Appendix to "Incentivizing Negative Emissions Through Carbon Shares"

This appendix gives the full equations for the model, which follows DICE-2016R (Nordhaus, 2017). The only modifications are to change the horizon, to allow uncertainty about a damage parameter, to allow negative emissions to begin as soon as that uncertainty is resolved, and to update the carbon cycle and climate system. Table A-1 reports the values of the model parameters.

The DICE model is a Ramsey growth model coupled to a climate module. An infinitely lived representative agent aims to maximize the sum of the stream of discounted utility from consuming output. The timestep is Δ years and the horizon is here 400 years, or $\bar{t} = 400/\Delta$ periods.[30] I follow DICE-2016R in setting $\Delta = 5$. The initial year is 2015, denoted here as time 0. At time 0, the policymaker chooses the abatement rate μ_t and savings rate s_t to maximize a utilitarian expected welfare function of consumption C_t and population L_t:

$$\max_{\{\mu_t, s_t\}_{t=0}^{\bar{t}-1}} E_0 \left[\sum_{t=0}^{\bar{t}-1} \frac{1}{(1+\rho)^{\Delta\, t}} L_t \, u(C_t; L_t) \right], \tag{Welfare}$$

where expectations are taken at the time 0 information set. Per-period utility is:

$$u(C_t; L_t) = \frac{(C_t/L_t)^{1-\eta}}{1-\eta}, \tag{Utility}$$

with $\eta \geq 0, \neq 1$ is the inverse of the elasticity of intertemporal substitution and also the coefficient of relative risk aversion. Utility is discounted at annual rate ρ. As described below, the policymaker chooses abatement and savings rates as functions of information about damages (i.e., as closed-loop policies), not as functions of time.

To produce time t gross output Y_t^g, the agent combines capital K_t with labor L_t and technology A_t in a Cobb-Douglas production function:

$$Y_t^g = A_t \, (L_t/1000)^{1-\kappa} \, K_t^{\kappa}. \tag{Gross output}$$

[30] The horizon in DICE-2016R is 500 years. Shortening the horizon to 400 years does not sacrifice much but helps when optimizing under uncertainty because the number of controls becomes large.

Some of this output is lost to damages caused by surface warming T_t, so that output net of damages is given by

$$Y_t^n = Y_t^g \left[1 - \min\{0.95, d_t \left[T_t\right]^2\}\right]. \qquad \text{(Net output)}$$

The parameter d_t is constant and known prior to 2065, with value d. It is also constant from 2065 on, with value $\tilde{d}$. The policymaker does not know $\tilde{d}$ until 2065. In the DICE damage specification, $d = 0.00236$ and the distribution of $\tilde{d}$ is lognormal with mean 0.00236. The standard deviation of $\ln \tilde{d}$ is 1.286, from Appendix C.1 of Lemoine (2021). That calibration fits a distribution to the Pindyck (2019) expert survey of losses from climate change in fifty years after adjusting for uncertainty about warming. The expert damage specification increases both d and the mean of $\tilde{d}$ to 0.0228 in order to match the survey results and truncates the distribution from above at 0.1132 (see Lemoine, 2021). I cap the losses in any one period at 95%.

The policymaker allocates net output to consumption C_t, investment I_t, or spending Ψ_t on emission abatement. Industrial emissions (net of abatement) per timestep are:

$$E_t^I = \Delta\, \sigma_t (1 - \mu_t) Y_t^g, \qquad \text{(Industrial emissions)}$$

where σ_t is the emission intensity of production at time t. Emissions E_t (net of abatement) per timestep are

$$E_t = E_t^I + \Delta\, E_t^{\sim I}, \qquad \text{(Emissions)}$$

where $E_t^{\sim I}$ gives (exogenous) annual emissions from deforestation. Cumulative industrial emissions up to each time t are constrained by the stock of available carbon:

$$\sum_{t=0}^{\tau} \left[400 + \max\{0, E_t^I\}\right] \leq 6000 \quad \text{for all } \tau \in [0, T-1], \qquad \text{(Cumulative fossil constraint)}$$

where E_t^I is measured in Gt C. The cost of abating fraction μ_t of industrial emissions is

$$\Psi_t = \psi_t Y_t^g [\mu_t]^{a_2}. \qquad \text{(Abatement cost)}$$

The carbon tax is equal to marginal abatement cost. I constrain $\mu_t \leq 1$ prior to 2065.[31]

The economy's resource constraint is:

$$C_t + I_t + \Psi_t \leq Y_t^n. \qquad \text{(Resource constraint)}$$

Capital depreciates at annual rate δ_K:

$$K_{t+1} = K_t\,(1-\delta_k)^{\Delta} + \Delta\, I_t. \qquad \text{(Capital)}$$

Annual investment is determined by the savings rate s_t:

$$I_t = s_t[Y_t^n - \Psi_t]. \qquad \text{(Investment)}$$

The final fifty years' savings rate is fixed at 0.2583.

The model's exogenous economic processes are

$$\begin{aligned}
L_{t+1} =& L_t \left(\frac{L_\infty}{L_t}\right)^{g_L\,\Delta/5}, && \text{(Population)}\\
A_{t+1} =& A_t/(1-g_{A,t})^{\Delta/5}, && \text{(Production technology)}\\
g_{A,t+1} =& g_{A,0}\, e^{-\Delta\,(t+1)\,\delta_A}. && \text{(Production technology growth rate)}
\end{aligned}$$

The model's exogenous climate-related processes are

$$\begin{aligned}
\sigma_{t+1} &= \sigma_t e^{\Delta\, g_{\sigma,t}}, && \text{(Gross emissions per unit of output)}\\
g_{\sigma,t+1} &= g_{\sigma,t}(1+\delta_\sigma)^{\Delta}, && \text{(Growth rate of gross emissions per unit of output)}\\
\psi_{t+1} &= \frac{a_1(1-g_\psi)^{t\,\Delta/5}\,\sigma_{t+1}}{1000\,a_2}, && \text{(Abatement cost coefficient)}\\
E_{t+1}^{\sim I} &= E_0^{\sim I}\,(1-g_E)^{(t+1)\,\Delta/5}, && \text{(Emissions from deforestation)}\\
EF_{t+1} &= EF_0 + (EF_{100} - EF_0)\min\{\Delta\, t/(5*17), 1\}. && \text{(Non-}CO_2\text{ forcing)}
\end{aligned}$$

I now describe the carbon cycle and climate model, both of which deviate from DICE-2016R. The carbon cycle follows Joos et al. (2013, Table 5), as recommended and compiled

[31] In DICE-2016R, $\mu_t \leq 1$ for the first 145 years and $\mu_t \leq 1.2$ afterward.

by Dietz et al. (2020).[32] That carbon cycle has

$$\boldsymbol{M}_{t+1} = \boldsymbol{\Lambda}^{\Delta} M_t + \boldsymbol{b} E_t \qquad \text{(Carbon reservoirs)}$$

where $\boldsymbol{M}$ is a 4×1 vector of atmospheric carbon reservoirs. The coefficient matrices are:

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.9975 & 0 & 0 \\ 0 & 0 & 0.9730 & 0 \\ 0 & 0 & 0 & 0.7927 \end{bmatrix}$$

and

$$\boldsymbol{b} = \begin{bmatrix} 0.2173 \\ 0.2240 \\ 0.2824 \\ 0.2763 \end{bmatrix}.$$

The year 2015 values (in Gt C) are

$$\boldsymbol{M}_0 = \begin{bmatrix} 588 + 139.1 \\ 90.2 \\ 29.2 \\ 4.2 \end{bmatrix}, \qquad \text{(Carbon starting value)}$$

where 588 Gt C is the stock of preindustrial carbon.

The parameters of the climate model come from Geoffroy et al. (2013), as compiled by Dietz et al. (2020). Additional atmospheric carbon dioxide (CO_2) increases radiative forcing $F_t(\boldsymbol{M}_t)$, which measures additional energy at the earth's surface due to CO_2 in the atmosphere. Forcing is

$$F_t(\boldsymbol{M}_t) = f_{2x} \frac{\ln\left(\sum_{i=1}^{4} M_t^i / 588\right)}{\ln(2)} + EF_t, \qquad \text{(Forcing)}$$

where i indicates element i of $\boldsymbol{M}_t$, EF_t is exogenous forcing from non-CO_2 greenhouse gases

[32] Dietz et al. (2020) additionally recommend using the FAIR model to capture carbon cycle feedbacks, but doing so would further increase the complexity of an already nontrivial optimization problem.

(defined above), and f_{2x} is forcing induced by doubling CO_2. Surface temperature evolves as

$$T^s_{t+1} = T^s_t + \frac{\Delta}{5}\phi_1 \left[F_{t+1}(\boldsymbol{M}_{t+1}) - \lambda T^s_t - \phi_3 \left(T^s_t - T^o_t\right)\right]. \qquad \text{(Surface temperature)}$$

Ocean temperature evolves as

$$T^o_{t+1} = T^o_t + \frac{\Delta}{5}\phi_4 \left[T^s_t - T^o_t\right]. \qquad \text{(Ocean temperature)}$$

Steady-state warming from doubled carbon dioxide ("climate sensitivity") is $f_{2x}/\lambda = 3.1$°C.

I solve the model by searching over contingent trajectories for μ_t, s_t, K_t, $\boldsymbol{M}_t$, T^s_t, and T^o_t, treating the transition equations as constraints. With this form, I can supply an analytic gradient for the objective and an analytic Jacobian for the constraints. I approximate the distribution over $\tilde{d}$ using quadrature with 5 nodes.[33] The trajectories are contingent because they vary by quadrature node. I solve the model in Matlab. When optimizing the full model, I search over 2,880 values. When simulating the distribution of future outcomes, I use the year 2065 state reached along the optimal trajectory (defined by policy optimized under uncertainty) and take 1,000 draws from the damage distribution.

I calculate the bond required by each draw from the damage distribution by combining the optimal year 2065 emission tax (as chosen upon learning the value of $\tilde{d}$ with the strip of pre-2065 charges. I calculate the pre-2065 charges by perturbing year 2015 emissions and calculating the change in each period's welfare.

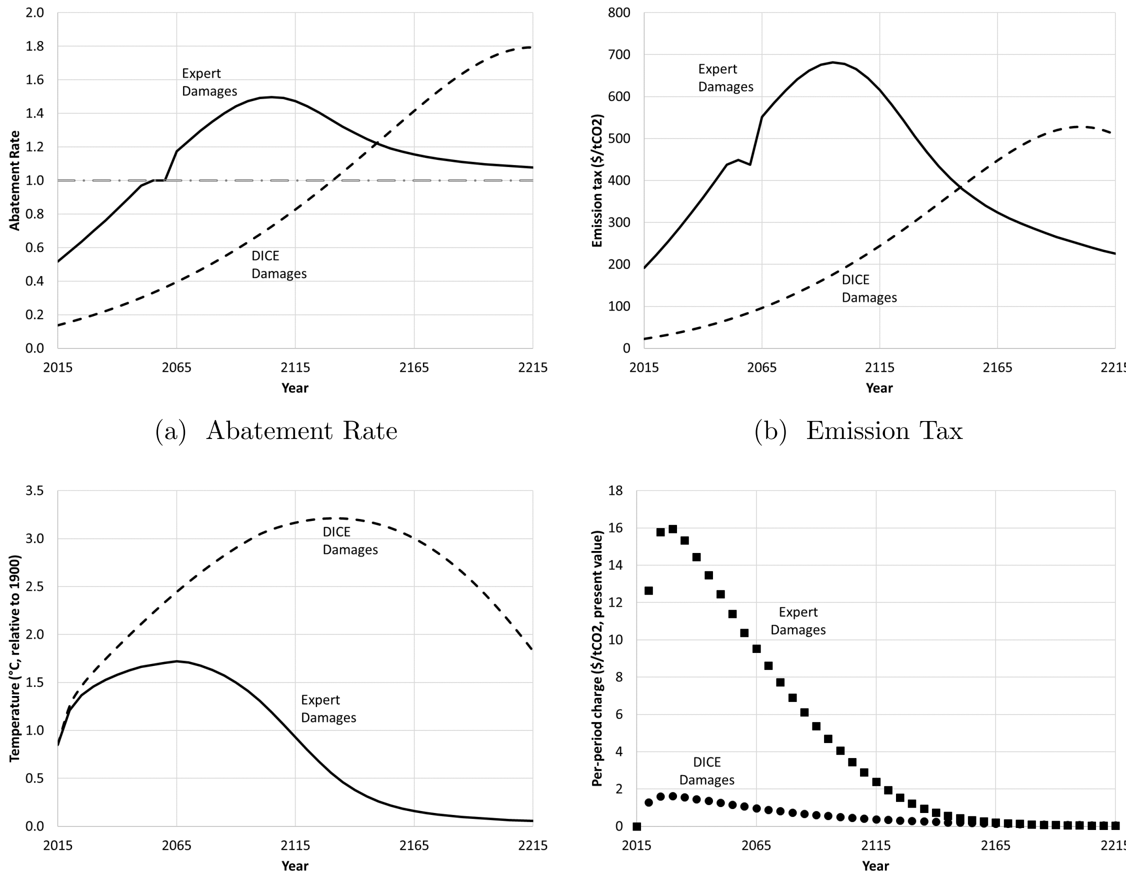
Figure A-1 reports the abatement, emission tax, and temperature trajectories in a deterministic model in which the damage parameter is fixed to its mean at all times (i.e., $\tilde{d} = d$). Negative emissions occur in midcentury in the case with expert damages and occur early in the next century in the case with DICE damages.[34] Near-term abatement is greater in the case with expert damages, but long-term abatement is reduced because early abatement leaves a smaller stock of atmospheric carbon. The emission tax trajectory reveals corre-

[33] I use the compecon toolbox to obtain Gaussian quadrature nodes for non-truncated distributions (Miranda and Fackler, 2002) and use the Fortran90 version of truncated_normal_rule (available at `http://people.math.sc.edu/Burkardt/c_src/truncated_normal_rule/truncated_normal_rule.html`) to obtain quadrature nodes for truncated distributions.

[34] The kink in the case with expert damages arises because the pre-2065 constraint that abatement be weakly less than 100% briefly binds. The tax declines over this interval because exogenously improving technology gradually reduces the tax needed to obtain 100% abatement.

sponding effects on the implied emission price. Negative emissions eventually undo some warming in both calibrations, with the expert damage calibration never allowing warming to exceed 2 degrees C.

The lower right panel explodes the tax into the strip of per-period marginal damages. These are the optimal rental charges per 5-year timestep and are also the optimal sequence of per-period damage charges that would be implemented under the carbon share policy. The sum of each set of points equals the optimal emission tax. The charges increase over the next decades as current emissions translate into warming and as higher temperatures interact with convex damages. The charges eventually decline due to the effect of discounting, the eventual decline in temperature, and the decay of initial emissions. The charges spread the emission tax's upfront payment over more than a century, with the peak charges comprising only a small fraction of the optimal emission tax.



(a) Abatement Rate

(b) Emission Tax

(c) Temperature

(d) Rental Charge

Figure A-1: Optimal trajectories in deterministic versions of each damage calibration.

Table A-1: Parameters

Parameter	Value	Description
Δ	5	Timestep (years)
$\bar{t}$	80	Horizon (periods)
A_0	5.115	Initial production technology
$g_{A,0}$	0.076	Initial growth rate of production technology, per five years
δ_A	0.005	Annual decline in growth rate of production technology
L_0	7403	Year 2015 population (millions)
L_∞	11500	Asymptotic population (millions)
g_L	0.134	Rate of approach to asymptotic population level, per five years
σ_0	0.0955	Initial emission intensity of output (Gt C per trillion 2010$)
$g_{\sigma,0}$	-0.0152	Initial annual growth rate of emission intensity
δ_σ	-0.001	Annual change in growth rate of emission intensity
a_1	2016.7	Cost of backstop technology in 2015 (2010$ per ton of C)
a_2	2.6	Abatement cost function exponent
g_ψ	0.025	Decline rate of backstop cost, per five years
$E_0^{\sim I}$	0.71	Initial emissions from deforestation (Gt C per year)
g_E	0.115	Decline rate of deforestation emissions, per five years
EF_0	0.5	Year 2015 non-CO_2 forcing (W/m^2)
EF_{100}	1	Year 2100 non-CO_2 forcing (W/m^2)
κ	0.3	Capital share in production
δ_K	0.1	Annual capital depreciation rate
ρ	0.015	Annual utility discount rate
η	1.45	Inverse of elasticity of intertemporal substitution; also RRA
ϕ_1	0.386	Warming delay parameter
ϕ_3	0.73	Parameter governing transfer of heat from ocean to surface
ϕ_4	0.034	Parameter governing transfer of heat from surface to ocean
f_{2x}	3.503	Forcing from doubling CO_2 (W/m^2)
λ	1.13	Forcing per degree warming ([W/m^2]/°C)
$d, \tilde{d}$	see text	Damage parameters

Continued on next page

Table A-1 – continued from previous page

Parameter	Value	Description
K_0	223	Year 2015 capital (trillion 2010$)
$\boldsymbol{M}_0$	see text	Year 2015 carbon reservoirs (Gt C)
T_0^s	0.85	Year 2015 surface temperature (°C, wrt 1900)
T_0^o	0.0068	Year 2015 lower ocean temperature (°C, wrt 1900)

References from the Appendix

Dietz, Simon, Frederick van der Ploeg, Armon Rezai, and Frank Venmans (2020) "Are economists getting climate dynamics right and does it matter?" Working Paper 8122, CESIfo.

Geoffroy, O., D. Saint-Martin, D. J. L. Olivié, A. Voldoire, G. Bellon, and S. Tytéca (2013) "Transient climate response in a two-layer energy-balance model. Part I: Analytical solution and parameter calibration using CMIP5 AOGCM experiments," *Journal of Climate*, Vol. 26, No. 6, pp. 1841–1857.

Joos, F., R. Roth, J. S. Fuglestvedt, G. P. Peters, I. G. Enting, W. von Bloh, V. Brovkin, E. J. Burke, M. Eby, N. R. Edwards, T. Friedrich, T. L. Frölicher, P. R. Halloran, P. B. Holden, C. Jones, T. Kleinen, F. Mackenzie, K. Matsumoto, M. Meinshausen, G.-K. Plattner, A. Reisinger, J. Segschneider, G. Shaffer, M. Steinacher, K. Strassmann, K. Tanaka, A. Timmermann, and A. J. Weaver (2013) "Carbon dioxide and climate impulse response functions for the computation of greenhouse gas metrics: a multi-model analysis," *Atmos. Chem. Phys. Discuss.*, Vol. 13, No. 5, pp. 2793–2825.

Lemoine, Derek (2021) "The climate risk premium: How uncertainty affects the social cost of carbon," *Journal of the Association of Environmental and Resource Economists*, No. forthcoming.

Miranda, Mario J. and Paul L. Fackler (2002) *Applied Computational Economics and Finance*, Cambridge, Massachusetts: MIT Press.

Nordhaus, William D. (2017) "Revisiting the social cost of carbon," *Proceedings of the National Academy of Sciences*, Vol. 114, No. 7, pp. 1518–1523.

Pindyck, Robert S. (2019) "The social cost of carbon revisited," *Journal of Environmental Economics and Management*, Vol. 94, pp. 140–160.