A Theory of Voluntary Testing and Self-isolation in an Ongoing Pandemic

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— ONLINE APPENDIX —

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Proof of Proposition 1.

Using the steady-state utilities we find that $U^{SI} > U^{GO}$ when $y < Y_1 \equiv \phi\delta\left(x+z\right)/(1+\phi\delta^2)$. We can see that $\frac{dY_1}{dz}$, $\frac{dY_1}{dx} > 0$. Moreover,

$$\frac{dY_1}{d\phi} = \frac{\delta\left(x+z\right)\left(1+\phi\delta^2\right) - \phi\delta\left(x+z\right)\delta^2}{\left[1+\phi\delta^2\right]^2} = \frac{\delta\left(x+z\right)}{\left[1+\phi\delta^2\right]^2} > 0.$$

□

Derivation of Contagiousness Curve (CO).

We know from Proposition 1 that for $y < Y_1$ individuals choose SI, and therefore never go out. For $y \geq Y_1$ individuals choose GO. Let $\varpi_t^{GO|0}$ denote the fraction of these individuals at time t that is not infected and goes out, $\varpi^{GO|1}$ the fraction that is infected (but asymptomatic) and goes out, and ϖ^{ILL} the fraction that is ill and stays home. By definition, $\varpi_t^{GO|0} + \varpi_t^{GO|1} + \varpi_t^{ILL} = 1$. Moreover, everyone who is ill in t was asymptomatic in $t-1$: $\varpi_t^{ILL} = \varpi_{t-1}^{GO|1}$. The portion ϕ of people that were uninfected in $t-1$ and went out, are infected (but asymptomatic) in t: $\varpi_t^{GO|1} = \phi\varpi_{t-1}^{GO|0}$. In the steady-state we have $\varpi^k \equiv \varpi_t^k = \varpi_{t-1}^k = \varpi_{t-2}^k = ...$, $k = \{GO|0, GO|1, ILL\}$. Consequently, the equilibrium is defined by (i) $\varpi^{GO|0} + \varpi^{GO|1} + \varpi^{ILL} = 1$, (ii) $\varpi^{ILL} = \varpi^{GO|1}$, and (iii) $\varpi^{GO|1} = \phi\varpi^{GO|0}$. Using (ii) in (i) we get $\varpi^{GO|0} = 1 - 2\varpi^{GO|1}$. Using this in (iii) yields $\varpi^{GO|1} = \frac{\phi}{1+2\phi}$. Consequently, $\varpi^{GO|0} = \frac{1}{1+2\phi}$.

Using the expressions for $\varpi^{GO|0}$ and $\varpi^{GO|1}$, we get

$$\mu_{CO}(\phi) = \frac{\frac{\phi}{1+2\phi}\left[1-\Omega\left(Y_1\right)\right]}{\frac{1+\phi}{1+2\phi}\left[1-\Omega\left(Y_1\right)\right]} = \frac{\phi}{1+\phi}.$$

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Note that $\mu_{CO}(0) = 0$. Moreover,

$$\frac{d\mu_{CO}(\phi)}{d\phi} = \frac{1}{[1+\phi]^2} > 0.$$

Proof of Proposition 2.

Solving $\phi_{IN}(\mu)$ for μ we get

$$\mu_{IN}(\phi) = \frac{1}{1-\lambda}\left[1-(1-\phi)^{\frac{1}{M}}\right].$$

Note that $\mu_{IN}(0) = 0$, $\mu_{IN}(1-\lambda^M) = 1$, and

$$\frac{d\mu_{IN}(\phi)}{d\phi} = \frac{1}{1-\lambda}\frac{1}{M}(1-\phi)^{\frac{1}{M}-1} > 0.$$

Thus, $\mu_{IN}(1-\lambda^M) = 1 > \phi > \mu_{CO}(\phi)$. Moreover, we have $\mu_{IN}(\phi) < \mu_{CO}(\phi)$ for small ϕ when

$$\left.\frac{d\mu_{IN}(\phi)}{d\phi}\right|_{\phi=0} < \left.\frac{d\mu_{CO}(\phi)}{d\phi}\right|_{\phi=0} = \delta(x+z).$$

This condition is equivalent to

$$M > \widetilde{M} \equiv \frac{1}{1-\lambda}\left[\frac{d\mu_{CO}(\phi)}{d\phi}|_{\phi=0}\right]^{-1} = \frac{1}{1-\lambda}\left[\delta(x+z)\right]^{-1}.$$

Thus, for $M > \widetilde{M}$ there exists a stable equilibrium with $\phi^* > 0$. For $M \leq \widetilde{M}$ the only equilibrium is $\phi^* = 0$. Finally note that $\frac{d\mu_{IN}(\phi)}{dM} < 0$ and $\frac{d\mu_{IN}(\phi)}{d\lambda} > 0$. □

Proof of Proposition 3.

Recall from Proof of Proposition 1 that $U^{SI} > U^{GO}$ when $y < Y_1 = \phi\delta(x+z)/(1+\phi\delta^2)$. Moreover, comparing the other expected utilities we get

$$U^{SI} > U^{TE} \text{ if } y < \widehat{Y}_1 \equiv \frac{\phi\delta[\beta x + z] + c}{\overline{\phi}(1-\alpha\overline{\delta}) + \phi(\beta+\delta^2)},$$

$$U^{TE} > U^{GO} \text{ if } y > \widehat{Y}_2 = \frac{\phi\delta\left[1-\delta\left(\overline{\phi}\left(1-\alpha\overline{\delta}\right)+\phi\delta^2\right)\right](x+z) - [\phi\delta(\beta x+z)+c]\left[\overline{\delta}+\phi\delta\left(1-\delta^2\right)\right]}{1-\overline{\phi}\left(1-\alpha\overline{\delta}\right)(1+\phi\delta) - \phi\left[\phi\delta^3 + \left(1-\delta\left(\overline{\phi}+\phi\delta^2\right)\right)\beta\right]}.$$

Note that $\widehat{Y}_1$ is defined by $U^{TE} - U^{SI} = 0$, where $U^{SI} = 0$. Implicitly differentiating $\widehat{Y}_1$ w.r.t. $k = c, \alpha, \beta$, we get $\frac{d\widehat{Y}_1}{dk} = -\frac{\partial U^{TE}}{\partial k} / \frac{\partial U^{TE}}{\partial y}$, $k = c, \alpha, \beta$, with $\frac{\partial U^{TE}}{\partial k} < 0$ and $\frac{\partial U^{TE}}{\partial y} > 0$. Thus, $\frac{d\widehat{Y}_1}{dk} > 0$. Moreover, $\widehat{Y}_2$ is defined by $U^{GO} - U^{TE} = 0$. Implicit differentiating $\widehat{Y}_2$ w.r.t. $k = c, \alpha, \beta$,

$$\frac{d\widehat{Y}_2}{dk} = \frac{\overbrace{\frac{\partial U^{TE}}{\partial k}}^{<0}}{\frac{(1+\phi\delta^2)}{\overline{\delta}+\phi\delta(1-\delta^2)} - \underbrace{\frac{\left[\overline{\phi}\left(1-\alpha\overline{\delta}\right)+\phi\delta^2\right]+\phi\beta}{1-\delta\left[\overline{\phi}\left(1-\alpha\overline{\delta}\right)+\phi\delta^2\right]}}_{\equiv T(\alpha,\beta)}} \quad k = c, \alpha, \beta.$$

$T(\alpha, \beta)$ is decreasing in α and increasing in β. Consequently, it is sufficient to evaluate the denominator at $\alpha = 0$ and $\beta = 1$ to show that it is positive. We get

$$T(0,1) = \frac{\left(1+\phi\delta^2\right)}{\overline{\delta}+\phi\delta\left(1-\delta^2\right)}.$$

Thus, the denominator is strictly positive for all $\alpha, \beta \in [0,1)$. Consequently, $\frac{d\widehat{Y}_2}{dk} < 0$.

Next, we derive the optimal choices for different values of z. Suppose for a moment that c is sufficiently low, so that some individuals choose TE (i.e., $\widehat{Y}_2 > \widehat{Y}_1$ for some $z > 0$). For $c = 0$ and $z = 0$ we find that $\widehat{Y}_1(z=0) < \widehat{Y}_2(z=0)$ if

$$0 < \overline{\phi}\left(1-\alpha\overline{\delta}\right) - \beta\overline{\phi} + \phi\delta^2\left(1-\beta\right).$$

Note that the RHS of this condition is decreasing in α and β. Evaluating this condition at the highest possible values of α and β, $\alpha = \beta = 1/2$, this condition becomes

$$0 < \overline{\phi}\left(1-\frac{1}{2}\overline{\delta}\right) - \frac{1}{2}\overline{\phi} + \frac{1}{2}\phi\delta^2 \quad \Leftrightarrow \quad 0 < \overline{\phi}\left(1-\overline{\delta}\right) + \phi\delta^2,$$

which is clearly satisfied. Thus, $\widehat{Y}_1(z=0) < \widehat{Y}_2(z=0)$ for all $\alpha, \beta \in [0, 1/2)$. Next, we show that $Y_1(z=0) > \widehat{Y}_1(z=0)$ for $c = 0$, which is equivalent to

$$\overline{\phi}\left(1-\alpha\overline{\delta}\right) + \phi\left(\beta+\delta^2\right) > \beta\left(1+\phi\delta^2\right) \quad \Leftrightarrow \quad 1 > \beta + \alpha\overline{\delta}.$$

Note that $\alpha, \beta < \frac{1}{2}$, so that this condition as satisfied. Thus, $Y_1(z=0) > \widehat{Y}_1(z=0)$ for $c = 0$. Moreover, $\frac{d\widehat{Y}_1}{dz} > \frac{dY_1}{dz}$ if

$$\frac{\phi\delta}{\overline{\phi}\left(1-\alpha\overline{\delta}\right)+\phi\left(\beta+\delta^2\right)} > \frac{\phi\delta}{1+\phi\delta^2} \quad \Leftrightarrow \quad 1 > \overline{\phi}\left(1-\alpha\overline{\delta}\right) + \phi\beta.$$

Note that the RHS is decreasing in α and increasing in β. Thus, if this condition is satisfied for $\alpha = 0$ and $\beta = \frac{1}{2}$, then is it also satisfied for all $\alpha, \beta \in [0, \frac{1}{2})$. Evaluating this condition at $\alpha = 0$ and $\beta = \frac{1}{2}$, we get $0 > -\frac{1}{2}\phi$, which is clearly satisfied. Thus, $\frac{d\widehat{Y}_1}{dz} > \frac{dY_1}{dz}$. This implies that for sufficiently low c, there exists a threshold z^*, so that $Y_1(z^*) = \widehat{Y}_1(z^*)$. At $z = z^*$ we have $U^{GO} = U^{TE} = U^{SI} = 0$, and therefore, $Y_1(z^*) = \widehat{Y}_1(z^*) = \widehat{Y}_2(z^*)$, with $\widehat{Y}_1(z) < \widehat{Y}_2(z)$ for $z < z^*$. Consequently, individuals choose (i) SI for all $y < \widehat{Y}_1$, (ii) TE for all $\widehat{Y}_1 \leq y < \widehat{Y}_2$, and (iii) GO for all $y \geq \widehat{Y}_2$. For $z \geq z^*$ no one chooses TE.

Finally, z^* is defined by $\widehat{Y}_1(z^*) - \widehat{Y}_2(z^*) = 0$, or equivalently $\widehat{Y}_1(z^*) - Y_1(z^*) = 0$ (as $U^{GO}(z^*) = U^{TE}(z^*) = U^{SI} = 0$); see Proof of Proposition 1 for Y_1). Implicit differentiating z^* w.r.t. c we get

$$\frac{dz^*}{dc} = -\frac{\frac{d\widehat{Y}_1}{dc} - \frac{dY_1}{dc}}{\frac{d\widehat{Y}_1}{dz} - \frac{dY_1}{dz}},$$

where $\frac{d\widehat{Y}_1}{dc} > 0$ and $\frac{dY_1}{dc} = 0$. Moreover, we have $\frac{d\widehat{Y}_1}{dz} > \frac{dY_1}{dz}$, because

$$\frac{\phi\delta}{\overline{\phi}\left(1 - \alpha\overline{\delta}\right) + \phi\left(\beta + \delta^2\right)} > \frac{\phi\delta}{1 + \phi\delta^2} \quad \Leftrightarrow \quad \phi\left(1 - \beta\right) > -\overline{\phi}\alpha\overline{\delta},$$

which is clearly satisfied. Consequently, $\frac{dz^*}{dc} < 0$. Moreover, note that $\lim_{c\to\infty} \widehat{Y}_1 = \infty$ and $\lim_{c\to\infty} \widehat{Y}_2 = -\infty$. Thus, there exists a threshold $\widehat{c}$ so that $z^*(\widehat{c}) = 0$. For $c \geq \widehat{c}$ we have $\widehat{Y}_1(z) > \widehat{Y}_2(z)$ for all $z > 0$; in this case no individual chooses TE for all $z \geq 0$. □

Proof of Proposition 4.

Consider the individuals that choose TE (i.e., $\widehat{Y}_1 \leq y < \widehat{Y}_2$ for $z < z^*$). Let $\eta_t^{GO|0}$ denote the fraction of these individuals at time t that is not infected and goes out (correct negative test), $\eta_t^{GO|1}$ the fraction that is infected and goes out (incorrect negative test), $\eta_t^{SI|0}$ the fraction that is not infected and self-isolates (incorrect positive test), $\eta_t^{SI|1}$ the fraction that is infected and self-isolates (correct positive test), and η_t^{ILL} the fraction that is ill and stays home. By definition, $\eta_t^{GO|0} + \eta_t^{GO|1} + \eta_t^{SI|0} + \eta_t^{SI|1} + \eta_t^{ILL} = 1$. Moreover, everyone who is ill in t was infected (but asymptomatic) in $t-1$: $\eta_t^{ILL} = \eta_{t-1}^{GO|1} + \eta_{t-1}^{SI|1}$. Everyone going out and being infected in t, went out in $t-1$, got infected, and got a false-negative test: $\eta_t^{GO|1} = \phi\beta\eta_{t-1}^{GO|0}$. Everyone self-isolating and being infected in t, went out in $t-1$, got infected, and got a correct positive test: $\eta_t^{SI|1} = \phi\left(1-\beta\right)\eta_{t-1}^{GO|0}$. Everyone self-isolating in t and being uninfected, went out in $t-1$, did not get infected, and got an incorrect positive test: $\eta_t^{SI|0} = \overline{\phi}\alpha\eta_{t-1}^{GO|0}$. In the steady-state we have $\eta^k \equiv \eta_t^k = \eta_{t-1}^k = \eta_{t-2}^k = ...$, $k = \{GO|0, GO|1, SI|0, SI|1, ILL\}$. Thus, the equilibrium is defined by the following equations: (i) $\eta^{GO|0} + \eta^{GO|1} + \eta^{SI|0} + \eta^{SI|1} + \eta^{ILL} = 1$, (ii) $\eta^{ILL} = \eta^{GO|1} + \eta^{SI|1}$, (iii) $\eta^{GO|1} = \phi\beta\eta^{GO|0}$, (iv) $\eta^{SI|1} = \phi\left(1-\beta\right)\eta^{GO|0}$, (v) $\eta^{SI|0} = \overline{\phi}\alpha\eta^{GO|0}$. Using (ii), (iii), (iv), and (v), in (i), we get $\eta^{GO|0} = \frac{1}{1+2\phi+\overline{\phi}\alpha}$. Consequently, $\eta^{GO|1} = \frac{\phi\beta}{1+2\phi+\overline{\phi}\alpha}$.

The CO-curve is then given by

$$\mu_{CO}(\phi) = \frac{\frac{\phi}{1+2\phi}\left[1-\Omega\left(\widehat{Y}_2\right)\right]+\frac{\phi\beta}{1+2\phi+\overline{\phi}\alpha}\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1\right)\right]}{\frac{1+\phi}{1+2\phi}\left[1-\Omega\left(\widehat{Y}_2\right)\right]+\frac{1+\phi\beta}{1+2\phi+\overline{\phi}\alpha}\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1\right)\right]}$$

$$= \phi\frac{1+2\phi+\overline{\phi}\alpha+\beta(1+2\phi)\Theta}{(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta}, \quad \Theta = \frac{\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1\right)}{1-\Omega\left(\widehat{Y}_2\right)}. \text{(A.1)}$$

We get

$$\frac{d\mu_{CO}(\phi)}{dk} = \frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_2}\frac{\partial\widehat{Y}_2}{\partial k}+\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_1}\frac{\partial\widehat{Y}_1}{\partial k}+\frac{\partial\mu_{CO}(\phi)}{\partial k}, \quad k=c,\alpha,\beta.$$

Note that

$$\frac{\partial\mu_{CO}(\phi)}{\partial\Theta} = \phi\frac{\beta(1+2\phi)\left[(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta\right]}{\left[(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta\right]^2} - \phi\frac{\left[1+2\phi+\overline{\phi}\alpha+\beta(1+2\phi)\Theta\right]\left[(1+\phi\beta)(1+2\phi)\right]}{\left[(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta\right]^2}.$$

This is negative if

$$\beta(1+2\phi)\left[(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta\right] < \left[1+2\phi+\overline{\phi}\alpha+\beta(1+2\phi)\Theta\right]\left[(1+\phi\beta)(1+2\phi)\right]$$
$$\Leftrightarrow \beta(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right) < (1+\phi\beta)\left(1+2\phi+\overline{\phi}\alpha\right)$$
$$\Leftrightarrow \beta < 1.$$

This is clearly satisfied. Thus, $\frac{\partial\mu_{CO}(\phi)}{\partial\Theta} < 0$. We can also see that $\frac{\partial\Theta}{\partial\widehat{Y}_2} > 0$ and $\frac{\partial\Theta}{\partial\widehat{Y}_1} < 0$. Thus, $\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_2} < 0$ and $\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_1} > 0$. We also know from Proposition 3 that $\frac{\partial\widehat{Y}_2}{\partial k} < 0$ and $\frac{\partial\widehat{Y}_1}{\partial k} > 0$, $k=c,\alpha,\beta$. And we can see that $\frac{\partial\mu_{CO}(\phi)}{\partial c} = 0$. Moreover,

$$\frac{\partial\mu_{CO}(\phi)}{\partial\alpha} = \phi\frac{\overline{\phi}\left[(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta\right]-\left[1+2\phi+\overline{\phi}\alpha+\beta(1+2\phi)\Theta\right](1+\phi)\overline{\phi}}{\left[(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta\right]^2}.$$

This is positive if

$$(1+\phi\beta)(1+2\phi)\Theta > \beta(1+2\phi)\Theta(1+\phi) \quad \Leftrightarrow \quad 1>\beta,$$

which is clearly satisfied. Thus, $\frac{\partial \mu_{CO}(\phi)}{\partial \alpha} > 0$. Moreover,

$$\begin{aligned}\frac{\partial \mu_{CO}(\phi)}{\partial \beta} &= \phi \frac{(1+2\phi)\Theta\left[(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta\right]}{\left[(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta\right]^2} \\ &\quad -\phi\frac{\left[1+2\phi+\overline{\phi}\alpha+\beta(1+2\phi)\Theta\right]\left[\phi(1+2\phi)\Theta\right]}{\left[(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta\right]^2}.\end{aligned}$$

This is positive if

$$\begin{aligned}(1+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(1+\phi\beta)(1+2\phi)\Theta &> \left[1+2\phi+\overline{\phi}\alpha+\beta(1+2\phi)\Theta\right]\phi \\ &\Leftrightarrow \quad \left(1+2\phi+\overline{\phi}\alpha\right)+(1+2\phi)\Theta>0,\end{aligned}$$

which is always satisfied. Thus, $\frac{\partial \mu_{CO}(\phi)}{\partial \beta} > 0$. All this implies that $\frac{d\mu_{CO}(\phi)}{dk} > 0$, $k = c, \alpha, \beta$.□

Proof of Proposition 5.

Let $\omega(y)$ denote the pdf of y. Note that $U^{SI} = 0$. Taking the total derivative of W w.r.t. k, $k = c, \alpha, \beta$, we get

$$\frac{dW}{dk} = \frac{\partial W}{\partial \phi}\frac{d\phi^*}{dk} + \frac{\partial W}{\partial k}.$$

Recall from Proposition 4 that $\frac{d\phi^*}{dk} > 0$. Moreover,

$$\begin{aligned}\frac{\partial W}{\partial \phi} &= \left[U^{TE}\left(\widehat{Y}_2\right)-U^{GO}\left(\widehat{Y}_2\right)\right]\omega\left(\widehat{Y}_2\right)\frac{d\widehat{Y}_2}{d\phi}-U^{TE}\left(\widehat{Y}_1\right)\omega\left(\widehat{Y}_1\right)\frac{d\widehat{Y}_1}{d\phi}+\int_{\widehat{Y}_1}^{\widehat{Y}_2}\frac{\partial U^{TE}}{\partial\phi}\Omega(y) \\ &\quad +\int_{\widehat{Y}_2}^{\infty}\frac{\partial U^{GO}}{\partial \phi}\Omega(y).\end{aligned}$$

At $y = \widehat{Y}_1$ we have $U^{TE}\left(\widehat{Y}_1\right) = U^{SI} = 0$, and at $y = \widehat{Y}_2$ we have $U^{TE}\left(\widehat{Y}_2\right) = U^{GO}\left(\widehat{Y}_2\right)$. Moreover,

$$\frac{\partial U^{GO}}{\partial \phi} = \frac{\left[\delta^2 y-\delta(x+z)\right]\left[\overline{\delta}+\phi\delta\left(1-\delta^2\right)\right]-\left[\left(1+\phi\delta^2\right)y-\phi\delta(x+z)\right]\delta\left(1-\delta^2\right)}{\left[\overline{\delta}+\phi\delta\left(1-\delta^2\right)\right]^2},$$

which is negative if

$$\left[\delta^2 y-\delta(x+z)\right]\left[\overline{\delta}+\phi\delta\left(1-\delta^2\right)\right] < \left[\left(1+\phi\delta^2\right)y-\phi\delta(x+z)\right]\delta\left(1-\delta^2\right).$$

This condition can be simplified to $0 < (1-\delta)\,y + (x+z)\,\overline{\delta}$, which is always satisfied. Thus, $\frac{\partial U^{GO}}{\partial \phi} < 0$. Recall that U^{TE} is defined by (4). Let

$$\begin{aligned}F &\equiv -c+\overline{\phi}\overline{\alpha}\left(y+\delta U^{TE}\right)+\overline{\phi}\alpha\delta\left(y+\delta U^{TE}\right)+\phi\overline{\beta}\left(\delta\left(-z+\delta\left(y+\delta U^{TE}\right)\right)\right) \\ &\quad +\phi\beta\left(y+\delta\left(-x-z+\delta\left(y+\delta U^{TE}\right)\right)\right)-U^{TE}.\end{aligned}$$

We get

$$\frac{\partial F}{\partial U^{TE}} = \overline{\phi}\overline{\alpha}\delta + \overline{\phi}\alpha\delta\delta + \phi\overline{\beta}\delta\delta\delta + \phi\beta\delta\delta\delta - 1 = -\left[1 - \delta + \phi\delta\left(1 - \delta^2\right) + \overline{\phi}\alpha\delta\left(1 - \delta\right)\right] < 0.$$

Moreover, defining $A \equiv y + \delta U^{TE}$, we get

$$\frac{\partial F}{\partial \phi} = -\overline{\alpha}A - \alpha\delta A + \overline{\beta}\left(\delta\left(-z + \delta A\right)\right) + \beta\left(y + \delta\left(-x - z + \delta A\right)\right).$$

We can show that this is negative if

$$\overline{\alpha}A + \alpha\delta A + \delta\left(z - \delta A\right) + \beta\left(\delta x - y\right) > 0.$$

Note that the LHS is increasing z. Thus, if this condition is satisfied for $z = 0$, it is also satisfied for all $z > 0$. Evaluating this condition at $z = 0$, we get

$$\overline{\alpha}A + \alpha\delta A - \delta^2 A + \beta\left(\delta x - y\right) > 0.$$

We know that $\delta x > y$ for individuals that choose TE. Thus, this condition is satisfied if

$$\overline{\alpha}A + \alpha\delta A - \delta^2 A > 0 \quad \Leftrightarrow \quad 1 - \alpha\left(1 - \delta\right) - \delta^2 > 0.$$

The LHS is decreasing in α. Thus, we can evaluate this condition at the highest possible α, $\alpha = 1/2$, and get

$$\underbrace{1 + \delta - 2\delta^2}_{\equiv \Psi(\delta)} > 0.$$

Note that $\Psi(\delta)$ is increasing in δ for $\delta < 1/4$, and decreasing in δ for $\delta > 1/4$. Moreover, we have $\Psi(0) = 1 > 0$ and $\Psi(1) = 0$. Thus, this condition is satisfied for all $\delta \in [0, 1)$, so that $\frac{\partial F}{\partial \phi} < 0$. Consequently,

$$\frac{\partial U^{TE}}{\partial \phi} = -\frac{\overbrace{\frac{\partial F}{\partial \phi}}^{<0}}{\underbrace{\frac{\partial F}{\partial U^{TE}}}_{<0}} < 0.$$

This implies that $\frac{\partial W}{\partial \phi} < 0$.

Likewise, noting that $\frac{\partial U^{GO}}{\partial k} = 0$, we get

$$\frac{\partial W}{\partial k} = \left[U^{TE}\left(\widehat{Y}_2\right) - U^{GO}\left(\widehat{Y}_2\right)\right]\omega\left(\widehat{Y}_2\right)\frac{d\widehat{Y}_2}{dk} - U^{TE}\left(\widehat{Y}_1\right)\omega\left(\widehat{Y}_1\right)\frac{d\widehat{Y}_1}{dk} + \int_{\widehat{Y}_1}^{\widehat{Y}_2}\frac{\partial U^{TE}}{\partial k}d\Omega(y).$$

Again, at $y = \widehat{Y}_1$ we have $U^{TE}\left(\widehat{Y}_1\right) = U^{SI} = 0$, and at $y = \widehat{Y}_2$ we have $U^{TE}\left(\widehat{Y}_2\right) = U^{GO}\left(\widehat{Y}_2\right)$. Moreover, recall from Proposition 3 that $\frac{d\widehat{Y}_1}{dk} > 0$ and $\frac{d\widehat{Y}_2}{dk} < 0$. We can also immediately see that $\frac{\partial U^{TE}}{\partial k} < 0$. Thus, $\frac{\partial W}{\partial k} < 0$. All this implies that $\frac{dW}{dk} < 0$, $k = c, \alpha, \beta$.$\square$

Proof of Proposition 6.

First, recall from Proposition 3 that $\frac{\partial \widehat{Y}_1(\alpha,\beta)}{\partial k} > 0$ and $\frac{\partial \widehat{Y}_2(\alpha,\beta)}{\partial k} < 0$, $k = \alpha, \beta$. Thus, $\widehat{Y}_1(0,0) < \widehat{Y}_1(\alpha,\beta)$, and $\widehat{Y}_2(0,0) > \widehat{Y}_2(\alpha,\beta)$, for $\alpha, \beta > 0$.

The number of individuals that go out (GO) and are infected, is given by

$$N_1^{GO} = \varpi^{GO|1}\left[(1-\xi)\left[1-\Omega\left(\widehat{Y}_2(\alpha,\beta)\right)\right] + \xi\left[1-\Omega\left(\widehat{Y}_2(0,0)\right)\right]\right].$$

The number of individuals that test (TE) and are infected (false negative test), is

$$N_1^{TE} = \eta^{GO|1}\left[(1-\xi)\left[\Omega\left(\widehat{Y}_2(\alpha,\beta)\right) - \Omega\left(\widehat{Y}_1(\alpha,\beta)\right)\right] + \xi\left[\Omega\left(\widehat{Y}_2(0,0)\right) - \Omega\left(\widehat{Y}_1(0,0)\right)\right]\right].$$

The total number of people going out (GO and TE with negative tests), is given by

$$\begin{aligned}\overline{N} &= \left(\varpi^{GO|0} + \varpi^{GO|1}\right)\left[(1-\xi)\left[1-\Omega\left(\widehat{Y}_2(\alpha,\beta)\right)\right] + \xi\left[1-\Omega\left(\widehat{Y}_2(0,0)\right)\right]\right] \\ &\quad + \left(\eta^{GO|0} + \eta^{GO|1}\right)\left[(1-\xi)\left[\Omega\left(\widehat{Y}_2(\alpha,\beta)\right) - \Omega\left(\widehat{Y}_1(\alpha,\beta)\right)\right] + \xi\left[\Omega\left(\widehat{Y}_2(0,0)\right) - \Omega\left(\widehat{Y}_1(0,0)\right)\right]\right].\end{aligned}$$

The CO-curve is therefore given by

$$\mu_{CO}(\phi) = \frac{N_1^{GO} + N_1^{TE}}{\overline{N}}.$$

We get

$$\begin{aligned}\frac{d\mu_{CO}(\phi)}{d\xi} &= \frac{\left[-\varpi^{GO|1}\Delta\Omega\left(\widehat{Y}_2\right) + \eta^{GO|1}\left[\Delta\Omega\left(\widehat{Y}_1\right) + \Delta\Omega\left(\widehat{Y}_2\right)\right]\right]\overline{N}}{\left[\overline{N}\right]^2} \\ &\quad - \frac{\left[N_1^{GO} + N_1^{TE}\right]\left[-\left(\varpi^{GO|0} + \varpi^{GO|1}\right)\Delta\Omega\left(\widehat{Y}_2\right) + \eta^{GO|1}\left[\Delta\Omega\left(\widehat{Y}_2\right) + \Delta\Omega\left(\widehat{Y}_1\right)\right]\right]}{\left[\overline{N}\right]^2},\end{aligned}$$

where $\Delta\Omega\left(\widehat{Y}_1\right) = \Omega\left(\widehat{Y}_1(0,0)\right) - \Omega\left(\widehat{Y}_1(\alpha,\beta)\right) < 0$ and $\Delta\Omega\left(\widehat{Y}_2\right) = \Omega\left(\widehat{Y}_2(0,0)\right) - \Omega\left(\widehat{Y}_2(\alpha,\beta)\right) > 0$. Using the expressions for N_1^{GO}, N_1^{TE}, and $\overline{N}$, we find that $\frac{d\mu_{CO}(\phi)}{d\xi} < 0$ if

$$\begin{aligned}&\left[-\varpi^{GO|1}\Delta\Omega\left(\widehat{Y}_2\right) + \eta^{GO|1}C\right]\left[\left(\varpi^{GO|0} + \varpi^{GO|1}\right)\left[A - \xi\Delta\Omega\left(\widehat{Y}_2\right)\right] + \left(\eta^{GO|0} + \eta^{GO|1}\right)[B + \xi C]\right] \\ &< \left[\varpi^{GO|1}\left[A - \xi\Delta\Omega\left(\widehat{Y}_2\right)\right] + \eta^{GO|1}[B + \xi C]\right]\left[-\left(\varpi^{GO|0} + \varpi^{GO|1}\right)\Delta\Omega\left(\widehat{Y}_2\right) + \left(\eta^{GO|0} + \eta^{GO|1}\right)C\right],\end{aligned}$$

where

$$\begin{aligned} A &= 1-\Omega\left(\widehat{Y}_2(\alpha,\beta)\right)>0 \\ B &= \Omega\left(\widehat{Y}_2(\alpha,\beta)\right)-\Omega\left(\widehat{Y}_1(\alpha,\beta)\right)>0 \\ C &= \Delta\Omega\left(\widehat{Y}_2\right)-\Delta\Omega\left(\widehat{Y}_1\right)>0. \end{aligned}$$

Simplifying this condition we get

$$\begin{aligned} \eta^{GO|1}\left[AC+B\Delta\Omega\left(\widehat{Y}_2\right)\right]\left(\varpi^{GO|0}+\varpi^{GO|1}\right) &< \varpi^{GO|1}\left[AC+B\Delta\Omega\left(\widehat{Y}_2\right)\right]\left(\eta^{GO|0}+\eta^{GO|1}\right). \\ \Leftrightarrow \quad \frac{\varpi^{GO|0}+\varpi^{GO|1}}{\varpi^{GO|1}} &< \frac{\eta^{GO|0}+\eta^{GO|1}}{\eta^{GO|1}}. \end{aligned}$$

Using the expressions for $\varpi^{GO|0}$, $\varpi^{GO|1}$, $\eta^{GO|0}$, and $\eta^{GO|1}$ (see the derivation of the CO-curve in this appendix, and Proof of Proposition 4), we can write this condition as

$$\frac{1+\phi}{\phi}<\frac{1+\phi\beta}{\phi\beta} \quad \Leftrightarrow \quad \beta(1+\phi)<(1+\phi\beta) \quad \Leftrightarrow \quad \beta<1,$$

which is clearly satisfied. Thus, $\frac{d\mu_{CO}(\phi)}{d\xi}<0.\square$

Proof of Proposition 7.

Re-deriving the expected utility U^{GO} we get

$$U^{GO}=\frac{\left(1+\phi\delta^2\right)(1-\tau)y-\phi\delta(x+z)}{\overline{\delta}+\phi\delta\left(1-\delta^2\right)}.$$

The expected utilities U^{TE} and $U^{SI}=0$ do not depend on τ, and therefore do not change. We then find that $U^{TE}>U^{GO}$ if

$$y>\widehat{Y}_2(\tau)\equiv\frac{\phi\delta\left[1-\delta\left(\overline{\phi}\left(1-\alpha\overline{\delta}\right)+\phi\delta^2\right)\right](x+z)-\left[\phi\delta(\beta x+z)+c\right]\left(\overline{\delta}+\phi\delta\left(1-\delta^2\right)\right)}{1-\overline{\phi}\left(1-\alpha\overline{\delta}\right)(1+\phi\delta)-\phi\left[\phi\delta^3+\left(1-\delta\left(1-\phi+\phi\delta^2\right)\right)\beta\right]-\tau\left(1+\phi\delta^2\right)\left[1-\delta\overline{\phi}\left(1-\alpha\overline{\delta}\right)\right]}.$$

Using (A.1) (see Proof of Proposition 4) we get

$$\frac{d\mu_{CO}(\phi)}{d\tau}=\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_2}\frac{\partial\widehat{Y}_2}{\partial\tau}.$$

We know from Proof of Proposition 4 that $\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_2}<0$. Moreover, we can immediately see that $\frac{\partial\widehat{Y}_2(\tau)}{\partial\tau}>0$. Consequently, $\frac{d\mu_{CO}(\phi)}{d\tau}<0.\square$

Proof of Proposition 8.

Using the steady-state utilities we find that $U^{CI} > U^{PI}$ when $y < \widehat{Y}_1' \equiv \phi z/(1-\phi\bar{\delta})$. Moreover, we have $U^{PI} > U^{TE}$ when

$$y < \widehat{Y}_1'' \equiv \frac{\left[1-\left(1-\phi\bar{\delta}\right)\delta^2\right]\left[\phi\delta\left(\beta x+z\right)+c\right]-\left[1-\delta\left(\overline{\phi}\left(1-\alpha\bar{\delta}\right)+\phi\delta^2\right)\right]\delta\phi z}{\overline{\phi\delta}\left(1-\alpha\right)+\left[1-\left(1-\phi\bar{\delta}\right)\delta^2\right]\phi\beta}.$$

We can immediately see that $\frac{d\widehat{Y}_1'}{dk}=0$, $k=c,\alpha,\beta$. Moreover, noting that $\widehat{Y}_1''$ is defined by $U^{TE}-U^{PI}=0$, we get

$$\frac{d\widehat{Y}_1''}{dk} = -\frac{\overbrace{\frac{\partial U^{TE}}{\partial k}}^{<0}}{\underbrace{\frac{\overline{\phi}\left(1-\alpha\bar{\delta}\right)+\phi\delta^2+\phi\beta}{1-\delta\left[\overline{\phi}\left(1-\alpha\bar{\delta}\right)+\phi\delta^2\right]}}_{\equiv S(\alpha,\beta)}-\frac{\delta\left(1-\phi\bar{\delta}\right)}{1-\left(1-\phi\bar{\delta}\right)\delta^2}}, \quad k=c,\alpha,\beta.$$

Note that $S(\alpha,\beta)$ is decreasing in α and increasing in β. Thus, to show that the denominator is positive, it is sufficient to evaluate the denominator at $\alpha=1$ and $\beta=0$. We then find that $S(1,0)=\frac{\delta\left(1-\phi\bar{\delta}\right)}{1-\left(1-\phi\bar{\delta}\right)\delta^2}$. This implies that the denominator is strictly positive for all $\alpha,\beta\in[0,1/2)$. Consequently, $\frac{d\widehat{Y}_1''}{dk}>0$.

Next, we derive the optimal choices for different values of y. Note that $\widehat{Y}_1'(z=0)=\widehat{Y}_1''(z=\alpha=\beta=c=0)=0$. We also know that $\frac{d\widehat{Y}_1''}{dk}>0$, $k=\alpha,\beta,c$. Thus, $\widehat{Y}_1'(z=0)<\widehat{Y}_1''(z=0)$ for $\alpha,\beta,c>0$. Moreover, it is easy to see that $\frac{d\widehat{Y}_1'}{dz}>0$. Thus, there exists a z' such that $\widehat{Y}_1'(z')=\widehat{Y}_1''(z')$, and therefore, $U^{TE}(z')=U^{PI}(z')=U^{CI}=0$. The threshold z' is defined by $U^{PI}(z')=U^{CI}=0$, which implies that $z'>0$. Moreover, note that at $z=z'$ we have $U^{TE}(z')=U^{PI}(z')=U^{CI}=0<U^{GO}(z')$, so that $\widehat{Y}_1'(z')=\widehat{Y}_1''(z')<\widehat{Y}_2(z')$. This implies that $z'<z^*$. Consequently, for $z\leq z'$, individuals choose (i) SI for all $y<\widehat{Y}_1'$, (ii) PI for all $\widehat{Y}_1'\leq y<\widehat{Y}_1''$, (iii) TE for all $\widehat{Y}_1''\leq y<\widehat{Y}_2$, and (iv) GO for all $y\geq\widehat{Y}_2$. □

Effects of c, α, and β on the CO-Curve with Partial Isolation (PI).

We know from Proposition 8 that for $z\leq z'$ and $\widehat{Y}_1'\leq y<\widehat{Y}_1''$, individuals choose PI. Let φ_t^{GO} denote the fraction of these individuals at time t that go out, φ_t^{SI} the fraction that self-isolates, and φ_t^{ILL} the fraction that is ill and stays home. By definition, $\varphi_t^{GO}+\varphi_t^{SI}+\varphi_t^{ILL}=1$. Moreover, everyone who went out in $t-1$ is self-isolating in t: $\varphi_t^{SI}=\varphi_{t-1}^{GO}$. Individuals that were ill in $t-1$ and recovered, and individuals that went out in $t-2$ but did not get infected (and only self-isolated in $t-1$), go out in t: $\varphi_t^{GO}=\varphi_{t-1}^{ILL}+(1-\phi)\varphi_{t-2}^{GO}$. In the steady-state we have $\varphi^k\equiv\varphi_t^k=\varphi_{t-1}^k=\varphi_{t-2}^k=...$, $k=\{GO,ILL,SI\}$. Thus, the equilibrium is defined by the following equations: (i) $\varphi^{GO}+\varphi^{SI}+\varphi^{ILL}=1$, (ii) $\varphi^{SI}=\varphi^{GO}$, and (iii) $\varphi^{GO}=\varphi^{ILL}+(1-\phi)\varphi^{GO}$. Solving (i) for φ^{ILL}, and using this expression with (ii) in (iii), we get $\varphi^{GO}=\frac{1}{2+\phi}$.

The CO-curve is then given by

$$\begin{aligned}
\mu_{CO}(\phi) &= \frac{\frac{\phi}{1+2\phi}\left[1-\Omega\left(\widehat{Y}_2\right)\right]+\frac{\phi\beta}{1+2\phi+\overline{\phi}\alpha}\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]}{\frac{1+\phi}{1+2\phi}\left[1-\Omega\left(\widehat{Y}_2\right)\right]+\frac{1+\phi\beta}{1+2\phi+\overline{\phi}\alpha}\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]+\frac{1}{2+\phi}\left[\Omega\left(\widehat{Y}_1''\right)-\Omega\left(\widehat{Y}_1'\right)\right]} \\
&= \frac{\phi m\left[1-\Omega\left(\widehat{Y}_2\right)\right]+\phi\beta k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]}{(1+\phi)\,m\left[1-\Omega\left(\widehat{Y}_2\right)\right]+(1+\phi\beta)\,k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]+\underbrace{\frac{mk}{2+\phi}\left[\Omega\left(\widehat{Y}_1''\right)-\Omega\left(\widehat{Y}_1'\right)\right]}_{\equiv\Psi}},
\end{aligned}$$

where $m=\left(1+2\phi+\overline{\phi}\alpha\right)$ and $k=(1+2\phi)$. We can see that $\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_1'}>0$. Moreover,

$$\begin{aligned}
\frac{\partial\mu_{CO}(\phi)}{\partial\Omega\left(\widehat{Y}_2\right)} &= \frac{\left[-\phi m+\phi\beta k\right]\left[(1+\phi)\,m\left[1-\Omega\left(\widehat{Y}_2\right)\right]+(1+\phi\beta)\,k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]+\Psi\right]}{\left[(1+\phi)\,m\left[1-\Omega\left(\widehat{Y}_2\right)\right]+(1+\phi\beta)\,k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]+\Psi\right]^2} \\
&\quad -\frac{\left[\phi m\left[1-\Omega\left(\widehat{Y}_2\right)\right]+\phi\beta k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]\right]\left[-(1+\phi)\,m+(1+\phi\beta)\,k\right]}{\left[(1+\phi)\,m\left[1-\Omega\left(\widehat{Y}_2\right)\right]+(1+\phi\beta)\,k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]+\Psi\right]^2}.
\end{aligned}$$

This is negative if

$$\begin{aligned}
&\phi\beta k\,(1+\phi)\,m\left[1-\Omega\left(\widehat{Y}_2\right)\right]-\phi m\,(1+\phi\beta)\,k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]+\left[-\phi m+\phi\beta k\right]\Psi \\
&<(1+\phi\beta)\,k\phi m\left[1-\Omega\left(\widehat{Y}_2\right)\right]-(1+\phi)\,m\phi\beta k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right],
\end{aligned}$$

which can be simplified to

$$-\left[1-\beta+2\phi\,(1-\beta)+\overline{\phi}\alpha\right]\Psi<[1-\beta]\,mk\left[1-\Omega\left(\widehat{Y}_2\right)\right]+[1-\beta]\,mk\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right].$$

The LHS is negative, while the RHS is positive. Thus, $\frac{\partial\mu_{CO}(\phi)}{\partial\Omega\left(\widehat{Y}_2\right)}<0$. This also implies that $\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_2}<0$.

Next, we can write the CO-curve as

$$\mu_{CO}(\phi)=\frac{\overbrace{\phi\,(2+\phi)\,m\left[1-\Omega\left(\widehat{Y}_2\right)\right]+\phi\,(2+\phi)\,\beta k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]}^{\equiv T_1}}{\underbrace{(1+\phi)\,(2+\phi)\,m\left[1-\Omega\left(\widehat{Y}_2\right)\right]+(1+\phi\beta)\,(2+\phi)\,k\left[\Omega\left(\widehat{Y}_2\right)-\Omega\left(\widehat{Y}_1''\right)\right]+mk\left[\Omega\left(\widehat{Y}_1''\right)-\Omega\left(\widehat{Y}_1'\right)\right]}_{\equiv T_2}}.$$

We get

$$\frac{\partial\mu_{CO}}{\partial\Omega\left(\widehat{Y}_1''\right)}=k\frac{-\phi\,(2+\phi)\,\beta T_2+T_1\left[(1+\phi\beta)\,(2+\phi)-m\right]}{\left[T_2\right]^2}.$$

Using $m = \left(1 + 2\phi + \overline{\phi}\alpha\right)$ we find that this is positive if

$$\begin{aligned} T_1 \left[(1+\phi\beta)(2+\phi) - \left(1+2\phi+\overline{\phi}\alpha\right)\right] &> \phi(2+\phi)\beta T_2 \\ \Leftrightarrow \quad T_1\overline{\phi}(1-\alpha) &> \phi(2+\phi)\beta \underbrace{[T_2 - T_1]}_{>0}. \end{aligned}$$

Note that this condition is satisfied for a sufficiently small β. Thus, $\frac{\partial \mu_{CO}}{\partial \Omega(\widehat{Y}_1'')} > 0$, and therefore $\frac{\partial \mu_{CO}}{\partial \widehat{Y}_1''} > 0$, for a sufficiently small β.

Totally differentiating $\mu_{CO}(\phi)$ w.r.t. c,

$$\frac{d\mu_{CO}(\phi)}{dc} = \underbrace{\frac{\partial \mu_{CO}(\phi)}{\partial \widehat{Y}_2}}_{<0} \frac{d\widehat{Y}_2}{dc} + \underbrace{\frac{\partial \mu_{CO}(\phi)}{\partial \widehat{Y}_1''}}_{>0 \text{ for small } \beta} \frac{d\widehat{Y}_1''}{dc} + \underbrace{\frac{\partial \mu_{CO}(\phi)}{\partial \widehat{Y}_1'}}_{>0} \frac{d\widehat{Y}_1'}{dc} + \underbrace{\frac{\partial \mu_{CO}(\phi)}{\partial c}}_{=0}.$$

Recall from Proposition 8 that $\frac{d\widehat{Y}_1''}{dk} > 0$ and $\frac{d\widehat{Y}_1'}{dk} = 0$, $k = c, \alpha, \beta$. Moreover, we know from Proposition 3 that $\frac{d\widehat{Y}_2}{dk} < 0$. Thus, $\frac{d\mu_{CO}(\phi)}{dc} > 0$ for sufficiently small β.

Next, we get

$$\frac{d\mu_{CO}(\phi)}{d\alpha} = \underbrace{\frac{\partial \mu_{CO}(\phi)}{\partial \widehat{Y}_2}}_{<0} \underbrace{\frac{d\widehat{Y}_2}{d\alpha}}_{<0} + \underbrace{\frac{\partial \mu_{CO}(\phi)}{\partial \widehat{Y}_1''}}_{>0 \text{ for small } \beta} \underbrace{\frac{d\widehat{Y}_1''}{d\alpha}}_{>0} + \underbrace{\frac{\partial \mu_{CO}(\phi)}{\partial \widehat{Y}_1'}}_{>0} \underbrace{\frac{d\widehat{Y}_1'}{d\alpha}}_{=0} + \frac{\partial \mu_{CO}(\phi)}{\partial \alpha}.$$

Writing the CO-curve as

$$\mu_{CO}(\phi) = \frac{\overbrace{\phi\left(1+2\phi+\overline{\phi}\alpha\right)(2+\phi) + \phi\beta(1+2\phi)(2+\phi)\Theta}^{\equiv F_1}}{\underbrace{(1+\phi)(2+\phi)\left(1+2\phi+\overline{\phi}\alpha\right) + (1+\phi\beta)(2+\phi)(1+2\phi)\Theta + \left(1+2\phi+\overline{\phi}\alpha\right)(1+2\phi)\Gamma}_{\equiv F_2}},$$

where

$$\Theta = \frac{\Omega\left(\widehat{Y}_2\right) - \Omega\left(\widehat{Y}_1''\right)}{1 - \Omega\left(\widehat{Y}_2\right)}, \quad \Gamma = \frac{\Omega\left(\widehat{Y}_1''\right) - \Omega\left(\widehat{Y}_1'\right)}{1 - \Omega\left(\widehat{Y}_2\right)},$$

we get

$$\frac{\partial \mu_{CO}(\phi)}{\partial \alpha} = \frac{\phi\overline{\phi}(2+\phi)F_2 - F_1\left[(1+\phi)(2+\phi)\overline{\phi} + \overline{\phi}(1+2\phi)\Gamma\right]}{[F_2]^2}.$$

This is positive if

$$\begin{aligned} \phi\overline{\phi}(2+\phi)F_2 &> F_1\left[(1+\phi)(2+\phi)\overline{\phi} + \overline{\phi}(1+2\phi)\Gamma\right] \\ \Leftrightarrow \quad & \phi F_2 > F_1\left[1 + \phi + \frac{1+2\phi}{2+\phi}\Gamma\right]. \end{aligned}$$

Using F_1 and F_2 we can write this condition as

$$\begin{aligned}&(1+\phi)(2+\phi)\left(1+2\phi+\overline{\phi}\alpha\right)+(2+\phi)(1+2\phi)\Theta+\left(1+2\phi+\overline{\phi}\alpha\right)(1+2\phi)\Gamma\\&>\left(1+2\phi+\overline{\phi}\alpha\right)(2+\phi)\left[(1+\phi)+\frac{1+2\phi}{2+\phi}\Gamma\right]+\beta(1+2\phi)(2+\phi)\Theta+\beta(1+2\phi)\Theta(1+2\phi)\Gamma,\end{aligned}$$

which can be simplified to

$$1-\beta>\beta\frac{1+2\phi}{2+\phi}\Gamma.$$

This condition is satisfied for a sufficiently small β. Thus, $\frac{\partial\mu_{CO}(\phi)}{\partial\alpha}>0$ when β is sufficiently small.

Finally,

$$\frac{d\mu_{CO}(\phi)}{d\beta}=\underbrace{\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_2}}_{<0}\underbrace{\frac{d\widehat{Y}_2}{d\beta}}_{<0}+\underbrace{\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_1''}}_{>0\text{ for small }\beta}\underbrace{\frac{d\widehat{Y}_1''}{d\beta}}_{>0}+\underbrace{\frac{\partial\mu_{CO}(\phi)}{\partial\widehat{Y}_1'}}_{>0}\underbrace{\frac{d\widehat{Y}_1'}{d\beta}}_{=0}+\frac{\partial\mu_{CO}(\phi)}{\partial\beta}.$$

We get

$$\frac{\partial\mu_{CO}(\phi)}{\partial\beta}=\frac{\phi(1+2\phi)(2+\phi)\Theta F_2-F_1\phi(2+\phi)(1+2\phi)\Theta}{[F_2]^2}.$$

This is positive if

$$\phi(1+2\phi)(2+\phi)\Theta F_2-F_1\phi(2+\phi)(1+2\phi)\Theta\quad\Leftrightarrow\quad F_2>F_1,$$

which is satisfied, so that $\frac{\partial\mu_{CO}(\phi)}{\partial\beta}>0$. Consequently, $\frac{d\mu_{CO}(\phi)}{d\beta}>0$ for sufficiently small β.

Proof – Endogenous Encounters and Shielding.

Suppose that M is a function of ϕ, with $M'(\phi)<0$. Using (2) we can differentiate μ w.r.t. ϕ:

$$\frac{d\mu}{d\phi}=\frac{1+\overbrace{\ln(1-(1-\lambda)\mu)}^{<0}[1-(1-\lambda)\mu]^M\overbrace{M'(\phi)}^{<0}}{(1-\lambda)M[1-(1-\lambda)\mu]^{M-1}}>0.$$

Now suppose that λ is a function of ϕ, with $\lambda'(\phi)>0$. Using (2) we get

$$\frac{d\mu}{d\phi}=\frac{1+M[1-(1-\lambda)\mu]^M\mu\overbrace{\lambda'(\phi)}^{>0}}{(1-\lambda)M[1-(1-\lambda)\mu]^{M-1}}>0.$$

Overall this implies that the IN-curve becomes steeper when either M or λ depends on ϕ. However, this does not change the basic equilibrium structure.