A Toolkit for Solving Models with a Lower Bound on Interest Rates of Stochastic Duration

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A Appendix

A.1 Proof of Lemma 1

Proof. The system can be written as:

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} E_t Zt+1 \\ P_t \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} Z_t \\ P_{t-1} \end{bmatrix}$$

$$\begin{bmatrix} A_1 & A_2 & -B_1 & -B_2 \\ A_3 & A_4 & -B_3 & -B_4 \end{bmatrix} \begin{bmatrix} E_t Z_{t+1} \\ P_t \\ Z_t \\ P_{t-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_2 & -B_1 & -B_2 & A_1 \\ A_4 & -B_3 & -B_4 & A_3 \end{bmatrix} \begin{bmatrix} P_t \\ Z_t \\ P_{t-1} \\ E_t Z_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A.1}$$

Let the row reduced echelon form of $\begin{bmatrix} A_2 & -B_1 & -B_2 & A_1 \\ A_4 & -B_3 & -B_4 & A_3 \end{bmatrix}$ be $\begin{bmatrix} I & 0 & -C_1 & -C_2 \\ 0 & I & -C_3 & -C_4 \end{bmatrix}$

The system will then be:

$$\begin{bmatrix} I & 0 & -C_1 & -C_2 \\ 0 & I & -C_3 & -C_4 \end{bmatrix} \begin{bmatrix} P_t \\ Z_t \\ P_{t-1} \\ E_t Z_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P_t \\ Z_t \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{bmatrix} P_{t-1} \\ E_t Z_{t+1} \end{bmatrix} \tag{A.2}$$

At time $t+1$, recall that $j \equiv k_\tau - (t-\tau)$, we know that:

$$\begin{aligned} P_{t+1} &= G^{2,j-1} P_t \\ Z_{t+1} &= D^{2,j-1} P_t \end{aligned}$$

It follows that $E_t Z_{t+1} = E_t D^{2,j-1} P_t = D^{2,j-1} P_t$. By substituting this result in (A.2) we can solve the system:

$$\begin{bmatrix} P_t \\ Z_t \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{bmatrix} P_{t-1} \\ E_t Z_{t+1} \end{bmatrix}$$
$$\begin{bmatrix} P_t \\ Z_t \end{bmatrix} = \begin{bmatrix} C_1 P_{t-1} + C_2 D^{2,j-1} P_t \\ C_3 P_{t-1} + C_4 D^{2,j-1} P_t \end{bmatrix}$$
$$\begin{bmatrix} P_t \\ Z_j \end{bmatrix} = \begin{bmatrix} \left(I - C_2 D^{2,j-1}\right)^{-1} C_1 P_{j-1} \\ \left(C_3 + C_4 D^{2,j-1} \left(I - C_2 D^3\right)^{-1} C_1\right) P_{t-1} \end{bmatrix}$$
$$\begin{bmatrix} P_j \\ Z_j \end{bmatrix} = \begin{bmatrix} G^{2,j} P_{t-1} \\ \left(C_3 + C_4 D^{2,j-1} G^{2,j}\right) P_{t-1} \end{bmatrix} \tag{A.3}$$
$$\begin{bmatrix} G^{2,j} \\ D^{2,j} \end{bmatrix} = \begin{bmatrix} \left[I - C_2 D^{2,j-1}\right]^{-1} C_1 \\ C_3 + C_4 D^{2,j-1} G^{2,j} \end{bmatrix} \tag{A.4}$$

□

A.2 Code Setup

In this Section we explain how to set up codes for the toolkit for the New Keynesian model under OCP as explained in Section 4. The solution algorithm is generated in three functions that have to be run sequentially. The essential inputs of those functions are (1) the matrices that define the model; (2) a parameter structure that is described below; (3) a configuration structure that is described below. There are several optional inputs that we explain at the end of the Section.

A.2.1 Variables and Matrices

Our toolkit requires the model to be cast in the form specified Equation 5. In addition, a specific ordering of variables and equations is necessary for the code to recognise state variables, shocks and the equation subject to the occasionally binding constraint. In particular, the user should order elements in the vector ξ_t such that the jump variables come first, the nominal rate should be between jump variables and predetermined variables, predetermined variables should follow, and finally shock variables should be at the end of the vector.[35] Furthermore, the order of rows in the matrices A and B must satisfy that the last equation is the one that is slack when in regimes 1 and 2 (e.g. a policy rule for the central bank), and the block preceding it is made by as many identity rows as there are shocks (i.e. equations in the form $\mathbb{E}_t u_{t+1} = u_t$).

As an example, consider Equations (12)-(18) in the absorbing state, where (16) is binding. The system can

[35]Note that if the system has constants, one can implement them in 2 ways. First, the user could define a predetermined variable – along with its own initial value – that never change value across time. A second way is to increase the shock vector to include a component that does not change value across regimes.

be written as $A\,\mathbb{E}_t\xi_{t+1} = B\,\xi_t$,[36] where $\xi_t \equiv \begin{bmatrix}\hat{Y}_t & \pi_t & i_t & \phi_{1,t-1} & \phi_{2,t-1} & r^n_{t-1} & u_t\end{bmatrix}'$ and

$$A = \begin{bmatrix} 1 & \sigma & 0 & 0 & 0 & \sigma & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & \kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & \sigma & 0 & 0 & 0 & 0 \\ -\kappa & 1 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & -\frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{\beta}\sigma & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.5}$$

A.2.2 Parameters structure

The parameter structure (we name it *param*) needs to contain the following elements.

1. A scalar element μ, equal to the Markov probability described in the text.
2. Two vector elements s_h and s_l equal to the values of the two-state Markov process in the absorbing and crisis states. The order needs to be the same as in the vector ξ_t.
3. A scalar element N_S, that is equal to the number of state variables in ξ_t, including the exogenous shocks.

A.2.3 Configuration structure

The configuration structure (we name it *config*) needs to contain the following elements.

1. A scalar element τ_{max}, equal to the time at which the Markov shock reverts to its absorbing state with probability 1, conditional on being in the crisis state at $\tau_{max} - 1$.
2. A scalar element $max_{length2}$, equal to the maximum length of regime 2. This is a shortcut to save computing time. The toolkit will warn the user if regime 2 requires a higher value for $max_{length2}$.
3. A scalar element *bound*, equal to lower bound in the inequality constraint, e.g. 0 for the zero lower bound.
4. A scalar element *mono*. *mono*=1 means toolkit will run under the assumption that the vector k represents a monotone function.

A.2.4 The general setup

The user should construct the elements described above and set the code as follows.

```
[D_3,G_3,D_3a]                        = regime3(AAA,BBB,param);
[D_2,G_2]                             = regime2(AAA,BBB,D_3a,param,config);
[D_1,G_1, ResM, max_k,k,T_tilde]      = ...
        regime1(AAA,BBB,D_3a,D_3,D_2,G_3,G_2,param,config);
impulseresponse
```

[36] We suggest to use AAA and BBB in place of A and B in order to avoid coding conflicts with the inputs in other scripts.

A.2.5 The function **regime3.m**

The function `regime3.m` takes as inputs the matrices AAA and BBB, as well as the parameters structure `param`.

```
[D_3,G_3,D_3a] = regime3(AAA,BBB,param);
```

This function provides transition matrices that are then used in the other functions.

A.2.6 The function **regime2.m**

The function `regime2.m` takes as inputs the matrices AAA and BBB, the parameters structure `param`, the configuration structure *config*, as well as one output (D_{3a}) from the function `regime3.m`.

```
[D_2,G_2] = regime2(AAA,BBB,D_3a,param,config);
```

This function provides transition matrices that are then used in the function `regime1.m`.

A.2.7 The function **regime1.m**

The function `regime1.m` takes as inputs the matrices AAA and BBB, the parameters structure `param`, the configuration structure `config`, as well as the output from the functions `regime3.m` and `regime2.m`.

```
[D_1,G_1, ResM, max_k,k,T_tilde] = ...
        regime1(AAA,BBB,D_3a,D_3,D_2,G_3,G_2,param,config);
```

This function provides a 3 dimensional matrix $ResM$, a scalar max_k, the vector k, the scalar $\tilde{T}$, and the transition matrices in `regime1.m`.

The dimensions of $ResM$ are time, variables, and contingencies. For instance, the element $ResM(5,1,8)$ contains the value of the variable in position 1 in ξ_t, at time 5, for contingency 8.

The vector k is a vector that links contingencies and their respective duration of regime 2. max_k is the maximum value across k, and the scalar $\tilde{T}$ is the period at which regime 1 starts.[37]

Optional inputs There are several optional parameters to `regime1.m`. The user can choose to have any combination of the following:

Option	Input	Description
`verbose`	0 (default) | 1	display real-time diagnostics from the search algorithm
`k_input`	vector	impose arbitrary k
`T_tilde_input`	scalar	input a value for $\tilde{T}$
`R0_search`	0 | 1 (default)	search for $\tilde{T}$

The input of a vector k shall be taken with some caution. Notice that if k has too low entries (e.g. there is too little stimulus), the toolkit will force k to change because regime 3 would feature a violation of the inequality constraint. On the other hand, shall one want to impose a k that is large enough (for instance in the case of a fixed horizon forward guidance experiment), the user should be aware it is necessary to shut down the search for $\tilde{T}$ at the same time to fulfil the purpose. For the third optional input, an issue

[37]To be precise, the scalar $\tilde{T}$ is the period at which the regime that follows regime 0 starts. In most cases this will be regime 1. However, it can be the case that the code switches to regime 2 or 3 immediately after regime 0. This is for example the case if the ELB is never binding.

arises when the user inputs a value for $\tilde{T}$ that is too large. The toolkit will not find a solution because of the algorithm.[38] To avoid this, one should shut down the search for $\tilde{T}$ as well. In this case the toolkit will give a solution that features the value of $\tilde{T}$ chose by the user (otherwise the default value is 1). Note that this solution may violate the inequality constraint in regime 0, exactly because the algorithm for the search of $\tilde{T}$ is shut down. Shutting down the regime 0 algorithm, and setting `T_tilde_input` to τ_{max} corresponds to the case of the ELB never binding.

A.2.8 The script `impulseresponse.m`

This script generates impulse response functions as weighted averages of the evolution of each variable across all contingencies, weighted by the ex-ante probability of the shock reverting to the absorbing state. The script generates a two-dimensional matrix, where each row is a period and column is a variable. Notice that the matrix `ResM` contains variables in *levels*, so percentage point variations can be obtained by simply multiplying by 100. In addition in some cases, as for example for inflation, the variable is defined on a quarterly basis, so it needs to be multiplied by 4 to yield annualised variations.

A.2.9 The function `graphing.m`

In addition, we provide a basic graphics function to produce plots for IRFs as in Section 4. The function has the following required inputs:

- the matrix *IR* containing the impulse response functions, as generated by `impulseresponse.m`;
- a variables structure, that contains the position of each variable;
- a scalar for the horizon of the plots;

This default inputs will produce the average impulse response function for all variables in the model, averaging IRFs across contingencies.

Optional inputs Options for this function allow to plot only a subset of variables and to superimpose impulse response functions for a specific set of contingencies:

Option	Input	Description
`variables`	cell array	plot a subset of variables listed in cell array
`cont_data`	matrix *ResM*	to be used in conjunction with `cont_num`. Provides IRFs for each contingency
`cont_num`	vector	selects contingencies to plot

The line of code below shows an example.

```
graphing(IR,var,30,'variables',{'pi','y','i'},'cont_data',ResM,'cont_num',1:30)
```

This line produces the impulse response function for variables π, y and i, as well as the IRFs specific to contingencies 1 to 30.

A.3 Data

Output shows deviation of real GDP from the linear trend of real GDP estimated on 2000Q1-2007Q2 sample. Inflation shows chain-type price index of personal consumption expenditures excluding food and energy, percentage change to previous quarter, annualised. The interest rate is the Federal Funds

[38] See Section 3 for more details.

Rate. With the exception of nominal GDP in Figure 8, data series for the price level, NGDP, cumulated NGDP deviations and Dual Mandate index are constructed from output and inflation series.[39]

A.4 Additional welfare metrics

In the Tables providing performance metrics for the policy rules we calculate a volatility index for selected variables z_t as the following.

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (z_t - \bar{z})^2 \tag{A.6}$$

A.5 FRBNY DSGE model

The FRBNY DSGE model was developed for policy analysis at the Federal Reserve Bank of New York and builds on several milestone papers in the DSGE literature.[40] The model features nominal wage and price rigidities, variable capital utilisation, costs of adjusting investment, habit formation in consumption and credit frictions. In total, the equilibrium conditions of the model include 17 equations.[41]

The equilibrium conditions are taken one for one from Del Negro, Giannoni and Patterson (2013). To implement the model in our toolkit, we make different assumptions on the shock structure. Del Negro, Giannoni and Patterson (2013) include eight structural shocks, each following an *AR(1)* process, and two i.i.d. monetary policy shocks. We assume that the shocks are perfectly correlated and follow a two-state Markov Process with absorbing state. The applications in this paper feature two structural shocks: a preference shock, $\hat{b}_t$ and a cost push shock, $\tilde{\lambda}_{f,t}$. The preference shock scales the overall per period utility and acts as a negative shock to the natural rate of interest in our experiment. The cost push shock enters the Phillips curve and is used to target a specific drop in inflation.

The policy rule proposed in Del Negro, Giannoni and Patterson (2013) is the following:[42]

$$R_t = \max \left\{ 1; \rho_R R_{t-1} + (1-\rho_R) \left(\varphi_\pi \sum_{j=0}^{3} \hat{\pi}_{t-j} + \varphi_y \sum_{j=0}^{3} (\hat{y}_{t-j} - \hat{y}_{t-j-1}) + \ln \bar{R} \right) \right\} \tag{A.7}$$

were R_t is the (gross) nominal interest rate, π_t inflation rate, y_t output gap and the remaining are parameters. All hatted variables are in log-deviation from steady state, while steady state variables are denoted by a bar. The policy rule in this model has a standard form: The central bank sets the interest rate according to a function of (lagged) terms of inflation and output as well as the nominal interest rate of the previous period. If this number turns out to be negative, the nominal interest rate is equal to zero, the lower bound. In regimes 1 and 2, the lower bound part of the policy rule will be in effect. In regimes 0 and 3, the endogenous part of the policy rule is part of the equilibrium conditions and has to hold at a candidate solution.

The model includes several predetermined variables introducing inertial dynamics into the model. A consequence of this is that we can no longer impose the monotonicity of k: As the shock is on for longer,

[39]Series identifiers are GDPC1 (output), GDP (nominal output), PCEPILFE (price level), DPCCRV1Q225SBEA (inflation) and FEDFUNDS (Federal Funds rate). Data retrieved from FRED (2020), on January 29, 2020, except data for inflation, which was retrieved on October 1, 2020.

[40]The FRBNY DSGE Model is explained in detail in Del Negro et al. (2013). We implement a slightly different version of the model which is presented in Del Negro, Giannoni and Patterson (2013). We decide in favor of this version because it features a preference shock.

[41]The model includes several lagged terms. Setting up the full model with our toolkit, we count 15 state variables.

[42]We will commonly refer to this rule as the "FRBNY rule".

the state variables, like capital, approach their new 'steady state'. As they do this, it can turn out to be optimal to keep the interest rate at the lower bound for a shorter period as contingencies get higher, i.e. a hump shaped k.[43]

A.5.1 NYFRB Model description

We refer the reader to the Appendix in Del Negro, Giannoni and Patterson (2013).

A.5.2 Calibration

To parametrise our model, we choose the posterior means as reported in Del Negro, Giannoni and Patterson (2013) as parameter values.

There are only five exceptions: the transition probability of the two-state Markov shock, μ, the values of the two shocks in the low state, $\hat{b}_L$ and $\tilde{\lambda}_{f,L}$, the discount factor β and the steady state inflation rate $\bar{\pi}$. The first three values are chosen to minimise the quadratic distance to three targets: maximum drop of output of 8.5%, a maximum drop of inflation when at the ELB to a value of 1.5% and an expected duration at the ELB at the point in time of hitting the ELB of 4 quarters. The first two are motivated by observed values during the Great Recession. Expected duration at the ELB of 4 quarters we take from Blue Chip survey of forecaster, see Aspen Publishers 2008-12.

We use the model with the FRBNY policy rule, Equation (A.7), to do the calibration. Importantly, the targets have to be matched for a realisation of the shock that implies 28 quarters at the ELB, as observed in the data. In our calibration this corresponds to the two-state Markov shock switching to the high state in period 32, i.e. contingency 32.[44] The discount factor β is chosen to match a steady state natural rate of real interest of 0%. A real rate of 0% corresponds to the lower bound of December 2019 FOMC long run projections.[45] Finally, the steady state inflation rate is set to 2%. Table A.1 shows the values of the variables we target in the data and the model, respectively, and the implied values for the parameters. Furthermore, the shock variables $\hat{\mu}$, $\hat{z}$, $\hat{\chi}$, $\hat{\psi}$, $\hat{g}$, ϵ^R, ϵ^R_k, $\tilde{\sigma}_\omega$ are all set to zero.

Since the model has non-zero steady state inflation, we adjust the welfare loss function used to compare the performance of the policy rules to:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [(\pi_t - \bar{\pi})^2 + \lambda \hat{Y}_t^2] \tag{A.8}$$

The Fixed Length Forward Guidance Rule (FLFG) in the FRBNY model is the following:

$$R_t = \begin{cases} 1 & \text{for } t \leq \nu \\ \text{Rule (A.7)} \ \ (\textit{FRBNY Rule}) & \text{otherwise} \end{cases} \tag{A.9}$$

Note that R_t is the gross nominal interest rate, so the rule imposes the ZLB in the case of $t \leq \nu$. We choose $\nu = 6$ as it minimises welfare loss according to the loss function A.8 with equal weights. We impose $\tilde{T} = 1$ in this rule, meaning that the interest rate goes to zero immediately in this rule.

[43] It should also be mentioned that we do not prove analytically that there is a unique k and henceforth a unique equilibrium in the model. We have not encountered any case of multiple equilibria when experimenting with the model.

[44] The solution features 4 periods in regime 0, i.e. $\tilde{T} = 5$ and $k_{32} = 1$.

[45] See Board of Governors of the Federal Reserve System (2019).

Target	Data	Model	Parameter	Value
$\min \pi_t\|_{ELB}$	1.5	1.486	$\tilde{\lambda}_{f,L}$	0.0015
$\min \hat{Y}_t$	-8.5	-8.49	$\hat{b}_L$	-0.105
$\mathbb{E}(ELB)\|_{ELB}$	4	3.95	μ	0.736
$\bar{r}$	0.0	0.0	β	1.0042
$\bar{\pi}$	2.0	2.0		

Table A.1: Calibration results. See text for details. $\mathbb{E}(ELB)|_{ELB}$ is in quarters. Note: First three parameters are calibrated simultaneously to hit targets. Changing one of the three parameters will affect all three targets. Last two parameters have one-to-one relation with respective target. $\bar{\pi}$ is set directly.

Acronym	Source	Standard Values	Optimal Values
TTR	Nakov (2008)	$\phi_\pi = 1.5$ $\phi_y = 0.5$	N/A
TTR-1	Nakov (2008)	$\phi_\pi = 1.5$ $\phi_y = 0.5$	$\phi_\pi = 1.14$ $\phi_y = 0.28$
TTR+1	Nakov (2008)	$\phi_\pi = 1.5$ $\phi_y = 0.5$	N/A
TTRS	Nakov (2008)	$\phi_\pi = 1.5$ $\phi_y = 0.5$ $\phi_i = 0.8$	$\phi_\pi = 51.50$ $\phi_y = 48.77$ $\phi_i = 0.98$
TTRP	Wolman (2005)	$\phi_p = 1.5$ $\phi_y = 0.5$	$\phi_p = 1.23$ $\phi_y = 1.23$
ATR	Reifschneider and Williams (2000)	$\alpha = 1$ $\phi_\pi = 1.5$ $\phi_y = 0.5$	$\alpha = 92.09$ $\phi_\pi = 100$ $\phi_y = 34.23$
SUP	Rotemberg and Woodford (1999)	$\phi^{SUP} = 1.28$	$\phi^{SUP} = 1.48$
AIT	Reifschneider and Wilcox (2019)	$\phi^{AIT} = 5$	$\phi^{AIT} = 18.7$

Table A.2: Policy rules, standard parametrisation, and optimal parametrisation. The Table reports, for each Taylor-type policy rule, the parameter values used for the simulations. The second column reports values used in the literature as well as the source. The last column reports optimal values, that minimise the welfare loss (25).In TTR and TTR+1 the standard values are already optimal.

A.6 Results if ELB constraint not imposed

In this section we present results for both the Simple New Keynesian model of Section 4.5 and the FRBNY DSGE model of Section 5 for the case of not imposing the ELB constraint. This means the nominal interest rate can go arbitrarily negative. Apart from dropping the ELB assumption, the parametrisation will be identical to the baseline experiments in the main text. Due to the linearity of the model (if the ELB is not imposed), the results also tell us how the rules perform in response to small enough shocks that do not make the ELB bind.[46]

A.6.1 Simple New Keynesian model without imposing the ELB

Table A.3 shows results if we replicate the rule comparison of Section 4.5 but do not impose the ELB. The parametrisation is exactly the same as the one used to derive the results shown in Table 2, except that the ELB constraint is not imposed.

First let us note that under optimal commitment, deeply negative interest rates are implemented. The nominal rate drops to almost -5%. Comparing optimal commitment with and without imposing the ELB, we get that in the latter case the welfare loss is about 40% lower.

When it comes to the relative ranking of the rules, we see that our proposed rules HD-NGDPT and SDTR perform very well. The only rule outperforming both is a price level targeting rule (PLT), which coincides with optimal commitment in the case of no ELB imposed.[47] NGDPT and HD-NGDPT overlap, which is due to the ability of the central bank to keep nominal GDP on target at all times if it is not constrained by the ELB. Finally, rules ATR and SUP lose some appeal once we do not impose the ELB.

[46] Figures for this section are available from the authors upon request.
[47] This result is shown in Eggertsson and Woodford (2003).

	Welfare Loss (1)	Volatility x (2)	Volatility π (3)	Volatility i (4)	Impact x (5)	Impact π (6)
OCP	$5.196\,10^{-4}$	$3.497\,10^{-3}$	$3.011\,10^{-4}$	$4.697\,10^{-3}$	−0.233	2.914
PANEL A: baseline rules						
OCP	1.000	1.000	1.000	1.000	1.000	1.000
TTR	2.156	3.923	0.873	0.250	16.673	0.739
HD-NGDPT	1.258	2.404	0.427	1.283	2.401	0.730
SDTR	1.821	2.165	1.571	0.264	12.406	0.991
ATR	2.156	3.923	0.873	0.250	16.673	0.739
SUP	2.147	2.904	1.597	0.128	17.239	0.941
PANEL B: additional rules						
PLT	1.000	1.000	1.000	1.000	1.000	1.000
NGDPT	1.258	2.404	0.427	1.283	2.401	0.730
TTRP	6.643	15.777	0.012	0.060	36.881	−0.143
TTRS-1	2.414	4.091	1.196	0.109	20.730	0.797
TTR-1	2.155	3.807	0.955	0.220	18.079	0.764
AIT	2.532	5.779	0.175	0.451	17.672	0.426

Table A.3: Some metrics for selected interest rate rules in the simple two-equation NK model in the presence of a correlated cost push shock without imposing the ELB. All rows except the first show values normalised with respect to the optimal commitment policy (OCP, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights; Columns (2)-(4) report a summary measure of deviations of the endogenous variables from target, computed according to Equation (A.6); Finally, Columns (5) and (6) show the response on impact, in annual percentage points, of the output gap and inflation to a natural interest rate shock and a correlated cost push shock of the same size as in Table 2. Rule calibration reported in Table (A.2). The model is calibrated with the standard EW (2003) parameter values reported in footnote 21. FLFG coincides with TTR if the ELB constraint is not imposed.

A.6.2 FRBNY DSGE model without imposing the ELB

Table A.4 shows results if we replicate the rule comparison of Section 5 but do not impose the ELB. The parametrisation is exactly the same as the one used to derive the results shown in Table 3, except that the ELB constraint is not imposed.

As in the simple two-equation NK model, we can get deep negative rates in this case. While the nominal rate does not drop far below zero under the FRBNY Rule, rule PLT lowers the nominal rate to almost -15% for a period of more than 7 years. The welfare loss is not much lower under the FRBNY Rule if the ELB is not imposed, but drops by more than 96% if one compares the best performing rule when the ELB is not imposed (rule PLT) to the FRBNY Rule with the ELB constraint active.

When it comes to the relative ranking of the rules, we see that our proposed rules HD-NGDPT and SDTR perform very well. The only rule outperforming both is again a price level targeting rule (PLT), which does not come as a surprise given the result in the simple NK model of the previous section.[48] NGDPT and HD-NGDPT again overlap, for the same reasons stated in Section A.6.1.

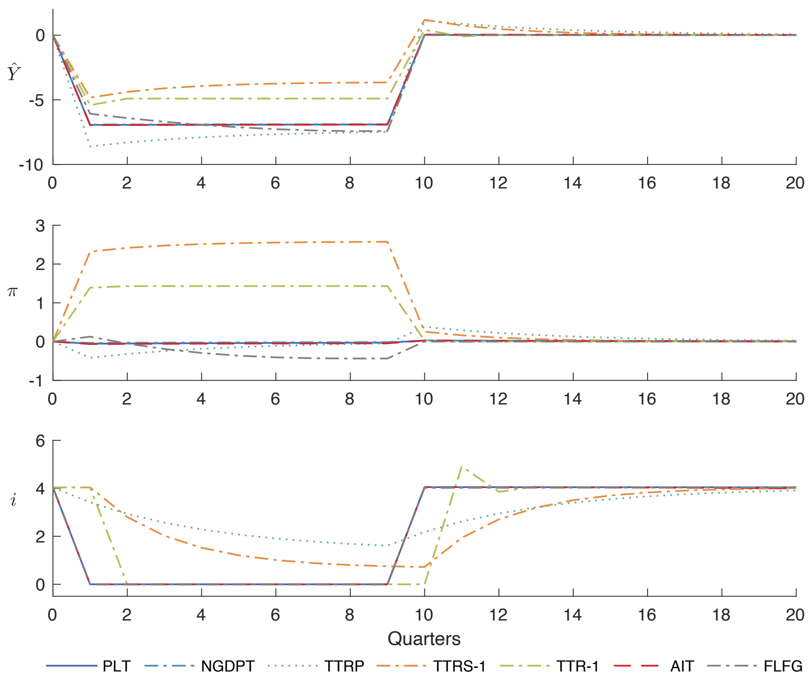
[48]Note however that we are not aware of a derivation that shows that PLT is equivalent to optimal commitment in the FRBNY DSGE model.

	Welfare Loss (1)	Volatility $\hat{Y}$ (2)	Volatility π (3)	Volatility i (4)	Impact $\hat{Y}$ (5)	Impact π (6)
FRBNY Rule	$8.216\ 10^{-4}$	0.013	$1.3433\ 10^{-5}$	$7.2973\ 10^{-5}$	−2.523	2.809
PANEL A: baseline rules						
FRBNY Rule	1.000	1.000	1.000	1.000	1.000	1.000
HD-NGDPT	0.058	0.025	2.052	68.381	0.116	1.130
SDTR	0.082	0.042	2.498	322.130	0.495	1.157
ATR	0.392	0.372	1.570	5.250	0.717	1.059
SUP	0.288	0.254	2.336	5.129	0.598	1.157
PANEL B: additional rules						
PLT	0.040	0.003	2.256	77.405	0.030	1.136
NGDPT	0.058	0.025	2.052	68.381	0.116	1.130
TTRP	0.871	0.869	1.003	1.100	0.955	1.035
TTRS-1	0.406	0.386	1.564	2.598	0.798	1.076
TTR	0.392	0.372	1.570	5.250	0.717	1.059
TTR-1	0.368	0.348	1.600	4.590	0.735	1.066
AIT	0.440	0.424	1.354	4.156	0.753	1.064
TTR+1	0.436	0.418	1.527	6.192	0.710	1.051

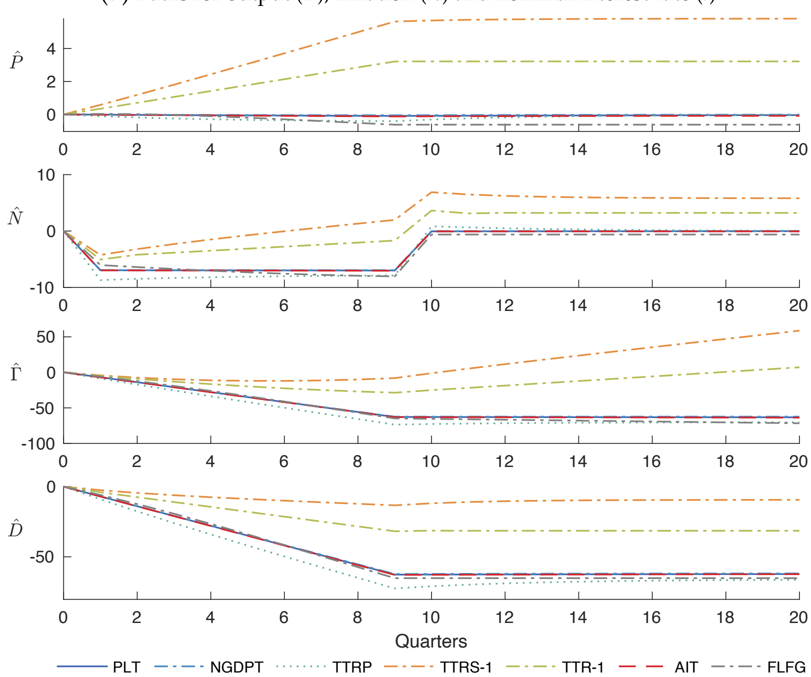
Table A.4: Some metrics for interest rate rules in the FRBNY model in the presence of a preference shock and correlated cost push shock without imposing the ELB. All rows except the first show values normalised with respect to the modified Taylor rule in Del Negro, Giannoni and Patterson (2013) (FRBNY Rule, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights and an inflation target, see Equation (A.8); Columns (2)-(4) report a summary measure of deviations of the endogenous variables from target, computed according to Equation (A.6); Finally, Columns (5) and (6) show the response on impact, in annual percentage points, of the output gap and inflation to a preference shock and a correlated cost push shock of the same size as in Table 3. See Section A.5.2 for details on calibration. The list of acronyms is detailed in Table 1. FLFG coincides with FRBNY Rule if the ELB constraint is not imposed.

A.7 Robustness checks

A.7.1 Simple New Keynesian model with -0.5% inflation drop – Additional policy rules and impulse responses

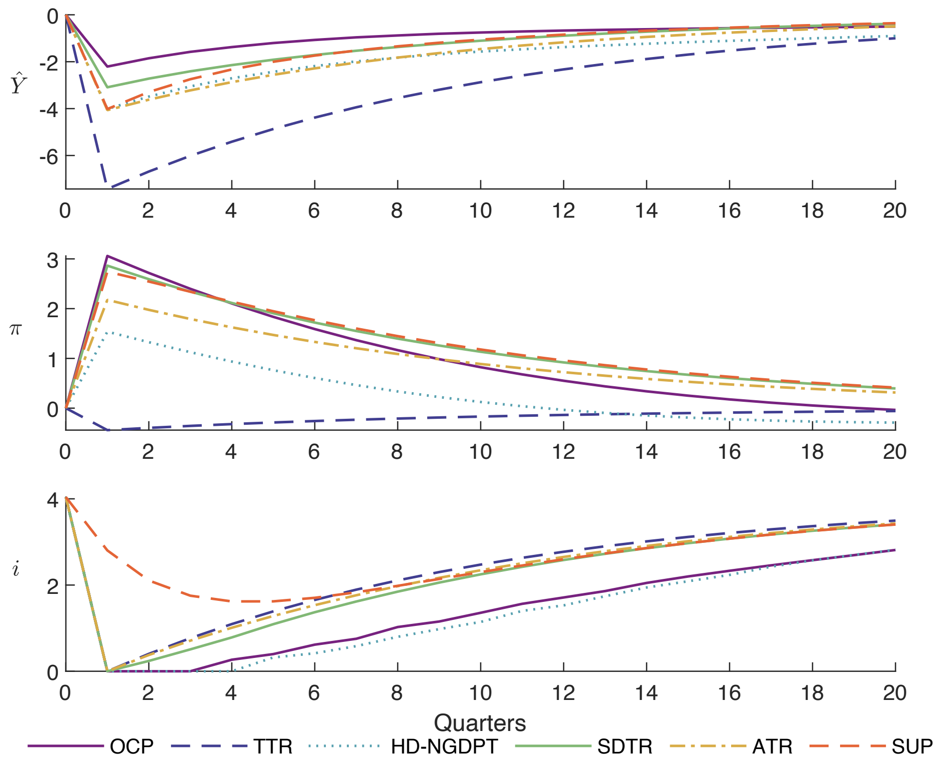


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

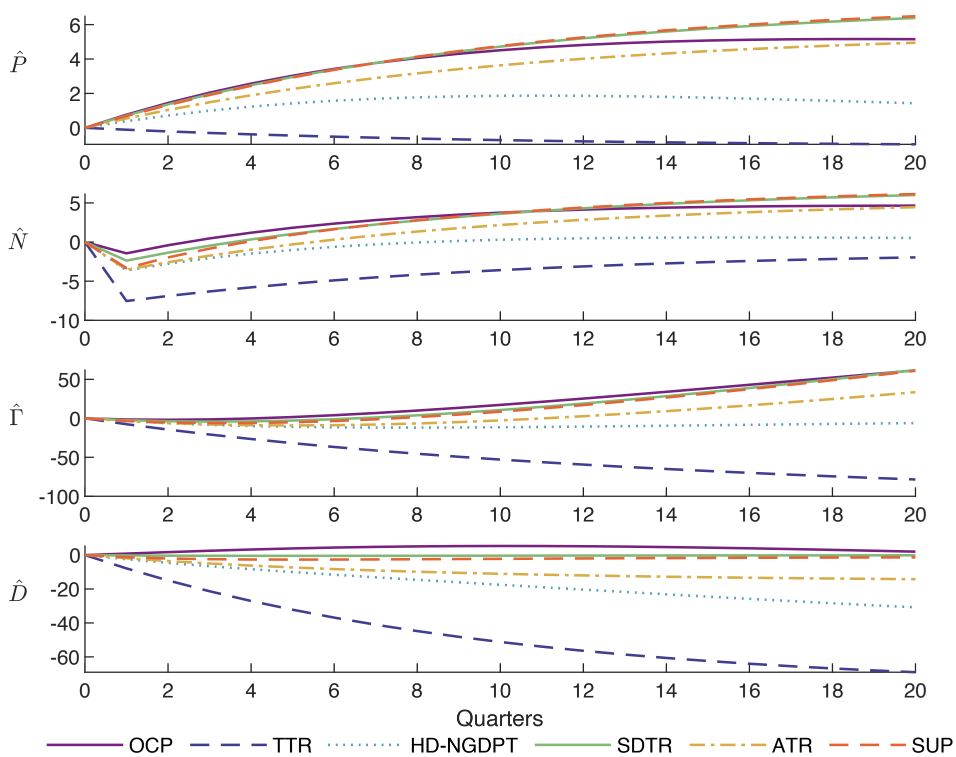


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.1: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.

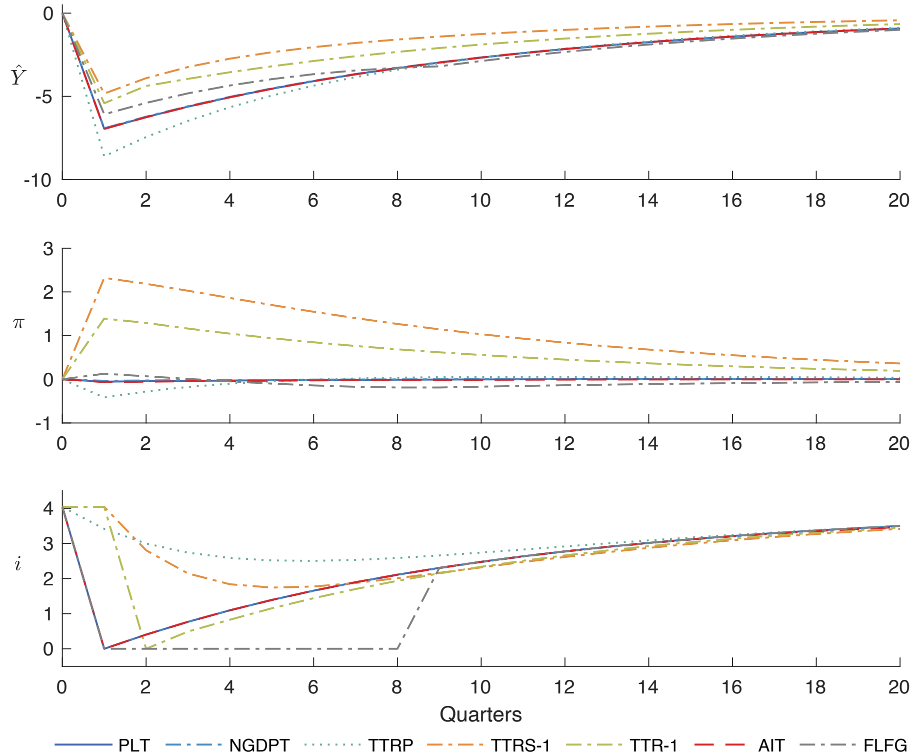


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

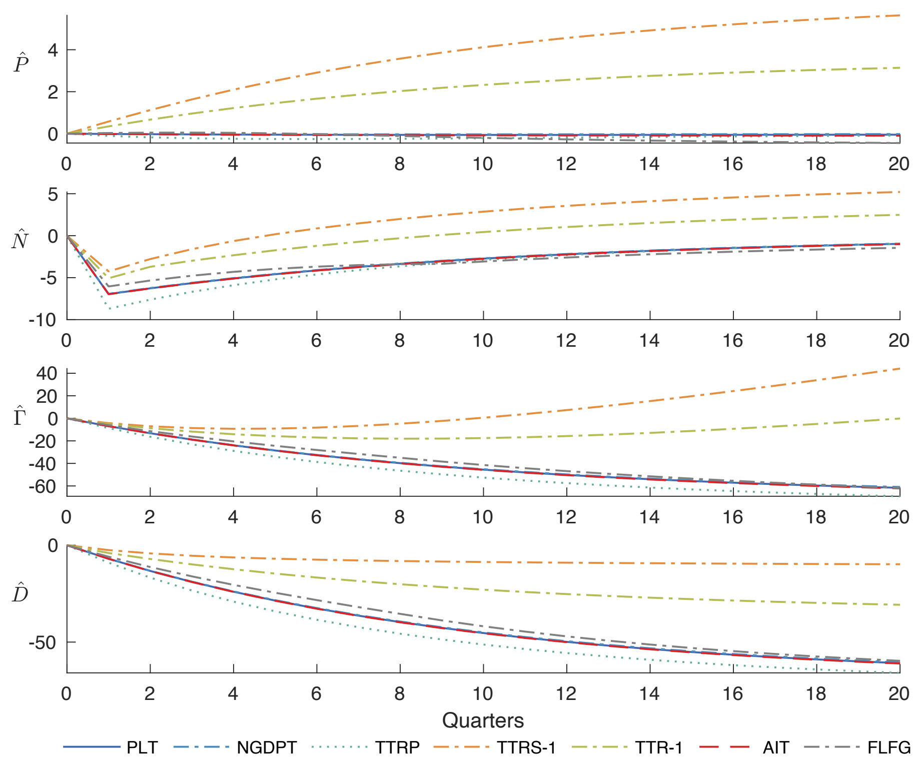


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.2: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.



(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)



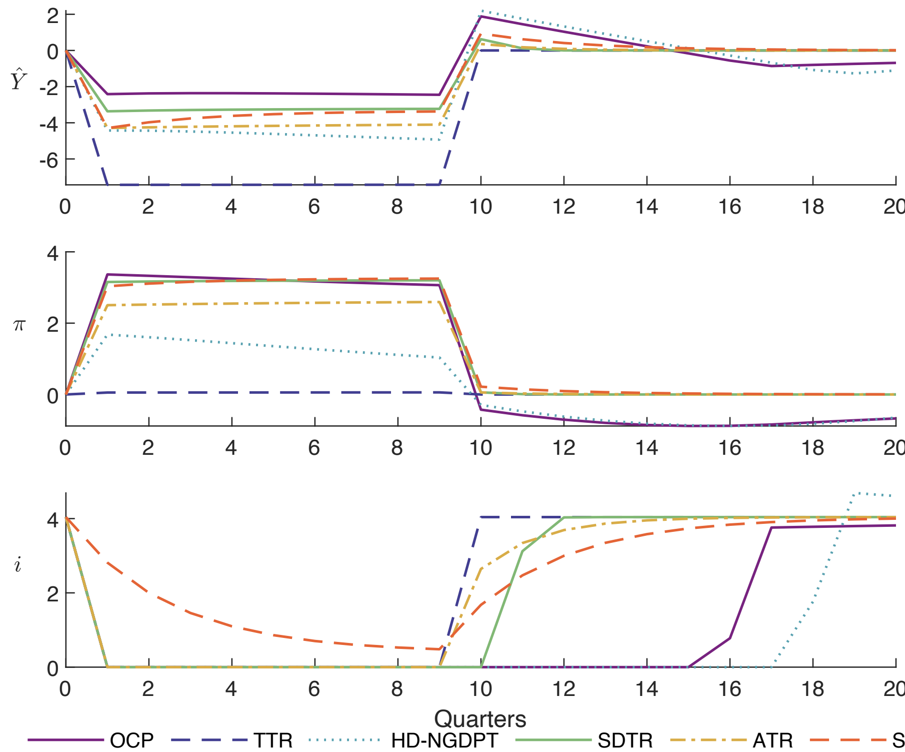
(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.3: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.

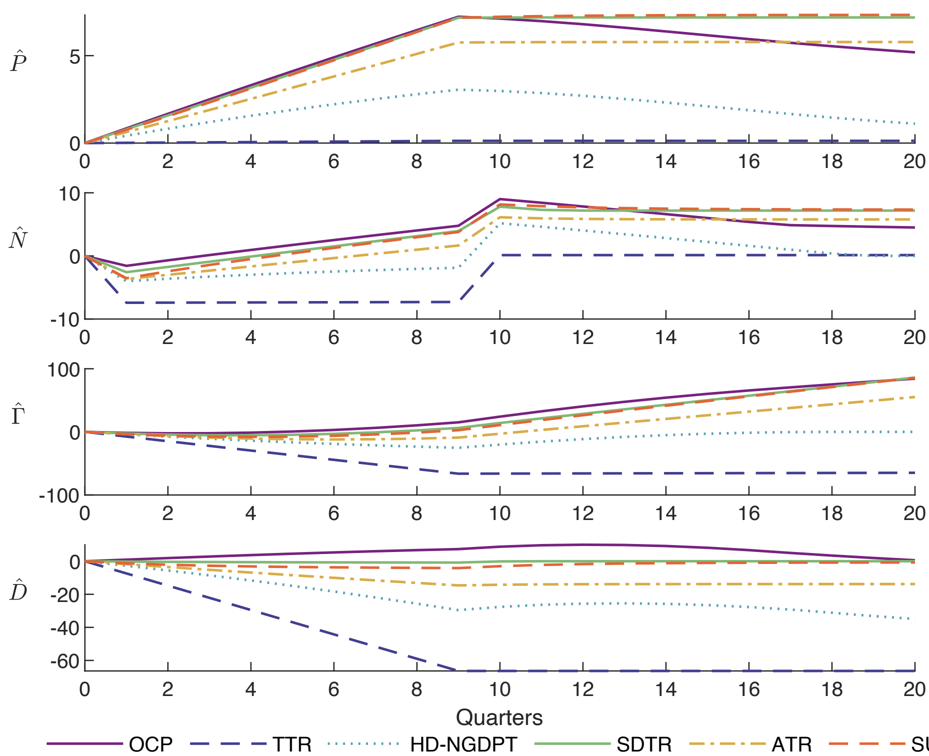
A.7.2 Simple New Keynesian model with 0% inflation drop

	Welfare Loss (1)	$\mathbb{E}_0[\tau + k_\tau - \tilde{T}]$ (2)	Volatility x (3)	Volatility π (4)	Volatility i (5)	Impact x (6)	Impact π (7)
OCP	$9.938\,10^{-4}$	15.471	$6.409\,10^{-3}$	$5.932\,10^{-4}$	$1.438\,10^{-3}$	−2.415	3.366
PANEL A: baseline rules							
OCP	1.000	1.000	1.000	1.000	1.000	1.000	1.000
TTR	3.150	0.646	7.814	0.000	0.645	3.077	0.016
HD-NGDPT	1.577	1.108	3.613	0.203	1.098	1.832	0.499
SDTR	1.191	0.693	1.512	0.974	0.700	1.393	0.937
ATR	1.361	0.646	2.437	0.634	0.653	1.775	0.744
SUP	1.335	0.000	1.860	0.980	0.411	1.778	0.902
PANEL B: additional rules							
PLT	3.213	0.646	7.971	0.000	0.645	3.105	0.000
NGDPT	3.213	0.646	7.971	0.000	0.645	3.105	0.000
TTRP	4.082	0.000	10.119	0.006	0.186	3.824	−0.121
TTRS-1	1.469	0.000	2.496	0.775	0.352	2.094	0.787
TTR-1	1.622	0.640	3.500	0.354	0.641	2.243	0.560
AIT	3.213	0.646	7.971	0.000	0.645	3.105	0.000
FRBNY Rule	67.154	0.782	124.781	28.239	1.700	12.431	−5.177
FLFG	2.908	0.795	7.196	0.012	0.799	2.518	0.185

Table A.5: Some metrics for selected interest rate rules in the simple two-equation NK model in the presence of a correlated cost push shock. All rows except the first show values normalised with respect to the optimal commitment policy (OCP, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights; Column (2) displays the unconditional expected duration of the Zero Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output gap and inflation to a natural interest rate shock and a correlated cost push shock such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). Rule calibration reported in Table (A.2). The model is calibrated with the standard EW (2003) parameter values reported in footnote 21.

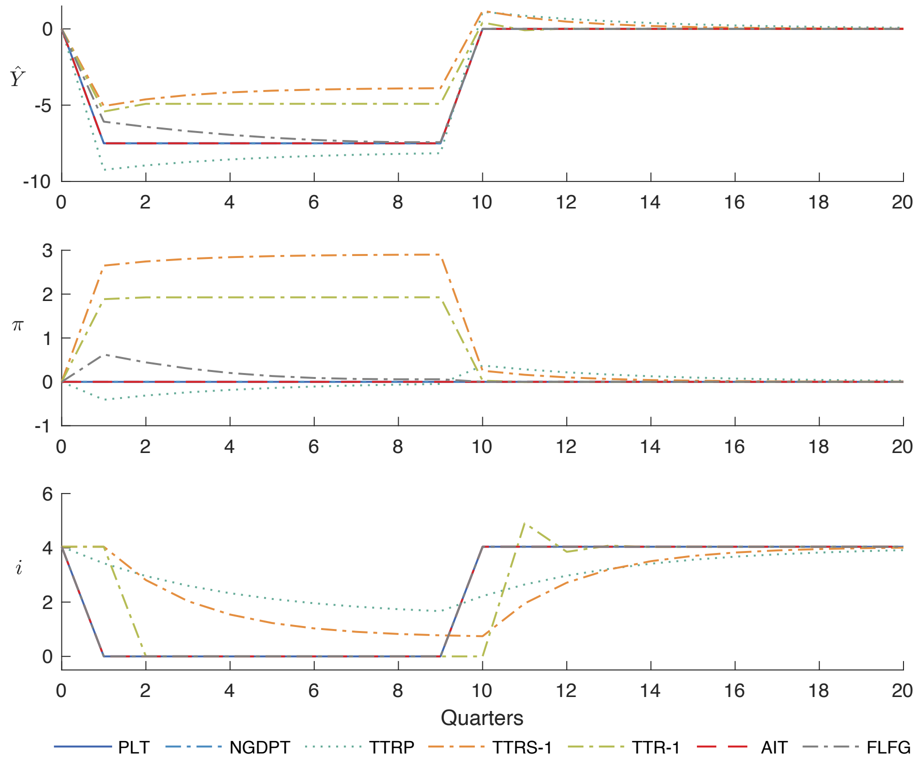


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

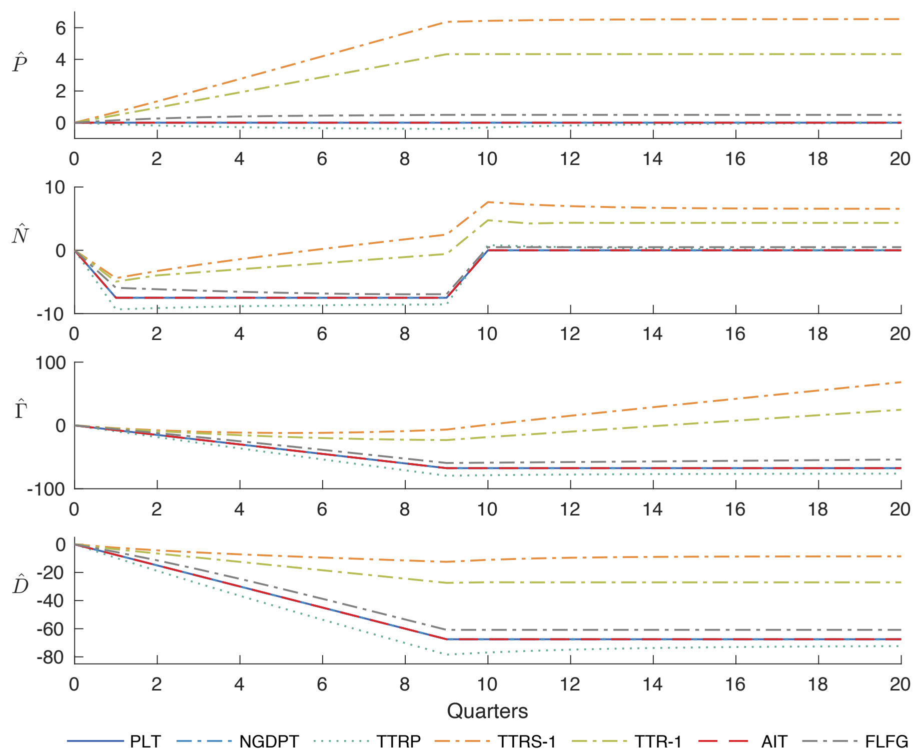


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.4: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.

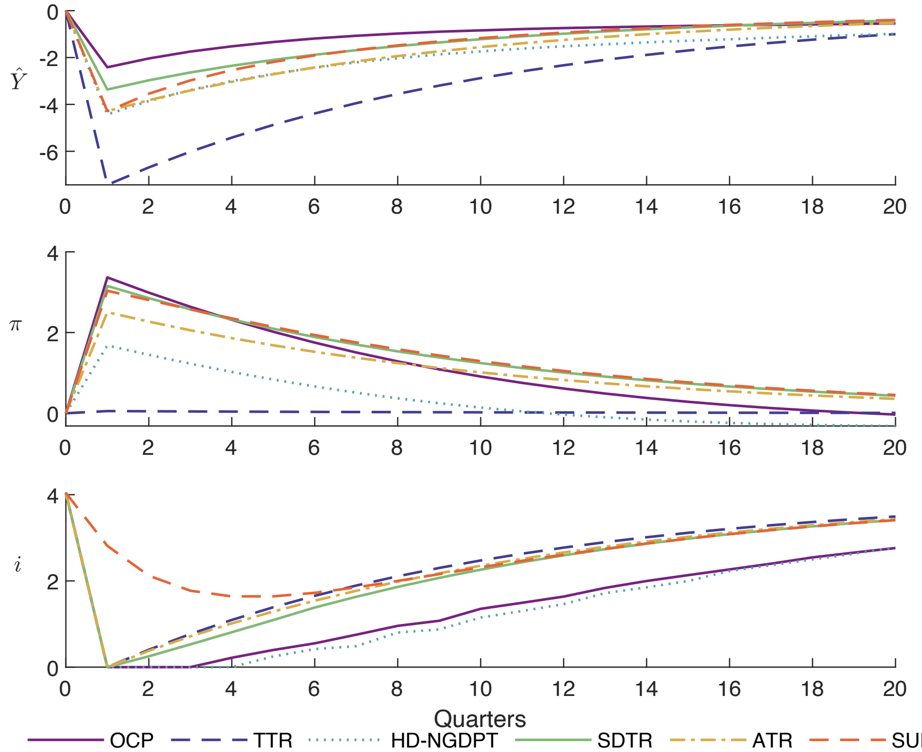


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

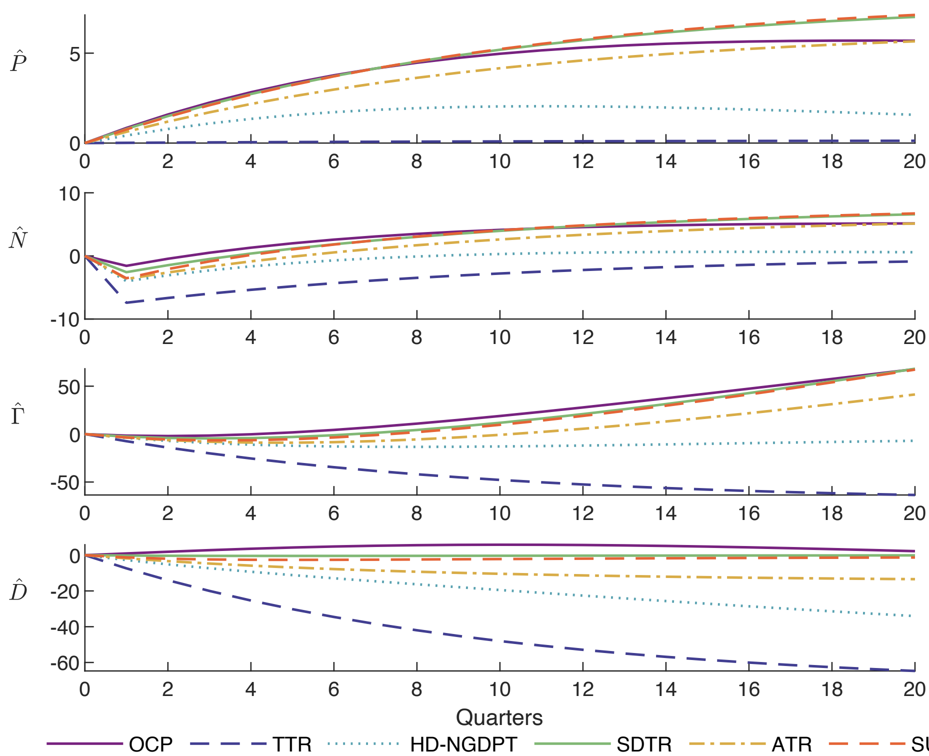


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.5: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.

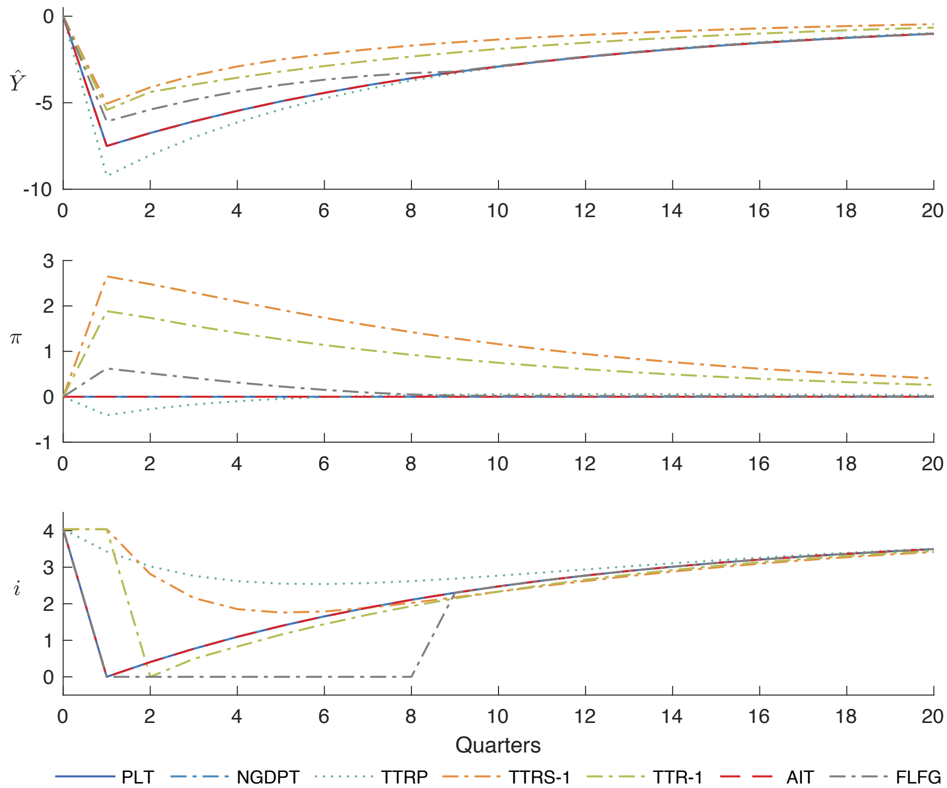


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

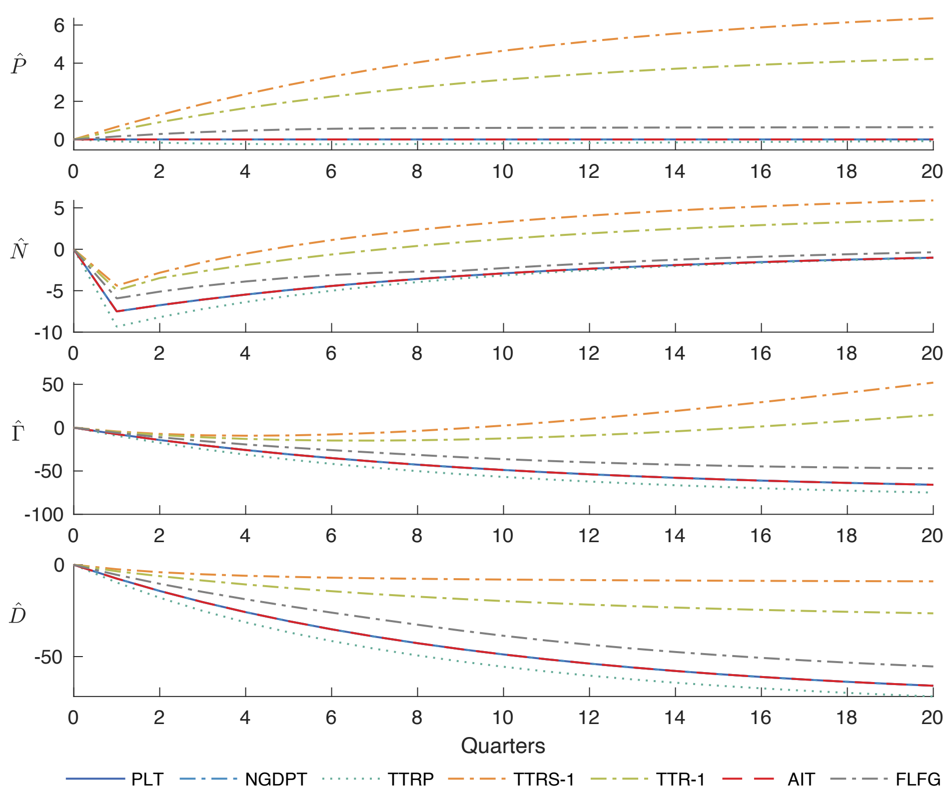


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.6: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.



(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)



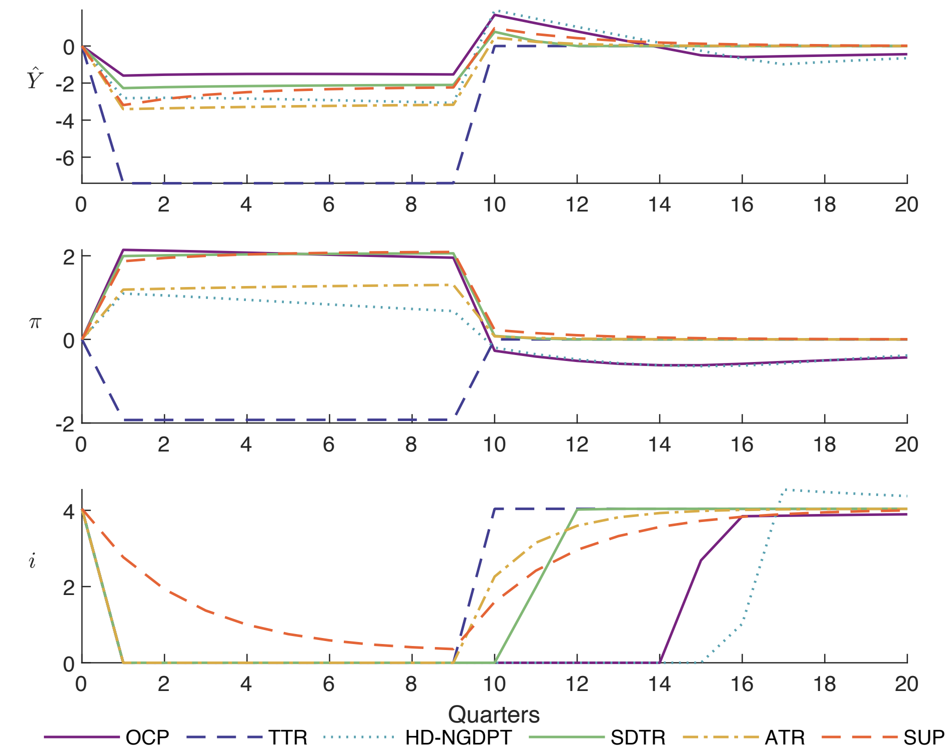
(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.7: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation remains constant under a Truncated Taylor Rule (TTR). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.

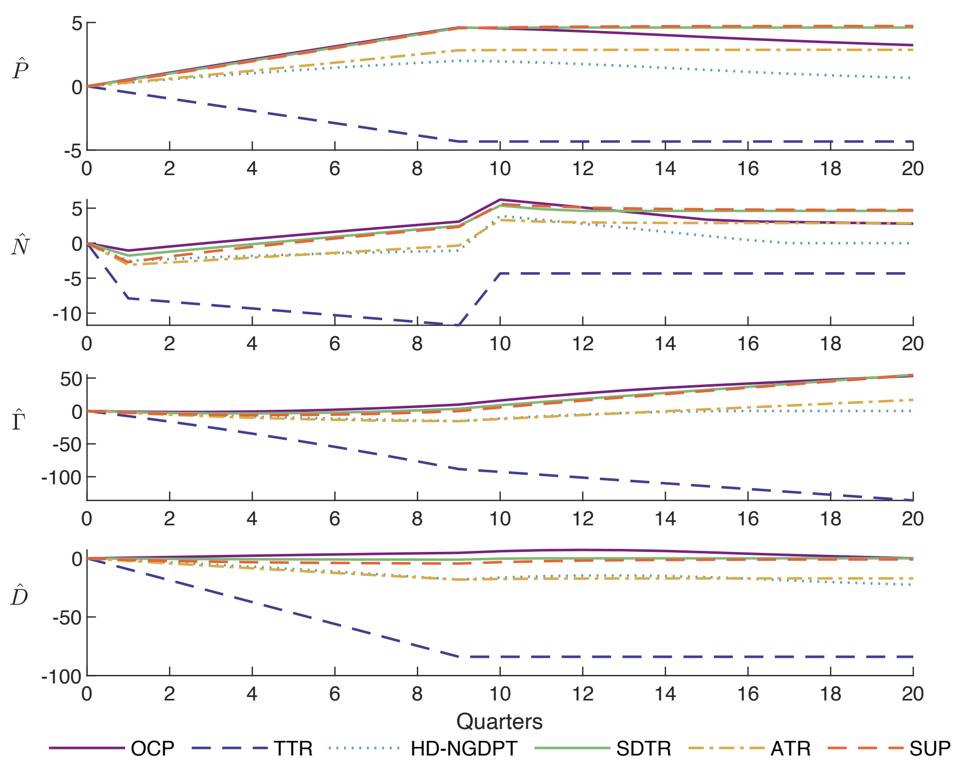
A.7.3 Simple New Keynesian model with -2% inflation drop

	Welfare Loss (1)	$\mathbb{E}_0[\tau + k_\tau - \tilde{T}]$ (2)	Volatility x (3)	Volatility π (4)	Volatility i (5)	Impact x (6)	Impact π (7)
OCP	$4.162\,10^{-4}$	14.073	$2.784\,10^{-3}$	$2.422\,10^{-4}$	$1.326\,10^{-3}$	−1.597	2.141
PANEL A: baseline rules							
OCP	1.000	1.000	1.000	1.000	1.000	1.000	1.000
TTR	7.975	0.711	17.869	0.865	0.699	4.638	−0.900
HD-NGDPT	1.537	1.096	3.372	0.217	1.083	1.762	0.512
SDTR	1.202	0.768	1.514	0.978	0.774	1.422	0.931
ATR	1.647	0.711	3.405	0.383	0.714	2.133	0.556
SUP	1.428	0.000	2.056	0.977	0.477	1.997	0.873
PANEL B: additional rules							
PLT	3.660	0.711	8.745	0.006	0.699	3.303	−0.101
NGDPT	3.504	0.711	8.377	0.002	0.699	3.229	−0.064
TTRP	4.645	0.000	11.085	0.017	0.242	4.179	−0.208
TTRS-1	1.796	0.000	3.523	0.555	0.403	2.603	0.623
TTR-1	3.336	0.703	7.976	0.001	0.695	3.377	−0.045
AIT	3.737	0.711	8.923	0.011	0.700	3.327	−0.121
FRBNY Rule	15.284	0.729	30.660	4.234	1.157	5.849	−1.549
FLFG	7.323	0.874	16.451	0.764	0.866	3.793	−0.635

Table A.6: Some metrics for selected interest rate rules in the simple two-equation NK model in the presence of a correlated cost push shock. All rows except the first show values normalised with respect to the optimal commitment policy (OCP, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights; Column (2) displays the unconditional expected duration of the Zero Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output gap and inflation to a natural interest rate shock and a correlated cost push shock such that output falls by 7.5% and inflation by 2% under a Truncated Taylor Rule (TTR). Rule calibration reported in Table (A.2). The model is calibrated with the standard EW (2003) parameter values reported in footnote 21.

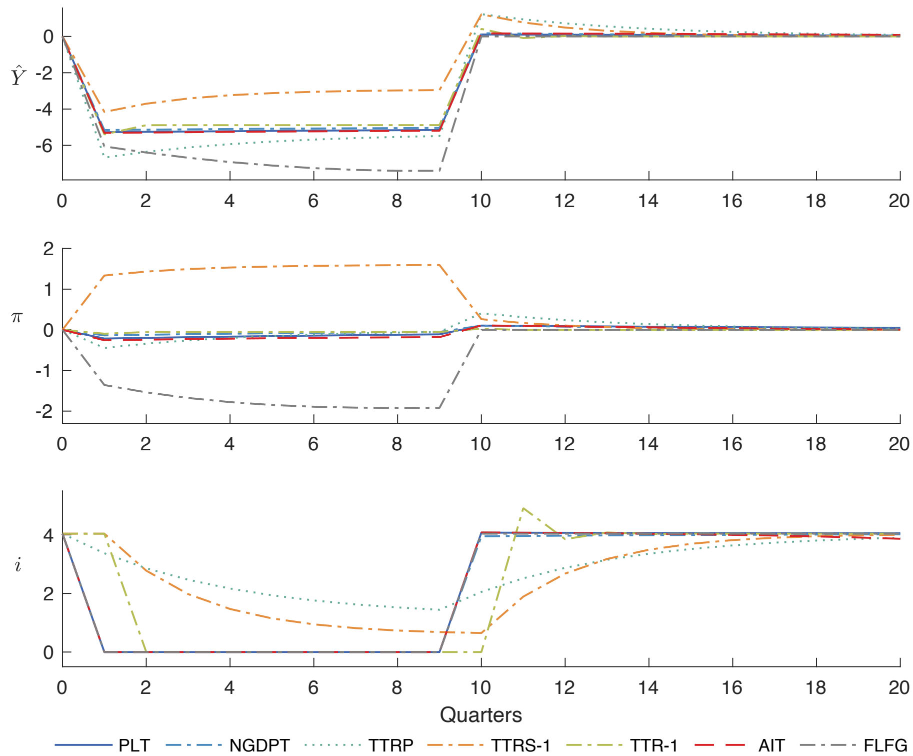


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

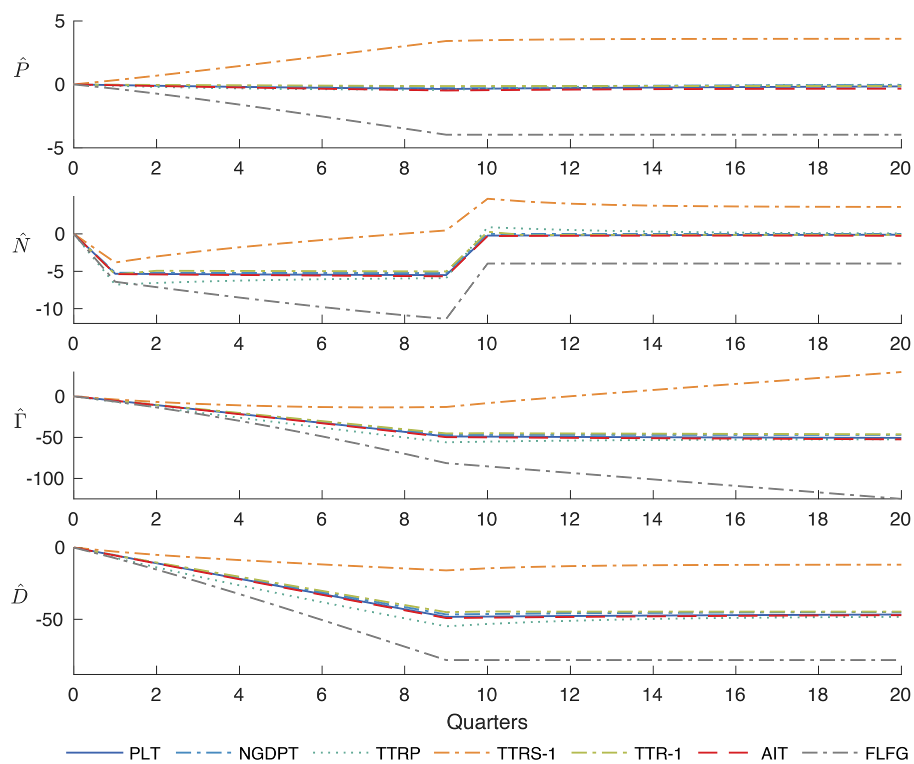


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.8: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 2% constant under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.

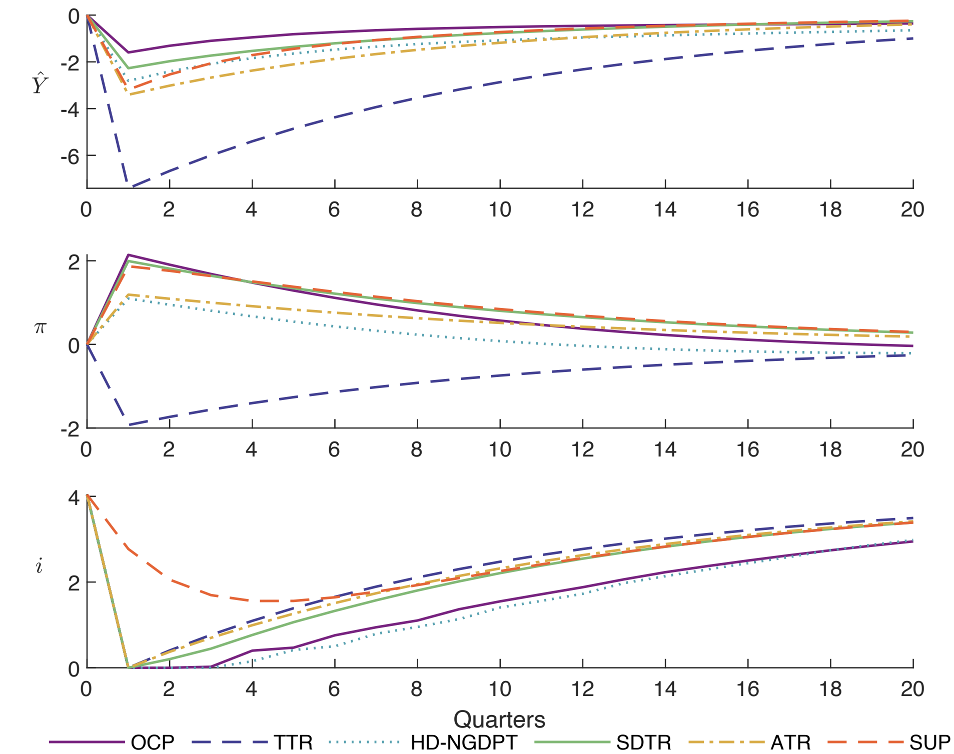


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

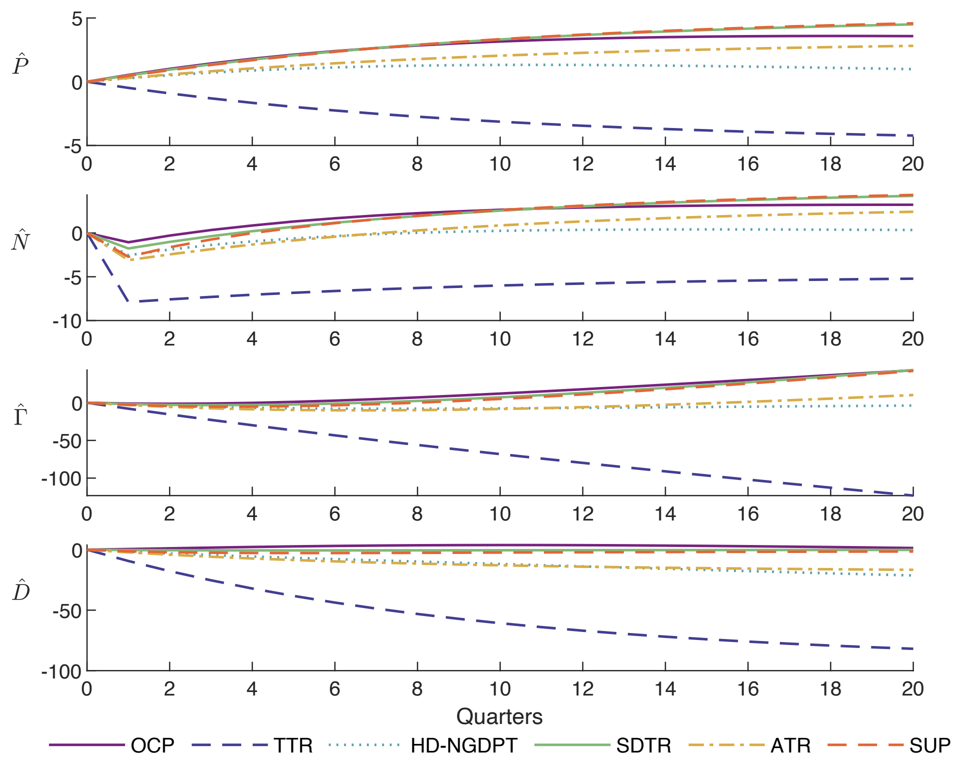


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.9: Dynamic response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 2% under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.

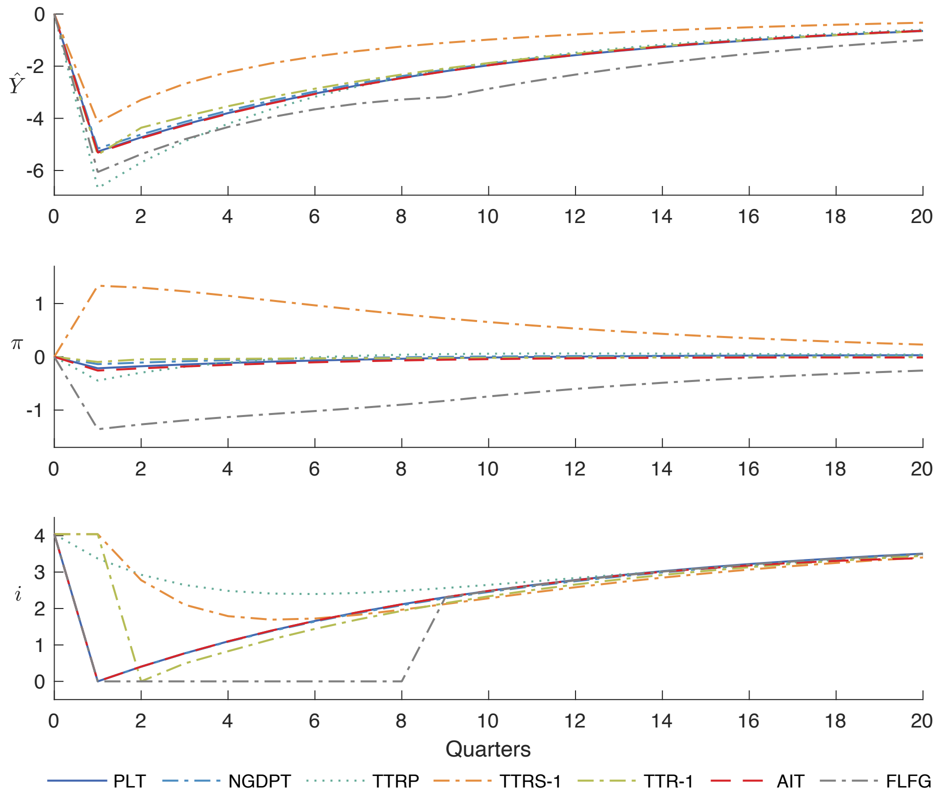


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

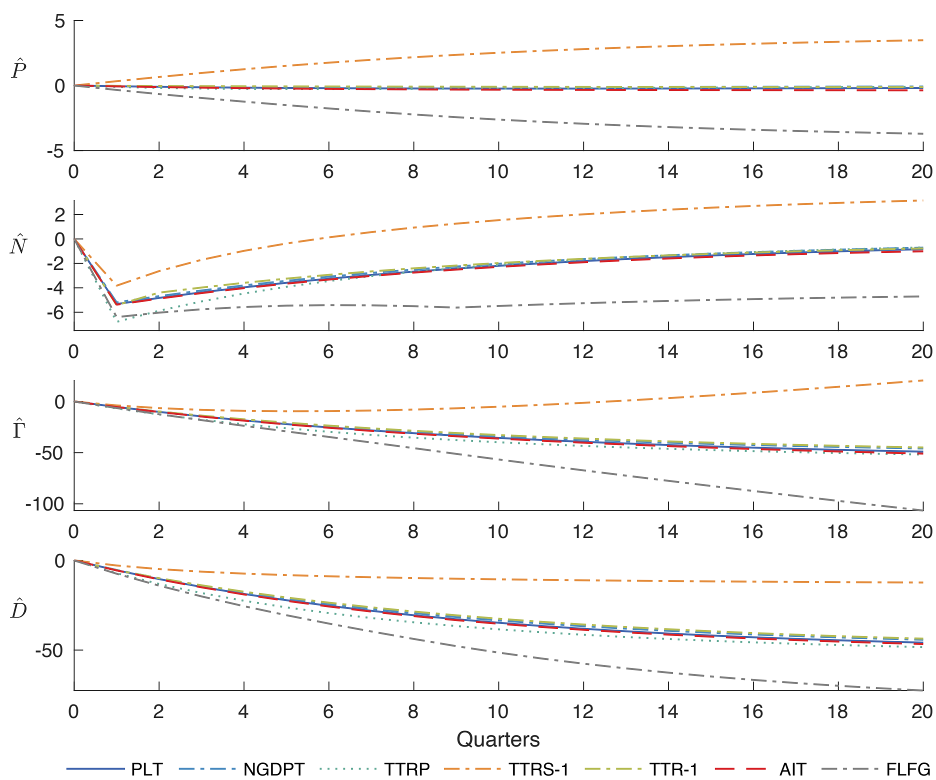


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.10: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 2% under a Truncated Taylor Rule (TTR). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.



(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)



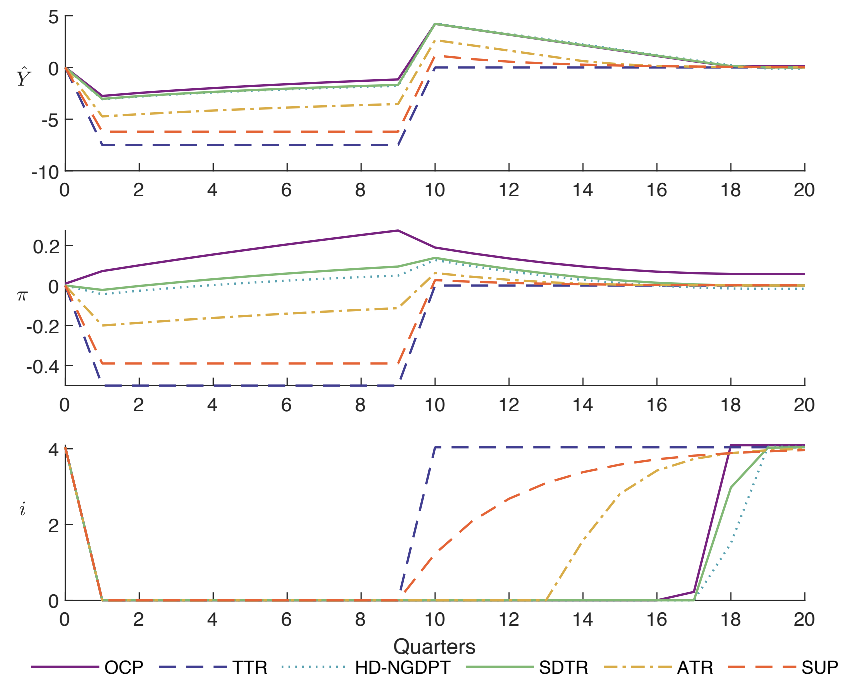
(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.11: Average impulse response of a natural interest rate shock and a correlated cost push shock in a simple two-equation NK model, under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 2% under a Truncated Taylor Rule (TTR). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.

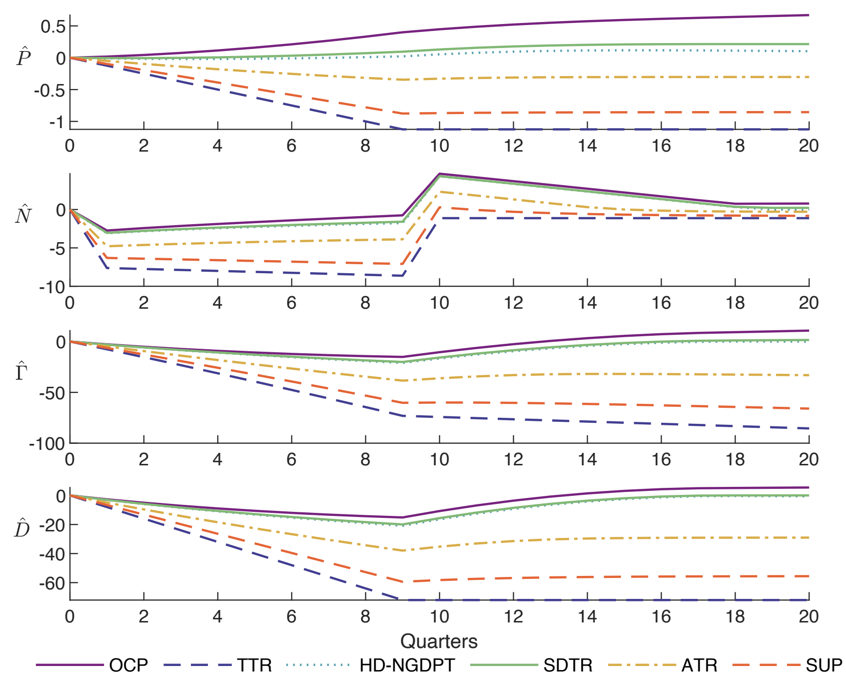
A.7.4 Simple New Keynesian model with -0.5% inflation drop – Price rigidity

	Welfare Loss (1)	$\mathbb{E}_0[\tau + k_\tau - \tilde{T}]$ (2)	Volatility x (3)	Volatility π (4)	Volatility i (5)	Impact x (6)	Impact π (7)
OCP	$4.896\,10^{-4}$	16.549	$7.661\,10^{-3}$	$1.081\,10^{-5}$	$1.524\,10^{-3}$	−2.755	0.072
PANEL A: baseline rules							
OCP	1.000	1.000	1.000	1.000	1.000	1.000	1.000
TTR	6.550	0.604	6.668	1.312	0.608	2.722	−6.929
HD-NGDPT	1.137	1.048	1.162	0.034	1.036	1.116	−0.600
SDTR	1.085	1.020	1.108	0.079	1.018	1.102	−0.304
ATR	1.876	0.794	1.916	0.105	0.824	1.719	−2.769
SUP	4.520	0.604	4.604	0.797	0.664	2.255	−5.398
PANEL B: additional rules							
PLT	6.328	0.604	6.445	1.133	0.608	2.692	−6.608
NGDPT	4.492	0.604	4.584	0.439	0.608	2.394	−4.589
TTRP	4.133	0.000	4.222	0.183	0.227	2.716	−3.517
TTRS-1	1.522	0.000	1.552	0.173	1.010	1.820	−2.890
TTR-1	5.874	0.598	5.981	1.149	0.605	2.740	−6.528
AIT	5.431	0.604	5.533	0.911	0.611	2.547	−5.984
FRBNY Rule	23.603	0.620	23.399	32.647	1.449	4.404	−8.159
FLFG	6.084	0.744	6.193	1.253	0.753	2.294	−6.257

Table A.7: Some metrics for selected interest rate rules in the simple two-equation NK model with increased price rigidity (lower κ). All rows except the first show values normalised with respect to the optimal commitment policy (OCP, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights; Column (2) displays the unconditional expected duration of the Zero Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output gap and inflation to a natural interest rate shock such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). Rule calibration reported in Table (A.2). Remaining parameters are calibrated according to standard EW (2003) values reported in footnote 21.

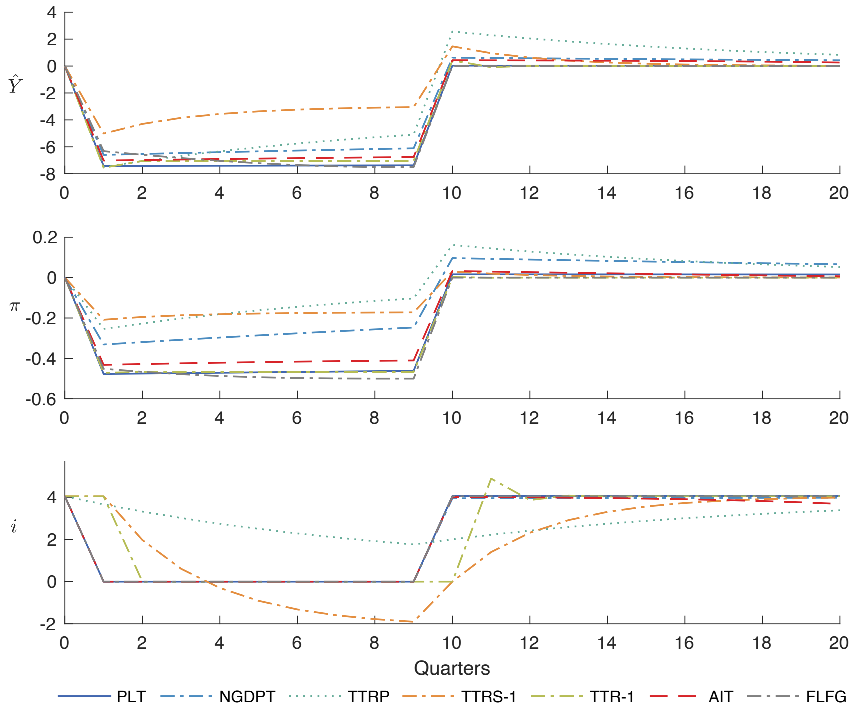


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

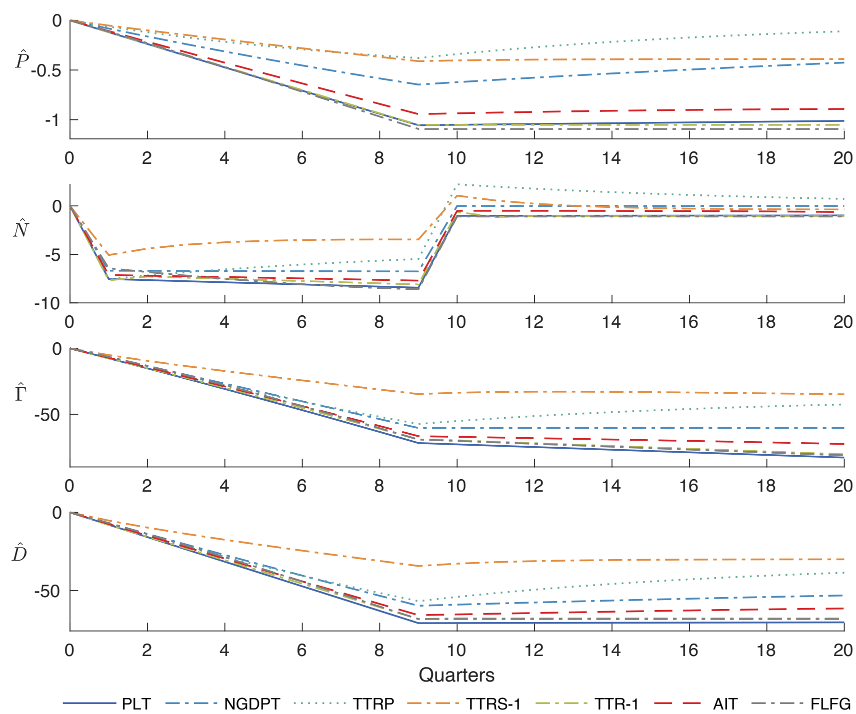


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.12: Dynamic response of a natural interest rate shock in a simple two-equation NK model with increased price rigidity (lower κ), under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. Remaining parameters are calibrated according to standard EW (2003) values reported in footnote 21.

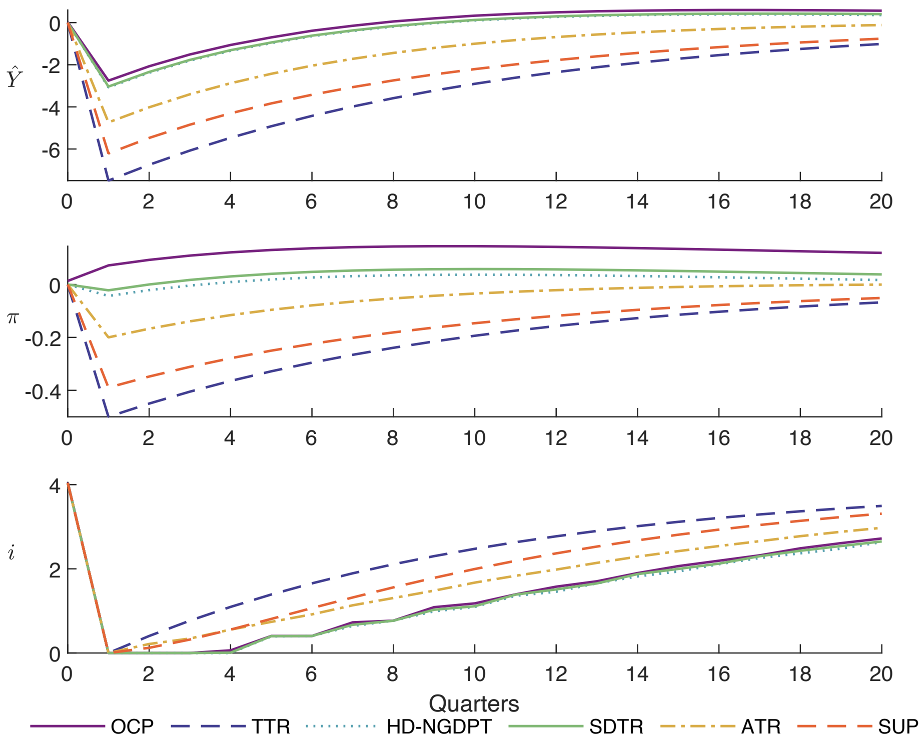


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

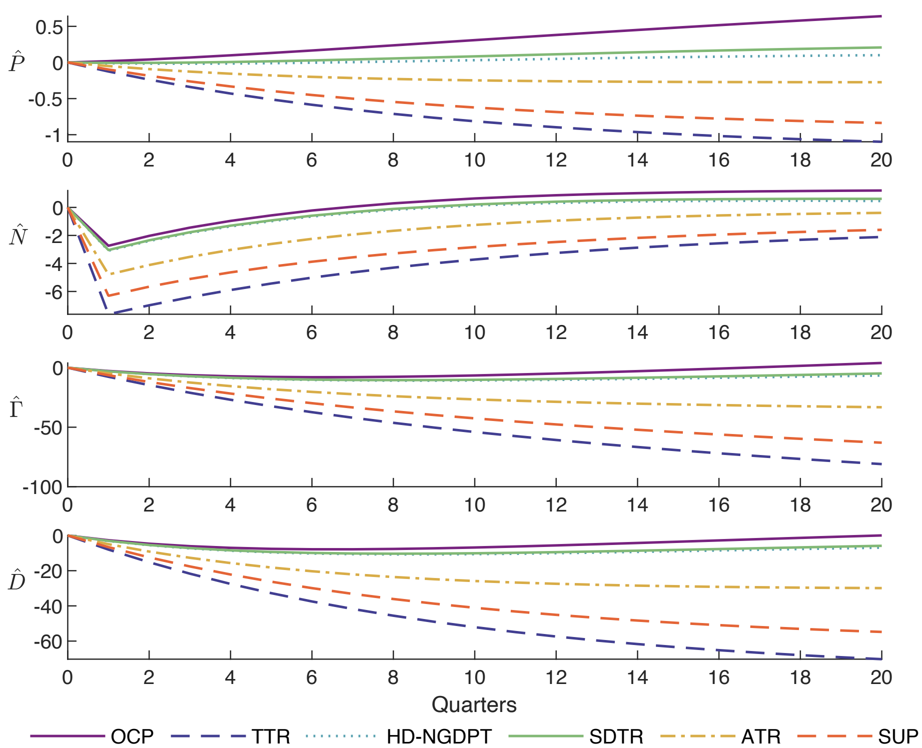


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.13: Dynamic response of a natural interest rate shock in a simple two-equation NK model with increased price rigidity (lower κ), under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The natural interest rate reverts to the absorbing state after 10 quarters (10th contingency). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1.Remaining parameters are calibrated according to standard EW (2003) values reported in footnote 21.

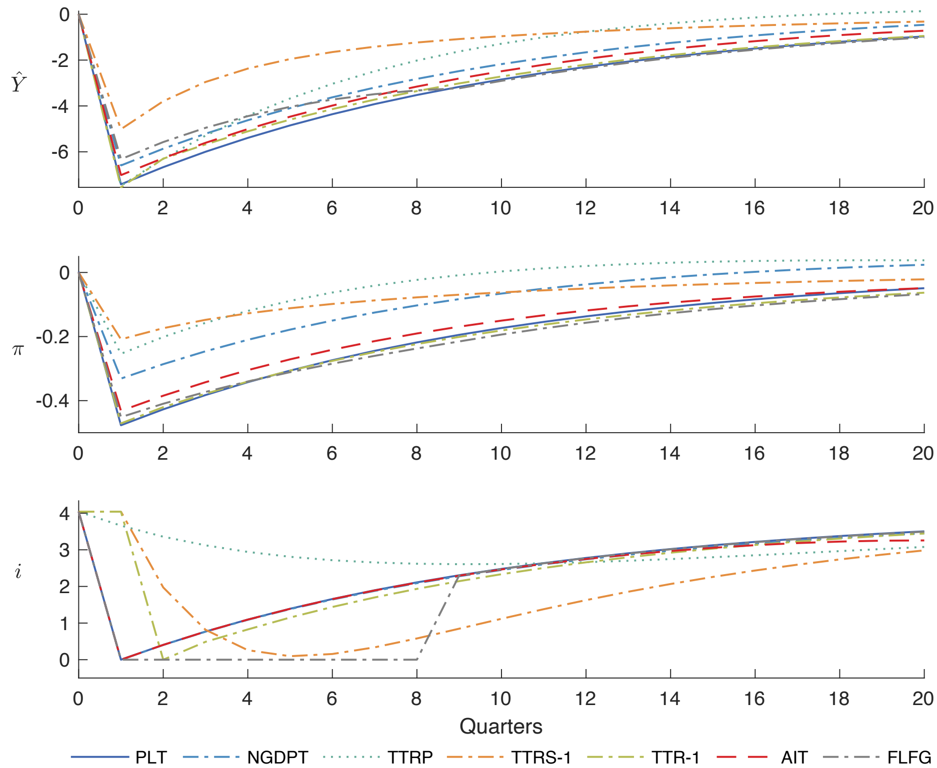


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

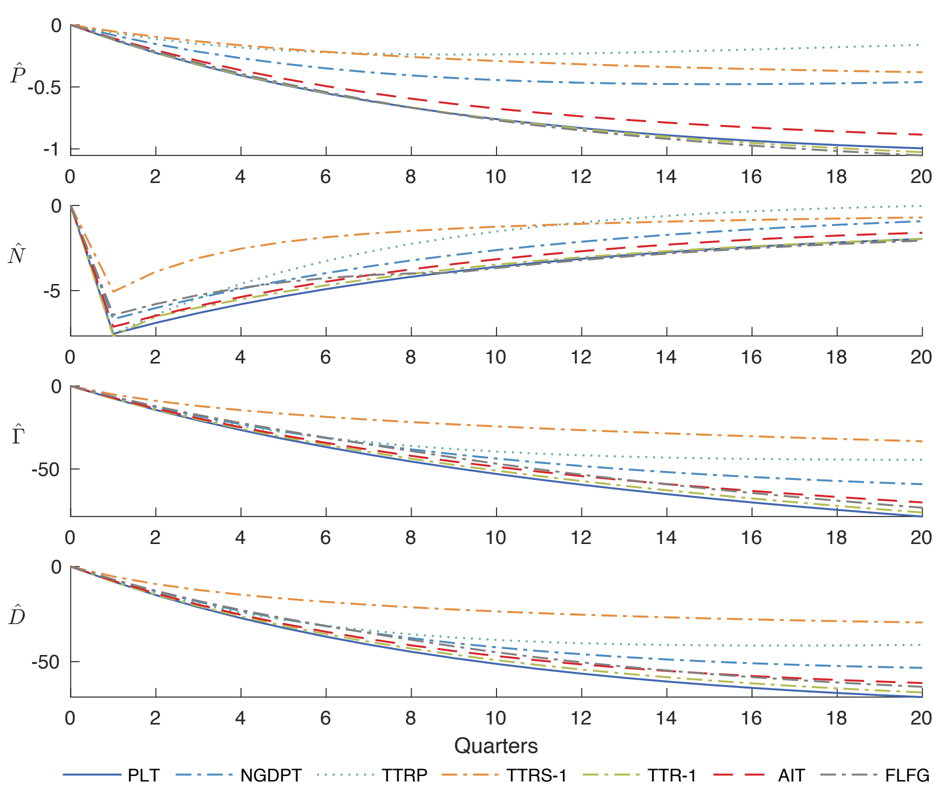


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.14: Average impulse response of a natural interest rate shock in a simple two-equation NK model with increased price rigidity (lower κ), under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.



(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)



(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.15: Average impulse response of a natural interest rate shock in a simple two-equation NK model with increased price rigidity (lower κ), under different policy rules. Shocks are calibrated such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). The vertical axes report percentage deviations from steady state (annualised figures). The vertical axis for i reports annualised percentage points. The list of acronyms is detailed in Table 1. The parametrisation is reported in Table A.2.

A.7.5 Simple New Keynesian model with -0.5% inflation drop – optimised policy rules

	Welfare Loss (1)	$\mathbb{E}_0[\tau + k_\tau - \tilde{T}]$ (2)	Volatility x (3)	Volatility π (4)	Volatility i (5)	Impact x (6)	Impact π (7)
OCP	$8.252\,10^{-4}$	15.257	$5.356\,10^{-3}$	$4.904\,10^{-4}$	$1.411\,10^{-3}$	−2.208	3.059
PANEL A: baseline rules							
OCP	1.000	1.000	1.000	1.000	1.000	1.000	1.000
TTR	3.800	0.655	9.335	0.022	0.657	3.364	−0.144
HD-NGDPT	1.568	1.099	3.563	0.207	1.094	1.818	0.502
SDTR	1.194	0.703	1.514	0.975	0.716	1.400	0.936
ATR	1.169	0.709	1.070	1.237	0.721	1.191	1.053
SUP	1.181	0.590	1.101	1.235	0.689	1.147	1.068
PANEL B: additional rules							
PLT	3.294	0.655	8.118	0.000	0.657	3.145	−0.018
NGDPT	3.267	0.655	8.054	0.000	0.657	3.132	−0.011
TTRP	3.269	0.655	8.058	0.000	0.657	3.133	−0.012
TTRS-1	1.353	0.000	0.295	2.076	0.647	0.882	1.358
TTR-1	1.732	0.649	3.845	0.289	0.651	2.360	0.504
AIT	3.278	0.655	8.079	0.000	0.657	3.138	−0.014
FLFG	3.105	1.019	7.391	0.180	1.023	1.558	0.597

Table A.8: Some metrics for selected interest rate rules in the simple two-equation NK model with optimised policy rules. All rows except the first show values normalised with respect to the optimal commitment policy (OCP, first row). Column (1) reports the welfare loss computed from a quadratic loss function for the central bank with equal weights; Column (2) displays the unconditional expected duration of the Zero Lower Bound (regimes 1 and 2); Columns (3)-(5) report a summary measure of deviations of the endogenous variables from target, computed according to Equation A.6; Finally, Columns (6) and (7) show the response on impact, in annual percentage points, of the output gap and inflation to a natural interest rate shock such that output falls by 7.5% and inflation by 0.5% under a Truncated Taylor Rule (TTR). Rule calibration reported in Table (A.2). The model is calibrated with the standard EW (2003) parameter values reported in footnote 21.

A.7.6 FRBNY DSGE Model

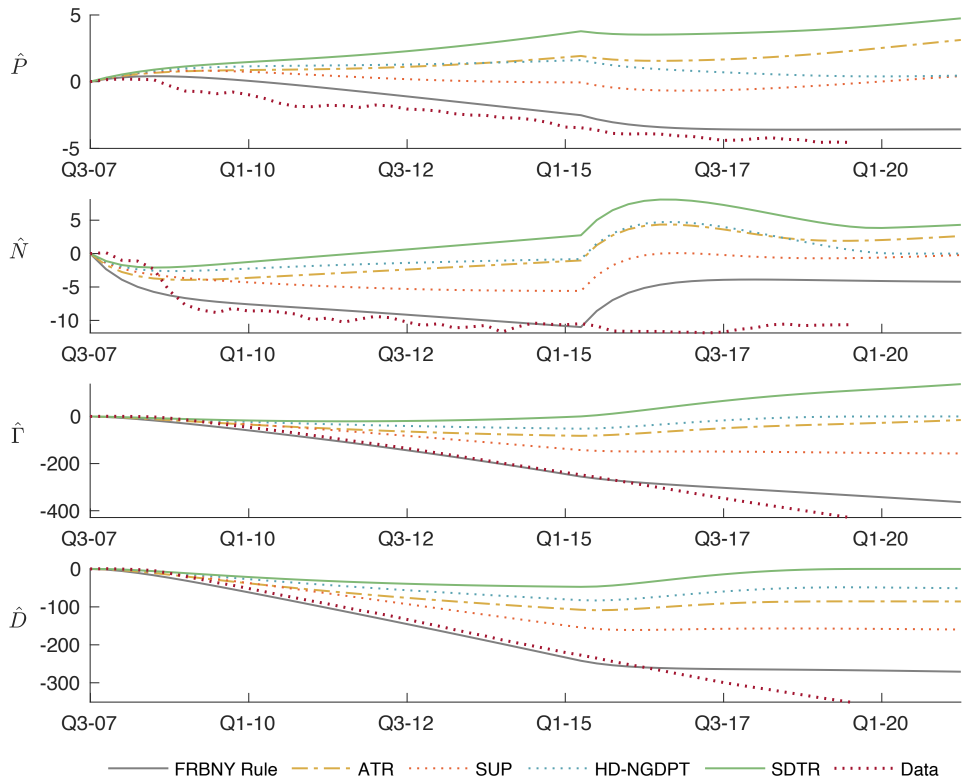


Figure A.16: Dynamic response to a preference shock and a correlated cost push shock in FRBNY model, under baseline policy rules. Coloured lines show paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$). The two-state Markov shocks switch to low state in Q4-07 and revert to the absorbing state after 32 quarters (32nd contingency). The vertical axis for P reports percent deviations from its trend. The vertical axis for $\hat{N}$ reports deviations from detrended steady state, in percentage points (annualised figures). The vertical axes for $\hat{\Gamma}$ and $\hat{D}$ report deviations from initial levels (Q3-2007 = 0). The horizontal axis shows quarter and calendar year. See Section A.5.2 for calibration. The list of acronyms is detailed in Table 1. FRBNY Rule refers to Equation (A.7).

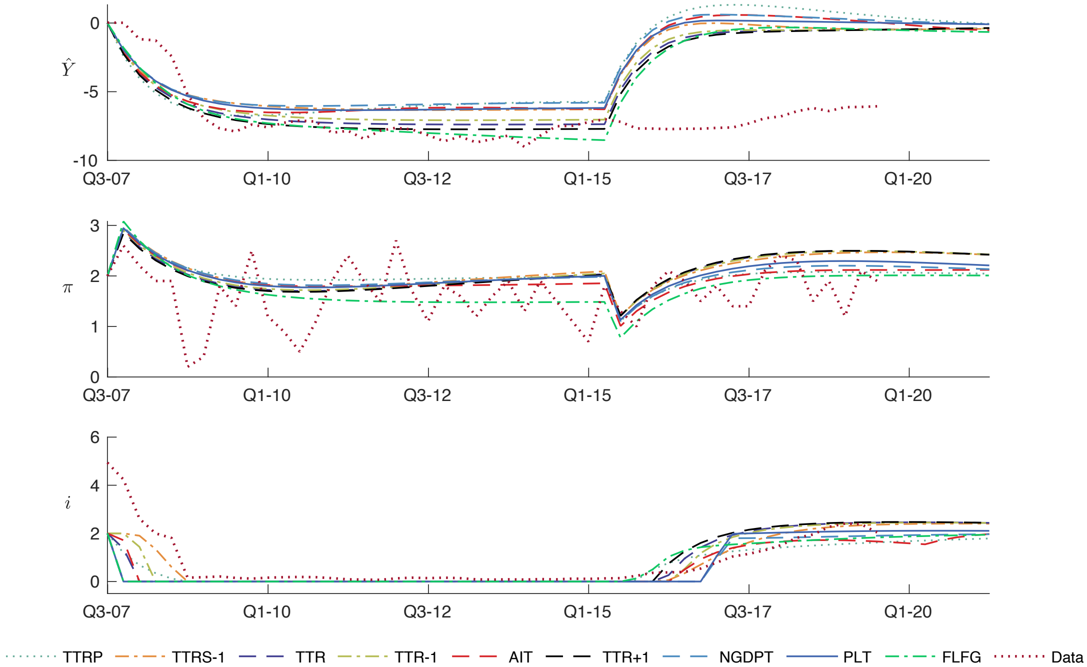


Figure A.17: Dynamic response to a preference shock and a correlated cost push shock in FRBNY model, under additional policy rules. Coloured lines show paths for output ($\hat{Y}_t$), inflation (π), the nominal interest rate (i), and nominal GDP ($\hat{N}$). Dotted red line is data. The two-state Markov shocks switches to low state in Q4-07 and reverts to the absorbing state after 32 quarters (32nd contingency). The vertical axes for $\hat{Y}_t$ and $\hat{N}$ report deviations from detrended steady state, in percentage points (annualised figures). The vertical axes for π and i report annualised percentage points. The horizontal axis shows quarter and calendar year. See Section A.3 for details on data and Section A.5.2 for calibration. The list of acronyms is detailed in Table 1.

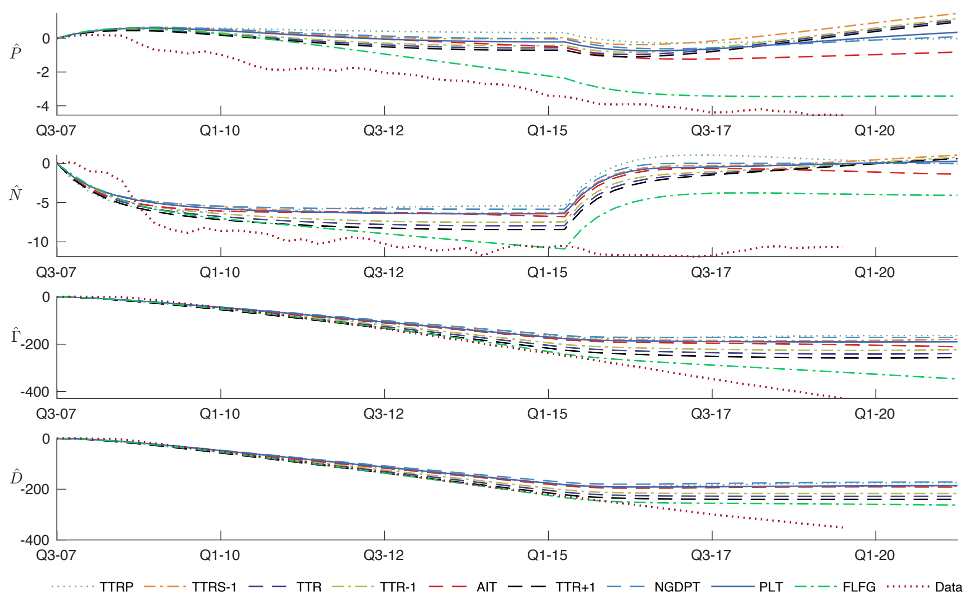
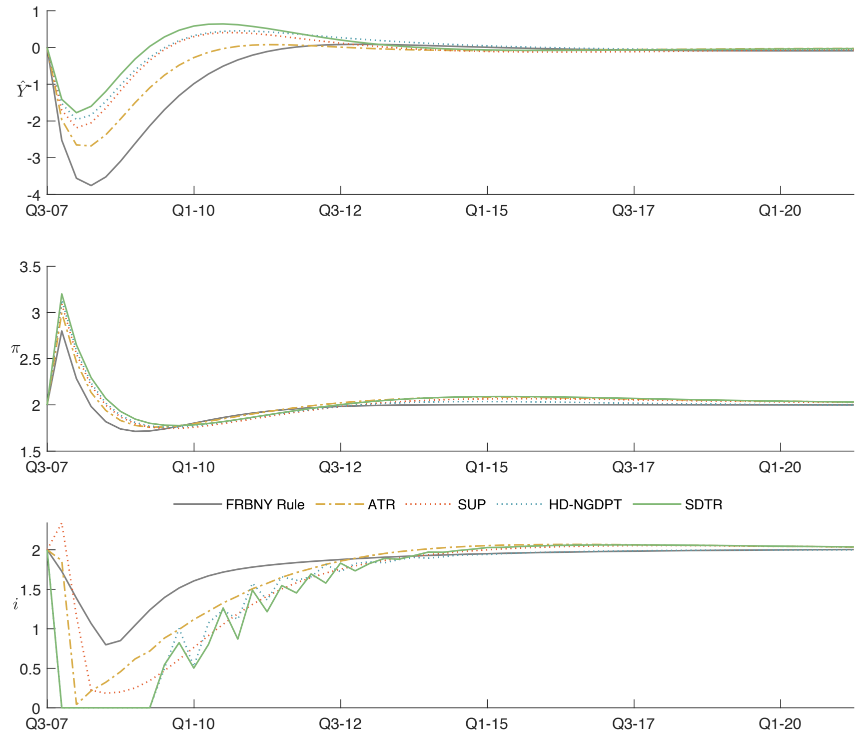
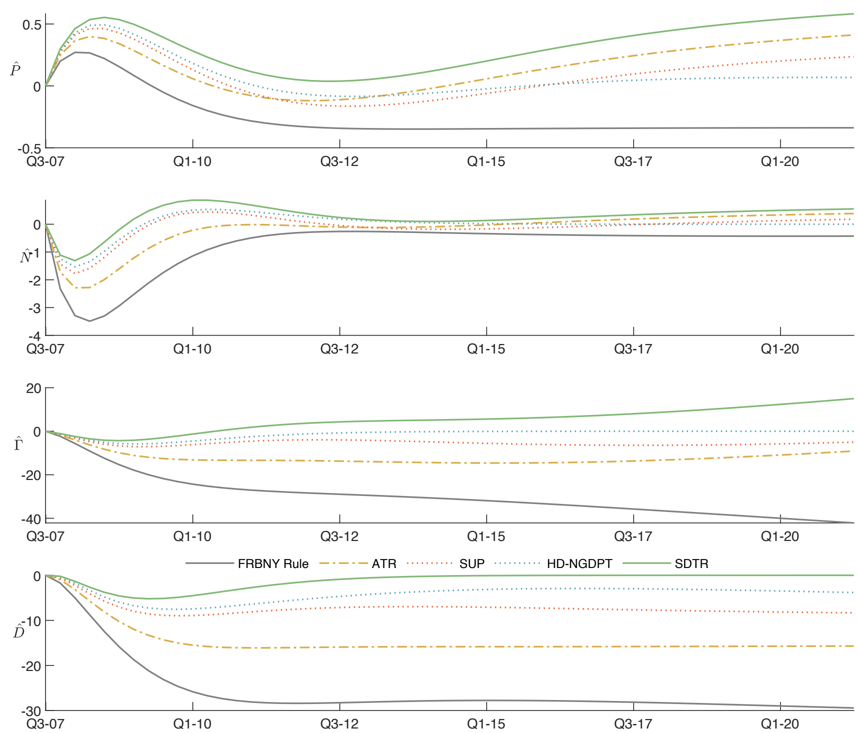


Figure A.18: Dynamic response to a preference shock and a correlated cost push shock in FRBNY model, under additional policy rules. Coloured lines show paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$). The two-state Markov shocks switch to low state in Q4-07 and revert to the absorbing state after 32 quarters (32nd contingency). The vertical axis for P reports percent deviations from its trend. The vertical axis for $\hat{N}$ reports deviations from detrended steady state, in percentage points (annualised figures). The vertical axes for $\hat{\Gamma}$ and $\hat{D}$ report deviations from initial levels (Q3-2007 = 0). The horizontal axis shows quarter and calendar year. The horizontal axis shows quarter and calendar year. See Section A.5.2 for calibration. The list of acronyms is detailed in Table 1.

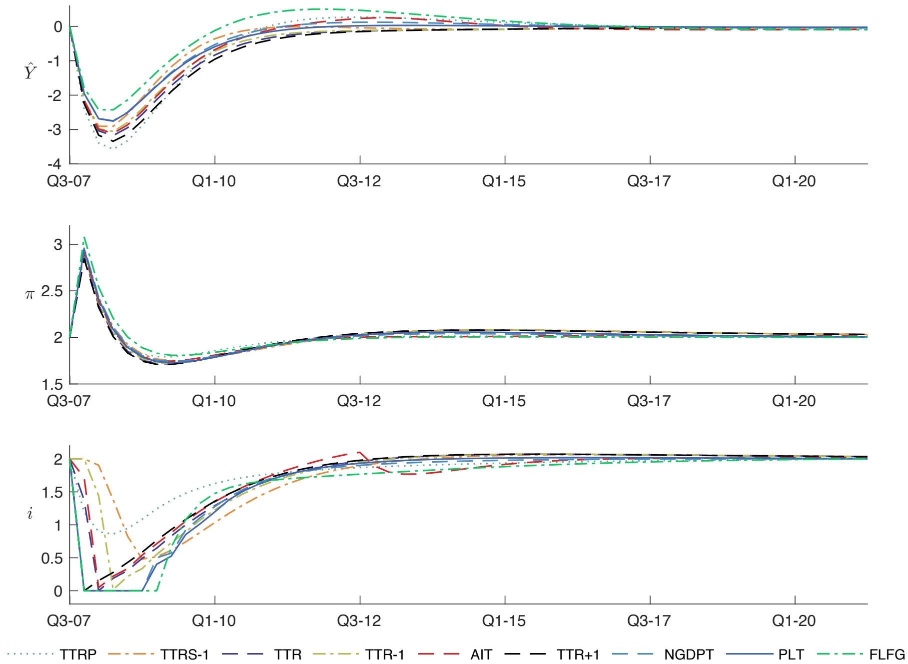


(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)

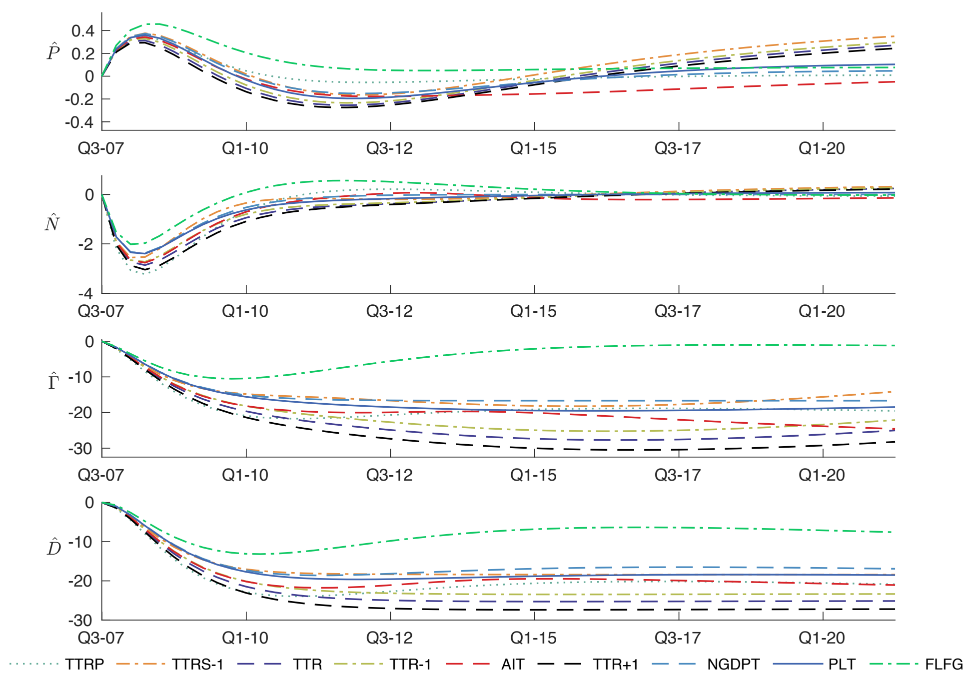


(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.19: Average impulse response to a preference shock and a correlated cost push shock in FRBNY model, under baseline policy rules. The vertical axis for $\hat{Y}$ reports deviations from detrended steady state, in percentage points (annualised figures). The vertical axes for π and i report annualised percentage points. The vertical axis for P reports percent deviations from its trend. The vertical axis for $\hat{N}$ reports deviations from detrended steady state, in percentage points (annualised figures). The vertical axes for $\hat{\Gamma}$ and $\hat{D}$ report deviations from initial levels (Q3-2007 = 0). The horizontal axis shows quarter and calendar year. See Section A.5.2 for calibration. The list of acronyms is detailed in Table 1. FRBNY rule refers to Equation (A.7).



(a) Paths for output ($\hat{Y}$), inflation (π) and nominal interest rate (i)



(b) Paths for price level ($\hat{P}$), nominal output ($\hat{N}$), cumulated nominal output ($\hat{\Gamma}$) and the Dual Mandate index ($\hat{D}$)

Figure A.20: Average impulse response to a preference shock and a correlated cost push shock in FRBNY model, under additional policy rules. The vertical axis for $\hat{Y}$ reports deviations from detrended steady state, in percentage points (annualised figures). The vertical axes for π and i report annualised percentage points. The vertical axis for P reports percent deviations from its trend. The vertical axis for $\hat{N}$ reports deviations from detrended steady state, in percentage points (annualised figures). The vertical axes for $\hat{\Gamma}$ and $\hat{D}$ report deviations from initial levels (Q3-2007 = 0). The horizontal axis shows quarter and calendar year. See Section A.5.2 for calibration. The list of acronyms is detailed in Table 1.

B Examples

This section presents a series of examples that are designed to show the features of the toolkit, and to guide the user in the setup required to solve a specific model. The examples are built around the simple two-equation model described by Equations 14-15 and a backward-looking Taylor rule with interest rate smoothing (TTRS).

All example files set the function `regime1.m` in verbose mode, in order to provide the user with more information on the inner workings of our toolkit.

Example 1 shows how to use the code in its most basic form. We suggest the user to start with that first, as its configuration is often referenced to by subsequent examples.

B.1 Example 1

This example shows how to solve a simple model with state variables.

B.1.1 Model

The equations describing the model are:

$$\mathbb{E}_t x_{t+1} + \sigma \mathbb{E}_t \pi_{t+1} = x_t + \sigma \left(i_t - r_t^n \right) \tag{A.10}$$

$$\beta \mathbb{E}_t \pi_{t+1} = \pi_t - \kappa x_t \tag{A.11}$$

$$0 = -i_t + \max\{0, \phi_i i_{t-1} + (1 - \phi_i)(r^* + \phi_\pi \pi_{t-1} + \phi_x x_{t-1})\} \tag{A.12}$$

where r_t^n evolves as a two-state Markov process with constant transition probability μ. Equations A.10-A.12 are stated with all expectational variables on the left-hand side, and current pre-determined variables and constants on the right-hand side, so that they conform to the linear system in Equation 5. Note how A.12 is the equation subject to the bound, so it will be listed as last in the script declaring all the equations (which we conventionally call `equations.m`).

B.1.2 Variables

Our toolkit requires the model to be cast in the form specified by Equation 5. We want to plot, together with the other endogenous variables, impulse responses for the interest rate implied by TTRS in A.12; hence, we define an ancillary jump variable tracking the implied rate: $i_t^{imp} \equiv \phi_i i_{t-1} + (1 - \phi_i)(r^* + \phi_\pi \pi_{t-1} + \phi_x x_{t-1})$. As a consequence, the vector of variables ξ_t will be:

$$\xi_t \equiv \left[x_t \ \pi_t \ i_t^{imp} \ i_t \ x_{t-1} \ \pi_{t-1} \ i_{t-1} \ r^* \ r_t^n \right]'$$

The ordering of variables is as important for the correct operation of our toolkit as it was for equations. As noted in A.2.1, the user should first include jump variables (in this example x_t, π_t and i_t^{imp}), then the variable subject to the occasionally binding constraint (i_t) followed by predetermined variables ($x_{t-1}, \pi_{t-1}, i_{t-1}$), constants ($r^*$) and finally shocks ($r_t^n$). This is achieved by generating a structure called `vars`, which lists as fields the position of each variable in ξ_t (in our codes and examples, we build this structure in the script `variables.m`)

B.1.3 Equations

A very similar structure (generated in our codes in `equations.m`) declares the ordering for the equations. This object does not contain any actual mathematical specification, but rather it simply links labels for each equation (for example `nkpc` or `policy`) to the intended ordering. Once again, the equation subject to the occasionally binding constraint A.12 is ordered last. Here it is called `rule` and listed as number 9. In addition, the identities specifying the shocks (only r_t^n in this example) are to be included second to last. The other seven equations are, in ascending order: equations A.10 and A.11, three equations defining the lagged terms x_{t-1}, π_{t-1} and i_{t-1},[49] one equation defining i_t^{imp} and finally one equation for the constant term r^*. In our implementation of the model, this last term is introduced as a permanent shock, so the equation specifying it is in the form $\mathbb{E}_t r_{t+1}^* = r_t^*$. This will be the first component of what the toolkit treats as Markov disturbances. Section B.1.5 provides further details on how to parametrise it so that it takes a constant value across all regimes.

The coefficients of the linearised system in A and B in 5 are specified in `matrices.m`. Since the matrices are quite sparse, the code initialises them at zero and fills in the non-zero elements in the subsequent lines. Our codes makes use of the structures defined in `equations.m` and `variables.m` to locate the position of each variable in every equation. The user is free to specify numerical values for the coefficients directly, or to obtain them from a separate structure dedicated to the model calibration (we take the latter route, see `parameters.m`). The resulting matrices are as follows:

$$A \equiv \begin{bmatrix} 1 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 & \sigma \\ 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 1 & 0 & 0 & \sigma & 0 & 0 & 0 & 0 & 0 \\ -\kappa & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & (1-\phi_i)\phi_x & (1-\phi_i)\phi_\pi & \phi_i & 1-\phi_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & (1-\phi_i)\phi_x & (1-\phi_i)\phi_\pi & \phi_i & 1-\phi_i & 0 \end{bmatrix} \tag{A.13}$$

Once again, the last row in A and B is the policy rule subject to the occasionally binding constraint. The fourth column contains coefficients for the jump variable subject to the constraint (i_t). The last two columns report coefficients for the shocks: the second to last is a permanent shock used to include a constant term r^*, while the last column is the proper forcing term r_t^n, which follows a two-state Markov process.

B.1.4 Initial conditions

The toolkit allows for optional initial conditions only for the pre-determined variables. To simplify the syntax and help the user keep track of what initial values are introduced, these conditions are declared as a column vector of the same length as ξ_t, stored in the field `param.init_cond`. If the field is omitted, the toolkit simply initialises it at zero. In this example, we introduce an initial condition $i_{-1} = \frac{1}{\beta} - 1$.[50] This is implemented in the last line of the script `parameters.m`. As the constant term r^* is introduced in the form of a permanent shock, no further initial conditions are necessary. In the first line of the same

[49] These simple equations are in the form $\mathbb{E}_t y_{t+1} = x_t$, where $y_t = x_{t-1}$. They link the current variable x_t with its lagged value y_t.
[50] Note: this condition applies to the predetermined variable i_{t-1} at $t = 0$, not on the jump variable i_t.

script we also declare the number of state variables, which we pass to the toolkit via the field `param.NS`. In the present example we have three predetermined variables and two shocks, so $N_S = 5$.

B.1.5 Specifying shocks

Finally, we need to parametrise the values for the two-state Markov shock in the high (steady state) and low state (or crisis state), as well as the value for the constant term r^*. The toolkit handles these values as separate vectors for each state. In other words, we need to specify a vector for the values that all shocks in the model take in the low state, and a separate one for the values of each shock in the high state. These vectors are fields in the structure `param` and are called `param.sl` and `param.sh`, respectively. In this example, the first shock is actually a constant term r^*, so it will take the same value $r^* = \frac{1}{\beta} - 1$ in both states of the world. The natural rate r_t^n drops to -0.5% in the low state and reverts back to r^* in steady state. Therefore our two vectors will be as follows:

$$\texttt{param.sl} = \begin{bmatrix} \frac{1}{\beta} - 1 \\ -0.005 \end{bmatrix}, \qquad \texttt{param.sh} = \left(\frac{1}{\beta} - 1\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{A.14}$$

We create these vectors in the script `parameters.m`, in the section named *Shocks*.

This concludes the instructions and customisation specific to this example. For the other parts of `ex_1.m` that are required by the toolkit as standard, we refer the reader to the guide at Section A.2.

B.2 Example 2

This example is a minor variation of Example 1, and shows how to solve a model under the assumption that the ELB becomes binding at an exogenous period $t = \tilde{T}$. This is implemented by passing two optional parameters to the function `regime1.m`: a value for $\tilde{T}$ itself (option `T_tilde_input`) and a flag to exclude the search for the length of regime 0 in equilibrium (`R0_search`). In this configuration, equilibrium conditions are not satisfied in regime 0. Consequently, the inequality constraint $i_t \geq 0$ is initially violated and the nominal interest rate turns negative in some periods $t < \tilde{T}$ under our calibration.

B.2.1 Model

As in Section B.1.1.

B.2.2 Variables

As in Section B.1.2.

B.2.3 Equations

As in Section B.1.3.

B.2.4 Initial conditions

As in Section B.1.4.

B.2.5 Specifying shocks

As in Section B.1.5

B.2.6 Additional parameters

To force the ELB to be binding from a pre-specified period, we need to include two optional parameters to the inputs of the function `regime1.m`:

1. `R0_search` set to 0, to disable the search for a $\tilde{T}$ that satisfies equilibrium conditions;
2. `T_tilde_input` set to a scalar, in our example $t = 5$, to provide a numerical value for the exogenous start of Regime 1.

For the other parts of `ex_2.m` that are required by the toolkit as standard, we refer the reader to the guide at Section A.2.

B.3 Example 3

This example shows how to impose an exogenous vector k. As in Example 2, it is implemented as an optional configuration of `regime3.m`, where the user provides a value for k and excludes the search for the equilibrium $\tilde{T}$. For simplicity, in what follows we will work with a constant $k_\tau = 5$ for all τ, so that $k = [5 \ \ldots \ 5]$. This policy prescribes $i_t = 0$ for five additional periods after the natural rate r_t^n has returned to the absorbing state s_h, with no regards to the contingency when the change in regime happens.

B.3.1 Model

As in Section B.1.1.

B.3.2 Variables

As in Section B.1.2.

B.3.3 Equations

As in Section B.1.3.

B.3.4 Initial conditions

As in Section B.1.4.

B.3.5 Specifying shocks

As in Section B.1.5

B.3.6 Additional parameters

To force regime 2 to last for five periods, the user has to include two optional parameters to the inputs of the function `regime1.m`:

1. `R0_search` set to 0, to disable the search for a $\tilde{T}$ that satisfies equilibrium conditions;
2. `k_input` set to a row vector of length τ_{max}, in our example $k = [5 \ \ldots \ 5]$, specifying the duration of regime 2 in every contingency τ.

For the other parts of `ex_3.m` that are required by the toolkit as standard, we refer the reader to the guide at Section A.2.

B.4 Example 4

This example shows how to solve a model under a deterministic AR(p) shock. For ease of exposition, what follows describes the simplest possible case, where $p = 1$. The implementation is based on setting the parameter $\mu = 1$, so that the two-state Markov disturbance reverts to steady state at $t = 2$ with probability one. Hence, the shock acts as a one-off unanticipated innovation (ε_t) for the deterministic autoregressive shock of the model (r_t^n). The shock itself and its law of motion are then specified as a pre-determined variable. The resulting impulse response is then identical to the second contingency $\tau = 2$, since all others have zero probability.

B.4.1 Model

The equations describing the model are the same as in Example 1 (A.10-A.12), plus the *AR(1)* process for the natural real interest rate:

$$r_{t+1}^n = (1-\rho)r^* + \rho r_t^n + \varepsilon_t \tag{A.15}$$

where ε_t evolves as a two-state Markov process with constant transition probability $\mu = 1$. Note that equation A.15 will be listed fourth to last in the script declaring all the equations (`equations.m`), as r_t^n is now treated as a pre-determined variable.

B.4.2 Variables

Since we now have an AR(1) process for r_t^n, the vector of variables ξ_t will be:

$$\xi_t \equiv \left[x_t \; \pi_t \; i_t^{imp} \; i_t \; x_{t-1} \; \pi_{t-1} \; i_{t-1} \; r_t^n \; r^* \; \varepsilon_t\right]'$$

where ε_t evolves as a two-state Markov process with degenerate transition probability $\mu = 1$. The variable for the natural rate r_t^n is now ordered third to last, among the other pre-determined states x_{t-1}, π_{t-1} and i_{t-1}. We refer the reader to B.1.2 for further details on the implementation.

B.4.3 Equations

The matrices declared in the script `matrices.m` are as follows:

$$A \equiv \begin{bmatrix} 1 & \sigma & 0 & 0 & 0 & 0 & 0 & \sigma & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 1 & 0 & 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 \\ -\kappa & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & (1-\phi_i)\phi_x & (1-\phi_i)\phi_\pi & \phi_i & 0 & 1-\phi_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & r^* & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & (1-\phi_i)\phi_x & (1-\phi_i)\phi_\pi & \phi_i & 0 & 1-\phi_i & 0 \end{bmatrix} \tag{A.16}$$

The only substantive difference with the matrices in B.1.3 is the presence of the equation for the AR(1) process for r_t^n in the fourth-to-last line of A and B.

B.4.4 Initial conditions

Since r_t^n is now a pre-determined variable, it requires an initial condition for the value it assumes in regime 0. For simplicity, we choose the steady-state value $r_{-1}^n = \frac{1}{\beta} - 1$. As in B.1.4, this will be declared at the end of the script `parameters.m`.

B.4.5 Specifying shocks

We modify the vectors specified in B.1.5 so that the two-state Markov disturbance ε_t has no effect on the model after the first period. Trivially, this is done by setting its steady-state value to zero. Our vectors of shocks will be therefore:

$$\texttt{param.sl} = \begin{bmatrix} \frac{1}{\beta} - 1 \\ -0.005 \end{bmatrix}, \qquad \texttt{param.sl} = \begin{bmatrix} \frac{1}{\beta} - 1 \\ 0 \end{bmatrix} \tag{A.17}$$

B.4.6 Additional parameters

For the other parts of `ex_4.m` that are required by the toolkit as standard, we refer the reader to the guide at Section A.2. Since the model is deterministic, it should be noted that it can be solved by setting $\tau_{max} = 2$, as the only relevant contingency is $\tau = 2$.