently, $\mathbb{E}_D[D]$), this expression can be interpreted as a regression equation in which $\frac{\varepsilon_{di}}{A\mathbb{V}ar[D]}$ corresponds to an error term. Therefore, it is possible to recover from observables the explained sum of squares, which corresponds to

$$ESS \equiv \frac{\sigma_h^2}{\left(A\mathbb{V}ar\left[D\right]\right)^2},$$

where ESS denotes the explained sum of squares.

I then propose the following estimator for δ^{NF}:

$$\hat{\delta}^{NF} = 1 - \frac{ESS}{TSS}. \tag{51}$$

Note that this estimator recovers the following ratio

$$1 - \frac{ESS}{TSS} = \frac{\sigma_d^2 - 2\rho\sigma_d\sigma_h}{\sigma_d^2 + \sigma_h^2 - 2\rho\sigma_d\sigma_h}, \tag{52}$$

which is always positive. By comparing Equation (52) with Equation (19), it becomes evident that $\hat{\delta}^{NF}$ may be biased when $\rho \neq 0$.[31] Formally, the relative bias of this estimator can be expressed as follows

$$\frac{\hat{\delta}^{NF} - \delta^{NF}}{\delta^{NF}} = -\frac{\rho}{\frac{\sigma_d}{\sigma_h} - \rho}. \tag{53}$$

Therefore, when $\rho = 0$, the estimator $\hat{\delta}^{NF}$, as defined in Equation 51 is an unbiased estimator of the share of non-fundamental trading. The magnitude of the bias is increasing in the magnitude of ρ. Note that when $\rho < 0$, the maximum relative bias is bounded by $\frac{1}{\frac{\sigma_d}{\sigma_h}+1}$, When $\rho > 0$, the bias could be arbitrarily large when $\frac{\sigma_d}{\sigma_h} \approx \rho$, but those are the cases in which $\delta^{NF} \approx 0$, so the overall impact on the optimal tax may still be small.

Conceptually, when beliefs and hedging motives are uncorrelated, it is possible to use observed investors' portfolio allocations and hedging needs to find the relevant ratio that determines the share of non-fundamental trading. However, when beliefs and hedging motives are correlated, there will be confounding effects unless one can observe individual beliefs too. Equation (53) is useful to understand the bias that arises in those cases.

C.3 Auxiliary results

The following auxiliary results are useful. These are well-known properties of the normal distribution — see, for instance, Greene (2003). Let's respectively denote by $\phi(\cdot)$ and $\Phi(\cdot)$ the p.d.f. and c.d.f. of the standard normal distribution.

Fact 1. *If $X \sim N\left(\mu, \sigma^2\right)$, then*

$$\mathbb{E}\left[X \left| X > a\right.\right] = \mu + \sigma\lambda\left(\alpha\right), \quad \text{where} \quad \lambda\left(\alpha\right) = \frac{\phi\left(\alpha\right)}{1-\Phi\left(\alpha\right)} \quad \text{and} \quad \alpha = \frac{a-\mu}{\sigma}.$$

Fact 2. *If Y and Z have a bivariate normal distribution with means μ_y and μ_z, variances σ_y^2 and σ_z^2, and a correlation coefficient ρ_{yz}, then*

$$\mathbb{E}\left[Y \left| Z > a\right.\right] = \mu_y + \rho_{yz}\sigma_y\lambda\left(\alpha_z\right), \quad \text{where} \quad \lambda\left(\alpha_z\right) = \frac{\phi\left(\alpha_z\right)}{1-\Phi\left(\alpha_z\right)} \quad \text{and} \quad \alpha_z = \frac{a-\mu_z}{\sigma_z}$$

$$\mathbb{E}\left[Y \left| Z < a\right.\right] = \mu_y + \rho_{yz}\sigma_y\lambda_-\left(\alpha_z\right), \quad \text{where} \quad \lambda_-\left(\alpha_z\right) = -\frac{\phi\left(\alpha_z\right)}{\Phi\left(\alpha_z\right)} \quad \text{and} \quad \alpha_z = \frac{a-\mu_z}{\sigma_z}.$$

Fact 3. *The function $\lambda\left(\alpha\right) = \frac{\phi(\alpha)}{1-\Phi(\alpha)}$, which corresponds to the hazard rate of the normal distribution, is also known as the inverse Mills ratio. It satisfies the following properties:*

1. $\lambda\left(0\right) = \sqrt{\frac{2}{\pi}}$, $\lambda\left(\alpha\right) \geq 0$, $\lambda\left(\alpha\right) > \alpha$, $\lambda'\left(\alpha\right) > 0$, and $\lambda''\left(\alpha\right) > 0$.
2. $\lim_{\alpha\to-\infty}\lambda\left(\alpha\right) = \lim_{\alpha\to-\infty}\lambda'\left(\alpha\right) = 0$, and $\lim_{\alpha\to\infty}\lambda'\left(\alpha\right) = 1$.
3. $\lambda\left(\alpha\right) < \frac{1}{\alpha} + \alpha$, $\lambda'\left(\alpha\right) = \frac{\phi'(\alpha_i)}{1-\Phi(\alpha_i)} + \left(\lambda\left(\alpha_i\right)\right)^2 = \lambda\left(\alpha\right)\left(\lambda\left(\alpha\right) - \alpha\right) > 0$, $\lambda'\left(\alpha\right) < 1$, and $\lambda''\left(\alpha\right) \geq 0$.

[31] The rest of the argument assumes that ESS and TSS can be estimated without bias.

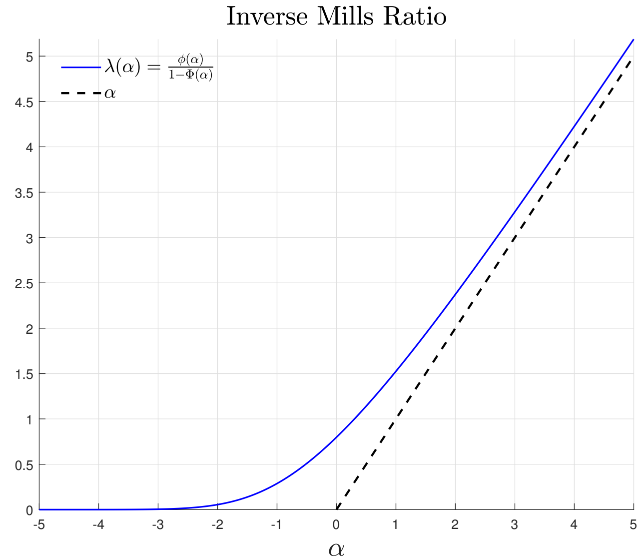


Figure A.5: Inverse Mills ratio

Note: Figure A.5 shows the inverse Mills ratio for the normal distribution: $\lambda(\alpha) = \frac{\phi(\alpha)}{1-\Phi(\alpha)}$.

D Proofs and derivations: Section 6

Proposition 5. (General utility and arbitrary beliefs)

Proof. a) It is useful to express Equations (21) and (22) as follows

$$1 - R\mathbb{E}_p[m_i] = R(\mathbb{E}_i[m_i] - \mathbb{E}_p[m_i]) \tag{54}$$

$$P_1 - \mathbb{E}_p[m_i D] = \mathbb{E}_i[m_i D] - \mathbb{E}_p[m_i D] - \tau \operatorname{sgn}(\Delta X_{1i}) P_1, \tag{55}$$

where the notation $\mathbb{E}_p[\cdot]$ represents expectations computed using the planner's belief.

The individual marginal welfare impact of a tax change from the planner's perspective is given by

$$\frac{dV_i^p}{d\tau} = U_i'(C_{1i}) \left(-P_1 \frac{dX_{1i}}{d\tau} - \frac{dY_{1i}}{d\tau} - \frac{dP_1}{d\tau}\Delta X_{1i} + \frac{d\hat{T}_{1i}}{d\tau} \right) + \beta_i \mathbb{E}_p \left[U_i'(C_{2i}) \left(D\frac{dX_{1i}}{d\tau} + R\frac{dY_{1i}}{d\tau} + \frac{dR}{d\tau}Y_{1i} \right) \right].$$

After normalizing by the marginal value of wealth, we can express $\frac{\frac{dV_i^p}{d\tau}}{U_i'(C_{1i})}$ as follows

$$\begin{aligned}
\frac{\frac{dV_i^p}{d\tau}}{U_i'(C_{1i})} &= (-P_1 + \mathbb{E}_p[m_i D]) \frac{dX_{1i}}{d\tau} + (-1 + \mathbb{E}_p[m_i] R) \frac{dY_{1i}}{d\tau} - \frac{dP_1}{d\tau}\Delta X_{1i} + \frac{dR}{d\tau}Y_{1i} + \frac{d\hat{T}_{1i}}{d\tau} \\
&= (\mathbb{E}_p[m_i D] - \mathbb{E}_i[m_i D] + \tau \operatorname{sgn}(\Delta X_{1i}) P_1) \frac{dX_{1i}}{d\tau} \\
&\quad + R(\mathbb{E}_p[m_i] - \mathbb{E}_i[m_i]) \frac{dY_{1i}}{d\tau} + \frac{dR}{d\tau}Y_{1i} - \frac{dP_1}{d\tau}\Delta X_{1i} + \frac{d\hat{T}_{1i}}{d\tau},
\end{aligned}$$

where the last line uses Equations (54) and (55). Finally, we can exploit market clearing and the fact the tax revenue is rebated to aggregate across investors and compute the aggregate welfare impact of a tax change from the planner's perspective, given by

$$\begin{aligned}
\int \frac{\frac{dV_i^p}{d\tau}}{U_i'(C_{1i})} dF(i) &= \int_{i \in \mathcal{T}(\tau)} (\mathbb{E}_p[m_i D] - \mathbb{E}_i[m_i D] + \tau \operatorname{sgn}(\Delta X_{1i}) P_1) \frac{dX_{1i}}{d\tau} dF(i) \\
&\quad + R \int (\mathbb{E}_p[m_i] - \mathbb{E}_i[m_i]) \frac{dY_{1i}}{d\tau} dF(i).
\end{aligned}$$

This expression is the counterpart to Equation (11) in the text, with the introduction of a new term capturing differences in the valuations of the risk-free asset between the investors' and the planner.

Consequently, at an interior optimum, the optimal tax must satisfy

$$\tau^* = \frac{\int_{i\in\mathcal{T}(\tau^*)} \left(\mathbb{E}_i\left[m_i(\tau^*)D\right] - \mathbb{E}_p\left[m_i(\tau^*)D\right]\right)\frac{dX_{1i}}{d\tau}(\tau^*)\,dF(i) + R(\tau^*)\int\left(\mathbb{E}_i\left[m_i(\tau^*)\right] - \mathbb{E}_p\left[m_i(\tau^*)\right]\right)\frac{dY_{1i}}{d\tau}(\tau^*)\,dF(i)}{\int_{i\in\mathcal{T}(\tau^*)} P_1(\tau^*)\operatorname{sgn}(\Delta X_{1i}(\tau^*))\frac{dX_{1i}}{d\tau}(\tau^*)\,dF(i)},$$

which can be expressed as follows

$$\tau^* = \underbrace{\frac{\int_{i\in\mathcal{T}(\tau^*)}\left(\mathbb{E}_i\left[m_i(\tau^*)\frac{D}{P_1(\tau^*)}\right] - \mathbb{E}_p\left[m_i(\tau^*)\frac{D}{P_1(\tau^*)}\right]\right)\frac{dX_{1i}}{d\tau}(\tau^*)\,dF(i)}{2\int_{i\in\mathcal{B}(\tau^*)}\frac{dX_{1i}}{d\tau}(\tau^*)\,dF(i)}}_{=\frac{\Omega^r_{\mathcal{B}(\tau^*)}-\Omega^r_{\mathcal{S}(\tau^*)}}{2}} +$$

$$+\underbrace{\frac{\int_{i\in\mathcal{B}^f(\tau^*)}\frac{dY_{1i}}{d\tau}(\tau^*)\,dF(i)}{\int_{i\in\mathcal{B}(\tau^*)}\frac{dX_{1i}}{d\tau}(\tau^*)\,dF(i)}}_{=\theta(\tau^*)}\underbrace{\frac{\int\left(\mathbb{E}_i\left[m_i(\tau^*)R(\tau^*)\right] - \mathbb{E}_p\left[m_i(\tau^*)R(\tau^*)\right]\right)\frac{dY_{1i}}{d\tau}(\tau^*)\,dF(i)}{2\int_{i\in\mathcal{B}^f(\tau^*)}\frac{dY_{1i}}{d\tau}(\tau^*)\,dF(i)}}_{\frac{\Omega^f_{\mathcal{B}(\tau^*)}-\Omega^f_{\mathcal{S}(\tau^*)}}{2}}.$$

Exploiting market clearing for both the risky and the safe asset, we can therefore define

$$\Omega^r_{\mathcal{B}(\tau^*)} \equiv \int_{i\in\mathcal{B}(\tau^*)}\left(\mathbb{E}_i\left[m_i(\tau^*)\frac{D}{P_1(\tau^*)}\right] - \mathbb{E}_p\left[m_i(\tau^*)\frac{D}{P_1(\tau^*)}\right]\right)\underbrace{\frac{\frac{dX_{1i}}{d\tau}(\tau^*)}{\int_{i\in\mathcal{B}(\tau^*)}\frac{dX_{1i}}{d\tau}(\tau^*)\,dF(i)}}_{=\omega_i^{\mathcal{B}}(\tau^*)}dF(i)$$

$$\Omega^r_{\mathcal{S}(\tau^*)} \equiv \int_{i\in\mathcal{S}(\tau)}\left(\mathbb{E}_i\left[m_i(\tau^*)\frac{D}{P_1(\tau^*)}\right] - \mathbb{E}_p\left[m_i(\tau^*)\frac{D}{P_1(\tau^*)}\right]\right)\underbrace{\frac{\frac{dX_{1i}}{d\tau}(\tau^*)}{\int_{i\in\mathcal{S}(\tau^*)}\frac{dX_{1i}}{d\tau}(\tau^*)\,dF(i)}}_{=\omega_i^{\mathcal{S}}(\tau^*)}dF(i).$$

Similarly, we can define

$$\Omega^f_{\mathcal{B}(\tau^*)} \equiv \int_{i\in\mathcal{B}^f(\tau^*)}\left(\left(\mathbb{E}_i\left[m_i(\tau^*)R(\tau^*)\right] - \mathbb{E}_p\left[m_i(\tau^*)R(\tau^*)\right]\right)\right)\underbrace{\frac{\frac{dY_{1i}}{d\tau}(\tau^*)}{\int_{i\in\mathcal{B}^f(\tau^*)}\frac{dY_{1i}}{d\tau}(\tau^*)\,dF(i)}}_{=\omega_i^{\mathcal{B}f}(\tau^*)}dF(i)$$

$$\Omega^f_{\mathcal{S}(\tau^*)} \equiv \int_{i\in\mathcal{S}^f(\tau^*)}\left(\mathbb{E}_i\left[m_i(\tau^*)R(\tau^*)\right] - \mathbb{E}_p\left[m_i(\tau^*)R(\tau^*)\right]\right)\underbrace{\frac{\frac{dY_{1i}}{d\tau}(\tau^*)}{\int_{i\in\mathcal{S}^f(\tau^*)}\frac{dY_{1i}}{d\tau}(\tau^*)\,dF(i)}}_{=\omega_i^{\mathcal{S}f}(\tau^*)}dF(i),$$

which shows the result. Note that $\theta(\tau^*)$ corresponds to the differential in volume sensitivities to tax changes between the risky asset and risk-free asset, that is

$$\theta(\tau^*) = \frac{\frac{d\mathcal{V}^s(\tau^*)}{d\tau}}{\frac{d\mathcal{V}^r(\tau^*)}{d\tau}},$$

where $\mathcal{V}^r(\tau^*) = \int_{i\in\mathcal{B}^r(\tau^*)}\Delta X_{1i}(\tau^*)\,dF(i)$ and $\mathcal{V}^s(\tau^*) = \int_{i\in\mathcal{B}^f(\tau^*)}\Delta Y_{1i}(\tau^*)\,dF(i)$. □

Investors' demand/volume approximation Starting from Equation (22), it is possible to recover investors' demand in the baseline model, after approximating investors' date 2 marginal utility around its mean. Formally, note that we can rewrite Equation (22) as follows

$$\begin{aligned} P_1R(1+\operatorname{sgn}(\Delta X_{1i})\tau) &= \mathbb{E}_i\left[\frac{U_i'(W_{2i})}{\mathbb{E}_i\left[U_i'(W_{2i})\right]}D\right] \\ &= \mathbb{E}_i[D] + \mathbb{C}ov\left[\frac{U_i'(W_{2i})}{\mathbb{E}_i\left[U_i'(W_{2i})\right]}, D\right]. \end{aligned} \tag{56}$$

A first-order approximation of $U_i'(W_{2i})$ around its mean implies that

$$U_i'(W_{2i}) \approx U_i'(\mathbb{E}[W_{2i}]) + U_i''(\mathbb{E}[W_{2i}])(W_{2i} - \mathbb{E}[W_{2i}]),$$

which at the same time implies that

$$\frac{U_i'(W_{2i})}{\mathbb{E}_i[U_i'(W_{2i})]} \approx 1 + \frac{U_i''(\mathbb{E}[W_{2i}])}{U_i'(\mathbb{E}[W_{2i}])}(W_{2i} - \mathbb{E}[W_{2i}]) \approx 1 - A_i(W_{2i} - \mathbb{E}[W_{2i}]),$$

where $A_i = -\frac{U_i''(\mathbb{E}[W_{2i}])}{U_i'(\mathbb{E}[W_{2i}])}$. Consequently, we can express $\mathbb{C}ov\left[\frac{U_i'(W_{2i})}{\mathbb{E}_i[U_i'(W_{2i})]}, D\right]$ as follows

$$\begin{aligned}\mathbb{C}ov\left[\frac{U_i'(W_{2i})}{\mathbb{E}_i[U_i'(W_{2i})]}, D\right] &\approx \mathbb{C}ov[1 - A_i(W_{2i} - \mathbb{E}[W_{2i}]), D] \\ &\approx -A_i(\mathbb{C}ov[M_{2i}, D] + X_{1i}\mathbb{V}ar[D]).\end{aligned}$$

Substituting into Equation (56), it immediately follows that

$$X_{1i} \approx \frac{\mathbb{E}_i[D] - A_i\mathbb{C}ov[M_{2i}, D] - P_1R(1 + \text{sgn}(\Delta X_{1i})\tau)}{A_i\mathbb{V}ar[D]},$$

which corresponds to Equation (5) in the text. Consequently, thee volume decomposition in Proposition 2 also remains valid in approximate terms.

E Additional results

In this section, I provide extended remarks on several important issues that are only briefly mentioned in the body of the paper.

E.1 Extended comparison to alternative criteria

It is easier to illustrate the comparison between the welfare criterion in this paper and those of Gilboa, Samuelson and Schmeidler (2014) and Brunnermeier, Simsek and Xiong (2014) by exploring two different scenarios. Both scenarios can be seen as the following special case of the general model studied in the paper.

Environment Consider a environment in which there are two (types of) investors, indexed by $i = \{A, B\}$ — there should be no confusion between investor's A index and the risk aversion coefficient. All investors have identical risk aversion, so $A_i = A$, $\forall i$, and start with the same asset holdings, $X_{0i} = Q$, $\forall i$. Also, $M_{1i} = 0$ and $\mathbb{E}[M_{2i}] - \frac{A_i}{2}\mathbb{V}ar[M_{2i}] = 0$, $\forall i$. Investors' beliefs and hedging motives respectively correspond to

$$\begin{aligned}\mathbb{E}_A[D] &= \mu_d + \varepsilon_d \quad \text{and} \quad A\mathbb{C}ov[M_{2A}, D] = \varepsilon_h \\ \mathbb{E}_B[D] &= \mu_d - \varepsilon_d \quad \text{and} \quad A\mathbb{C}ov[M_{2B}, D] = -\varepsilon_h.\end{aligned}$$

Hence, when $\varepsilon_d \geq 0$, as assumed in both scenarios, A investors will be relatively more optimistic than B investors, and would demand more of the risky asset. When $\varepsilon_h \leq 0$, A investors also have a higher demand for the risky asset relative to B investors, but this time because of hedging reasons. As in the body of the paper, I measure investors' welfare through their certainty equivalents, given by $\mathbb{E}[W_{2i}] - \frac{A}{2}\mathbb{V}ar[W_{2i}]$. In this environment, the certainty equivalents from the planner's perspective and the investors' perspective are respectively given by

$$\begin{aligned}V_i^p &= (\mathbb{E}_p[D] - A\mathbb{C}ov[M_{2i}, D])X_{1i} - \frac{A}{2}\mathbb{V}ar[D](X_{1i})^2 \\ V_i^i &= (\mathbb{E}_i[D] - A\mathbb{C}ov[M_{2i}, D])X_{1i} - \frac{A}{2}\mathbb{V}ar[D](X_{1i})^2.\end{aligned}$$

Figure A.6 shows the marginal welfare impact of an increase in X_{1A} in both cases, given by

$$\frac{dV_i^p}{dX_{1i}} = \mathbb{E}_p[D] - A\mathbb{C}ov[M_{2i}, D] - A\mathbb{V}ar[D] X_{1i} \tag{57}$$

$$\frac{dV_i^i}{dX_{1i}} = \mathbb{E}_i[D] - A\mathbb{C}ov[M_{2i}, D] - A\mathbb{V}ar[D] X_{1i}. \tag{58}$$

The three allocations worth discussing are summarized in Table 1. First, the no-trade allocation, characterized by $X_{1A} = X_{1B} = Q$. Second, the laissez-faire competitive equilibrium allocation, characterized by $X_{1A} = Q + \frac{\varepsilon_d - \varepsilon_h}{A\mathbb{V}ar[D]}$ and $X_{1B} = Q + \frac{-\varepsilon_d + \varepsilon_h}{A\mathbb{V}ar[D]}$. Finally, the competitive equilibrium under the optimal tax τ^*, characterized by $X_{1A} = Q + \frac{-\varepsilon_h}{A\mathbb{V}ar[D]}$ and $X_{1B} = Q + \frac{\varepsilon_h}{A\mathbb{V}ar[D]}$. In this last case, the optimal tax corresponds to $\tau^* = \frac{\varepsilon_d}{P_1}$. In any competitive equilibrium, the equilibrium price for any tax rate is given by $P_1 = \mu_d - A\mathbb{V}ar[D] Q$.

Allocations	Risky Asset Holdings X_{1A}	Risky Asset Holdings X_{1B}
No-trade	Q	Q
Competitive Equilibrium (laissez-faire)	$Q + \frac{\varepsilon_d - \varepsilon_h}{A\mathbb{V}ar[D]}$	$Q + \frac{-\varepsilon_d + \varepsilon_h}{A\mathbb{V}ar[D]}$
Competitive Equilibrium ($\tau = \tau^*$)	$Q + \frac{-\varepsilon_h}{A\mathbb{V}ar[D]}$	$Q + \frac{\varepsilon_h}{A\mathbb{V}ar[D]}$

Table 1: Relevant allocations

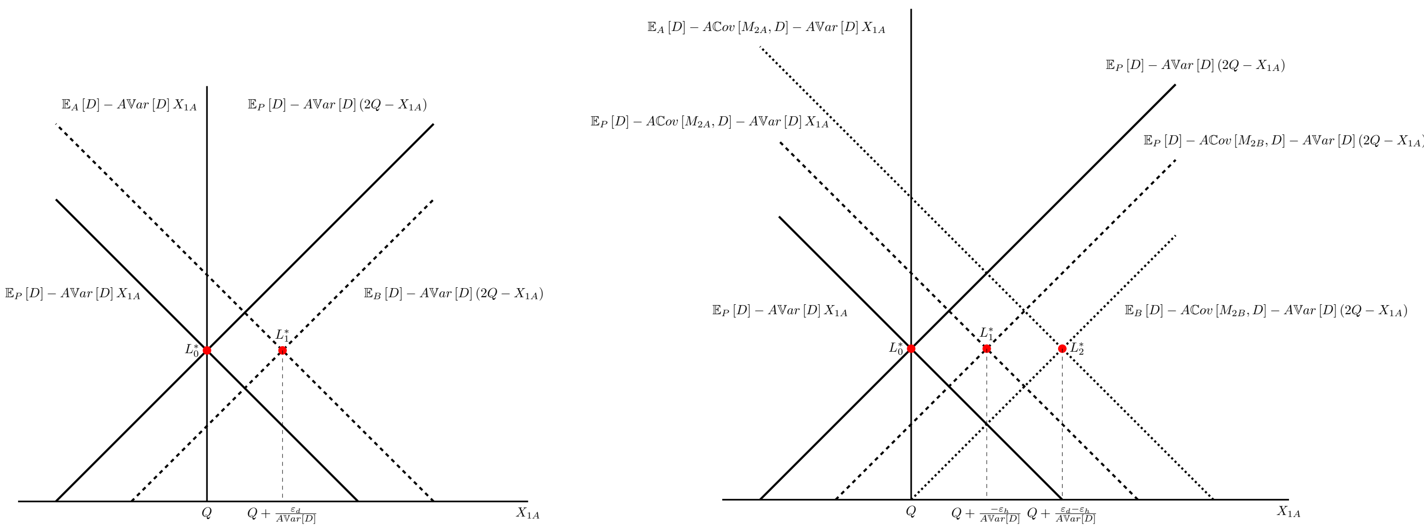
The first scenario considered here (pure betting) exclusively features non-fundamental trading. The second scenario has both fundamental and non-fundamental trading.

E.1.1 Pure betting scenario ($\varepsilon_d \geq 0$, $\varepsilon_h = 0$)

In the pure betting scenario, consistent with Lemma 4, the optimal tax implements the no-trade allocation. In this case, the no-betting Pareto criterion of Gilboa, Samuelson and Schmeidler (2014) does not rank the no-trade/optimal tax allocation relative to the laissez-faire allocation. It is evident that both investors prefer the laissez-faire allocation (given by the intersection point L_1^* in Figure A.6a) to the no-trade/optimal tax allocation (given by the intersection point L_0^* in Figure A.6a) when computing welfare using their own beliefs — this follows immediately by invoking the First Welfare Theorem in this context. However, for any single belief welfare assessment, investors prefer the no-trade allocation to the laissez-faire allocation. Therefore, the laissez-faire allocation fails to no-betting Pareto dominate the no-trade allocation, since it fails the second requirement of that criterion. At the same time, the no-trade/optimal tax allocation fails to no-betting Pareto dominate the laissez-faire allocation, since it fails the first requirement of that criterion.

In this case, using the criterion of Brunnermeier, Simsek and Xiong (2014), the optimal tax allocation is the best belief-neutral efficient allocation within the set of competitive equilibria with transfers for a specific set of welfare weights. In this particular scenario, the no-trade allocation is also a belief-neutral Pareto Efficient allocation, since it can be found as the solution to a planning problem. The latter is not a general result. If we included some investors who only trade for hedging reasons, the optimal-tax allocation would fail to be belief-neutral Pareto Efficient, although it would still select the best belief-neutral efficient allocation within the set of competitive equilibria with transfers for a specific set of welfare weights.

Finally, because single-belief welfare assessments are independent of the belief chosen, allocations that satisfy the no-betting Pareto dominance criterion also satisfy the Unanimity Pareto criterion proposed by Gayer et al. (2014), since these assessments are valid in particular for the beliefs of each of the investors in the economy. This remark also applies to the scenario discussed next.



(a) Pure betting scenario

(b) Fundamental/Non-fundamental trading scenario

Figure A.6: Extended comparison with GSS and BSX

Note: Figure A.6 shows the marginal welfare impact of increasing X_{1A} for both A (downward sloping lines) and B investors (upward sloping lines) from the perspective of the planner (solid lines) and from the perspective of each individual set of investors (dashed and dotted lines). Figure A.6a illustrates the pure betting scenario ($\varepsilon_d > 0, \varepsilon_h = 0$). Figure A.6b illustrates the alternative scenario in which $\varepsilon_d \geq 0, \varepsilon_h \leq 0$. The intersection of the solid lines corresponds to the solution of the planner's problem. The intersection of the dashed lines corresponds to the laissez-faire equilibrium, which in this case can be computed as the solution to a planning problem in which investors' beliefs are respected. The intersection of the dotted lines in Figure A.6b corresponds to the allocation under the optimal tax. Both plots are drawn under the assumption that $\mathbb{E}_p[D] = \mu_d$, but note that varying $\mathbb{E}_p[D]$ only shifts the curves vertically, which has no impact on the ranking of allocations.

E.1.2 Fundamental/Non-fundamental trading scenario ($\varepsilon_d \geq 0$, $\varepsilon_h \leq 0$)

In this scenario, trading volume is still positive under the optimal tax. The optimal tax allocation (given by the intersection point L_1^* in Figure A.6b) fails to no-betting Pareto dominate the laissez-faire equilibrium allocation (given by the intersection point L_2^* in Figure A.6b) and vice versa, following a similar logic to the one described in the pure-betting case. Investors prefer the laissez-faire allocation using their own beliefs, but there is no single belief assessment that prefers the laissez-faire allocation to the optimal tax allocation. However, in this scenario, the optimal tax allocation will no-betting Pareto dominate the no-trade allocation, since the two conditions of the criterion are satisfied. First, any single-belief assessment prefers the optimal tax allocation to no-trade. Second, all investors prefer the optimal tax allocation to no trading using their own beliefs.

As in the previous case, the optimal tax allocation is the best belief-neutral efficient allocation within the set of competitive equilibria with tax for some welfare weights and also characterizes a belief-neutral Pareto Efficient allocation. As discussed above, the optimal tax allocation would fail to be belief-neutral Pareto Efficient if we introduce any additional heterogeneity.

Even though I have chosen these two scenarios to illustrate the relation between the criterion used in the paper with the criteria of Gilboa, Samuelson and Schmeidler (2014) and Brunnermeier, Simsek and Xiong (2014), the Pareto notions in both papers will in general fail to be able to rank the optimal tax allocation characterized in this paper once there is rich heterogeneity among investors. In particular, different investors with a different fundamental/non-fundamental trading motive mix will disagree on the desirability of different tax levels. Similarly, including a set of optimistic sellers/pessimistic buyers will immediately generate winners and losers from any policy.

E.2 Transaction taxes vs. transaction subsidies

It is worth discussing two different issues that emerge when investors face transaction subsidies instead of taxes. First, I compare the properties of an investor's portfolio problem when they face a positive tax ($\tau > 0$) versus a subsidy ($\tau < 0$). Next, I show that trading subsidies can be implemented when paid on the net change of asset holdings over a given period, but cannot be implemented when paid on every purchase or sale.

Figure A.7 illustrates the objective function of an investor, defined in Equation (23), for different values of the tax/subsidy: $\tau = \{-0.4, 0, 0.4\}$. As formally shown above, exploiting Equations (24) and (25), the problem faced by investors features a concave kink when $\tau > 0$ and a convex kink when $\tau < 0$. When $\tau > 0$, investors may find optimal not to trade for some primitives, as in the case considered in the figure. When $\tau < 0$, investors always find optimal to trade for any set of primitives.

Next, I study the implementation problems that may arise under specific forms of trading subsidies. Throughout the paper, as implied by Equation (3), the tax/subsidy base is the change in an investor's net asset position between dates 0 and 1, given by $|\Delta X_{1i}| = |X_{1i} - X_{0i}|$. Now, I assume instead that the tax/subsidy base is given by the sum of net purchases and net sales of the risky asset. Formally, investor i's budget/wealth accumulation constraint is given by

$$W_{2i} = N_{2i} + X_{1i}D + \left(X_{0i}P_1 - X_{1i}P_1 - \tau P_1 \left(B_{1i} + S_{1i}\right) + T_{1i}\right),$$

where $\Delta X_{1i} = X_{1i} - X_{0i} = B_{1i} - S_{1i}$, and $B_{1i} \geq 0$ and $S_{1i} \geq 0$. That is, an investor who starts with x_0 shares, buys x_b shares and sells x_s shares, ends up with $x_1 = x_0 + x_b - x_s$ shares, so the tax/subsidy under the new assumption is $\tau P_1 \left(|x_b| + |x_s|\right)$ dollars, while the tax/subsidy under the assumption sustained in most of the paper is instead $\tau P_1 \left|x_b - x_s\right|$ dollars (equivalently, $\tau P_1 \left|x_1 - x_0\right|$).

Formally, an investor now chooses X_{1i}, B_{1i}, and S_{1i} in order to maximize

$$\begin{aligned} J\left(X_{1i}\right) = &\left[\mathbb{E}_i\left[D\right] - A_i \mathbb{C}ov\left[M_{2i}, D\right] - P_1\right] X_{1i} - \tau P_1 \left(|B_{1i}| + |S_{1i}|\right) - \frac{A_i}{2} \mathbb{V}ar\left[D\right] X_{1i}^2 \\ &- \eta\left(X_{1i} - X_{0i} - B_{1i} + S_{1i}\right) + \eta^B B_{1i} + \eta^S S_{1i}, \end{aligned}$$

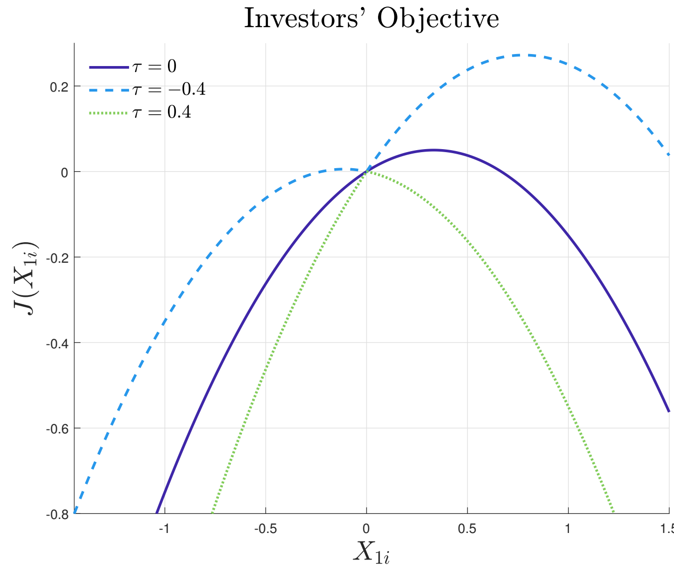


Figure A.7: Investors' objective for $\tau \gtreqless 0$

Note: Figure A.7 illustrates the objective function of an investor, $J(X_{1i})$, defined in Equation (23), for different values of the tax/subsidy: $\tau = \{-0.4, 0, 0.4\}$. The parameters used to draw these plots are $X_{0i} = 0$, $\mathbb{E}_i[D] = 1.3$, $\mathbb{V}ar[D] = 0.9$, $\mathbb{C}ov[M_{2i}, D] = 0$, $A = 1$, $P_1 = 1$, and $T_{1i} = 0$. These values are chosen to highlight the shape of the investors' objective, and are unrelated to those used for the quantitative exercise in Section 5 of the paper.

where $\eta \gtreqless 0$ denotes the Lagrange multiplier on the constraint $X_{1i} - X_{0i} = B_{1i} - S_{1i}$, and $\eta^B \geq 0$ and $\eta^S \geq 0$ respectively denote the Lagrange multipliers on the non-negativity constraints on buying and selling choices.

The first-order conditions to this problem are given by

$$\frac{\partial J}{\partial X_{1i}} = \mathbb{E}_i[D] - A_i \mathbb{C}ov[M_{2i}, D] - P_1 - A_i \mathbb{V}ar[D] X_{1i} - \eta = 0$$
$$\frac{\partial J}{\partial B_{1i}} = -\tau P_1 + \eta + \eta^B = 0$$
$$\frac{\partial J}{\partial S_{1i}} = -\tau P_1 - \eta + \eta^S = 0.$$

Note that $\frac{\partial^2 J}{\partial X_{1i}^2} = -A_i \mathbb{V}ar[D] < 0$, guaranteeing a well defined interior optimum for X_{1i}, but also that the problem is linear in B_{1i} and S_{1i}. Let's consider first the case in which $\tau > 0$. In that case, we can add the first-order conditions for B_{1i} and S_{1i}, which imply that $\eta^S + \eta^B = 2\tau P_1$. Therefore, at least one of the non-negativity constraints is binding at an optimum. Let's assume that it is optimal for an investor to be a buyer, in that case, $\eta^S > 0$ and $\eta^B = 0$, which implies that $X_{1i} - X_{0i} = B_{1i}$ and that X_{1i} is given by

$$\mathbb{E}_i[D] - A_i \mathbb{C}ov[M_{2i}, D] - P_1 - \tau P_1 - A_i \mathbb{V}ar[D] X_{1i} = 0,$$

exactly as implied by Equation (5). A parallel argument can be made when $\eta^S > 0$ and $\eta^B = 0$.

Let's consider next the more interesting subsidy case, in which $\tau < 0$. In that case, one can show that a perturbation in which an investor buys and sells a unit of the asset (often called a "wash trade") increases the investor objective by

$$\frac{\partial J}{\partial B_{1i}} + \frac{\partial J}{\partial S_{1i}} = -2\tau P_1 + \eta^B + \eta^S > 0,$$

which is a strictly positive value whenever $B_{1i} > 0$ and $S_{1i} > 0$. Consequently, we have found a feasible perturbation that increases the objective function everywhere, so an investor finds optimal to choose $B_{1i} = S_{1i} = \infty$, which achieves a value of ∞ and rules out the existence of an equilibrium. At the same time, there is a well defined interior solution for X_{1i}. Note that there are two interior optimum candidates for X_{1i} (as illustrated by the dashed line in Figure A.7), one in which $\frac{\partial J}{\partial X_{1i}} = 0$ is solved with $\eta = \tau P_1$, and another one in which $\eta = -\tau P_1$. Intuitively, under the new formulation for the tax/subsidy base, even though the size of net trades ΔX_{1i} is pinned down by the same forces as when the tax/subsidy base are net trades, investors can engage on "wash trades" to obtain an infinite profit by trading. These results show that the model predicts that all investors would execute wash trades, regardless of whether their trading motives are fundamental or non-fundamental.

E.3 Marginal welfare impact when $\mathbb{E}_p[D] \neq \mu_d$

Figure A.8 shows the normalized individual welfare impact of a tax change when the planner's mean belief $\mathbb{E}_p[D]$ is different from the average mean belief of investors μ_d. Using the same parameterization as in Figure 2, only the set of left plots, which show the normalized individual welfare impact from the planner's perspective $\frac{dV_i^p}{d\tau}$, change. Formally, note that we can express $\frac{\frac{dV_i^p}{d\tau}}{P_1 Q}$ as follows

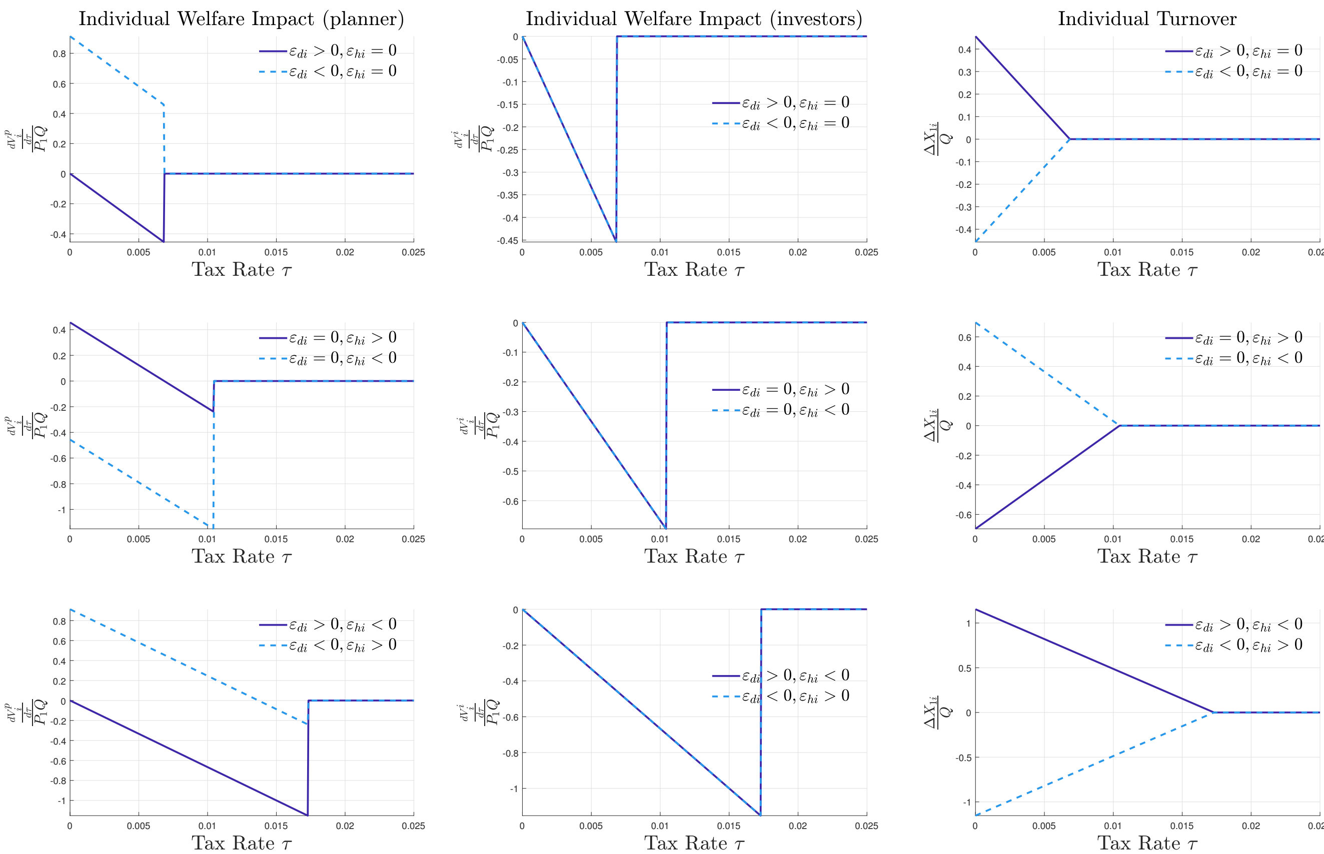


Figure A.8: Marginal welfare impact when $\mathbb{E}_p[D] \neq \mu_d$

Note: Figure A.8 is the counterpart of Figure 2 when assuming that the planner's belief is different from the average belief, that is, $\mathbb{E}_p[D] \neq \mu_d$. Specifically, Figure A.8 assumes that the price-normalized planner's belief $\frac{\mathbb{E}_p[D]}{P_1}$ is one standard deviation above the average on the distribution of investors. Figure 3 applies unchanged in this scenario. Note that the middle and right plots are identical to those in Figure 2.
The top row plots show a buyer ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} = 0$) and a seller ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} = 0$) who only trade for non-fundamental reasons. The middle row plots show a buyer ($\varepsilon_{hi} = 0$ and $\varepsilon_{di} < 0$) and a seller ($\varepsilon_{hi} = 0$ and $\varepsilon_{di} > 0$) who only trade for fundamental reasons. The bottom plots show a buyer ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} < 0$) who buys for non-fundamental and fundamental reasons and a seller ($\varepsilon_{hi} < 0$ and $\varepsilon_{di} > 0$) who sells for non-fundamental and fundamental reasons. The values of ε_{di} and ε_{hi} correspond to one standard deviation of the distributions of $\frac{\varepsilon_{di}}{P_1}$ and $\frac{\varepsilon_{hi}}{P_1}$, respectively. All plots in Figure A.8 use the baseline calibration from Section 5.

$$\frac{\frac{dV_i^p}{d\tau}}{P_1 Q} = \left[\frac{\mathbb{E}_p\left[D\right] - \mu_d + \overbrace{\mu_d - \mathbb{E}_i\left[D\right]}^{-\varepsilon_{di}}}{P_1} + \operatorname{sgn}\left(\Delta X_{1i}\right)\tau \right] \frac{\frac{dX_{1i}}{d\tau}}{Q} + \frac{\frac{d\tilde{T}_{1i}}{d\tau}}{P_1 Q}, \tag{59}$$

where $\frac{d\tilde{T}_{1i}}{d\tau} = \frac{dT_{1i}}{d\tau} - \frac{d(\tau P_1|\Delta X_{1i}|)}{d\tau}$ is set to zero in both Figures 2 and A.8. Since Figure A.8 assumes that $\frac{\mathbb{E}_p[D]}{P_1} - \frac{\mu_d}{P_1} > 0$, Equation (59) clearly shows why the planner perceives a systematically higher positive welfare impact of a tax change for sellers than for buyers. This occurs because the term $\left[\frac{\mathbb{E}_p[D]-\mu_d}{P_1}\right]\frac{\frac{dX_{1i}}{d\tau}}{Q}$ takes negative values for buyers, but positive values for sellers. Importantly, the aggregate welfare impact of a tax change, which still corresponds to the one shown in Figure 3, and consequently the optimal tax, are invariant to the level of the planner's mean belief, as shown in Lemma 2 and Proposition 1.

E.4 Uniform rebate rule

In order to clearly illustrate how taxes impact investors' welfare through changes in equilibrium allocations, the illustration of investors' individual marginal welfare impact in Figure 2 and others in this paper assume that the planner follows an individually-targeted rebate rule. Under that rule, the net rebate received by each investor for every tax rate τ is 0, which implies that $\frac{d\tilde{T}_{1i}}{d\tau} = 0$.

In this section, I assume that the planner follows a uniform rebate rule, in which every investor receives the same rebate, given by the average tax payment among all investors. Formally, the rebate received by an investor i is given by $T_{1i} = \frac{\int \tau P_1|\Delta X_{1i}|dF(i)}{\int dF(i)}$, which only depends on aggregate variables. It is evident that this revenue rule is revenue neutral in the aggregate, since $\int \tilde{T}_{1i} dF\left(i\right) = 0$. Consequently, the net rebate received by an investor i under a uniform rebate rule can be expressed as follows

$$\tilde{T}_{1i} = \frac{\int \tau P_1 \left|\Delta X_{1i}\right| dF\left(i\right)}{\int dF\left(i\right)} - \tau P_1 \left|\Delta X_{1i}\right|. \tag{60}$$

Note that the individual net rebate and the marginal impact of a tax change of an investor's net rebate can respectively be expressed, once normalized by the value of risky asset, as follows

$$\frac{\tilde{T}_{1i}}{P_1 Q} = \tau\left(2\frac{\mathcal{V}\left(\tau\right)}{Q} - \frac{\left|\Delta X_{1i}\right|}{Q}\right) = \tau\left(2\Xi\left(\tau\right) - \frac{\left|\Delta X_{1i}\right|}{Q}\right) \tag{61}$$

$$\frac{\frac{d\tilde{T}_{1i}}{d\tau}}{P_1 Q} = 2\left(\Xi\left(\tau\right) + \tau\frac{\frac{d\mathcal{V}}{d\tau}}{Q}\right) - \frac{\left|\Delta X_{1i}\right| + \tau\frac{d\left|\Delta X_{1i}\right|}{d\tau}}{Q}, \tag{62}$$

where the elements of the marginal impact on the rebate can be expressed as

$$\Xi\left(\tau\right) = \frac{\mathcal{V}\left(\tau\right)}{Q} = \left(1 - \Phi\left(\alpha_+\right)\right)\left(-\frac{\tau}{\Pi}\right) + \Xi\left(0\right)\sqrt{2\pi}\phi\left(\alpha_+\right)$$

$$\tau\frac{\frac{d\mathcal{V}}{d\tau}}{Q} = -\frac{\tau}{\Pi}\left(1 - \Phi\left(\alpha_+\right)\right),$$

while the elements of the marginal impact on investor i's tax liability are given by

$$\frac{\left|\Delta X_{1i}\right|}{Q} = \left|\frac{1}{\Pi}\left(\frac{\varepsilon_{di} - \varepsilon_{hi}}{P_1} - \operatorname{sgn}\left(\Delta X_{1i}\right)\tau\right)\right| \text{if } i \text{ is a buyer or seller}$$

$$\frac{\frac{d\left|\Delta X_{1i}\right|}{d\tau}}{Q} = \begin{cases} 0, & \text{if } i \text{ is inactive} \\ -\frac{1}{\Pi}, & \text{if } i \text{ is a buyer or seller.} \end{cases}$$

Figures A.9 and A.10 illustrate how using a uniform rebate rule changes the individual welfare impact of tax changes. The rows in Figures A.9 and A.10b parallel those in Figure 2. It's perhaps easiest to understand first the last column in Figure A.9. When $\tau = 0$, by construction the net rebate for all investors is also 0. When τ increases,

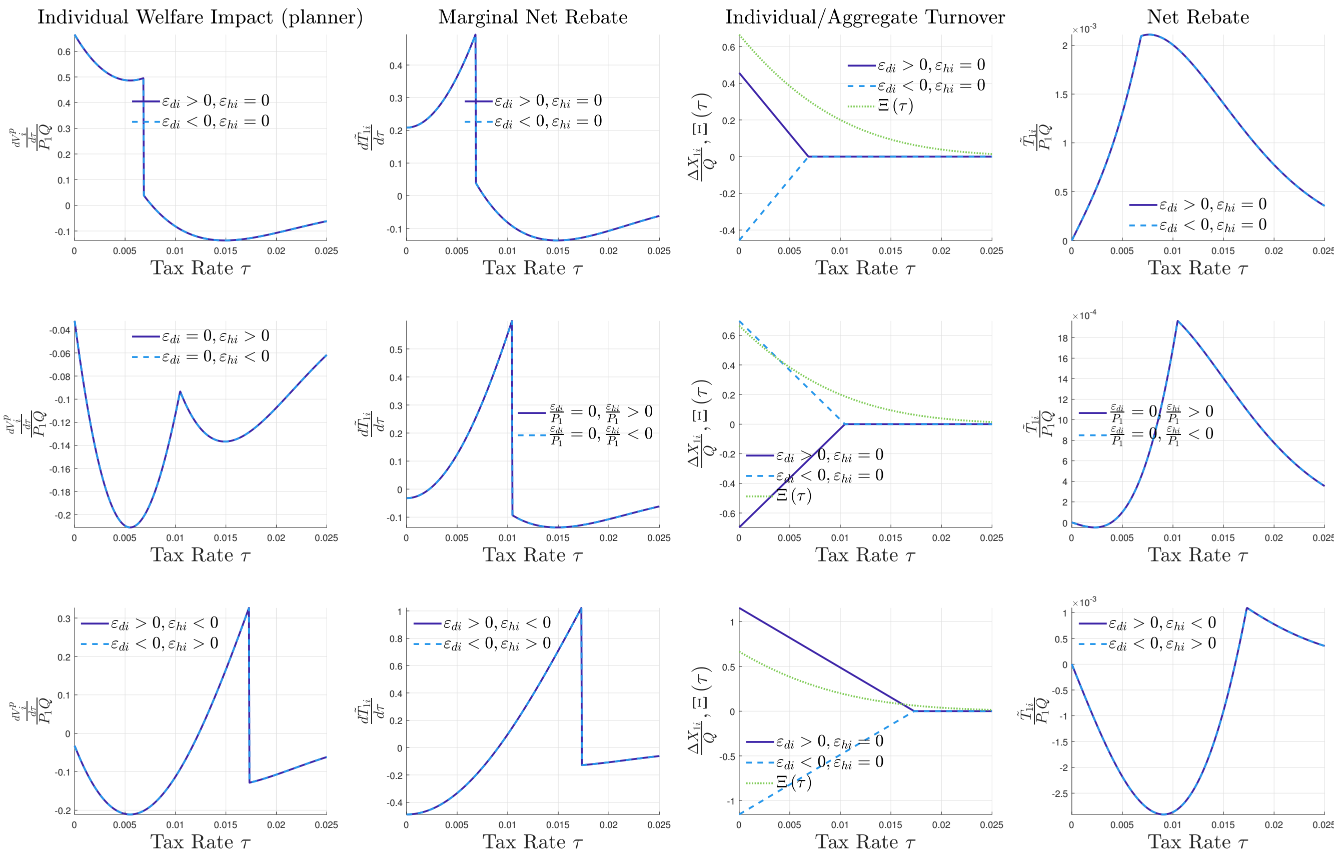
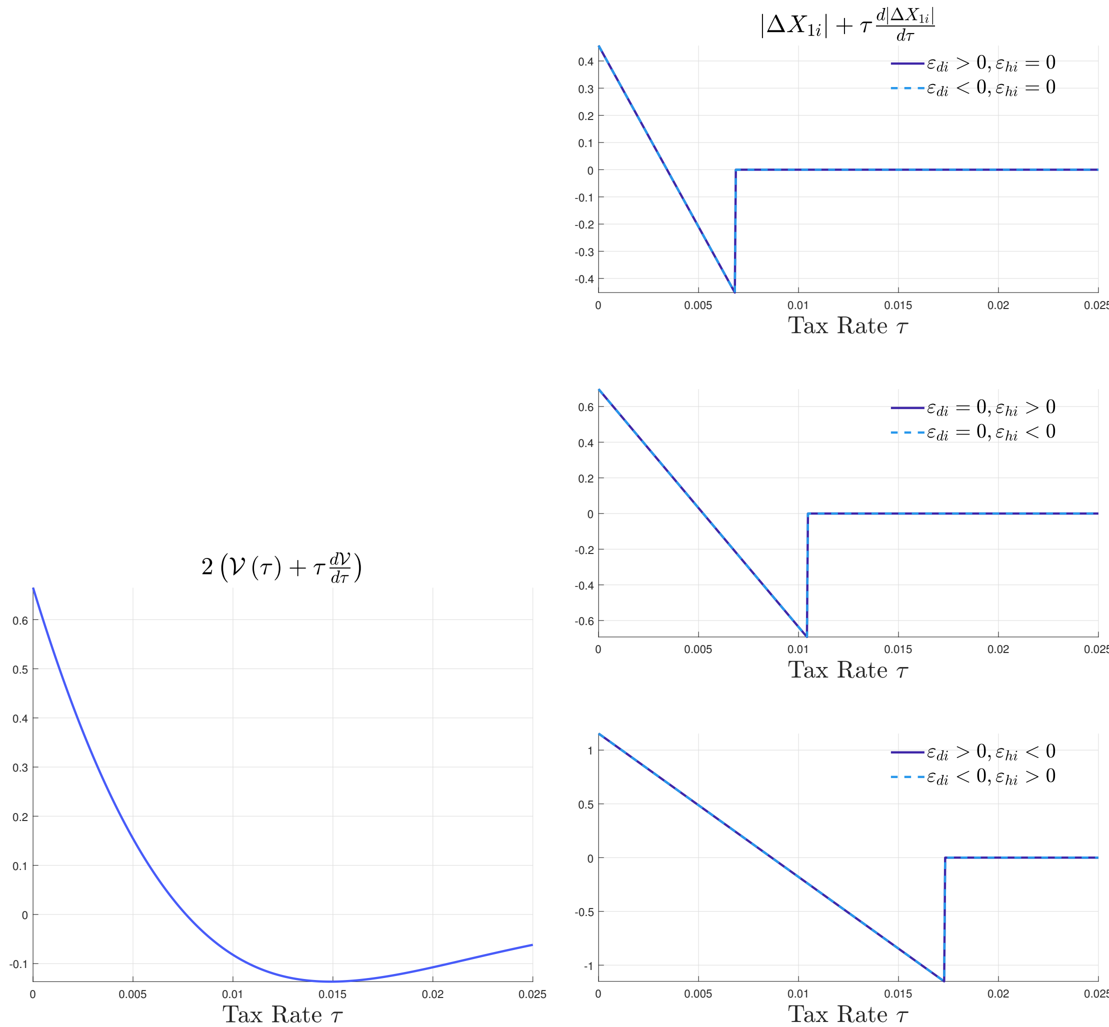


Figure A.9: Uniform rebate rule

Note: Figure A.9 is the counterpart of Figure 2 when instead of assuming an individually-targeted rebate rule that fully offsets the tax liability ($\frac{d\tilde{T}_{1i}}{d\tau} = 0$), it assumes a uniform rebate rule that distributes tax revenue equally among all investors, including those who do not trade, as described in Equation (60). The left plots show Equation (59), and the middle-left plots show Equation (62). The middle-right plots show individual and aggregate turnover, as defined in Equations (36) and (48), while the right plots show Equation (61).

The top row plots show a buyer ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} = 0$) and a seller ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} = 0$) who only trade for non-fundamental reasons. The middle row plots show a buyer ($\varepsilon_{hi} = 0$ and $\varepsilon_{di} < 0$) and a seller ($\varepsilon_{hi} = 0$ and $\varepsilon_{di} > 0$) who only trade for fundamental reasons. The bottom plots show a buyer ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} < 0$) who buys for non-fundamental and fundamental reasons and a seller ($\varepsilon_{hi} < 0$ and $\varepsilon_{di} > 0$) who sells for non-fundamental and fundamental reasons. The values of ε_{di} and ε_{hi} correspond to one standard deviation of the distributions of $\frac{\varepsilon_{di}}{P_1}$ and $\frac{\varepsilon_{hi}}{P_1}$, respectively. All plots in Figure A.9 use the baseline calibration from Section 5.



(a) Marginal average transfer impact

(b) Marginal individual tax impact

Figure A.10: Marginal net rebate decomposition

Note: Figures A.10a and A.10b show the components of Equation (62). Figure A.10a shows the marginal impact on the investors' rebate of an increase in the tax rate. This plot effectively traces an aggregate revenue Laffer curve. Figure A.10b shows the marginal impact on investors' tax liability of an increase in the tax rate. These plots effectively trace individual revenue Laffer curves.

The top row plots of Figure A.10b show a buyer ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} = 0$) and a seller ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} = 0$) who only trade for non-fundamental reasons. The middle row plots show a buyer ($\varepsilon_{hi} = 0$ and $\varepsilon_{di} < 0$) and a seller ($\varepsilon_{hi} = 0$ and $\varepsilon_{di} > 0$) who only trade for fundamental reasons. The bottom plots show a buyer ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} < 0$) who buys for non-fundamental and fundamental reasons and a seller ($\varepsilon_{hi} < 0$ and $\varepsilon_{di} > 0$) who sells for non-fundamental and fundamental reasons. The values of ε_{di} and ε_{hi} correspond to one standard deviation of the distributions of $\frac{\varepsilon_{di}}{P_1}$ and $\frac{\varepsilon_{hi}}{P_1}$, respectively. All plots in Figure A.10 use the baseline calibration from Section 5.

the net transfer received by each agent depends on his position in the distribution of trades. For instance, the transfer for the bottom row investors decreases with the tax rate for a while, since their trading position gets further away from the average trade initially. When the tax is sufficiently large, their net trading position becomes closer to the average, making their net rebate even positive. After they stop trading, the individual net rebate received by each investor is simply decreasing on the total amount of revenue raised. The second column of these plots represents these effects in marginal terms, while the first column of these plots simply combines the first column of plots in Figure 2 with the second column of plots in this Figure A.9.

Figure A.10b separately illustrates the two components of Equation (62). Figure A.10b shows the marginal impact of an increase in the tax rate on an individual investor's rebate. Intuitively, the average tax payment initially increases with the tax rate up to a point, in which higher taxes reduce total revenue, and consequently the rebates for each investor. Figure A.10a shows the marginal impact of an increase in the tax rate on an individual investor's tax liability. An initial increase in the tax rate τ increases the investors' tax liability, but further increases in τ will reduce the total tax liability, once investors trade less and less until becoming inactive, defining an individual Laffer curve.

E.5 Equilibrium price changes

While the results in Section 5 have been derived in a symmetric environment in which Assumption [S] holds, a change in taxes may have an impact on asset prices, as shown in Lemma 1. These price movements may differentially affect buyers and sellers, as well as the optimal tax. Figure A.11 illustrates such effects in an asymmetric environment with three groups of investors. Figure A.11 is designed to show how $\frac{dP_1}{d\tau}$ can both be increasing and decreasing in the tax rate for different values of τ depending on how the composition of investors varies with the tax level, as shown in Lemma 1.

In this economy, as shown by the top-left plot, the price of the risky asset initially falls with the tax rate, since the share of buyers is initially higher than the share of sellers in the economy. When τ is sufficiently large, most of group 2 investors, which correspond to a set of optimistic investors with low belief dispersion and were buying the risky asset, stop trading, as shown by the bottom right plot, so the share of sellers becomes higher than the share of buyers, as shown by the top middle plot. This makes the asset price increasing in the tax rate, which is consistent with Lemma 1. At $\tau = 0$, this economy matches the same share of non-fundamental trading volume, risk premium, and aggregate turnover as the calibrated economy studied in Section 5. In terms of the optimal tax, this calibrated economy, which features equilibrium price changes features an optimal tax somewhat higher, but comparable, to the one in Section 5 ($\tau^* = 0.39\%$).

More broadly, it may seem that the planner in this paper only seeks to improve the welfare of investors with distorted beliefs. That need not be the case, because of general equilibrium effects like the price effect considered here. This is an important observation since there is often greater support for policies that protect bystanders from the mistakes of others than for policies that protect people from their own mistakes. It is true that tax policy can only increase social welfare if some investors hold heterogeneous beliefs. However, which particular investors benefit or lose from the tax policy may depend on the type of general equilibrium effects illustrated. Although the paper focuses on aggregate efficiency, this general equilibrium spillovers will in principle affect all investors in the economy.

E.6 Welfare aggregation

Up to Section 6, the paper makes use of a welfare criterion that involves adding up the sum of investors' certainty equivalents. More generally, in Section 6, in order to facilitate the aggregation of investors' utilities, this paper assumes that the planner uses *uniform generalized social welfare weights*, using the "generalized social welfare weight" terminology introduced in Saez and Stantcheva (2016). Formally, under this aggregation criterion, the marginal impact on aggregate welfare of a tax change can be computed as the sum of the marginal impact on

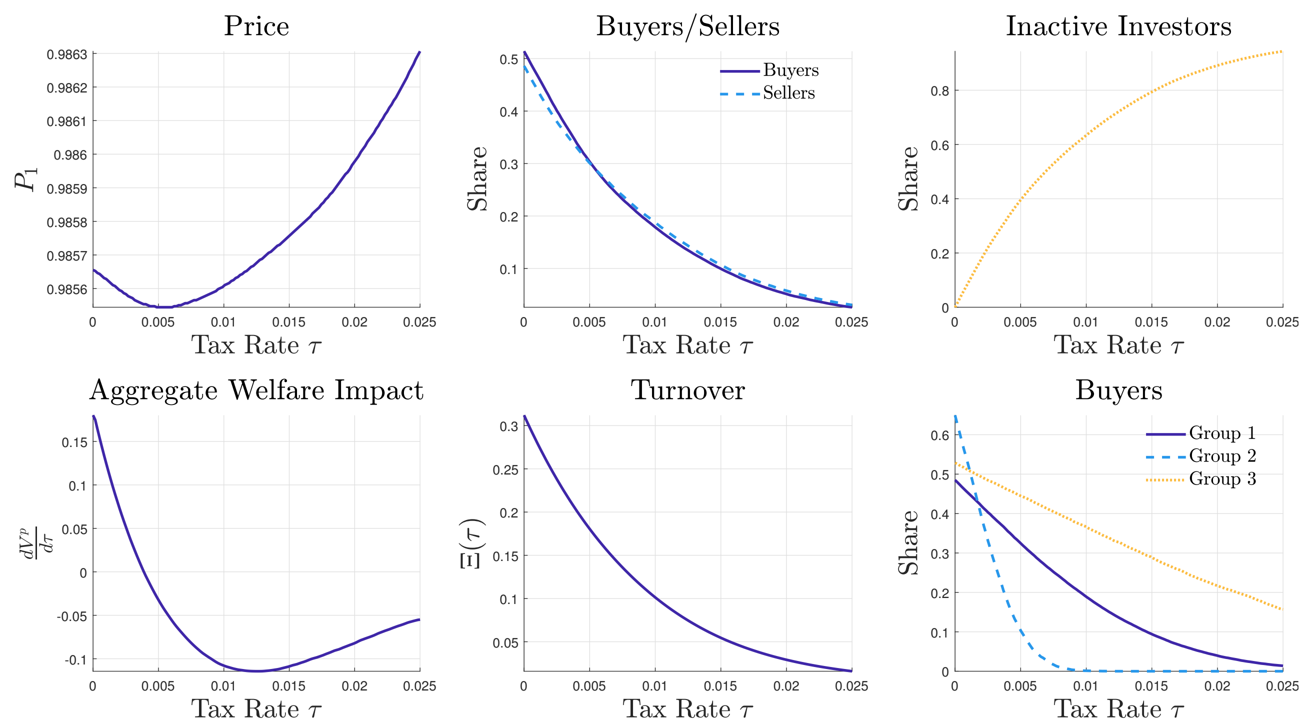


Figure A.11: Pecuniary effects

Note: The top left plot in Figure A.11 shows the equilibrium price P_1 as a function of the tax rate. The top middle plot shows the share of buyers and sellers in the economy as a function of the tax rate. The top right plot shows the share of inactive investors in the economy as a function of the tax rate. The bottom left plot shows the aggregate marginal welfare impact of a tax change. The bottom middle plot shows aggregate turnover. The bottom right plots shows the share of investors of each group who are net buyers of the risky asset as a function of the tax rate.

This simulation is calibrated to match the same three targets used in the quantitative assessment in Section 5. In particular, it features a 0.3 share of non-fundamental trading volume, a 1.5% risk premium, and an aggregate laissez-faire turnover of 31%. This calibration has three groups of investors. Group 1, with 75% of investors, has $\mu_d = 1$ and $\sigma_d = 0.0064$, group 2, with 15% of investors, has $\mu_d = 1.0016$ and $\sigma_d = 0.0016$, and group 3, with 10% of investors, has $\mu_d = 1.0024$ and $\sigma_d = 0.0128$. The optimal tax is $\tau = 0.39\%$. These plots show the outcome of a simulation of the model with $N = 100,000$ investors.

investors' indirect utility, given by $\frac{dV_i}{d\tau}$ and measured in $\frac{\text{utils}}{\text{tax rate}}$, normalized by the investors' marginal value of unit of wealth, given by $U_i'(C_{1i})$ in the case with initial consumption (similarly defined in the case without out) and measured in $\frac{\text{utils}}{\text{dollars}}$, that is

$$\frac{dV^p}{d\tau} = \int \frac{\frac{dV_i^p}{d\tau}}{U_i'(C_{1i})} dF(i),$$

which is measured in $\frac{\text{dollars}}{\text{tax rate}}$. Normalizing investor i's indirect utility by the marginal utility of consumption allows us to add up the marginal impact of a tax change for each investor measured in dollars, which allows for a meaningful aggregation process. Under this welfare criterion, if $\frac{dV^p}{d\tau} > 0$ for a given level of τ, the winners of the policy can always locally compensate the losers in dollar terms, from the planner's perspective. Hence, local welfare comparisons under this approach become similar to those in problems with quasi-linearity or transferable utility. This is a natural assumption in models of corrective taxation with concave utility, and corresponds to a local Kaldor-Hicks interpretation. See Weyl (2019) for further references.

Note that this approach can be mapped to the conventional approach in which a planner maximizes a weighted utilitarian social welfare function built by adding up agents' indirect utility with the different weights. Formally, once we have found the optimal tax using the set of uniform generalized social welfare weights, we can find conventional utilitarian (linear) social welfare weights given by $\lambda_i = \frac{1}{U_i'(C_{1i}(\tau^*))}$ such that the solution to a social welfare maximization problem using those welfare weights also finds that τ^* is the optimal tax. Proposition 4 in Saez and Stantcheva (2016) characterizes this equivalence in the context of an income taxation problem. That result can be trivially extended to the environment studied in this paper.

Finally, note that optimal tax characterizations would look the same for a planner with access to individual specific lump-sum transfers. Formally, a planner with access to ex-ante lump-sum transfers would be able to equalize the marginal value of a unit of wealth across all investors, making the normalization irrelevant to derive optimal tax formulas. The downside of allowing for ex-ante transfers is that those would violate the anonymity assumption sustained throughout the paper.

E.7 Altruism

An alternative defense of the welfare criterion comes from considering altruism. As shown in Figure 2, given his own belief and leaving aside price changes and net rebates, each investor perceives that a transaction tax reduces his individual welfare. However, if investors are at all altruistic towards others, they will agree on implementing a positive tax if the planner finds a positive tax to be optimal. An altruistic investor perceives that a small tax creates a first-order gain for all other investors in the economy, at the cost of a second-order private loss. This approach is consistent with the political philosophy tradition of deliberative democracy, in which individuals think and decide together about what serves the common interest, provided this common interest does not harm much any given individual. The approach followed by the planner in this paper fits legal traditions that consider speculation as fraudulent, because each individual perceives a gain at the expense of others, as well as religious precepts questioning gambling. I formalize this notion as follows.

Let us assume that investors are altruistic of degree $\alpha \in [0, 1]$. That is, they compute individual welfare as a linear combination between their on individualistic welfare and social welfare, as computed by the single-belief planner in this paper. More specifically, we'll assume that the planner's belief $\mathbb{E}_p[D]$ is exactly the same as investor i's, although as shown in the paper, the only key restriction is that social welfare is computed consistently using any single belief. Formally, denote by V_i^A the welfare of an altruistic investor with a degree of altruism α, in certainty equivalent terms. In that case, $V_i^A = (1-\alpha) V_i^i + \alpha V^p$, and consequently

$$\frac{dV_i^A}{d\tau} = (1-\alpha) \frac{dV_i^i}{d\tau} + \alpha \frac{dV^p}{d\tau}, \tag{63}$$

where $\frac{dV_i^i}{d\tau}$ denotes the marginal impact of a tax change from an individual's perspective, defined in Equation (41), and $\frac{dV_i^p}{d\tau}$ denotes the marginal impact of a tax change from the perspective of a hypothetical planner which uses the same belief as investor i, defined in Equation (40).

The actual optimal tax preferred by an altruistic investor varies with the degree of altruism α. However, it follows that every investor will prefer a positive tax whenever the planner in this paper prefers a positive tax. Formally,

$$\left.\frac{dV_i^A}{d\tau}\right|_{\tau=0} = \alpha \left.\frac{dV^p}{d\tau}\right|_{\tau=0},$$

since $\left.\frac{dV_i^i}{d\tau}\right|_{\tau=0} = 0$. Hence, whenever $\left.\frac{dV_i^p}{d\tau}\right|_{\tau=0} > 0$ is positive, it must be the case that $\left.\frac{dV_i^A}{d\tau}\right|_{\tau=0} > 0$. Intuitively, for a small tax, each individual investor perceives a second-order welfare loss individually but a first-order gain socially, so they all support a positive optimal tax. When τ is away from zero, every investor whose altruism degree α is less than one will prefer a smaller tax than the optimal tax chosen by the planner.

E.8 Linear combination between planner's belief and investors' beliefs

The results in the paper have been derived under the assumption that the planner maximizes welfare using a single distribution of payoffs to compute the welfare of all investors. It is straightforward to generalize the results to a planner that puts weight γ on social welfare computed with a single belief and weight $1-\gamma$ on social welfare computed respecting investors' individual beliefs. In that case, the new optimal tax τ_γ^* simply turns out to be a scaled version of the optimal tax characterized under the criterion proposed in this paper. Formally, the optimal tax for a planner who puts weight γ on social welfare as computed in this paper and weight $1-\gamma$ on social welfare as computed respecting investors' beliefs is given by

$$\tau_\alpha^* = \gamma \tau^*,$$

where τ^* is given by Equation (12). This result extends to the different extensions analyzed in this paper. This intermediate approach may be appealing to those who prefer a partially paternalistic welfare criterion.

E.9 Transaction taxes under incomplete markets

Even though the model in the paper features incomplete markets, the economy is constrained efficient since there is effectively a single market that clears. This section briefly describes how, beyond the role of differences in beliefs, the pecuniary impact of transaction taxes may by itself have a first-order effect on welfare.

Environment Consider an economy with investors indexed by i who solve the following dynamic problem

$$\max_{C_{ti}, X_{ti}, Y_{ti}} \mathbb{E}\left[\sum_{t=0}^{T} \beta^t U_i\left(C_{ti}\right)\right],$$

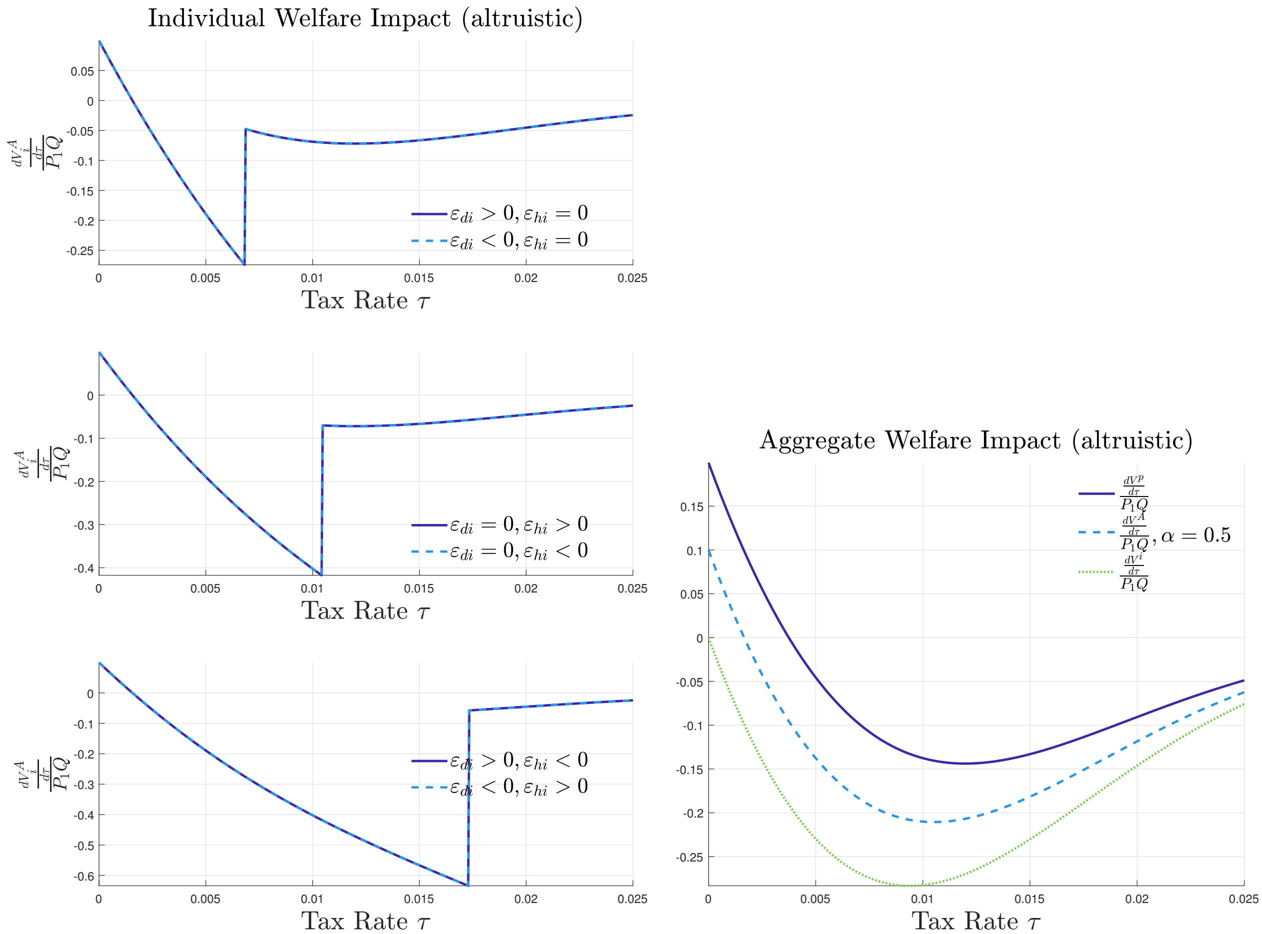
where

$$C_{ti} = M_{ti} + X_{t-1i}\left(P_t + D_t\right) + T_{ti} - \tau P_t \left|\Delta X_{ti}\right| - X_{ti} P_t + R Y_{t-1i} - Y_{ti},$$

where C_{ti} denotes investors' consumption, M_{ti} denotes investors' stochastic endowment, D_t denotes the stochastic payoff of the risky asset and P_t its price, $T_{ti} - \tau P_t \left|\Delta X_{ti}\right|$ denotes the net rebate received by investors, Y_{ti} and X_{ti} respectively denote investors' holdings of the risky and the safe asset, and R denotes the elastic risk-free rate.

To highlight the role of market incompleteness, I assume that both the planner and the investors' in the economy share the same beliefs. It is straightforward to include differences in beliefs here, as shown in Section 6. For simplicity, I consider an individually targeted rebate rule. The equilibrium definition is standard.

Results I simply focus here on the first-order of a transaction tax on social welfare, computed as in Section 6, using uniform generalized social welfare weights.



(a) Individual welfare impact (altruistic) (b) Aggregate welfare impact (altruistic)

Figure A.12: Altruism

Note: Figure A.12a shows the normalized individual marginal welfare impact of a tax change for specific investors from the perspective of an altruistic investor, as defined in Equation (63), for different values of τ. Figure A.12a is the counterpart of Figure 2, so the top row shows a buyer ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} = 0$) and a seller ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} = 0$) who only trade for non-fundamental reasons, the middle row plots shows a buyer ($\varepsilon_{hi} = 0$ and $\varepsilon_{di} < 0$) and a seller ($\varepsilon_{hi} = 0$ and $\varepsilon_{di} > 0$) who only trade for fundamental reasons, and the bottom plots show a buyer ($\varepsilon_{hi} > 0$ and $\varepsilon_{di} < 0$) who buys for non-fundamental and fundamental reasons and a seller ($\varepsilon_{hi} < 0$ and $\varepsilon_{di} > 0$) who sells for non-fundamental and fundamental reasons.
Figure A.12b shows the normalized the normalized aggregate marginal welfare impact of a tax change for a planner who maximizes investors' altruistic utility. The case with $\alpha = 1$ (solid blue line) corresponds to the case studied in the rest of the paper. The case $\alpha = 0$ (dotted green line) corresponds to the case in which individual welfare is compute respecting investors' beliefs. The intermediate case with $\alpha = 0.5$ is a linear combination between the two. When $\alpha = 1$, $\tau^* = 0.37\%$, when $\alpha = 0.5$, $\tau^* = 0.11\%$, while when $\alpha = 0$, $\tau^* = 0$.
All plots in Figure A.12 use the baseline calibration from Section 5.

Proposition 6. (Incomplete markets and common beliefs) *The marginal welfare impact of a tax change at $\tau = 0$, which can take any sign, is given by*

$$\left.\frac{dV^p}{d\tau}\right|_{\tau=0} = \int \sum_{t=0}^{T} \mathbb{E}\left[m_{ti} \Delta X_{ti} \left.\frac{dP_t}{d\tau}\right|_{\tau=0} \right] dF(i),$$

where m_{ti} denotes investor i's stochastic discount factor for date t, given by

$$m_{ti} = \frac{\beta^t U_i'(C_{ti})}{U_i'(C_{0i})}.$$

Proof. Note that

$$\frac{dV_i^p}{d\tau} = \mathbb{E}\left[\sum_{t=0}^{T} \frac{\beta^t U_i'(C_{ti})}{U_i'(C_{0i})} \frac{dC_{ti}}{d\tau} \right],$$

where $\frac{dC_{ti}}{d\tau} = \frac{dX_{t-1i}}{d\tau}(P_t + D_t) - X_{ti} P_t + R \frac{dY_{t-1i}}{d\tau} - \frac{dY_{ti}}{d\tau} - \Delta X_{ti} \frac{dP_t}{dt}$. Aggregating over all investors and using investors' individual optimality conditions immediately yields the result. ☐

When markets are incomplete, m_{ti} varies across the distribution of investors in every equilibrium. Since $\frac{dP_t}{d\tau}$ can be positive or negative, and $\int \Delta X_{ti} dF(i) = 0$, $\forall t$, it is straightforward to show that $\left.\frac{dV^p}{d\tau}\right|_{\tau=0}$ can take positive or negative values, which implies that a transaction tax or subsidy may or may not be welfare improving in an economy with incomplete markets. Whether a tax or subsidy is welfare improving will depend on the differences in investors' stochastic discount factors, changes in net trading positions, and the price impact of a tax. See Dávila and Korinek (2018) for a detailed discussion of these effects when characterizing constrained efficient allocations. If markets are complete when $\tau = 0$, then the sequence of m_{ti} is identical across investors, and $\left.\frac{dV^p}{d\tau}\right|_{\tau=0} = 0$, which is simple local version of the First Welfare Theorem. From the perspective of missing hedging markets, 6 also shows that price volatility is not the key variable to target. The correct target is whether transaction taxes modify prices to improve investors' insurance at the right times. More generally, note that pecuniary effects derived from wage changes or relative price changes in a multi-good economy would have similar effects.

E.10 General utility and arbitrary beliefs: quantitative assessment

Finally, it is worth exploring the quantitative implications of the model in the general non-linear case. In order to minimize the differences with the baseline calibration, I effectively fix investors' consumption at the initial date, so they only face a portfolio allocation problem and not a consumption-savings decision. I assume that investors' preferences are given by an isoelastic (CRRA) utility specification of the form $\frac{c^{1-\gamma}}{1-\gamma}$. All investors have the same initial asset holdings of the risky asset and have identical preferences. I also assume that M_{2i} and D are normally distributed and that investors have the same assessment about the unconditional first and second univariate moments of M_{2i}, as well as the variance of D. As in the baseline model, investors have different beliefs about the expected payoff of D, and they all have different hedging needs, modeled in this case in terms of their correlation coefficient between D and M_{2i}.[32]

In terms of solving the model, since there is no closed form to the portfolio choice problem in this case (even without transaction taxes), it is necessary to numerically solve investor's optimality condition, given by

$$\mathbb{E}_i\left[U_i'(W_{2i})\left(D - P_1\left(1 + \tau \operatorname{sgn}(\Delta X_{1i})\right)\right)\right] = 0.$$

I use quadrature methods, with 15 points along each dimension of D and M_{2i}, to approximate the investors' expectations, as well as the planner's. I assume that investors receive individually targeted rebates, so $W_{2i} = M_{2i} + X_{1i} D + (M_{1i} - P_1 \Delta X_{1i})$. Figure A.13 illustrates the results, which imply an optimal tax similar and of the same order of magnitude as the optimal tax in the baseline model.

[32]Since households' preferences feature an Inada condition, assuming that D and M_{2i} are normally distributed can be problematic in some calibrations. For the calibration used here, I ensure that consumption is always positive. Similarly, I ensure that the variance-covariance matrix of the random variables D and M_{2i} is positive semi-definite for each investor.

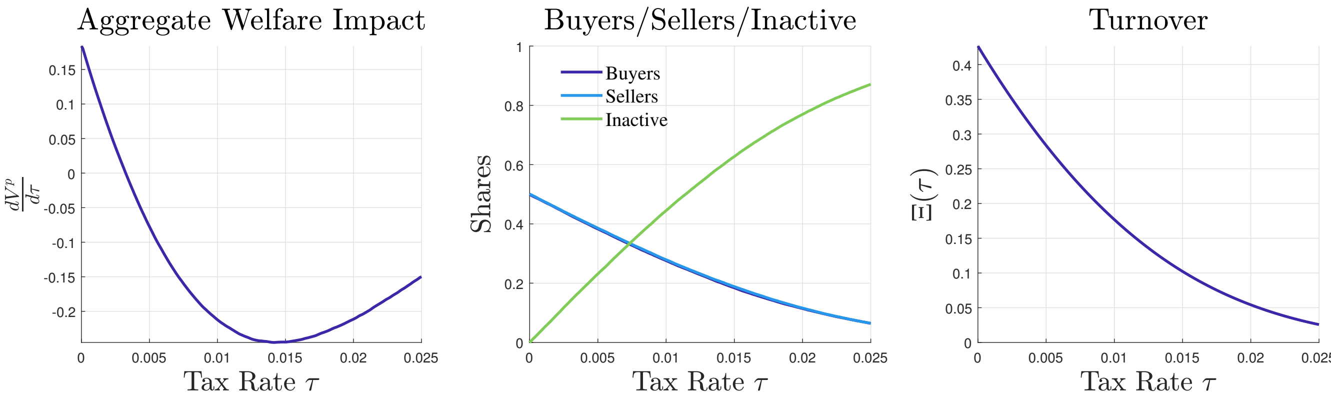


Figure A.13: Simulation with general preferences

Note: The left plot in Figure A.13 shows the aggregate welfare impact of a tax change for different values of τ. The middle plot shows the share of buyers, sellers, and investors who decide not trade for different values of τ. The right plots shows asset turnover for different values of τ. This simulation is calibrated to approximately match the same three targets used in the quantitative assessment in Section 5. In particular, it features a 34% share of non-fundamental trading volume (computed comparing a solution of the model with $\sigma_d = 0$ and $\sigma_h = 0$), a 1.53% risk premium, a volume semi-elasticity of -75, and a value of laissez-faire turnover of 42%. The optimal tax is $\tau = 0.33\%$. These plots show the outcome of a simulation of the model with $N = 100,000$ investors.

F Extensions

I now study multiple extensions of the benchmark model. Earlier versions of this paper included additional extensions.

F.1 Portfolio constraints: short-sale and borrowing constraints

Environment Although participants in financial markets face short-sale and borrowing constraints, investors in the baseline model face no restrictions when choosing portfolios. I now introduce trading constraints into the model as a pair of scalars for $\overline{g}_i$ and $\underline{g}_i$ for every investor i, such that

$$\underline{g}_i \leq X_{1i} \leq \overline{g}_i. \tag{64}$$

Both short-sale constraints and borrowing constraints are special cases of Equation (64). Short-sale constraints can be expressed as $X_{1i} \geq 0$, so $\underline{g}_i = 0$. Borrowing constraints can be mapped to a constraint of the form $X_{1i} \leq \overline{g}_i$.[33] Intuitively, an investor who wants to sufficiently increase his holdings of the risky asset must rely on borrowing. Hence, a borrowing limit is equivalent to an upper bound constraining the amount held of the risky asset.

Results and quantitative assessment The optimal portfolio is identical to the one in the baseline model, unless a constraint binds. In that case, X_{1i} equals the trading limit. Formally, the optimal portfolio decision for an investor is still given by Equation (5), unless the value of ΔX_{1i} implied by that equation violates the short-sale constraint, in which case $X_{1i} = \underline{g}_i$ or $X_{1i} = \overline{g}_i$The only substantial difference with the baseline is the expression that determines the equilibrium price, which is a slightly modified version of Equation (6) in the body of the paper. It is formally given by

$$P_1 = \frac{\int_{i \in \mathcal{U}} \frac{\mathbb{E}_i[D] - A_i \mathbb{C}ov[M_{2i}, D]}{A_i \mathbb{V}ar[D]} dF(i) - \int_{i \in \mathcal{T}} X_{0i} dF(i) - \int_{i \in \mathcal{C}} g_i(P_1) dF(i)}{\int_{i \in \mathcal{U}} \frac{1 + \text{sgn}(\Delta X_{1i})\tau}{A_i \mathbb{V}ar[D]} dF(i)},$$

where $\mathcal{T}, \mathcal{U}$, and $\mathcal{C}$ respectively denote the set of active investors, unconstrained investors, and constrained investors.

[33] An earlier version of this paper allowed for price dependent borrowing limits. In that case, the optimal tax must account for the pecuniary binding-constraint effects induced by tax changes.

Proposition 7. (Trading constraints) *The optimal financial transaction tax* τ^* *satisfies exactly the same expression as in Equation* (12)*.*

Proof. The marginal welfare impact of a tax change for investor i from the planner's perspective is given by

$$\frac{dV_i^p}{d\tau} = \left[\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right] + \text{sgn}\left(\Delta X_{1i}\right) P_1 \tau\right] \frac{dX_{1i}}{d\tau} - \Delta X_{1i} \frac{dP_1}{d\tau} + \frac{d\tilde{T}_{1i}}{d\tau}$$

for every investor, constrained or unconstrained. For this expression to be valid for constrained investors, it is essential that $\frac{dX_{1i}}{d\tau} = 0$ for those investors. Consequently, we can express aggregate welfare change as follows

$$\frac{dV^p}{d\tau} = \int_{i \in \mathcal{T}(\tau)} \left[-\mathbb{E}_i\left[D\right] + \text{sgn}\left(\Delta X_{1i}\right) P_1 \tau\right] \frac{dX_{1i}}{d\tau} dF\left(i\right),$$

as in Equation (29). The expression for the optimal tax follows immediately in that case. □

Intuitively, a marginal tax change at the optimum does not modify the portfolio allocation of constrained investors, leaving their welfare unchanged, i.e., for those investors $\frac{dX_{1i}}{d\tau} = 0$. Intuitively, investors who face trading constraints are infra-marginal for the determination of aggregate welfare.

Figure A.14 illustrates the role of short-sale constraints and how they affect the optimal tax within a model calibrated (without constraints and when $\tau = 0$) to match the share of non-fundamental trading volume, the level of turnover, and the risk premium, as in the quantitative assessment of the model in Section 5. Unsurprisingly, the top left plot in Figure A.14 shows that the equilibrium price when investors face short-sale constraints is higher than the equilibrium price when investors are unconstrained – this result is often attributed to Miller (1977). Since there are more buyers than sellers when investors are short-sale constrained, it is also natural to find that increases in the tax rate reduce the equilibrium price in the economy with short-sale constraints — the mechanism behind these effects was described in detail in Section E.5.

As expected, aggregate turnover is decreasing in the level of tax, while the share of inactive investors is increasing in the tax rate. In general, it is not possible to say in which direction introducing short-sale constraints shift the optimal tax, but it is possible to understand why in the model used in this section, the optimal tax is lower when there are short-sale constraints. In this case, starting from $\tau = 0$, the constraints will restrict the portfolio holdings of both fundamental and non-fundamental sellers. However, at the margin, only the non-fundamental/belief distortions part enters the welfare calculation of the planner. Since increasing the tax has now a smaller effect on restricting the trades of pessimists (since they already could not sell as much as they want and for them $\frac{dX_{1i}}{d\tau} = 0$), there is a lower marginal benefit from taxation and the optimal tax is lower.[34] Finally, the bottom right plot in Figure A.14 shows that higher tax rates reduce the number of short-sale constrained investors. In the limit, when τ is sufficiently large, all investors find optimal not to trade, so their short-sale constraints will necessarily be slack.

F.2 Pre-existing trading costs

Because actual investors face trading costs even when there are no taxes, one could wonder about the validity of the results derived around the point $\tau = 0$. Here, I show that the optimal tax formula is still valid as long as transaction costs are a mere compensation for the use of economic resources.[35]

[34]There is scope to explore the optimal determination of short-sale and borrowing constraints in a similar environment to the one considered here

[35]There is scope to study in more detail the interaction of trading costs in models that deliver endogenous bid-ask spreads in models with differentially informed investors, like that of Glosten and Milgrom (1985).

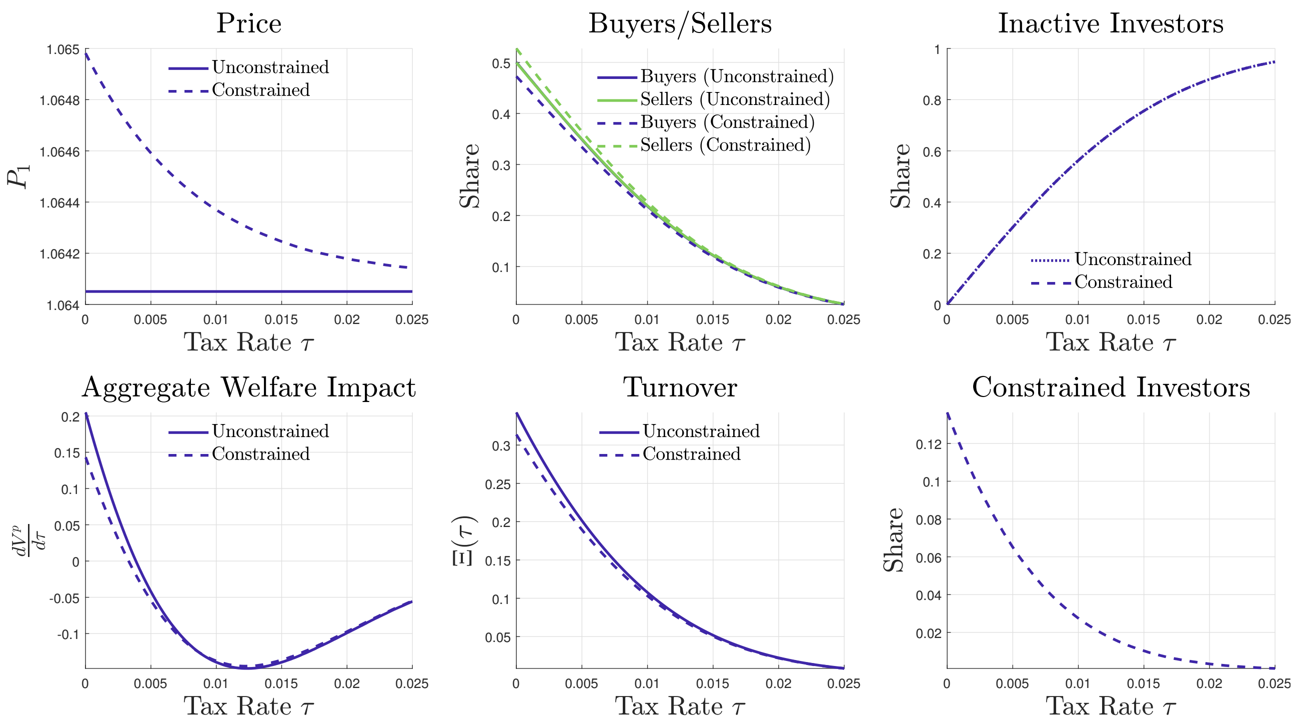


Figure A.14: Short-sale constraints

Note: The top left plot in Figure A.14 shows the equilibrium price P_1 as a function of the tax rate. The top middle plot shows the share of buyers and sellers in an economy that features short-sale constraints and in an economy in which all investors potentially face short-sale constraints. The top right plot shows the share of inactive investors in the economy as a function of the tax rate. The bottom left plot shows the aggregate marginal welfare impact of a tax change. The bottom middle plot shows aggregate turnover. The bottom right plots shows the share of investors of each group who are net buyers of the risky asset as a function of the tax rate.

This unconstrained model of this simulation is calibrated to match the same three targets used in the quantitative assessment in Section 5. In particular, it features a 0.3 share of non-fundamental trading volume, a 1.5% risk premium, and an aggregate laissez-faire turnover of 34%. The constrained results introduce a short-sale constraint in that calibration that satisfies Assumption [G], assuming $\mu_d = 1.08$, $\sigma_d = 0.0075$, and $\rho = 0$. The optimal tax is $\tau = 0.39\%$ in the case without constraints and $\tau = 0.33\%$ in the case in which investors face short-sale constraints. These plots show the outcome of a simulation of the model with $N = 100,000$ investors.

Environment Investors now face transaction costs, regardless of the value of τ. These represent costs associated with trading, like brokerage commissions, exchange fees, or bookkeeping costs. Investors must pay a quadratic cost, parameterized by α, a linear cost η on the number of shares traded, and a linear cost ψ on the dollar volume of the transaction. These trading costs are paid to a new group of investors (intermediaries), which facilitate the process of trading. Crucially, I assume that intermediaries make zero profits in equilibrium. Hence, wealth at date 2 for an investor i is now given by

$$W_{2i} = M_{2i} + X_{1i}D + X_{0i}P_1 - X_{1i}P_1 - |\Delta X_{1i}| P_1 (\tau + \psi) - \eta |\Delta X_{1i}| - \frac{\alpha}{2} (\Delta X_{1i})^2 + T_{1i}. \tag{65}$$

Tax revenues are rebated to investors through an arbitrary transfer, but not trading costs.

Results The demand for the risky asset takes a similar form as in the baseline model, featuring also an inaction region, now determined jointly by the trading costs and the transaction tax. The optimal portfolio given prices can be compactly written in the trade region as:

$$X_{1i} = \frac{\mathbb{E}_i [D] - A_i \mathbb{C}ov [M_{2i}, D] - P_1 (1 + \text{sgn} (\Delta X_{1i}) (\tau + \psi)) - \text{sgn} (\Delta X_{1i}) \eta + \alpha X_{0i}}{A_i \mathbb{V}ar [D] + \alpha}.$$

All three types of trading costs — quadratic, linear in shares and linear in dollar value — shift investors' portfolios towards their initial positions. The equilibrium price is a slightly modified version of Equation (6).

When calculating welfare, the planner takes into account that investors must incur these costs when trading — this is the natural constrained efficient benchmark. The optimal tax formula remains unchanged when investors face transaction costs, as long as these trading costs represent exclusively a compensation for the use of economic resources.

Proposition 8. (Pre-existing trading costs) *When investors face trading costs as specified in Equation (65), the optimal financial transaction tax τ^* satisfies exactly the same expression as in Equation (12).*

Proof. Given investors' optimal portfolios, stated in the main text, it is straightforward to derive the equilibrium price, which is given by

$$P_1 = \frac{\int_{i \in \mathcal{T}} \frac{\mathbb{E}_i[D] - A_i \mathbb{C}ov[M_{2i}, D] - \text{sgn}(\Delta X_{1i})\eta + \alpha X_{0i}}{A_i \mathbb{V}ar[D] + \alpha} dF(i) - \int_{i \in \mathcal{T}} X_{0i} dF(i)}{\int_{i \in \mathcal{T}} \frac{(1 + \text{sgn}(\Delta X_{1i})(\tau + \psi))}{A_i \mathbb{V}ar[D] + \alpha} dF(i)}.$$

The certainty equivalent for an investor i from the planner's perspective is given by

$$V_i^p = (\mathbb{E}_p [D] - A_i \mathbb{C}ov [M_{2i}, D] - P_1) X_{1i} + P_1 X_{0i} - |\Delta X_{1i}| P_1 \psi - \frac{\alpha}{2} (\Delta X_{1i})^2 - \frac{A_i}{2} \mathbb{V}ar [D] (X_{1i})^2 + \tilde{T}_{1i}.$$

Note that only the resources corresponding to the transaction tax are rebated back to investors. All resources devoted to transaction costs are a compensation for the use of resources, so the planner does not have to account for them explicitly, since they form part of a zero profit condition. Hence, the marginal change in welfare for an investor i is given by

$$\begin{aligned} \frac{dV_i^p}{d\tau} = & \left(\mathbb{E}_p [D] - A_i \mathbb{C}ov [M_{2i}, D] - P_1 - \text{sgn} (\Delta X_{1i}) P_1 \psi - \eta \, \text{sgn} (\Delta X_{1i})\right) \tfrac{dX_{1i}}{d\tau} \\ & - (\alpha \Delta X_{1i} + A \mathbb{V}ar [D] X_{1i}) \tfrac{dX_{1i}}{d\tau} - \Delta X_{1i} \tfrac{dP_1}{d\tau} + \tfrac{d\tilde{T}_{1i}}{d\tau}. \end{aligned}$$

Substituting investors' first order conditions and exploiting market clearing, we can express $\frac{dV^p}{d\tau}$ as follows

$$\frac{dV^p}{d\tau} = \int [\mathbb{E}_p [D] - \mathbb{E}_i [D] + \text{sgn} (\Delta X_{1i}) P_1 \tau] \frac{dX_{1i}}{d\tau} dF (i) .$$

Hence, the optimal tax has the same expression as in Proposition 1. ☐

The intuition behind Proposition 8 is similar to the baseline case. An envelope condition eliminates any term regarding transaction costs from $\frac{dV^p}{d\tau}$, because the planner must also face such costs, so the optimal tax looks identical to the one in the baseline model. This relies on the assumption that any economic profits made by the intermediaries who receive the transaction costs are zero — there cannot be economic rents. Proposition 8 does not imply that two overlapping authorities with taxation power would both impose the same τ^* twice. Assume for simplicity that they set taxes sequentially. The first authority would set the optimal tax according to Proposition 1, while the second authority, internalizing that the pre-existing tax is a mere transfer and does not correspond to a compensation for costs of trading, would set a zero tax. Alternatively, τ^* would characterize the sum of both taxes.

This result has further implications. First, although the optimal tax formula does not vary, an economy with transaction costs has less trade in equilibrium than one without transaction costs. Depending on whether this reduction in trading is of the fundamental type or not, the optimal tax may be larger or smaller. Transaction costs affect the optimal tax through changes in the identity of the marginal investors. Second, the mere existence of transaction costs does not provide a new rationale for further discouraging non-fundamental trading. Welfare losses must be traced back to wedges derived from portfolio distortions. Third, if transaction costs, that is, ψ, η, and α, were endogenously functions of τ, as in richer models of the market microstructure, the planner would have to take into account those effects when solving for optimal taxes.[36] For instance, if a transaction tax endogenously increases trading costs, the optimal tax may be very small. However, if endogenously determined transaction costs are efficiently determined, the envelope theorem would still apply, leaving Proposition 8 unchanged.

Remark. (Trading costs instead risk-sharing distortions) In the model considered in this paper, the planner perceives that trading on belief differences is costly because it distorts investors risk-sharing decisions. In a model in which investors are risk-neutral but feature physical or technological costs of trading, a planner would have an identical rationale to discourage trading. For instance, if we assume in this Section that investors face a quadratic trading but $A_i = 0$, $\forall i$, it is possible to effectively re-derive the same expressions for welfare impact and optimal taxes as in the body of the paper.

F.3 Imperfect tax enforcement

All the results in the paper have been derived under the assumption of perfect tax enforcement. I now show how introducing imperfect tax enforcement does not change the main qualitative predictions of the paper. I also show that imperfect enforcement is associated with lower optimal taxes.[37]

Environment Investors can now trade in two different markets, A and B. Market A captures existing venues for trading, and all trades in that market face a transaction tax τ. Market B seeks to represent trading venues that cannot be monitored by authorities. In market B, investors face instead a quadratic cost of trading, parameterized by α. When $\alpha \to 0$, avoiding the tax is costless, and all trades move to market B for any values of τ, so for regularity purposes, α must be sufficiently large, which is consistent with the empirical evidence discussed in the text. Assuming a different form of cost yields similar insights. Varying α modulates the costs of evasion. To simplify notation, at times I define $\hat{\alpha} = \frac{\alpha}{A_i \mathbb{V}ar[D]}$. Investors' initial endowments are X_{0i}^A market A shares and $X_{0i}^B = 0$ market B shares (without loss of generality).

Hence, wealth at date 2 for an investor i is now given by:

$$W_{2i} = M_{2i} + X_{1i}D - \Delta X_{1i}^A P_1^A - \tau \left|P_1^A\right| \left|\Delta X_{1i}^A\right| - \Delta X_{1i}^B P_1^B - \frac{\alpha}{2} \left(\Delta X_{1i}^B\right)^2 + T_{1i},$$

[36] The Walrasian approach of this paper does not capture market microstructure effects. There is scope for understanding how transaction taxes affect market making and liquidity provision in greater detail, introducing, for instance, imperfectly competitive investors, search, or network frictions. The results of this paper would still be present regardless of the specific trading microstructure.

[37] There is scope for further research understanding the specificities of tax avoidance/evasion when there is competition among exchanges, as in Santos and Scheinkman (2001).

where I define $X_{1i} = X_{1i}^A + X_{1i}^B$, same for $t = 0$. The transfer rebates tax revenues, but not the costs of trading in the B market, to investors. Note that the linear tax only affects trading in market A, while the quadratic cost only affects trading in market B.

Results Now investors must formulate demands for both markets. Investors' optimality conditions correspond to

$$X_{1i} = X_{1i}^A + X_{1i}^B = \frac{\left[\mathbb{E}_i\left[D\right] - A_i \mathbb{C}ov\left[M_{2i}, D\right] - P_1^A\right] - \tau\left|P_1^A\right| \operatorname{sgn}\left(\Delta X_{1i}^A\right)}{A_i \mathbb{V}ar\left[D\right]} \tag{66}$$

$$X_{1i} = X_{1i}^A + X_{1i}^B = \frac{\left[\mathbb{E}_i\left[D\right] - A_i \mathbb{C}ov\left[M_{2i}, D\right] - P_1^B\right]}{A_i \mathbb{V}ar\left[D\right]} - \hat{\alpha} X_{1i}^B. \tag{67}$$

Note that we generically expect $\Delta X_{1i}^B \neq 0$. Whenever $\Delta X_1^A \neq 0$, by combining Equations (66) and (67), X_1^B must satisfy

$$X_1^B = \frac{\tau}{\hat{\alpha}} \frac{P_1 \operatorname{sgn}\left(\Delta X_{1i}^A\right)}{A_i \mathbb{V}ar\left[D\right]}.$$

Whenever investors are inactive in market A, so $\Delta X_1^A = 0$, X_1^B is given by

$$X_{1i}^B = \frac{1}{1+\hat{\alpha}} \left(\frac{\left[\mathbb{E}_i\left[D\right] - A_i \mathbb{C}ov\left[M_{2i}, D\right] - P_1^B\right]}{A_i \mathbb{V}ar\left[D\right]} - X_{0i}^A \right).$$

Note that, regardless of the scenario, $\frac{dX_{1i}}{d\tau}$ is weakly lower in absolute value. To simplify the exposition, I impose Assumption [S] from now on, which guarantees that $P_1^A = P_1^B = P_1$, although similar results can be found for the general case.

Proposition 9. (Imperfect tax enforcement) *When investors can trade in an alternative market without facing the tax, the sign of the optimal is given by the sign of*

$$\left.\frac{dV^p}{d\tau}\right|_{\tau=0} = -\int \mathbb{E}_i\left[D\right] \left.\frac{dX_{1i}}{d\tau}\right|_{\tau=0} dF\left(i\right), \tag{68}$$

where $\frac{dX_{1i}}{d\tau} = \frac{dX_{1i}^A}{d\tau} + \frac{dX_{1i}^B}{d\tau}$. *The optimal financial transaction tax* τ^* *corresponds to*

$$\tau^* = \frac{\int \mathbb{E}_i\left[D\right] \frac{dX_{1i}}{d\tau} dF\left(i\right)}{\int \operatorname{sgn}\left(\Delta X_{1i}\right) P_1 \frac{dX_{1i}^A}{d\tau} dF\left(i\right)}. \tag{69}$$

Proof. After eliminating terms that do not affect the maximization problem, investors solve:

$$\max_{X_{1i}^A, X_{1i}^B} \left[\mathbb{E}_i\left[D\right] - A_i \mathbb{C}ov\left[M_{2i}, D\right] - P_1\right] \left(X_{1i}^A + X_{1i}^B\right) - \tau P_1 \left|\Delta X_{1i}^A\right| - \frac{A_i}{2} \mathbb{V}ar\left[D\right] \left(X_{1i}^A + X_{1i}^B\right)^2 - \frac{\alpha}{2} \left(\Delta X_{1i}^B\right)^2 + T_{1i},$$

with interior optimality conditions shown in the text.

The change in investors' certainty equivalents is given by

$$\frac{dV_i^p}{d\tau} = \left[\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right]\right] \frac{dX_{1i}}{d\tau} + \operatorname{sgn}\left(\Delta X_{1i}\right) P_1 \tau \frac{dX_{1i}^A}{d\tau} - \Delta X_{1i}^A \frac{dP_1^A}{d\tau} - \Delta X_{1i}^B \frac{dP_1^B}{d\tau} + \frac{d\tilde{T}_{1i}}{d\tau},$$

where the last two terms are zero under symmetry. Equations (68) and (69) follow immediately. □

Under imperfect tax enforcement, the numerator of the optimal tax formula accounts for the change in volume in both markets, while the denominator only accounts for changes in the market in which the optimal tax is paid. Intuitively, now the same tax change is less effective in reducing non-fundamental trades, so the optimal tax chosen by a planner is lower than before. Note that the condition that determines the sign of the optimal tax is identical with and without perfect enforcement after accounting for the total volume reduction in both markets. Since it can be shown that $\frac{dX_{1i}}{d\tau}$ has the same sign as in the model with perfect enforcement, the qualitative insights for the sign of the optimal tax go through unchanged.

F.4 Multiple risky assets

Environment The results of the baseline model extend naturally to an environment with multiple assets. Now there are J risky assets in fixed supply, in addition to the risk-free asset. The $J \times 1$ vectors of total shares, equilibrium prices and dividend payments are respectively denoted by $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{d}$.[38] Every purchase or sale of a risky asset faces an identical linear transaction tax τ. This is a further restriction on the planner's problem, since belief disagreements can vary across different assets, but the tax must be constant. Allowing for different taxes for different (groups of) assets is conceptually straightforward, following the logic of Section F.5.

The distribution of dividends $\mathbf{d}$ paid by the risky assets is a multivariate normal with a given mean and variance-covariance matrix $\mathbb{V}ar[\mathbf{d}]$. All investors agree about the variance, but an investor i believes that the mean of $\mathbf{d}$ is $\mathbb{E}_i[\mathbf{d}]$. We can thus write:

$$\mathbf{d} \sim_i N(\mathbb{E}_i[\mathbf{d}], \mathbb{V}ar[\mathbf{d}]),$$

where risk aversion A_i, and the vectors of initial asset holdings $\mathbf{x}_{0i}$, hedging needs $\mathbb{C}ov[M_{2i}, \mathbf{d}]$ and beliefs $\mathbb{E}_i[\mathbf{d}]$ are arbitrary across the distribution of investors. The wealth at $t = 2$ of an investor i is thus given by:

$$W_{2i} = M_{2i} + \mathbf{x}'_{1i}\mathbf{d} + \mathbf{x}'_{0i}\mathbf{p} - \mathbf{x}'_{1i}\mathbf{p} - |\mathbf{x}'_{1i} - \mathbf{x}'_{0i}|\,\mathbf{p}\tau + T_{1i}.$$

Results The first order condition (70) characterizes the solution of this problem for the set of assets traded:

$$\mathbf{x}_{1i} = (A_i \mathbb{V}ar[\mathbf{d}])^{-1}(\mathbb{E}_i[\mathbf{d}] - A_i \mathbb{C}ov[M_{2i}, \mathbf{d}] - \mathbf{p} - \hat{\mathbf{p}}_i\tau), \tag{70}$$

where $\hat{\mathbf{p}}_i$ is a $J \times 1$ vector where row j is given by $\text{sgn}(\Delta X_{1ij})\,p_j$ and p_j denotes the price of asset j. If an asset j is not traded by an investors i, then $X_{1ij} = X_{0ij}$. If asset returns are independent, the portfolio allocation to every asset can be determined in isolation. Equilibrium prices are the natural generalization of the baseline model.

Proposition 10. (Multiple risky assets) *The optimal financial transaction tax τ^* when investors can trade J risky assets is given by*

$$\tau^* = \sum_{j=1}^{J} \omega_j \tau_j^*, \tag{71}$$

with weights ω_j and individual-asset taxes τ_j^* given by $\omega_j \equiv \frac{p_j \int \text{sgn}(\Delta X_{1ij}) \frac{dX_{1ij}}{d\tau} dF(i)}{\sum_{j=1}^{J} p_j \int \text{sgn}(\Delta X_{1ij}) \frac{dX_{1ij}}{d\tau} dF(i)}$ *and* $\tau_j^* \equiv \frac{\int \frac{\mathbb{E}_i[D_j]}{p_j} \frac{dX_{1ij}}{d\tau} dF(i)}{\int \text{sgn}(\Delta X_{1ij}) \frac{dX_{1ij}}{d\tau} dF(i)}$.

Proof. After eliminating terms that do not affect the maximization problem, investors solve

$$\max_{\mathbf{x}_{1i}} \mathbf{x}'_{1i}(\mathbb{E}_i[\mathbf{d}] - A_i \mathbb{C}ov[M_{2i}, \mathbf{d}] - \mathbf{p}) - |\mathbf{x}'_{1i} - \mathbf{x}'_{0i}|\,\mathbf{p}\tau - \frac{A_i}{2}\mathbf{x}'_{1i}\mathbb{V}ar[\mathbf{d}]\,\mathbf{x}_{1i}.$$

Where I use $|\mathbf{x}'_{1i} - \mathbf{x}'_{0i}|$ to denote the vector of absolute values of the difference between both vectors. This problem is well-behaved, so the first order condition fully characterizes investors' optimal portfolios as long as they trade a given asset j

$$\mathbf{x}_{1i} = (A_i \mathbb{V}ar[\mathbf{d}])^{-1}(\mathbb{E}_i[\mathbf{d}] - A_i \mathbb{C}ov[M_{2i}, \mathbf{d}] - \mathbf{p} - \hat{\mathbf{p}}_i\tau),$$

where $\hat{\mathbf{p}}_i$ is a $J \times 1$ vector where a given row j is given by $\text{sgn}(\Delta X_{1ij})\,p_j$. If an asset j is not traded by an investor i, then $X_{1ij} = X_{0ij}$. The inaction regions are defined analogously to the one asset case. Note that there exists a way to write optimal portfolio choices only with matrix operations; however, the notation turns out to be more cumbersome. The equilibrium price vector is given by

$$\mathbf{p} \odot \int \left(1 + \frac{\mathbf{s}_i}{A_i}\tau\right) dF(i) = \int \frac{\mathbb{E}_i[\mathbf{d}]}{A_i} dF(i) - \int (\mathbb{C}ov[M_{2i}, \mathbf{d}] + \mathbb{V}ar[\mathbf{d}]\,\mathbf{x}_{0i})\, dF(i),$$

[38] I use bold lower-case letters to denote vectors but, for consistency, I keep the upper-case notation for holdings of a single asset.

where I denote the element-by-element multiplication (Hadamard product) as $y \odot z$ and use $\mathbf{s}_i$ to denote a $J \times 1$ vector given by $\text{sgn}(\Delta X_{1ij})$.

$$\frac{dV^p}{d\tau} = \int (\mathbb{E}[\mathbf{d}] - \mathbb{E}_i[\mathbf{d}] + \hat{\mathbf{p}}_i \tau)' \frac{d\mathbf{x}_{1i}}{d\tau} dF(i).$$

The marginal effect of varying taxes in social welfare is given by

$$\frac{dV^p}{d\tau} = \int \left[(\mathbb{E}[\mathbf{d}] - \mathbb{E}_i[\mathbf{d}] + \hat{\mathbf{p}}_i \tau)' \frac{d\mathbf{x}_{1i}}{d\tau} - (\mathbf{x}_{1i} - \mathbf{x}_{0i})' \frac{d\mathbf{p}}{d\tau} \right] dF(i).$$

This is a generalization of the one asset case. We can write in product notation

$$\int \sum_{j=1}^{J} (-\mathbb{E}_i[D_j] + \text{sgn}(\Delta X_{1ij}) p_j \tau) \frac{dX_{1ij}}{d\tau} dF(i) = 0.$$

So the optimal tax becomes

$$\tau^* = \frac{\sum_{j=1}^{J} \int \mathbb{E}_i[D_j] \frac{dX_{1ij}}{d\tau} dF(i)}{\sum_{j=1}^{J} \int \text{sgn}(\Delta X_{1ij}) p_j \frac{dX_{1ij}}{d\tau} dF(i)},$$

which can be rewritten as

$$\tau^* = \frac{\sum_{j=1}^{J} \int \text{sgn}(\Delta X_{1ij}) p_j \frac{dX_{1ij}}{d\tau} dF(i) \tau_j^*}{\sum_{j=1}^{J} \int \text{sgn}(\Delta X_{1ij}) p_j \frac{dX_{1ij}}{d\tau} dF(i)},$$

where $\tau_j^* = \frac{\int \frac{\mathbb{E}_i[D_j]}{p_j} \frac{dX_{1ij}}{d\tau} dF(i)}{\int \text{sgn}(\Delta X_{1ij}) \frac{dX_{1ij}}{d\tau} dF(i)}$. And by defining weights $\omega_j = \frac{\int \text{sgn}(\Delta X_{1ij}) p_j \frac{dX_{1ij}}{d\tau} dF(i)}{\sum_{j=1}^{J} \int \text{sgn}(\Delta X_{1ij}) p_j \frac{dX_{1ij}}{d\tau} dF(i)}$, we recover Equation (71). □

The formula for τ_j^* is identical to the one in an economy with a single risky asset. The optimal tax in a model with J risky assets is simply a weighted average of all τ_j^*. The weights are determined by the relative marginal changes in (dollar) volume. Those assets whose volume responds more aggressively to tax changes carry higher weights when determining the optimal tax and vice versa.

F.5 Asymmetric taxes/Multiple tax instruments

In the baseline model, the only instrument available to the planner is a single linear financial transaction tax, which applies symmetrically to all investors. However, the planner could set different (linear) taxes for buyers and sellers. Or, at least theoretically, even investor-specific taxes. In general, more sophisticated policy instruments bring the outcome of the planner's problem closer to the first-best, at the cost of increasing informational requirements.

Asymmetric taxes on buyers versus sellers Assume now that buyers pay a linear tax τ_B in the dollar volume of the transaction while sellers pay τ_S. Hence, the total tax liability is given by $(\tau_B + \tau_S) P_1 |\Delta X_{1i}|$. Outside of the inaction region, the optimal portfolio demand is given by

$$X_{1i} = \frac{\mathbb{E}_i[D] - A_i \mathbb{C}ov[M_{2i}, D] - P_1 (1 + \mathbb{I}[\Delta X_{1i} > 0] \tau_B + \mathbb{I}[\Delta X_{1i} < 0] \tau_S)}{A_i \mathbb{V}ar[D]},$$

where $\mathbb{I}[\cdot]$ denotes the indicator function. This expression differs from (5) in that buyers now face a different tax than sellers. The equilibrium price is a natural extension of the one in the baseline model.

Proposition 11. (Asymmetric taxes on buyers versus sellers) *The pair of optimal financial transaction taxes for buyers and sellers, τ_B^* and τ_S^*, is characterized by the solution of the following system of non-linear equations:*

$$\tau_B^* + \tau_S^* = \frac{\int \frac{\mathbb{E}_i[D]}{P_1} \frac{dX_{1i}}{d\tau_B} dF(i)}{\int_{\mathcal{B}} \frac{dX_{1i}}{d\tau_B} dF(i)}, \qquad \tau_B^* + \tau_S^* = \frac{\int \frac{\mathbb{E}_i[D]}{P_1} \frac{dX_{1i}}{d\tau_S} dF(i)}{\int_{\mathcal{B}} \frac{dX_{1i}}{d\tau_S} dF(i)}. \quad (72)$$

Proof. The budget/wealth accumulation constraint for an investor in this case can be expressed as:

$$W_{2i} = M_{2i} + X_{1i}D + X_{0i}P_1 - X_{1i}P_1 - \tau_B P_1 \left|\Delta X_{1i}\right|_+ - \tau_S P_1 \left|\Delta X_{1i}\right|_- + T_{1i}.$$

The first order condition becomes

$$X_{1i} = \frac{\mathbb{E}_i\left[D\right] - A_i \mathbb{C}ov\left[M_{2i}, D\right] - P_1\left(1 + \mathbb{I}\left[\Delta X_{1i} > 0\right]\tau_B + \mathbb{I}\left[\Delta X_{1i} < 0\right]\tau_S\right)}{A_i \mathbb{V}ar\left[D\right]}.$$

With an equilibrium price given by

$$P_1 = \frac{\int_{i\in\mathcal{T}}\left(\frac{\mathbb{E}_i[D] - A_i\mathbb{C}ov[M_{2i},D]}{A_i} - \mathbb{V}ar\left[D\right]X_{0i}\right)dF\left(i\right)}{\int_{i\in\mathcal{T}}\frac{1}{A_i} + \tau_B\int_{i\in\mathcal{B}}\frac{1}{A_i} - \tau_S\int_{i\in\mathcal{S}}\frac{1}{A_i}dF\left(i\right)}.$$

In this case, we can write: $X_{1i}\left(\tau_i, P_1\left(\{\tau_j\}\right)\right)$, where $\{\tau_j\}$ denotes a vector of taxes. This implies that $\frac{dX_{1i}}{d\tau_j} = \frac{\partial X_{1i}}{\partial \tau_j} + \frac{\partial X_{1i}}{\partial P_1}\frac{dP_1}{d\tau_j}$. The change in social welfare for an investor i when varying a tax τ_j, from a planner's perspective, is given by

$$\begin{aligned}
\frac{dV_i^p}{d\tau_j} &= \left(\mathbb{E}_p\left[D\right] - A_i\mathbb{C}ov\left[M_{2i}, D\right] - P_1 - A_i X_{1i}\mathbb{V}ar\left[D\right]\right)\frac{dX_{1i}}{d\tau_j} - \Delta X_{1i}\frac{dP_1}{d\tau_j} + \frac{d\tilde{T}_{1i}}{d\tau_j}\\
&= \left(\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right] + P_1\left(\mathbb{I}\left[\Delta X_{1i} > 0\right]\tau_B - \mathbb{I}\left[\Delta X_{1i} < 0\right]\tau_S\right)\right)\frac{dX_{1i}}{d\tau_j} - \Delta X_{1i}\frac{dP_1}{d\tau_j} + \frac{d\tilde{T}_{1i}}{d\tau_j}.
\end{aligned}$$

The marginal aggregate welfare change of a tax change is given by

$$\frac{dV^p}{d\tau_j} = \int\left(\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right] + P_1\left(\mathbb{I}\left[\Delta X_{1i} > 0\right]\tau_B - \mathbb{I}\left[\Delta X_{1i} < 0\right]\tau_S\right)\right)\frac{dX_{1i}}{d\tau_j}dF\left(i\right),$$

which is a generalization of Lemma 2. Under the usual differentiability and convexity assumptions, the optimal tax is characterized by $\frac{dV}{d\tau_j} = 0$, $\forall j$. This yields the following system of equation in the vector of taxes τ_B and τ_S

$$0 = \int\left(\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right] + P_1\,\mathrm{sgn}\left(\Delta X_{1i}\right)\tau_i\right)\frac{dX_{1i}}{d\tau_j}dF\left(i\right),\ \forall j.$$

We can write

$$\frac{dV^p}{d\tau_j} = \int\left(\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right]\right)\frac{dX_{1i}}{d\tau_j}dF\left(i\right) + P_1\left(\tau_B\int_{i\in\mathcal{B}}\frac{dX_{1i}}{d\tau_j}dF\left(i\right) - \tau_S\int_{i\in\mathcal{S}}\frac{dX_{1i}}{d\tau_j}dF\left(i\right)\right).$$

Using market clearing, we can find

$$\frac{dV^p}{d\tau_j} = -\int\mathbb{E}_i\left[D\right]\frac{dX_{1i}}{d\tau_j}dF\left(i\right) + P_1\left(\tau_B + \tau_S\right)\int_{i\in\mathcal{B}}\frac{dX_{1i}}{d\tau_j}dF\left(i\right),$$

which allows us to solve for $\tau_B + \tau_S$ as follows

$$\tau_B + \tau_S = \frac{\int\mathbb{E}_i\left[D\right]\frac{dX_{1i}}{d\tau_j}dF\left(i\right)}{P_1\int_{i\in\mathcal{B}}\frac{dX_{1i}}{d\tau_j}dF\left(i\right)},\ \forall_j.$$

In general this gives a system of non-linear equations in $\tau_B + \tau_S$. When there are two investors, the two equations become collinear, because of market clearing

$$\int\mathbb{E}_i\left[D\right]\frac{dX_{1i}}{d\tau_j}dF\left(i\right) = \left(\mathbb{E}_{\mathcal{B}}\left[D\right] - \mathbb{E}_S\left[D\right]\right)\int_{i\in\mathcal{B}}\frac{dX_{1i}}{d\tau_j}dF\left(i\right).$$

Hence, in that case, only the sum of taxes is pinned down, and must satisfy

$$\tau_B + \tau_S = \frac{\mathbb{E}_{\mathcal{B}}\left[D\right] - \mathbb{E}_{\mathcal{S}}\left[D\right]}{P_1}.$$

□

The economic forces that shape the optimal values for τ_B^* and τ_S^* are the same as in the baseline model. Once again, the planner's belief is irrelevant for the optimal policy, which shows that that results are not sensitive to the use of more sophisticated policy instruments. Intuitively, the change in portfolio allocations induced by a marginal change in any instrument must cancel out in the aggregate. Equation (72) provides intuition for why all taxes in the baseline model are divided by 2; in that case, there exists a single optimality condition and $2\tau^* = \tau_B^* + \tau_S^*$. As long as there are more than two investors, this system has at least a solution. When there are two investors, the system is indeterminate and only the sum $\tau_B^* + \tau_S^*$ is pinned down. In that case, $\tau_B^* + \tau_S^* = \frac{\mathbb{E}_{\mathcal{B}}[D] - \mathbb{E}_{\mathcal{S}}[D]}{P_1}$.

Individual taxes/First-best Assume now that the planner can set investor specific taxes. This is an interesting theoretical benchmark, despite being unrealistic. For simplicity, I now assume that there is a finite number N of (types of) investors in the economy.

Proposition 12. (Individual taxes/First-best)

a) The first-best can be implemented with a set of investor specific taxes given by

$$\tau_i^* = \operatorname{sgn}(\Delta X_{1i}) \frac{\mathbb{E}_i[D] - \Upsilon}{P_1}, \forall i = 1, \ldots, N, \tag{73}$$

where Υ is any real number; a natural choice for Υ is $\mathbb{E}_p[D]$.

b) The planner only needs $N-1$ taxes to implement the first-best in an economy with N investors.

Proof. a) In the case with I taxes and I investors, the first order conditions for the planner become

$$\frac{dV^p}{d\tau_j} = \sum_i \left(-\mathbb{E}_i[D] + P_1 \operatorname{sgn}(\Delta X_{1i}) \tau_i\right) \frac{dX_{1i}}{d\tau_j} F(i) = 0, \ \forall j.$$

This system of equations characterizes the set of optimal taxes. Note that one solution to this system is given by

$$-\mathbb{E}_i[D] + P_1 \operatorname{sgn}(\Delta X_{1i}) \tau_i = -F.$$

Where F is an arbitrary real number. Rearranging this expression we can find Equation (73).

b) Starting from the system of equations which characterizes the optimal set of taxes, we can write, using market clearing $F(j) \frac{dX_{1i}}{d\tau_j} + \sum_{i \neq j} \frac{dX_{1i}}{d\tau_j} F(i) = 0$, the following set of equations:

$$\sum_{i \neq j} \left(\mathbb{E}_j[D] - \mathbb{E}_i[D]\right) \frac{dX_{1i}}{d\tau_j} F(i) + P_1 \begin{pmatrix} -\operatorname{sgn}(\Delta X_{1j}) \tau_j \sum_{i \neq j} \frac{dX_{1i}}{d\tau_j} F(i) \\ + \sum_i \left(\operatorname{sgn}(\Delta X_{1i}) \tau_i \frac{dX_{1i}}{d\tau_j}\right) F(i) \end{pmatrix} = 0.$$

For all equations but for the one with respect to tax j. To show that this system only depends on $N-1$ taxes, we simply need to show that all $\frac{dX_{1i}}{d\tau_j}$ do not depend on the tax τ_j. Note that $\frac{dX_{1i}}{d\tau_j} = \frac{\partial X_{1i}}{\partial \tau_j} + \frac{\partial X_{1i}}{\partial P_1} \frac{dP_1}{d\tau_j}$. But when $i \neq j$ then $\frac{dX_{1i}}{d\tau_j}$ only depends on $\frac{dP_1}{d\tau_j}$ because $\frac{\partial X_{1i}}{\partial \tau_j}$ equals zero and $\frac{\partial X_{1i}}{\partial P_1}$ does not depend on τ_j. We just need to show that $\frac{dP_1}{d\tau_j}$ can be expressed as a function of all other taxes but τ_j. This can be easily shown combining the expressions used to show Lemma 1 with market clearing conditions. □

Proposition 12a) follows standard Pigouvian logic. The planner sets optimal individual taxes so that investors' portfolio choices replicate those of an economy with homogeneous beliefs. Note that the planner can use any belief Υ to implement the first-best allocation, as long as it is the same for all investors. In a production economy, the natural choice would be $\Upsilon = \mathbb{E}_p[D]$. Finally, because P_1 is a function of all taxes, Equation (73) also defines a system of non-linear equations.

Proposition 12b) shows that the first-best could be implemented with $N-1$ taxes. This occurs because the risky asset is in fixed supply. The logic behind this result is similar to Walras' law. For instance, when $N = 2$, a single tax which modifies directly the allocation of one of the investors necessarily changes the allocation of the other one through market clearing.

F.6 Production

The results derived so far rely on the assumption that assets are in fixed supply. I now study how optimal policies vary when financial markets determine production by influencing the intertemporal investment decision in a standard price-taking environment — this is the role explored in classic q-theory models.

Environment There is a new group of agents in the economy who were not present in the baseline model: identical competitive producers in unit measure. Producers are indexed by k and maximize well-behaved time separable expected utility, with flow utility given by $U_k(\cdot)$. They have exclusive access to a technology $\Phi(S_{1k})$, which allows them to issue or dispose of S_{1k} shares of the risky asset at date 1.[39] I refer to S_{1k}, which can be negative, as investment. The function $\Phi(\cdot)$ is increasing and strictly convex; that is, $\Phi'(\cdot) > 0$, $\Phi''(\cdot) > 0$. To ease the exposition, I assume throughout that $\Phi(S_{1k}) = \gamma_1 |S_{1k}| + \frac{\gamma_2}{2} |S_{1k}|^2$, with $\gamma_1, \gamma_2 > 0$. Producers are initially endowed with E_{1k} units of consumption good (dollars) and can only borrow or save in the risk-free asset at a (gross) rate $R = 1$. Their endowment E_{2k} at date 2 is stochastic and follows an arbitrary distribution.

To avoid distortions in primary markets, the planner does not tax the issuance of new shares. Importantly, market clearing is now given by $\int X_{1i} dF(i) = Q + S_{1k}$. Total output at date 2 in this economy is endogenous and given by $D(Q + S_{1k})$.

Positive results Producers thus maximize

$$\max_{C_{1k}, C_{2k}, S_{1k}, Y_k} U_k(C_{1k}) + \mathbb{E}[U_k(C_{2k})],$$

with budget constraints $Y_k + C_{1k} = E_{1k} + P_1^s S_{1k} - \Phi(S_{1k})$ and $C_{2k} = E_{2k} + Y_k$, where Y_k denotes the amount saved in the risk-free asset and P_1^s denotes the price faced by producers — the superscript s stands for supply. The optimality conditions for producers are given by

$$U_k'(C_{1k}) = \mathbb{E}[U_k'(C_{2k})] \quad \text{and} \quad P_1^s = \Phi'(S_{1k}).$$

The first condition is a standard Euler condition for the risk-free asset. The second condition provides a supply curve for the number of shares. Combining this supply curve with the portfolio choices of investors, generates the following equilibrium price:

$$P_1 = (1 - \alpha)\gamma_1 + \alpha P_1^e,$$

where the weight $\alpha \in [0, 1]$ — defined in the Appendix — is higher when the adjustment cost is very concave (γ_2 is large) and P_1^e is essentially the same expression for the price that would prevail in an exchange economy, which is given in Equation (6). Intuitively, the equilibrium price is a weighted average of the exchange economy price and γ_1, which is the replacement cost of the risky asset with linear adjustments costs.

Allowing for production does not affect those positive properties of the model that matter for the determination of the optimal tax. An increase in the transaction tax can increase, reduce, or keep equilibrium prices (and investment) constant, but all buyers buy less and all sellers sell less.

Normative results The marginal welfare impact of a tax change in producers' welfare from the planner's perspective, when measured in dollars, is given by

$$\begin{aligned} \frac{\frac{dV_k^p}{d\tau}}{U_k'(C_{1k})} &= \left[\frac{dP_1}{d\tau} S_{1k} + [P_1 - \Phi'(S_{1k})] \frac{dS_{1k}}{d\tau} - \frac{dY_k}{d\tau}\right] + \mathbb{E}\left[\frac{U_k'(C_{2k})}{U_k'(C_{1k})}\right] \frac{dY_k}{d\tau} \\ &= \mathbb{E}\left[\frac{U_k'(C_{2k})}{U_k'(C_{1k})}\right] \frac{dP_1}{d\tau} S_{1k}, \end{aligned}$$

[39] A "tree" analogy can be helpful here. Assume that a share of the risky asset (i.e., a tree) entitles the owner to a dividend payment D (fruit). Producers can plant new trees or chop them at a cost $\Phi(S_{1k})$, which they sell or buy at a price P_1. Producers would be willing to create trees until the marginal cost of producing a new tree/chopping and old tree $\Phi'(S_{1k})$ equals the marginal benefit of selling/buying P_1. For consistency, any normalization concerning Q must also normalize $\Phi(\cdot)$.

where the second line follows by substituting producers' optimality conditions. Intuitively, because producers do not pay taxes and invest optimally given prices, a marginal tax change only modifies their welfare through the distributive price effects on the shares they issue/repurchase. When P_1 is high, producers enjoy a better deal selling shares than when P_1 is low. The envelope theorem eliminates from $\frac{dV_k^p}{d\tau}$ the direct effects caused by changes in producers portfolio or investment choices.

Proposition 13 assumes that the planner accounts for producers' certainty equivalents, and characterizes the optimal tax.

Proposition 13. (Optimal tax in production economies) *The optimal tax in a production economy is given by*

$$\tau^* = \frac{\int \left(\frac{\mathbb{E}_p[D]-\mathbb{E}_i[D]}{P_1}\right) \frac{dX_{1i}}{d\tau} dF(i)}{-\int \operatorname{sgn}(\Delta X_{1i}) \frac{dX_{1i}}{d\tau} dF(i)} = (1-\omega)\,\tau^*_{exchange} + \omega \tau^*_{production}, \tag{74}$$

where $\tau^*_{exchange} = \frac{\zeta(\tau)\mathbb{C}ov_{F,\mathcal{T}}\left[\frac{\mathbb{E}_i[D]}{P_1}, \frac{dX_{1i}}{d\tau}\right]}{2\int_{i\in\mathcal{B}} \frac{dX_{1i}}{d\tau} dF(i)}$, $\tau^*_{production} = \frac{\mathbb{E}_p[D]-\mathbb{E}_{F,\mathcal{T}}[\mathbb{E}_i[D]]}{P_1}$ *and* $\omega < 1$ *is given in the Appendix (*ω *is small in magnitude when* $\frac{dS_{1k}}{d\tau} \approx 0$ *and close to unity when* $\left|\frac{dS_{1k}}{d\tau}\right|$ *is large).* $\mathbb{E}_{F,\mathcal{T}}[\mathbb{E}_i[D]]$ *denotes the average belief in the population of active investors,* $\mathbb{C}ov_{F,\mathcal{T}}\left[\mathbb{E}_i[D], \frac{dX_{1i}}{d\tau}\right]$ *denotes a cross-sectional covariance among active investors and* $\zeta(\tau) \equiv \int_{i\in\mathcal{T}} dF(i)$ *is the share of active investors.*

Proof. The expression for the asset price in Equation (75) now yields the following demand curve for risky asset shares

$$P_1^e = \frac{\int_{i\in\mathcal{T}} \left(\frac{\mathbb{E}_i[D]}{\mathcal{A}_i} - A\left(\mathbb{C}ov\left[M_{2i}, D\right] + \mathbb{V}ar\left[D\right] X_{0i}\right)\right) dF(i) - A\mathbb{V}ar\left[D\right] S_{1k}}{1 + \tau \int_{i\in\mathcal{T}} \frac{\operatorname{sgn}(\Delta X_{1i})}{\mathcal{A}_i} dF(i)} \tag{75}$$

The demand by investors for the risky asset is identical to the baseline model. The equilibrium price is now determined by the intersection of Equation (75) and the supply curve, given by $P_1^s = \gamma_1 + \gamma_2 S_{1k}$.

After writing the market clearing condition as $\int (X_{1i} - X_{0i})\, dF(i) = F(P_1)$, where $F(\cdot) = \Phi'^{-1}(\cdot)$ is an upward sloping function, we can derive $\frac{dP_1}{d\tau} = \frac{\int \frac{\partial X_{1i}}{\partial \tau} dF(i)}{F'(P_1) - \int \frac{\partial X_{1i}}{\partial P_1} dF(i)} = \frac{-P_1 \int \frac{\operatorname{sgn}(\Delta X_{1i})}{A_i \mathbb{V}ar[D]} dF(i)}{F'(P_1) + \int \frac{(1+\operatorname{sgn}(\Delta X_{1i})\tau)}{A_i \mathbb{V}ar[D]} dF(i)}$. $\frac{dP_1}{d\tau}$ can have any sign, depending on its numerator. We can write $\frac{dX_{1i}}{d\tau} = \frac{\partial X_{1i}}{\partial \tau}\varepsilon_i$, where ε_i, which is constant within buyers/sellers, can be expressed as $\varepsilon_i = 1 - (\operatorname{sgn}(\Delta X_{1i}) + \tau) \frac{1-H}{\frac{\mathbb{V}ar[D]F'(P_1)}{\int_B \frac{1}{A_i} dF(i)} + 1 + \tau + H(1-\tau)}$ and $H \equiv \frac{\int_{i\in\mathcal{S}} \frac{1}{A_i} dF(i)}{\int_{i\in\mathcal{B}} \frac{1}{A_i} dF(i)} \in (0, \infty)$. It is easy to show that $\varepsilon_i > 0$, which proves the result.

The marginal change in social welfare is given by

$$\frac{dV^p}{d\tau} = \int \left[\left(\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right] + \operatorname{sgn}\left(\Delta X_{1i}\right) P_1 \tau\right) \frac{dX_{1i}}{d\tau} - \Delta X_{1i} \frac{dP_1}{d\tau}\right] dF(i) + \frac{dP_1}{d\tau} S_{1k}$$

Using market clearing, the marginal change in social welfare can be expressed as follows

$$\frac{dV^p}{d\tau} = \int \left(\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right] + \operatorname{sgn}\left(\Delta X_{1i}\right) P_1 \tau\right) \frac{dX_{1i}}{d\tau} dF(i)$$

Solving for τ^* in the previous expression yields Equation (74). We can re-write the numerator of the optimal tax as

$$\begin{aligned}
\int \left(\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right]\right) \frac{dX_{1i}}{d\tau} dF(i) &= \zeta(\tau)\, \mathbb{E}_{F,\mathcal{T}}\left[\left(\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right]\right) \frac{dX_{1i}}{d\tau}\right] \\
&= \zeta(\tau) \begin{pmatrix} \mathbb{C}ov_{F,\mathcal{T}}\left[\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right], \frac{dX_{1i}}{d\tau}\right] \\ +\mathbb{E}_{F,\mathcal{T}}\left[\mathbb{E}_p\left[D\right] - \mathbb{E}_i\left[D\right]\right] \mathbb{E}_{F,\mathcal{T}}\left[\frac{dX_{1i}}{d\tau}\right] \end{pmatrix} \\
&= -\zeta(\tau)\, \mathbb{C}ov_{F,\mathcal{T}}\left[\mathbb{E}_i\left[D\right], \frac{dX_{1i}}{d\tau}\right] + \left(\mathbb{E}_p\left[D\right] - \mathbb{E}_{F,\mathcal{T}}\left[\mathbb{E}_i\left[D\right]\right]\right) \frac{dS_{1k}}{d\tau},
\end{aligned}$$

where we can define $\zeta(\tau) \equiv \int_{i\in\mathcal{T}} dF(i)$. This normalization by the number of active investors is necessary to use expectation and covariance operators. Using the fact that $\int_{i\in\mathcal{S}} \frac{dX_{1i}}{d\tau} dF(i) = \frac{dS_{1k}}{d\tau} - \int_{i\in\mathcal{B}} \frac{dX_{1i}}{d\tau} dF(i)$, the denominator in (74) can be expressed as $\int \operatorname{sgn}(\Delta X_{1i}) \frac{dX_{1i}}{d\tau} dF(i) = \int_{i\in\mathcal{B}} \frac{dX_{1i}}{d\tau} dF(i) - \int_{i\in\mathcal{S}} \frac{dX_{1i}}{d\tau} dF(i) = 2\int_{i\in\mathcal{B}} \frac{dX_{1i}}{d\tau} dF(i) - \frac{dS_{1k}}{d\tau}$. By substituting and rearranging the previous two expressions in the optimal tax formula, we can write τ^*as follows

$$\tau^* = \underbrace{\frac{-2\int_{i\in\mathcal{B}} \frac{dX_{1i}}{d\tau} dF(i)}{-2\int_{i\in\mathcal{B}} \frac{dX_{1i}}{d\tau} dF(i) + \frac{dS_{1k}}{d\tau}}}_{\equiv 1-\omega} \underbrace{\frac{-\zeta(\tau)\,\mathbb{C}ov_{F,\mathcal{T}}\left[\frac{\mathbb{E}_i[D]}{P_1}, \frac{dX_{1i}}{d\tau}\right]}{-2\int_{i\in\mathcal{B}} \frac{dX_{1i}}{d\tau} dF(i)}}_{\equiv \tau^*_{\text{exchange}}} + \underbrace{\frac{\frac{dS_{1k}}{d\tau}}{-2\int_{i\in\mathcal{B}} \frac{dX_{1i}}{d\tau} dF(i) + \frac{dS_{1k}}{d\tau}}}_{\equiv \omega} \underbrace{\frac{\mathbb{E}_p[D] - \mathbb{E}_{F,\mathcal{T}}[\mathbb{E}_i[D]]}{P_1}}_{\equiv \tau^*_{\text{production}}}.$$

□

Corollary. (τ^* may depend on the planner's belief) *The optimal financial transaction tax in a production economy depends on the distribution of payoffs assumed by the planner. However, if the planner uses the average belief across investors, that is, $\mathbb{E}_p[D] = \mathbb{E}_{F,\mathcal{T}}[\mathbb{E}_i[D]]$ at the optimum, the optimal tax is identical to the one in the exchange economy and independent of the belief used by the planner.*[40]

The numerator in Equation (74), which evaluated at $\tau = 0$ determines the sign of the optimal tax can be decomposed in two terms

$$\int (\mathbb{E}_p[D] - \mathbb{E}_i[D]) \frac{dX_{1i}}{d\tau} dF(i) = -\zeta(\tau) \underbrace{\mathbb{C}ov_{F,\mathcal{T}}\left[\mathbb{E}_i[D], \frac{dX_{1i}}{d\tau}\right]}_{\text{Belief dispersion}} + \underbrace{(\mathbb{E}_p[D] - \mathbb{E}_{F,\mathcal{T}}[\mathbb{E}_i[D]]) \frac{dS_{1k}}{d\tau}}_{\substack{\text{Aggregate belief difference}\times \\ \text{Investment response}}}. \quad (76)$$

Because the second term in Equation (76) is in general non-zero when $\tau = 0$, we can say that belief distortions in production economies have an additional first-order effect on welfare. Again asset prices do not appear in optimal tax formulas, despite playing a role in determining allocations. All welfare losses must be traced back to distortions in "quantities", either in portfolio allocations, captured by $\frac{dX_{1i}}{d\tau}$, or in production decisions, captured by $\frac{dS_{1k}}{d\tau}$.

Intuitively, the optimal tax corrects two wedges created by heterogeneous beliefs. First, given an amount of aggregate risk, the optimal tax seeks to reduce the asset holding dispersion induced by disagreement — some investors are holding too much risk and some others too little risk. This is the same mechanism present in exchange economies. Second, as long as the average belief differs from the one used by the planner, the level of production in the economy is too high (low) when investors are on average too optimistic (pessimistic). This provides a second rationale for taxation. Intuitively, the investors in the economy hold too much aggregate risk when they are on average optimistic or too little when they are pessimistic.[41]

Let's describe the sign of $\omega\tau^*_{\text{production}}$ next. Belief dispersion is not sufficient anymore to pin down the sign of the optimal tax, which now also depends on whether $\mathbb{E}_p[D] - \mathbb{E}_F[\mathbb{E}_i[D]]$ and $\frac{dS_{1k}}{d\tau}$ have the same or opposite signs. Intuitively, if a marginal tax increase reduces (increases) investment at the margin when investors are too optimistic (pessimistic), a positive tax is welfare improving, and vice versa. Table 5 summarizes the conditions that determine the sign of the term associated with production.

Unlike in the exchange economy, in which the independence of beliefs justifies that the belief dispersion term is negative, it is not obvious whether we should expect $\omega\tau^*_{\text{production}}$ to be positive or negative. For investment to be reduced (increased) at the margin by a tax increase, it has to be the case that the (risk aversion adjusted) mass of buyers is larger (smaller) than the mass of sellers. In principle, the relation between the average belief

[40]The average belief may change if there are changes in the composition of marginal investors. For the irrelevance result to hold without further qualifications, the average belief for marginal investors must be invariant to the level of τ.

[41]If there were many produced risky assets, the welfare losses would capture the idea that belief distortions misallocate real investment across sectors in the economy. These results are available upon request.

	Aggregate optimism	Aggregate pessimism
	$\mathbb{E}_F[\mathbb{E}_i[D]] > \mathbb{E}_p[D]$	$\mathbb{E}_F[\mathbb{E}_i[D]] < \mathbb{E}_p[D]$
$\int_{\mathcal{B}} \frac{1}{A_i} dF(i) > \int_{\mathcal{S}} \frac{1}{A_i} dF(i)$	$\omega\tau^*_{\text{production}} > 0$	$\omega\tau^*_{\text{production}} < 0$
$\int_{\mathcal{B}} \frac{1}{A_i} dF(i) < \int_{\mathcal{S}} \frac{1}{A_i} dF(i)$	$\omega\tau^*_{\text{production}} < 0$	$\omega\tau^*_{\text{production}} > 0$

Table 2: Sign of $\omega\tau^*_{\text{production}}$

distortion and the relative mass of buyers/sellers need not be linked, so the sign of $\omega\tau^*_{\text{production}}$ is theoretically ambiguous. Additional policy instruments, like short-sale of borrowing constraints, investment taxes, or active monetary policy can be used to target the production distortion induced by beliefs, allowing the transaction tax to be exclusively focused again on the dispersion of beliefs among investors.

Many informal discussions regarding the convenience of a transaction tax, following Tobin (1978), revolve around the notion that it would help reduce price volatility. Implicit in those discussions is the notion that high volatility is bad. The results in this section show that it is not price volatility, a variance, but whether investment (through prices) is lower when investors are optimistic and vice versa, a covariance, what captures the welfare consequences of a transaction tax in a production context.

The optimal tax can be expressed as a linear combination between the optimal tax in a (fictitious) exchange economy and the optimal tax in a (fictitious) production economy with a single investor with belief $\mathbb{E}_{F,\mathcal{T}}[\mathbb{E}_i[D]]$. The sensitivity of investment with respect to a tax change determines the relative importance of each term. Market clearing now implies that $\int \frac{dX_{1i}}{d\tau} dF(i) = \frac{dS_{1k}}{d\tau}$, which can take any positive or negative value. Hence, in production economies, the belief used by the planner to calculate welfare matters in general for the optimal policy. However, if the planner uses investors' average belief to calculate welfare, the belief used by the planner drops out of the optimal tax expression. Because of its importance, I state this result as a corollary of Proposition 13.

F.7 Tax on the number of shares

The baseline model assumes the tax is levied on the dollar value of a trade rather than on the number of shares traded to prevent investors from circumventing it by varying the effective number of shares traded through a reverse split. All results apply to taxes that depend on the number of shares with minor modifications.

When P_1 is exactly zero, a tax based on the dollar volume of the transaction is ineffective. However, a tax based on the number of shares traded $|\Delta X_{1i}|$ can be introduced to effectively tax the notional value of the contract. I extend here Proposition 1 to the case of taxes levied on the number of shares traded. In this case, the distinction between buyers and sellers is somewhat arbitrary, giving support to the idea that both sides of the market should face the same tax.

In the trade region, the optimal portfolio choice of an investor can be expressed as: $X_{1i} = \frac{\mathbb{E}_i[D] - A_i \mathbb{C}ov[M_{2i}, D] - P_1 - \text{sgn}(\Delta X_{1i})\tau}{A_i \mathbb{V}ar[D]}$. The equilibrium price becomes:

$$P_1 = \int_{i \in \mathcal{T}} \left(\frac{\mathbb{E}_i[D]}{\mathcal{A}_i} - A\left(\mathbb{C}ov[M_{2i}, D] - \mathbb{V}ar[D] X_{0i}\right) - \frac{\text{sgn}(\Delta X_{1i})}{\mathcal{A}_i}\tau \right) dF(i).$$

The price correction is now additive rather than multiplicative. The value of $\frac{dV}{d\tau}$ corresponds to $\frac{dV}{d\tau} = \int \left[-\mathbb{E}_i[D] + \text{sgn}(\Delta X_{1i})\tau\right] \frac{dX_{1i}}{d\tau} dF(i)$. The optimal tax now satisfies:

$$\tau^* = \frac{\int \mathbb{E}_i[D] \frac{dX_{1i}}{d\tau} dF(i)}{\int \text{sgn}(\Delta X_{1i}) \frac{dX_{1i}}{d\tau} dF(i)}.$$

This shows that optimal taxes in the paper are written in terms of returns because they are levied on the dollar value of the transaction. When they are levied on the number of shares, the dispersion in expected payoffs rather than the dispersion in expected returns becomes the welfare relevant variable.

F.8 Harberger calculation

The results derived so far rely on the assumption that the planner maximizes welfare using a single belief. However, it is straightforward to quantify the welfare loss induced by a tax increase assuming that all investors hold correct beliefs or that the planner assesses social welfare respecting individual beliefs. Under either of these assumptions, all trades are regarded as fundamental, so any tax induces a welfare loss. I derive a result analogous to Harberger (1964), whose triangle analysis can be traced back to Dupuit (1844).

Proposition 14. (Harberger (1964) revisited)

a) When investors hold identical beliefs or the planner respects individual beliefs when calculating social welfare, the marginal welfare loss generated by increasing the transaction tax at a level $\tilde{\tau}$, expressed as a money-metric (in dollars) at $t = 1$, is given by

$$\int \left.\frac{dV_i}{d\tau}\right|_{\tau=\tilde{\tau}} dF(i) = 2\tilde{\tau} P_1 \int_{i\in\mathcal{B}} \left.\frac{dX_{1i}}{d\tau}\right|_{\tau=\tilde{\tau}} dF(i) \leq 0, \tag{77}$$

where $i \in \mathcal{B}$ denotes that the integration is made only over the set of buyers and $\hat{V}_i$ denotes investors' certainty equivalents.

b) The marginal welfare loss of a small tax change around $\tau = 0$ can be approximated, using a second order Taylor expansion, by:

$$\mathcal{L}(\tau) \equiv dV = \int d\hat{V}_i\Big|_{\tau=0} dF(i) \approx \tau^2 P_1 \int_{i\in\mathcal{B}} \left.\frac{dX_{1i}}{d\tau}\right|_{\tau=0} dF(i)$$

Proof. a) When there are no belief differences between investors and the planner or when the planner assesses social welfare respecting individual beliefs, we can write the marginal change in welfare as a money-metric (divided by investors' marginal utility) as:

$$\left.\frac{dV_i}{d\tau}\right|_{\tau=\tilde{\tau}} = \text{sgn}(\Delta X_{1i})\,\tilde{\tau} P_1 \frac{dX_{1i}}{d\tau} - \Delta X_{1i} \frac{dP_1}{d\tau}$$

Adding up across all investors, and using the fact that $\int \text{sgn}(\Delta X_{1i}) \frac{dX_{1i}}{d\tau} dF(i) = 2 \int_{i\in\mathcal{B}} \frac{dX_{1i}}{d\tau} dF(i)$, we then recover Equation (77).

b) The result in a) is an exact expression. However, we can write a second order approximation around $\tau = 0$ of the marginal change in social welfare. Note all terms corresponding to terms-of-trade cancel out after imposing market clearing, so I do not consider them. The first term of the Taylor expansion is given above. The derivative of the second term of the Taylor expansion is given by: $\text{sgn}(\Delta X_{1i}) P_1 \frac{dX_{1i}}{d\tau} + \text{sgn}(\Delta X_{1i})\,\tau P_1 \frac{d^2 X_{1i}}{d\tau^2}$. Around $\tau = 0$, this becomes $\text{sgn}(\Delta X_{1i}) P_1 \frac{dX_{1i}}{d\tau}$. Hence, when $\tau = 0$ we can write:

$$\begin{aligned}
\int dV_i\big|_{\tau=0} dF(i) &\approx \int \text{sgn}(\Delta X_{1i})\,\tau P_1 \left.\frac{dX_{1i}}{d\tau}\right|_{\tau=0} dF(i)\,(d\tau) \\
&\quad + \frac{1}{2}\int \left.\left(\text{sgn}(\Delta X_{1i}) P_1 \frac{dX_{1i}}{d\tau} + \text{sgn}(\Delta X_{1i})\,\tau P_1 \frac{d^2 X_{1i}}{d\tau^2}\right)\right|_{\tau=0} dF(i)\,(d\tau)^2 \\
&= P_1 \tau^2 \int_{i\in\mathcal{B}} \left.\frac{dX_{1i}}{d\tau}\right|_{\tau=0} dF(i).
\end{aligned}$$

□

This result provides a measure of welfare losses as a function of observables for any tax intervention. Given the money-metric correction, investors in this economy are willing to pay $\mathcal{L}(\tau)$ dollars to prevent a change in the tax rate. Note that this happens to correspond to the marginal change in revenue raised. Equation (77) derives an upper bound for the size of the welfare losses induced by taxation in the case in which all trades are deemed to be fundamental.

Equation (77) resembles the classic Harberger (1964) result about welfare losses in the context of commodity taxation.[42] However, the welfare loss in this case is given by twice the size of the tax, because the portfolio holdings of both buyers and sellers are distorted. Taxing a commodity distorts the amount consumed of a given good, reducing welfare. Taxing financial transactions distorts portfolio allocations, inducing investors to hold more or less risk than they should, also reducing welfare. The distortion created by a tax (approximately) grows with the square in this context of the model studied in this paper.

F.9 Disagreement about other moments

Environment Motivated by Proposition 5, in the baseline model, investors only disagree about the expected value of the payoff of the risky asset. I now assume that investors also hold distorted beliefs about their hedging needs $\mathbb{C}ov_i[M_{2i}, D]$ and about the variance of the payoff of the risky asset $\mathbb{V}ar_i[D]$.

Results The optimality condition presented in Equation (5) applies directly, after using the individual beliefs of each investor. Hedging needs enter additively, but perceived individual variances modify the sensitivity of portfolio demands with respect to the baseline case.

Market clearing determines the equilibrium price, given now by:

$$P_1 = \frac{\int_{i\in\mathcal{T}} \left(\frac{\mathbb{E}_i[D]}{\mathcal{AV}_i} - AV\left(\beta_{ii} + X_{0i}\right)\right) dF(i)}{1 + \tau \int_{i\in\mathcal{T}} \frac{\mathrm{sgn}(\Delta X_{1i})}{\mathcal{AV}_i} dF(i)},$$

where $AV \equiv \left(\int_{i\in\mathcal{T}} \frac{1}{A_i \mathbb{V}ar_i[D]} dF(i)\right)^{-1}$ is the harmonic mean of risk aversion coefficients and perceived variances for active investors and $\mathcal{AV}_i \equiv \frac{A_i \mathbb{V}ar_i[D]}{AV}$ is the quotient between investor i risk aversion times perceived variance and the harmonic mean. I define the regression coefficient (beta) of individual endowments M_{2i} on payoffs D perceived by investors by $\beta_{ii} = \frac{\mathbb{C}ov_i[M_{2i},D]}{\mathbb{V}ar_i[D]}$. Again, $\mathcal{T}$ denotes the set of active investors.

Proposition 15. (Disagreement about second moments)

a) The marginal change in social welfare from varying the financial transaction tax when investors disagree about second moments is given by:

$$\frac{dV^p}{d\tau} = \int \left[\left(-r_i \mathbb{E}_i[D] - A_i \mathbb{C}ov[M_{2i}, D]\left(1 - \frac{\beta_{ii}}{\beta_i}\right) + P_1 r_i \left(1 + \mathrm{sgn}(\Delta X_{1i})\tau\right)\right) \frac{dX_{1i}}{d\tau}\right] dF(i), \tag{78}$$

where $r_i \equiv \frac{\mathbb{V}ar[D]}{\mathbb{V}ar_i[D]}$, $\beta_{ii} \equiv \frac{\mathbb{C}ov_i[M_{2i},D]}{\mathbb{V}ar_i[D]}$ *and* $\beta_i \equiv \frac{\mathbb{C}ov[M_{2i},D]}{\mathbb{V}ar[D]}$. *Note that* $r_i \in (0,\infty)$ *and* $\beta_i, \beta_{ii} \in (-\infty,\infty)$.

b) The optimal tax when investors disagree about second moments is given by:

$$\tau^* = \frac{\int \left(r_i \mathbb{E}_i[D] + A_i \mathbb{C}ov[M_{2i}, D]\left(1 - \frac{\beta_{ii}}{\beta_i}\right)\right) \frac{dX_{1i}}{d\tau} dF(i)}{P_1 \int \left(r_i \left(1 + \mathrm{sgn}(\Delta X_{1i})\right)\right) \frac{dX_{1i}}{d\tau} dF(i)}.$$

Proof. The optimal portfolio allocation for an investor i in his trade region is given by:

$$X_{1i} = \frac{\mathbb{E}_i[D] - A_i \mathbb{C}ov_i[M_{2i}, D] - P_1\left(1 + \mathrm{sgn}(\Delta X_{1i})\tau\right)}{A_i \mathbb{V}ar_i[D]}.$$

The marginal change in welfare for an investor i is given by:

$$\frac{dV_i^p}{d\tau} = \begin{bmatrix} \left(\mathbb{E}_p[D] - A_i \mathbb{C}ov[M_{2i}, D] - P_1\right) \frac{dX_{1i}}{d\tau} \\ -r_i\left(\mathbb{E}_i[D] - A_i \mathbb{C}ov_i[M_{2i}, D] - P_1\left(1 + \mathrm{sgn}(\Delta X_{1i})\tau\right)\right) \frac{dX_{1i}}{d\tau} - \Delta X_{1i} \frac{dP_1}{d\tau} \end{bmatrix}.$$

[42] Although this result is intuitive, to my knowledge, it had not been derived before in the context of a portfolio choice problem. See Auerbach and Hines Jr. (2002) for a comprehensive analysis of tax efficiency results and Sandmo (1985) for a survey of results on how taxation affects portfolio allocations.

Where $r_i \equiv \frac{\mathbb{V}ar[D]}{\mathbb{V}ar_i[D]}$. The change in social welfare can then be written as:

$$\frac{dV^p}{d\tau} = \int \left(-r_i \mathbb{E}_i [D] - A_i \mathbb{C}ov [M_{2i}, D] \left(1 - \frac{\beta_{ii}}{\beta_i}\right) + P_1 r_i (1 + \text{sgn}(\Delta X_{1i}) \tau)\right) \frac{dX_{1i}}{d\tau} dF(i).$$

Solving for τ in this equation, which corresponds to Equation (78) in the paper, delivers the expression for the optimal tax in Proposition 15b). □

The formula for the optimal tax now incorporates hedging needs and modifies the weights given to investors' beliefs. An investor with correct beliefs about second moments has $r_i = 1$ and $\beta_{ii} = \beta_i$; in that case, we recover Equation (12). When investors perceive a high variance, that is, r_i is close to 0, they receive less weight in the optimal tax formula. The opposite occurs when they perceive a low variance. Intuitively, lower perceived variances amplify distortions in expected payoffs, and vice versa.

As in the baseline model, the planner does not need to know the value of $\mathbb{E}_p [D]$ to implement the optimal tax. However, if investors hold distorted beliefs about their hedging needs, the planner needs to know explicitly the magnitude of the mistake. Intuitively, there is no mechanism in the model which cancels out the mistakes in hedging made by investors. The sign of the optimal tax depends directly on the errors made by investors when hedging.

There are two interesting parameters restrictions. First, when investors with correct expected payoffs and hedging betas, that is $\frac{\beta_{ii}}{\beta_i} = 1$, disagree about variances, the optimal tax τ^* turns out to be:

$$\tau^* = \frac{\mathbb{E}[D] \int r_i \frac{dX_{1i}}{d\tau} dF(i)}{P_1 \int r_i (1 + \text{sgn}(\Delta X_{1i})) \frac{dX_{1i}}{d\tau} dF(i)}.$$

The dispersion of variances, given by $\mathbb{C}ov_{F,\mathcal{T}} \left[r_i, \frac{dX_{1i}}{d\tau}\right]$, determines now the sign of the optimal tax. When r_i is constant (although not necessarily equal to one), the optimal tax becomes zero. This reinforces the intuition that belief dispersion is what matters for optimal taxes in an exchange economy. Intuitively, when buyers, with $\frac{dX_{1i}}{d\tau} < 0$, are relatively aggressive, that is, r_i is large, they are buying too much of the risky asset, so $\mathbb{C}ov_{F,\mathcal{T}} \left[r_i, \frac{dX_{1i}}{d\tau}\right]$ is negative and the optimal tax is positive, and vice versa.

Second, when investors have correct beliefs about the mean and the variance of expected returns, but hedge incorrectly, the optimal tax becomes:

$$\tau^* = \frac{\mathbb{V}ar[D] \int A_i (\beta_i - \beta_{ii}) \frac{dX_{1i}}{d\tau} dF(i)}{P_1 \int \text{sgn}(\Delta X_{1i}) \frac{dX_{1i}}{d\tau} dF(i)}.$$

The optimal tax now has the opposite sign of $\mathbb{C}ov_{F,\mathcal{T}} \left[A_i (\beta_i - \beta_{ii}), \frac{dX_{1i}}{d\tau}\right]$. Intuitively, when buyers, with $\frac{dX_{1i}}{d\tau} < 0$, overestimate their need for hedging and end up buying too much of the risky asset — this occurs when $\beta_i - \beta_{ii} < 0$ — the optimal tax is positive, and vice versa.