

# Online Appendix to “Uncertainty, Wages, and the Business Cycle”

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## Abstract

This Appendix gathers supplementary material to Cacciatore and Ravenna (2020).

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## A Nash Bargaining

The worker's surplus  $W_t$  is the difference between the worker's asset value of being employed,  $H_t$ , and the value of being unemployed,  $U_{u,t}$ . The value of employment is given by the current wage plus the expected future value of being matched to the firm. With probability  $1 - \lambda$  the match will survive, while with probability  $\lambda$  the worker will become unemployed in  $t + 1$ . As a result:

$$H_t = w_t^{flex} + E_t \{ \beta_{t,t+1} [(1 - \lambda)H_{t+1} + \lambda U_{u,t+1}] \}. \quad (\text{A-1})$$

The value of unemployment is given by:

$$U_{u,t} = h_p + b + E_t \{ \beta_{t,t+1} [p_t (1 - \lambda) H_{t+1} + (1 - (1 - \lambda)p_t) U_{u,t+1}] \}, \quad (\text{A-2})$$

where  $h_p + b$  is the flow value of unemployment (the value of non-market activity plus unemployment benefits), and  $p_t \equiv M_t/U_t$  is the probability of finding a job in period  $t$ .

The real wage  $w_t$  maximizes the following objective function:

$$J_t^{1-\eta} W_t^\eta - \Gamma_t.$$

The first-order condition with respect to  $w_t$  implies:

$$(1 - \eta) \left( \frac{W_t}{J_t} \right)^\eta \frac{\partial J_t}{\partial w_t} + \eta \left( \frac{W_t}{J_t} \right)^{\eta-1} \frac{\partial W_t}{\partial w_t} - \Gamma_{w,t} = 0,$$

where  $\Gamma_{w,t} \equiv \partial \Gamma_t / \partial w_t$ . Since  $\partial J_t / \partial w_t = -\partial W_t / \partial w_t = -1$ , we obtain:

$$(\eta - 1) \left( \frac{W_t}{J_t} \right)^\eta + \eta \left( \frac{W_t}{J_t} \right)^{\eta-1} - \Gamma_{w,t} = 0. \quad (\text{A-3})$$

Equation (A-3) can be rearranged as follows:

$$\eta J_t + (\eta - 1) W_t - \Gamma_{w,t} J_t^\eta W_t^{1-\eta} = 0.$$

Let  $\Lambda_t \equiv \Gamma_{w,t} J_t^\eta W_t^{1-\eta}$ . Therefore:

$$\eta J_t + (\eta - 1) W_t - \Lambda_t = 0. \quad (\text{A-4})$$

Using the first-order condition for vacancy posting, the firm's surplus can be written as

$$J_t = e^{Z_t} - w_t + \frac{\kappa}{q_t}. \quad (\text{A-5})$$

Recall that:

$$W_t = w_t - (b + h_p) + (1 - \lambda)(1 - p_t) E_t(\beta_{t,t+1} W_{t+1}).$$

Using (A-4), we have

$$W_t = \frac{\eta J_t}{1 - \eta} - \frac{\Lambda_t}{1 - \eta},$$

which implies

$$W_t = w_t - b + (1 - \lambda)(1 - p_t) E_t \left[ \beta_{t,t+1} \left( \frac{\eta J_{t+1}}{1 - \eta} - \frac{\Lambda_{t+1}}{1 - \eta} \right) \right],$$

or

$$W_t = w_t - b + \frac{\eta}{1 - \eta} (1 - p_t) \frac{\kappa}{q_t} - (1 - \lambda)(1 - p_t) E_t \left( \beta_{t,t+1} \frac{\Lambda_{t+1}}{1 - \eta} \right). \quad (\text{A-6})$$

Substituting (A-5) and (A-6) into the sharing rule, we finally obtain:

$$w_t = w_t^{flex} + \Omega_t,$$

where

$$w_t^{flex} \equiv \eta \left( e^{Z_t} + \kappa \frac{p_t}{q_t} \right) + (1 - \eta) b$$

and

$$\Omega_t \equiv -\Lambda_t + (1 - \lambda)(1 - p_t) E_t[\beta_{t,t+1} \Lambda_{t+1}].$$

## B Parameterization

Table 1 below summarizes the parameterization of the model

## C Fourth-Order Policy Functions and the Pruned State Space

### Unpruned Policy Functions

The set of equilibrium conditions can be written as

$$0 = E_t[f(y_{t-1}, y_t, y_{t+1}, u_t; \delta)], \quad (\text{A-7})$$

where the term  $f$  denotes a vector valued function, continuously  $n$ -times differentiable in all its arguments,  $y_t$  is the vector of endogenous variables, and  $u_t$  is the vector of exogenous, independently and identically distributed random variables. The auxiliary parameter  $\delta \in [0, 1]$  scales the uncertainty in the model:  $\delta = 1$  corresponds to the stochastic model, while  $\delta = 0$  represents the deterministic version of the model.

Consider a generic model solution of the form:

$$y_t = g(y_{s,t-1}, u_t; \delta), \quad (\text{A-8})$$

where  $y_{s,t-1}$  is the vector of state variables contained in  $y_t$ . Let  $\bar{y}$  be the solution to (A-7) when  $\delta = 0$  and  $u_t = 0$  for any  $t$ . We refer to  $\bar{y}$  as the deterministic steady state of the model. We approximate the unknown policy function  $g(\cdot)$  by successively differentiating (A-7) around the deterministic steady state and solving the resulting system for the unknown policy coefficients. Below, we denote partial derivatives with subscripts.

A fourth-order Taylor expansion of (A-7) implies the following fourth-order approximation to

TABLE 1: MODEL PARAMETERS

<i>Calibrated Parameters</i>	
Risk aversion	$\gamma = 2$
Discount factor	$\beta = 0.994$
Matching function elasticity	$\varepsilon = 0.4$
Workers' bargaining power	$\eta = 0.4$
Home production value	$b = 0.94$
Matching efficiency	$\chi = 0.87$
Vacancy cost	$k = 0.056$
Exogenous separation rate	$\lambda = 0.1$
<i>Posterior medians (95 percent set in brackets)</i>	
TFP Level innovation, std. deviation	$\sigma_z = -5.06$ [-5.24; -4.87]
TFP Level, persistence	$\rho_z = 0.76$ [0.68; 0.82]
TFP Volatility innovation, std. deviation	$\sigma_\sigma = 0.42$ [0.36; 0.49]
TFP Volatility, persistence	$\rho_\sigma = 0.63$ [0.47; 0.76]

the model policy functions:

$$\begin{aligned}
y_t^{(4)} &= \bar{y}_s + g_y dy_{s,t-1} + g_u u_t \\
&+ \frac{1}{2} \left[ g_{yy} (dy_{s,t-1})^{\otimes 2} + 2g_{yu} (dy_{s,t-1} \otimes u_t) + g_{uu} (u_t)^{\otimes 2} + g_{s2} \right] \\
&+ \frac{1}{6} \left[ g_{yyy} (dy_{s,t-1})^{\otimes 3} + 3g_{yyu} (dy_{s,t-1})^{\otimes 2} \otimes u_t + 3g_{yuu} dy_{s,t-1} \otimes (u_t)^{\otimes 2} + g_{uuu} (u_t)^{\otimes 3} \right. \\
&\quad \left. + 3(g_{ssy} dy_{s,t-1} + g_{ssu} u_t) + g_{s3} \right] \\
&+ \frac{1}{24} \left[ g_{yyyy} (dy_{s,t-1})^{\otimes 4} + 4g_{yyyu} (dy_{s,t-1})^{\otimes 3} u_t + 6g_{yyuu} (dy_{s,t-1})^{\otimes 2} (u_t)^{\otimes 2} \right. \\
&\quad + 4g_{yuuu} dy_{s,t-1} (u_t)^{\otimes 3} + g_{uuuu} (u_t)^{\otimes 4} \\
&\quad 6g_{ssyy} (dy_{s,t-1})^{\otimes 2} + 12g_{ssyu} (dy_{s,t-1} \otimes u_t) + 6g_{ssuu} (u_t)^{\otimes 2} \\
&\quad \left. + 4g_{sssy} dy_{s,t-1} + 4g_{sssu} u_t + g_{s4} \right],
\end{aligned}$$

where the superscript (4) denotes the order of the approximation,  $\otimes$  denotes the Kronecker product, and  $dy_{s,t-1} \equiv y_{s,t-1} - \bar{y}_s$  (the term  $\bar{y}_s$  is the vector of states at the deterministic steady state).

It is possible to define the above recursion for  $y_t$  as the sum of first-, second-, third-, and fourth-order terms:

$$dy_t \equiv y_t - \bar{y} = dy_t^{(1)} + dy_t^{(2)} + dy_t^{(3)} + dy_t^{(4)}, \quad (\text{A-9})$$

where  $dy_t^i \equiv y_t^i - \bar{y}$  ( $i = 1, 2, 3, 4$ ) represents the order- $i$  increment relative to the steady state, i.e., the contribution of the  $i^{\text{th}}$ -order approximation to  $dy_t$ . In particular, let  $dy_t^{(1)} \equiv y_t^{(1)} - \bar{y}$  be the first-order increment, i.e., the variation in  $y_t$  that is captured by a first-order approximation to the model policy functions. This increment is given by

$$dy_t^{(1)} = g_y dy_{s,t-1}^{(1)} + g_u u_t. \quad (\text{A-10})$$

Let the second-order increment be the difference between the first and second-order approximation:

$dy_t^{(2)} \equiv y_t^{(2)} - y_t^{(1)}$ . It is possible to show that:

$$\begin{aligned}
dy_t^{(2)} &= g_y dy_{s,t-1}^{(2)} + \frac{1}{2} g_{yy} \left( dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(2)} \right)^{\otimes 2} + g_{yu} \left( dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(2)} \right) \otimes u_t + \frac{1}{2} g_{uu} (u_t)^{\otimes 2} + \frac{1}{2} g_{s2}.
\end{aligned} \quad (\text{A-11})$$

The third-order increment is the difference between the second and the third order approximation:

$dy_t^{(3)} \equiv y_t^{(3)} - y_t^{(2)}$ . It is possible to show that:

$$\begin{aligned}
dy_t^{(3)} &= g_y dy_{s,t-1}^{(3)} + \frac{1}{2} g_{yy} \left[ 2 \left( dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(2)} \right) \otimes dy_{s,t-1}^{(3)} + \left( dy_{s,t-1}^{(3)} \right)^{\otimes 2} \right] + g_{yu} \left( dy_{s,t-1}^{(3)} \otimes u_t \right) \\
&+ \frac{1}{6} g_{yyy} \left( dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(2)} + dy_{s,t-1}^{(3)} \right)^{\otimes 3} + \frac{1}{6} g_{uuu} (u_t)^{\otimes 3} + \frac{1}{6} g_{s3} \\
&+ \frac{1}{2} g_{yyu} \left( dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(2)} + dy_{s,t-1}^{(3)} \right)^{\otimes 2} \otimes u_t + \frac{1}{2} g_{yuu} \left( dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(2)} + dy_{s,t-1}^{(3)} \right) \otimes (u_t)^{\otimes 2} \\
&+ \frac{1}{2} g_{ys} \left( dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(2)} + dy_{s,t-1}^{(3)} \right) + \frac{1}{2} g_{us} u_t.
\end{aligned} \tag{A-12}$$

Finally, the fourth-order increment is

$$\begin{aligned}
dy_t^{(4)} &= \frac{1}{24} g_{s4} + g_y dy_{t-1}^{4,s} + \\
&+ \frac{1}{2} g_{yy} dy_{t-1}^{3,s} \otimes \left[ dz_t^{(3)} + dz_t^{(2)} \right] + g_{yu} dy_{t-1}^{3,s} \otimes u_t \\
&+ \frac{1}{6} g_{yyy} \left( dz_t^{(3)} \right)^{\otimes 3} + \frac{1}{6} g_{uuu} (u_t)^{\otimes 3} + \frac{1}{6} g_{s3} \\
&+ \frac{1}{2} g_{yyu} \left[ \left( dz_t^{(3)} \right)^{\otimes 2} \otimes u_t \right] + \frac{1}{2} g_{yuu} \left[ dz_t^{(3)} \otimes (u_t)^{\otimes 2} \right] \\
&+ \frac{1}{2} g_{ys} dz_t^{(3)} + \frac{1}{2} g_{us} u_t,
\end{aligned} \tag{A-13}$$

where

$$\begin{aligned}
dz_t^{(2)} &= dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(2)}, \\
dz_t^{(3)} &= dz_t^{(2)} + dy_{s,t-1}^{(3)}, \\
dz_t^{(4)} &= dz_t^{(3)} + dy_{s,t-1}^{(4)}.
\end{aligned}$$

The incremental representation of the fourth-order approximation is the starting point to construct the pruning algorithm, described below.

### Pruned Policy Functions

Let variables with a hat denote pruned solutions. The pruning algorithm is defined by the following incremental recursion:

$$d\hat{y}_t \equiv y_t - \bar{y} = dy_t^1 + d\hat{y}_t^2 + d\hat{y}_t^3 + d\hat{y}_t^4,$$

where:

$$\begin{aligned}
dy_t^{(1)} &= g_y dy_{s,t-1}^{(1)} + g_u u_t, \\
d\tilde{y}_t^{(2)} &= g_y d\tilde{y}_{s,t-1}^{(2)} + \frac{1}{2} g_{yy} \left( dy_{s,t}^{(1)} \right)^{\otimes 2} + g_{yu} \left( dy_{s,t}^{(1)} \otimes u_t \right) + \frac{1}{2} g_{uu} (u_t)^{\otimes 2} + \frac{1}{2} g_{s2}, \\
d\tilde{y}_t^3 &= g_y d\tilde{y}_{s,t-1}^{(3)} + \frac{1}{2} g_{yy} d\tilde{y}_{s,t-1}^{(2)} \otimes \left[ dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(1)} \right] + g_{yu} d\tilde{y}_{s,t-1}^{(2)} \otimes u_t \quad (\text{A-14}) \\
&\quad + \frac{1}{6} g_{yyy} \left( dy_{s,t-1}^{(1)} \right)^{\otimes 3} + \frac{1}{6} g_{uuu} (u_t)^{\otimes 3} + \frac{1}{6} g_{s3} \\
&\quad + \frac{1}{2} g_{yyu} \left[ \left( dy_{s,t-1}^{(1)} \right)^{\otimes 2} \otimes u_t \right] + \frac{1}{2} g_{yuu} \left[ dy_{s,t-1}^{(1)} \otimes (u_t)^{\otimes 2} \right] \\
&\quad + \frac{1}{2} g_{ys} dy_{s,t-1}^{(1)} + \frac{1}{2} g_{us} u_t,
\end{aligned}$$

and

$$\begin{aligned}
d\tilde{y}_t^{(4)} &= \frac{1}{24} g_{s4} + g_y dy_{s,t-1}^{(4)} + \\
&\quad + \frac{1}{2} g_{yy} d\tilde{y}_{s,t-1}^{(3)} \otimes \left[ dy_{s,t}^{(1)} + dy_{s,t}^{(1)} \right] + g_{yu} d\tilde{y}_{s,t-1}^{(3)} \otimes u_t \\
&\quad + \frac{1}{6} g_{yyy} \left[ \left( dy_{s,t-1}^{(1)} \right)^{\otimes 2} \otimes d\tilde{y}_{s,t-1}^{(2)} \right] + \frac{1}{6} g_{uuu} (u_t)^{\otimes 3} + \frac{1}{6} g_{s3} \\
&\quad + \frac{1}{2} g_{yyu} \left( dy_{s,t-1}^{(1)} \otimes u_t \otimes d\tilde{y}_{s,t-1}^{(2)} \right) + \frac{1}{2} g_{yuu} \left[ d\tilde{y}_{s,t-1}^{(2)} \otimes (u_t)^{\otimes 2} \right] \\
&\quad + \frac{1}{2} g_{ys} dy_{s,t-1}^{(1)} + \frac{1}{2} g_{us} u_t.
\end{aligned}$$

## State Dependence of Second-Moments Shocks and Pruned Approximations

### *Third-Order Pruned Policy Functions*

As extensively discussed by [Fernandez-Villaverde and Rubio-Ramirez \(2010\)](#) and [Fernandez-Villaverde, Guerron-Quintana, and Rubio-Ramirez \(2010\)](#), the policy functions need to be approximated at least at the third order in order to study the transmission of uncertainty shocks. In particular, policy functions are invariant to the volatility of the shock processes with a first-order approximation, while the second-moment shock only enters as a cross-product with the exogenous level of productivity in a second-order approximation to the policy functions. To see this, notice that the first-order derivatives of the function  $g$  with respect to  $\sigma_t$  and  $u_{\sigma t}$  evaluated at the steady state are zero. The same occurs with the perturbation parameter  $\delta$ . In the second-order component of the solution, there is a constant that corrects for risk. However, since the second partial derivatives of

$g$  with respect to  $u_{\sigma t}$  and any other variable but  $u_{zt}$  are also zero, time-varying volatility enters into the solution only through the interaction term  $u_{zt}u_{\sigma t}$ . By contrast, in the third-order increment of the solution, the level of volatility enters without interacting with any other variable. Thus, in order to understand the dynamics triggered by an uncertainty shock we need to focus at the very least on the third-order increment,  $dy_t^{(3)}$  (and its pruned counterpart,  $d\tilde{y}_t^{(3)}$ ).

We now show that the pruned third-order approximation completely eliminates state dependence in the propagation of uncertainty shocks. We focus on the pure effects of uncertainty, i.e., we abstract from the effects of higher volatility on realized first-moment shocks. Since, as discussed above, uncertainty shocks have no direct impact on macroeconomic variables up to a second-order approximation, the pure effects of uncertainty are nil in the first- and second-order increment  $d\tilde{y}_t^{(1)}$  and  $d\tilde{y}_t^{(2)}$ . Therefore, the pure effect of uncertainty shocks only depends on:

$$\begin{aligned}
d\tilde{y}_t^{(3)} &= g_y d\tilde{y}_{s,t-1}^{(3)} + \frac{1}{2} g_{yy} d\tilde{y}_{s,t-1}^{(2)} \otimes \left[ dy_{s,t-1}^{(1)} + dy_{s,t-1}^{(1)} \right] + g_{yu} d\tilde{y}_{s,t-1}^{(2)} \otimes u_t & (A-15) \\
&+ \frac{1}{6} g_{yyy} \left( dy_{s,t-1}^{(1)} \right)^{\otimes 3} + \frac{1}{6} g_{uuu} (u_t)^{\otimes 3} + \frac{1}{6} g_{s3} \\
&+ \frac{1}{2} g_{yyu} \left[ \left( dy_{s,t-1}^{(1)} \right)^{\otimes 2} \otimes u_t \right] + \frac{1}{2} g_{yuu} \left[ dy_{s,t-1}^{(1)} \otimes (u_t)^{\otimes 2} \right] \\
&+ \frac{1}{2} g_{ys} dy_{s,t-1}^{(1)} + \frac{1}{2} g_{us} u_t.
\end{aligned}$$

It is clear from the above expression that the pure effect of an uncertainty shock only shows up linearly in the third-order increment through the term  $g_y d\tilde{y}_{s,t-1}^{(3)}$ . As a result, in the pruned, third-order approximation uncertainty shocks do not have state dependent effects.

#### ***Fourth-Order Pruned Policy Functions***

With a fourth-order approximation, uncertainty shocks have state-dependent effects. Recall the pruned fourth-order increment:

$$\begin{aligned}
d\tilde{y}_t^{(4)} &= \frac{1}{24} g_{s4} + g_y dy_{s,t-1}^{(4)} + \\
&+ \frac{1}{2} g_{yy} d\tilde{y}_{s,t-1}^{(3)} \otimes \left[ dy_{s,t}^{(1)} + dy_{s,t}^{(1)} \right] + g_{yu} d\tilde{y}_{s,t-1}^{(3)} \otimes u_t \\
&+ \frac{1}{6} g_{yyy} \left[ \left( dy_{s,t-1}^{(1)} \right)^{\otimes 2} \otimes d\tilde{y}_{s,t-1}^{(2)} \right] + \frac{1}{6} g_{uuu} (u_t)^{\otimes 3} + \frac{1}{6} g_{s3} \\
&+ \frac{1}{2} g_{yyu} \left( dy_{s,t-1}^{(1)} \otimes u_t \otimes d\tilde{y}_{s,t-1}^{(2)} \right) + \frac{1}{2} g_{yuu} \left[ d\tilde{y}_{s,t-1}^{(2)} \otimes (u_t)^{\otimes 2} \right] \\
&+ \frac{1}{2} g_{ys} dy_{s,t-1}^{(1)} + \frac{1}{2} g_{us} u_t.
\end{aligned}$$

State dependence reflects the fact that third-order increments of the state variables (which are affected by uncertainty shocks) interact with aggregate shocks through the terms

$$\frac{1}{2}g_{yy}d\tilde{y}_{s,t-1}^{(3)} \otimes \left[ dy_{s,t}^{(1)} + dy_{s,t}^{(1)} \right] + g_{yu}d\tilde{y}_{s,t-1}^{(3)} \otimes u_t.$$

However, recall the unpruned fourth-order increment:

$$\begin{aligned} dy_t^{(4)} = & \frac{1}{24}g_{s4} + g_y dy_{t-1}^{4,s} \\ & + \frac{1}{2}g_{yy}dy_{t-1}^{3,s} \otimes \left[ dz_t^{(3)} + dz_t^{(2)} \right] + g_{yu}dy_{t-1}^{3,s} \otimes u_t \\ & + \frac{1}{6}g_{yyy} \left( dz_t^{(3)} \right)^{\otimes 3} + \frac{1}{6}g_{uuu} (u_t)^{\otimes 3} + \frac{1}{6}g_{s3} \\ & + \frac{1}{2}g_{yyu} \left[ \left( dz_t^{(3)} \right)^{\otimes 2} \otimes u_t \right] + \frac{1}{2}g_{yuu} \left[ dz_t^{(3)} \otimes (u_t)^{\otimes 2} \right] \\ & + \frac{1}{2}g_{ys}dz_t^{(3)} + \frac{1}{2}g_{us}u_t. \end{aligned} \tag{A-16}$$

It is clear that pruning eliminates several terms accounting for state dependence in the impact of uncertainty shocks:

$$\frac{1}{6}g_{yyy} \left( dz_t^{(3)} \right)^{\otimes 3} + \frac{1}{2}g_{yy}dy_{t-1}^{3,s} \otimes \left[ dz_t^{(3)} + dz_t^{(2)} \right] + \frac{1}{6}g_{yyy} \left( dz_t^{(3)} \right)^{\otimes 3} + \frac{1}{2}g_{ys}dz_t^{(3)}.$$

For this reason, we study the impact of uncertainty shocks over the business cycle using the unpruned fourth-order Taylor approximation.

## D Particle Filter Estimation and Second-Moments

### Particle-Filter Estimation

Figure A.1 plots the priors distribution used in the estimation of the stochastic volatility model. We draw 1,000,000 times from the posterior. These draws are obtained after an extensive search for appropriate initial conditions and an additional 100,000 burn-in draws. We select the scaling matrix of the proposal density to induce the appropriate acceptance ratio of proposals. Each evaluation of the likelihood is performed using 40,000 particles.

We implemented standard tests of convergence of the simulations, both of the Metropolis-Hastings and of the particle filter. Finally, we also checked identifiability of the stochastic-volatility process by simulating data from the process and trying to recover the true parameters from this

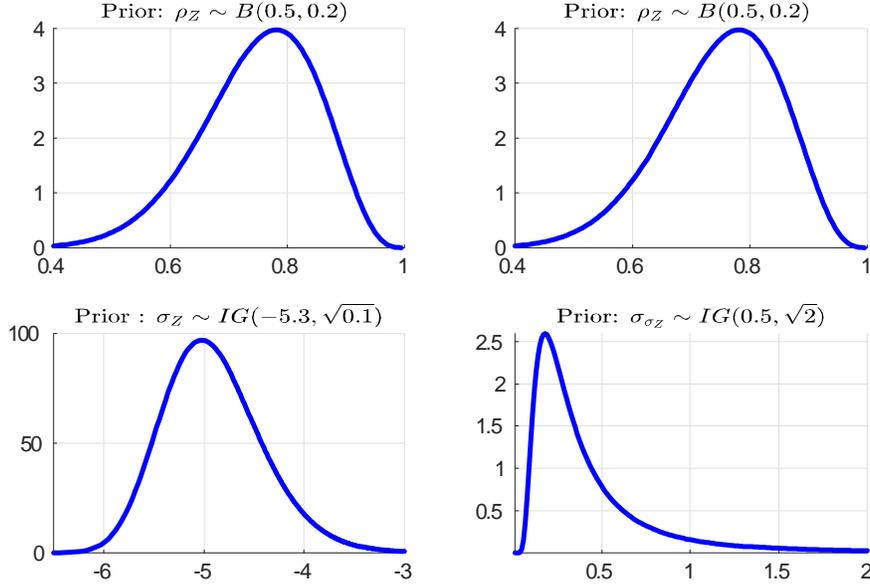


Figure A.1: prior distributions.  $B \equiv$  Beta distribution;  $IG \equiv$  Inverse Gamma distribution. Mean and standard deviation in parentheses.

artificial data.

## Second Moments

Table 2 presents second moments for the model and the data. We consider  $N = 5000$  simulations of length  $T = 250$  (with a burn-in of 2000 periods). We use pruned policy functions and HP-filter the simulated data. We report averages across the simulations.

TABLE 2: SECOND MOMENTS		
<i>Standard Deviations</i>	Data	Model
$\sigma_C/\sigma_Y$	0.80	0.56
$\sigma_I/\sigma_Y$	3.01	2.96
$\sigma_L/\sigma_Y$	0.79	0.77
$\sigma_W/\sigma_Y$	0.39	0.30
$\sigma_\omega/\sigma_Y$	0.89	1.01
<i>Cross-correlations</i>		
$corr(Y_t, C_t)$	0.88	0.77
$corr(Y_t, I_t)$	0.67	0.91
$corr(Y_t, L_t)$	0.78	0.68

## E Iterative Approximation Procedure to Parameterize the Penalty Function

### Steps of the Iteration

First, we simulate the model assuming unconstrained Nash wage bargaining to obtain the unconstrained ergodic wage distribution. We consider  $N = 5000$  simulations, each of length  $T = 250$  periods (with a burn-in of 2000 periods). We use this distribution to determine the frequency of wage outcomes eliminated by the OBC. We then perform the following iterative procedure:

1. Choose a candidate fourth-order polynomial  $\Gamma_{w,t}$ .
2. Choose  $\sigma_z$ .
3. Simulate the model with the OBC to obtain the constrained ergodic wage distribution.
4. Verify that the following criteria are met:
  - (a) The lowest wage value observed in the ergodic distribution corresponds to the 10<sup>th</sup> percentile of the simulated unconstrained ergodic wage distribution.
  - (b) The maximum value of the penalty function  $\Gamma_{w,t}$  is smaller than 1% of the steady-state wage ( $w$ ) for any  $w_t > w$ .
5. Repeat (1) to (4) until the criteria are met.

In the iterative procedure, we must rely on pruned policy functions to avoid explosive simulations. As shown in Appendix C, up to the third-order, pruning eliminates all the terms that capture the interaction between uncertainty shocks and state variables (after removing the effects of realized uncertainty). Pruning also removes most of the fourth-order terms capturing state-dependent effects of uncertainty shocks. As a result, key nonlinearities implied by the penalty function are not preserved in the pruned state space, making the output of the iterative procedure sensitive to the initial conditions. For this reason, we only consider first-moment shocks when simulating the pruned policy functions.

### Choice of the Candidate Polynomial $\Gamma_{w,t}$

Among the candidate functions for the penalty function  $\Gamma_{w,t}$  one possibility is to specify  $\Gamma_{w,t}$  as a fourth-order polynomial of the equilibrium value of the wage  $w_t$ . This specification can be written

as:

$$\Gamma_{w,t} = \sum_{i=0}^4 \alpha_i \Gamma_i[w_t(\mathbf{x}_t)], \quad (\text{A-17})$$

where  $\Gamma_i(\cdot)$  is a polynomial basis function, and  $w_t(\mathbf{x}_t)$  indicates that the approximated equilibrium law of motion for  $w_t$  is a fourth-order polynomial in the elements of the state vector  $\mathbf{x}_t = [\mathbf{x}_{t-1}^{en}, \mathbf{x}_t^{ex}]$ , where  $\mathbf{x}^{en}$  and  $\mathbf{x}^{ex}$  are, respectively, the vector of endogenous and exogenous state variables. Define  $\tilde{\Gamma}_{w,t}$  as the approximated equilibrium law of motion of the penalty function in equation (A-17). The polynomial  $\tilde{\Gamma}_{w,t}$ , is determined endogenously as part of the rational equilibrium solution to the model. As a result,  $\tilde{\Gamma}_{w,t}$  can be written as a fourth-order Taylor polynomial of the vector of states  $\mathbf{x}_t$ , i.e.,  $\tilde{\Gamma}_{w,t} = T_4(\mathbf{x}_t)$ . Notice that in general  $\tilde{\Gamma}_{w,t} \neq \Gamma_{w,t}$  unless the approximation  $\tilde{\Gamma}_{w,t}$  includes terms of order higher than the fourth.

An alternative specification for the penalty function assumes that  $\Gamma_{w,t}$  directly depends on the vector of states  $\mathbf{x}_t$ . Since in our simulations where we set  $\sigma_{z_t} = \sigma_z$ , the vector  $\mathbf{x}_t$  contains the level of productivity  $Z_t$ , employment  $L_{t-1}$ , new matches  $M_{t-1}$ , and capital  $K_{t-1}$ :

$$\Gamma_{w,t} = \sum_{i=0}^m \alpha_i \Gamma_i[\mathbf{x}_t], \quad (\text{A-18})$$

where for a polynomial of order  $n = 4$  and a vector  $x_t$  of size 4 we have  $m = (n+1)(n+2)(n+3)/6 = 20$ . Since  $\mathbf{x}_t$  only contains state variables, a fourth-order approximation of the penalty in (A-18) implies that  $\tilde{\Gamma}_{w,t}$  is identical to the candidate function  $\Gamma_{w,t}$  specified in equation (A-18)—the order of the Taylor expansion is the same as the order of the polynomial  $\Gamma_{w,t}$ . For this reason, we adopt the specification in (A-18) in the iterative procedure. This provides more robust results relative to the alternative candidate function in (A-17).<sup>1</sup>

The vector  $\alpha = [\alpha_i]_{i=0}^m$  has up to 20 non-zero elements. We make an initial guess for  $\alpha$  such that  $\Gamma_{w,t}$  is only function of productivity terms ( $Z_t$ ) and verify whether the optimized polynomial meets all the requirements detailed above. To complete step (1) in the iterative procedure, we assume ordinary polynomials as the basis functions  $\Gamma_i[x_t]$ . Since  $x_t = Z_t$ , we can select an initial candidate vector  $\alpha$  by either interpolating  $n + 1$  given couples  $(Z, Y)$ , where  $Y$  are the values taken by  $\Gamma_{w,t}$  for given value of  $Z$ , or by finding a polynomial that approximates a larger set of given couples  $(Z, Y)$ . In the simulation, we adopt the second approach.

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<sup>1</sup>This would not be the case if  $\mathbf{x}_t$  included the exogenous time-varying volatility  $\sigma_{Z_t}$ . However, it is important to notice that our specification of the penalty function accounts for the dynamics of  $\sigma_{z_t}$ , since the solution for the model includes higher-order terms in  $\sigma_{z_t}$  in the law of motion of the endogenous variable, including  $\Gamma_{w,t}$ .

We assume a uniform grid for the approximation points  $\mathbf{x} = Z$  defined by the vector

$$X = [(Z_0 - Z), \dots, (Z_k - Z), \dots, (Z_M - Z)],$$

where  $Z$  is the steady-state value of  $Z_t$  and  $M = 60$ . We determine the values in  $Y$  by using the following target function:

$$Y_k^L \equiv \frac{\phi}{\psi} \left[ 1 - e^{-\psi(Z_k - Z)} \right]. \quad (\text{A-19})$$

We then estimate the coefficients in equation (A-18) by regressing  $Y$  on  $X$  using non-linear least squares. Notice that the least squares algorithm does not require the target function to be neither continuous nor differentiable. However, we use  $Y_k^L$  because the approximation problem is better conditioned when choosing a smooth target function.<sup>2</sup>

The iterative procedure returns a vector  $\alpha = [1.8136, -0.0342 \times 10^4; 2.1795 \times 10^4; -44.3498 \times 10^4]$  for the quartic polynomial penalty function  $\Gamma_{w,t}$ . The corresponding values for  $\phi$  and  $\psi$  in the target function (A-19) are  $\phi = 520$  and  $\psi = 1$ . We discuss the accuracy of the penalty function approximation in the next section of the Appendix, showing numerically that in equilibrium the penalty function returns the desired relationship between  $\Gamma_{w,t}$  and  $w_t$  over the relevant state space of  $Z_t$ .

To conclude, notice that it is not surprising we find support for the initial guess that  $\Gamma_{w,t}$  can be expressed as a function of productivity terms  $Z_t$  only. The reason is that in the baseline search-and-matching model,  $Z_t$  has a very high correlation with the Nash-bargained wages.<sup>3</sup>

## F Accuracy

### Penalty Function Equilibrium Outcomes

Figure A.2 plots the ex-post the penalty function  $\Gamma_{w,t}$  correlates with  $w_t$ .

### OBC Binding with Higher Frequency

To assess how the frequency of the binding wage constraint affects the results, we analyze an economy where the wage constraint eliminates twice as many wage outcomes. Figure A.3 presents

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<sup>2</sup>Alternatively, we could assume Chebyshev polynomial as the basis functions  $\Gamma_i(w_t)$ , and the Chebyshev nodes as the interpolation points over an interval  $[w_{\min}, w_{\max}]$ . The candidate polynomial can be found by interpolating a target function, such as the function defined in equation (A-19). We experimented with this approach, and found that, as the least squares method, it is better conditioned when using a smooth function as a target.

<sup>3</sup>In our model without the OBC,  $\text{corr}(Z_t, w_t)$  is indeed higher than 0.99.

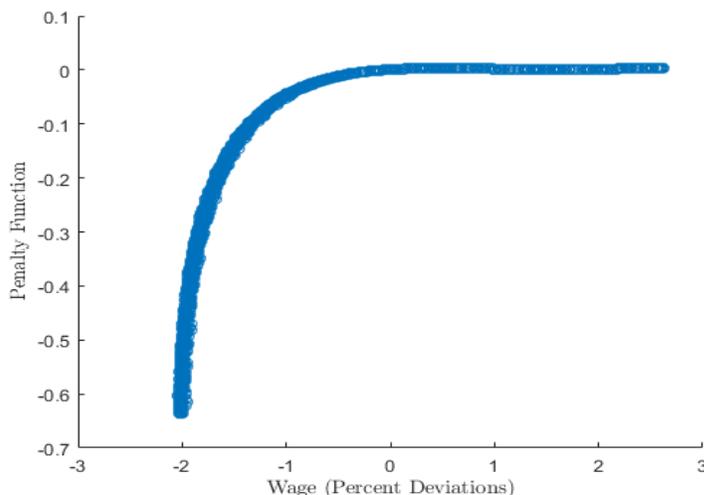


Figure A.2. Scatter plot of wage outcomes,  $w_t$ , and the level of the first-derivative of the penalty function,  $\Gamma_{w,t}$ . Variables are in percentage deviations from steady state.

the net effect of a one-standard deviation increase in uncertainty shock at the stochastic steady state. Relative to the baseline parameterization, the net effect of the uncertainty shock is

### Comparison to Taylor Expansion of Penalty Function

We compare our approximation method to a standard approach where the penalty function is locally approximated with a Taylor expansion of a differentiable penalty function (together with all the model equilibrium conditions). We consider one of the most widely used penalty functions, the Linex function, whose first-derivative is:

$$\Gamma_{w,t}^L \equiv \frac{\phi}{\psi} \left[ 1 - e^{-\psi(Z_t - Z)} \right].$$

The Linex has the advantage of being an entire function: As the order of the approximation increases, the Taylor approximation of  $\Gamma_{w,t}^L$ , denoted by  $\tilde{\Gamma}_{w,t}^L$ , converges to  $\Gamma_{w,t}^L$  over the whole domain. However, we show that up to the fourth order,  $\tilde{\Gamma}_{w,t}^L$  diverges substantially from  $\Gamma_{w,t}^L$  in regions of the state space that are of economic interest.

Figure A.4 compares the polynomial penalty function  $\Gamma_{w,t}$  in (A-18) derived using our global approximation to  $Y_k^L$  defined in equation (A-19), to the fourth-order Taylor expansion of  $\tilde{\Gamma}_{w,t}^L$ , using the same functional form and the same parameters values  $\phi = 520$  and  $\psi = 1$ . We inspect the behavior of these two alternative penalty functions as a function of the argument  $Z_t$ . In Figure A.4, the dotted line plots the fourth-order Taylor expansion of  $\Gamma_{w,t}^L$ , and the dashed line plots the

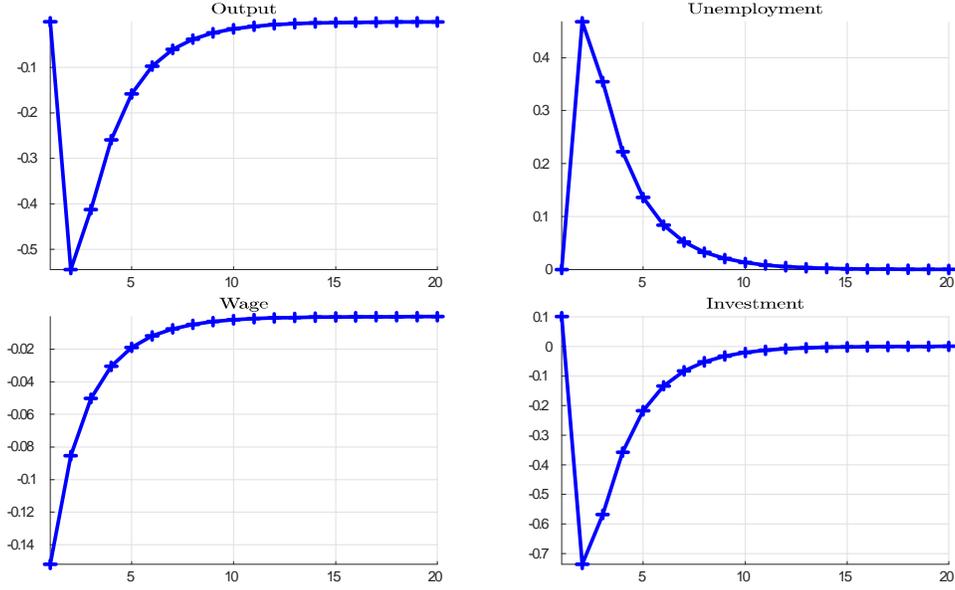


Figure A.3. One-standard deviation increase in the standard deviation of TFP. Alternative parameterization of the derivative of the penalty function: OBC binds 10% of the time on average.

polynomial penalty function  $\Gamma_{w,t}$  in (A-18) implied by the least square estimation. To facilitate the comparison, we scale the plots to center them at the steady-state value of productivity, normalized to 1. The y-axis reports the value of the first derivative of the penalty function as a fraction of the normalized steady-state productivity. The Taylor expansion of  $\Gamma_{w,t}^L$  implies that the penalty increases very steeply for values of productivity and wages below the steady state. However,  $\tilde{\Gamma}_{w,t}^L$  also leads to very large negative values of the penalty when productivity is just 1% *above* the steady state.

## G Three-Period Model

### Derivations

A firm-worker pair produces output  $y_t \equiv y(Z_t)$ , where  $Z_t$  is a random productivity draw. Let  $w_t$  be the wage payment accruing to the worker. Our assumptions imply that, at the time of the hiring decision ( $t = 0$ ), the expected stream of revenues and wage payments are, respectively:

$$\bar{y}_0 = E_0 \sum_{t=1}^2 y_t, \quad \bar{w}_0 = E_0 \sum_{t=1}^2 w_t, \quad (\text{A-20})$$

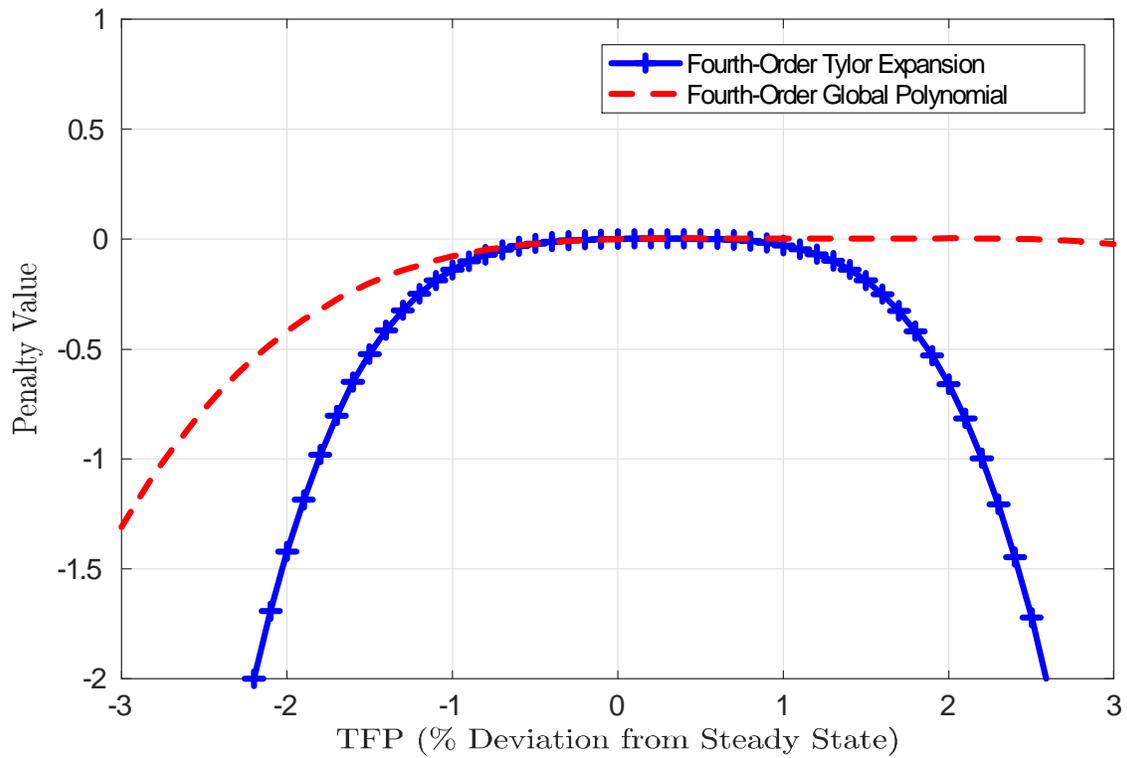


Figure A.4. Fourth-order Taylor expansion of the derivative of a Linex function with parameters  $\phi = 520$  and  $\psi = 1$  (*cross-marked line*); Polynomial approximation (*dashed line*).

where the price of output is normalized to one. Wage setting in periods 1 and 2 splits the surplus of the match between the firm and the worker according to a predetermined function of the productivity realization. We consider three alternative wage setting rules: (i) fixed wages, (ii) unconstrained Nash bargaining, and (iii) constrained Nash bargaining. Under fixed wages,  $w_1 = w_2 = w^{fixed}$ , where  $w^{fixed}$  is any feasible value in the bargaining set. When wages are not fixed, Nash bargaining sets a wage schedule conditional on the period- $t$  productivity realization. Under unconstrained Nash bargaining, the time- $t$  wage solves:

$$w_t^{flex} = \arg \max J_t^{1-\eta} W_t^\eta,$$

where  $J_t$  and  $W_t$  are, respectively, the value of the match to the firm and the to worker, conditional on time- $t$  information, and  $\eta \in (0, 1)$  is the worker's bargaining power. Under constrained Nash bargaining, the wage cannot fall below  $w_m$ , such that

$$w_t = \arg \max_{w_t \geq w_m} J_t^{1-\eta} W_t^\eta.$$

For a given wage protocol, we solve the model by backward induction. In the second period, the arguments of the Nash bargaining problem  $J_2$  and  $W_2$  only depend on  $Z_2$ , since the continuation value of the match is zero. For this reason,  $J_2 = y(Z_2) - w_2$  corresponds to the second-period profit, while  $W_2$  equals the second-period wage  $w_2$ . By contrast, in the first period, the values of the match to the firm and the worker depend on  $Z_1$  as well as on the expected flow values at time  $t = 2$ :

$$J_1 = y(Z_t) - w_1 + E_1(J_2),$$

$$W_1 = w_1 + E_1(W_2).$$

Job creation depends on the time-0 expected value of the stream of profits in periods 1 and 2, taking as given the wage renegotiation conditional on  $Z_t$ . To simplify the exposition, from now on we set  $\eta = 0.5$  and assume  $y_t = 2Z_t$ , where the c.d.f.  $F(Z_t)$  is uniform with support  $[Z^a, Z^b] \in \mathbb{R}^+$ .

We now discuss the effects of an exogenous increase in uncertainty about future productivity. First, we discuss the concavity of the profit function in period 2 under the three alternative wage setting protocols and its implications for the impact of an increase in uncertainty between periods 1 and 2. We then show that the profit-risk premium in period 2 affects the total expected value of

the match,  $\bar{y}_0 - \bar{w}_0$ .

**Proposition 1** *With unconstrained Nash bargaining or fixed wages, an increase in the variance of  $Z_2$  for given  $E(Z_2)$  does not affect period-2 expected profits. When a lower bound  $w_m$  constraints wage bargaining, period-2 expected profits fall. The fall is larger the higher the increase in variance of  $Z_2$ .*

With unconstrained Nash bargaining, the second-period wage and expected profits are respectively given by

$$w_2^{flex} = \frac{y_2(Z_2)}{2}, \quad (\text{A-21})$$

$$E_1(J_2) \equiv \int_{Z^a}^{Z^b} \frac{y_2(Z_2)}{2} dF(Z_2) = E(Z_2). \quad (\text{A-22})$$

Given our assumptions, profits are a linear function of productivity  $Z_2$ . Therefore, expected profits  $E_1(J_2)$  are unaffected by an increase in the variance of  $Z_2$ .

It is well known that in the baseline search-and-matching model, Nash bargaining implies a low elasticity of profits to productivity. [Shimer \(2005\)](#) notes this result ultimately explains the inability of the model to reproduce the observed volatility of vacancies and employment relative to output. In the literature about the ‘‘Shimer puzzle,’’ wage rigidity has been shown to amplify the impact of productivity shocks on profits and job creation. This result, however, does not extend to second-moment shocks, even when wages are fully rigid. In our partial-equilibrium model, setting the wage equal to any incentive-compatible constant value results in a profit function  $J_2$  with a higher first derivative with respect to productivity, but still linear in productivity. Thus, as in the case of unconstrained Nash bargaining, changes in uncertainty do not have any impact on  $E_1(J_2)$  with fixed wages.

Under constrained Nash bargaining,  $w_2 = \max\{w_2^{flex}, w_m\}$ . Define  $Z^m$  as the productivity threshold such that  $w_2 = w_m$  in (A-21). Expected profits  $E_1(J_2)$  can then be written as follows:

$$E_1(J_2) = \int_{Z^a}^{Z^m} [y_2(Z_2) - w_m] dF(Z_2) + \int_{Z^m}^{Z^b} \frac{y_2(Z_2)}{2} dF(Z_2). \quad (\text{A-23})$$

An increase in the variance of  $Z_2$  increases the range of values over which the two integrals are calculated and rescales the density function  $dF(Z_2)$ . As a consequence, the relative weight of the

productivity realizations across the two integrals in  $E_1(J_2)$  changes (except when  $Z^m$  is equal to the 50<sup>th</sup> percentile of  $Z_2$ ):  $Z^a$  and  $Z^b$  take different values, while  $Z^m$  is not affected by the variance increase. This in turn affects the expected profit, since the firm's surplus share is constant over  $[Z^b, Z^m]$  but decreasing in  $Z$  from  $Z^m$  to  $Z^a$ . As shown at the end of this section, an increase in the variance of  $Z_2$  always decreases the expected profit  $E_1(J_2)$ . The fall in profits is larger the higher the variance of  $Z_2$ . The results in Proposition 1 stem from the fact that the profit function and the distribution of  $Z_2$  satisfy two sufficient conditions. First, the distribution of  $Z_1$  second order stochastically dominates the distribution of  $Z_2$  conditional on a variance increase at time  $t = 2$ . Second, the wage constraint results in a concave profit function  $J_2$ .

Notice that if the profit function is not concave over the whole domain, a variance increase may lead to higher profits. We illustrate this argument with an example. Consider the limiting case where the first integral in equation (A-23) is equal to zero for  $Z < Z^m$ , and assume that workers receive all the surplus. As shown at the end of this section, these alternative assumptions imply that a marginal increase in the dispersion of productivity—measured by  $\sigma$ , the range of  $Z$ —lowers the expected profit  $E_1(J_2)$  only if  $Z^m$ ,  $w_m$ ,  $Z^a$ , and  $Z^b$  are such that:

$$\left\{ [E(Z)]^2 - Z^{m^2} \right\} > \frac{\sigma^2}{4}. \quad (\text{A-24})$$

Equation (A-24) implies that  $E_1(J_2)$  falls only if the productivity cutoff  $Z^m$  is sufficiently below the unconditional productivity average value,  $E(Z_2)$ . In general, an increase in uncertainty raises the probability that  $Z_2 < Z^m$ , implying that the set of outcomes  $[Z^a, Z^m]$  where the wage constraint is binding has a larger probability of occurring. Expected profits fall when this effect more than offsets the positive impact on expected profits induced by the larger upside potential of productivity realizations over the interval  $[Z^m, Z^b]$ , which occurs only when the condition in (A-24) is verified.

To summarize, the constraint  $w_2 = \max \left\{ w_2^{flex}, w_m \right\}$  results in a concave profit function with respect to productivity. This generates a profit-risk premium, which is increasing in the variance of  $Z$ . By contrast, since with unconstrained Nash-bargaining the profit function is linear in  $Z$ , an increase in uncertainty does not affect the profit-risk premium, which is always nil.

We now discuss under which conditions higher uncertainty about  $Z_2$  lowers the total stream of expected profits,  $(\bar{y}_0 - \bar{w}_0)$ , the relevant flow value to understand the response of job creation.

**Proposition 2** *When an increase in the variance of  $Z_2$  for given  $E(Z_2)$  lowers period-2 expected profits, the total stream of expected profits falls with constrained Nash bargaining.*

Under unconstrained Nash bargaining,  $w_t^{flex} = y(Z_t)/2 = Z_t$  for any  $t$ . Thus, total profits  $\bar{y}_0 - \bar{w}_0$  are linear in productivity, and higher uncertainty about future productivity does not change the time  $t = 0$  value of a match to the firm, leaving the firm's vacancy posting decision unaffected. Fixed wages,  $w_t = w$  for any time  $t$ , lead to the same conclusion.

Constrained Nash bargaining implies that  $w_1 = w_1^{flex} + [E_1(J_2) - E_1(W_2)]/2$ . Two observations are in order. First, since  $w_1 \neq w_1^{flex}$ , the wage payment in the first period differs from  $w_1^{flex}$  even when the wage constraint is not binding. Intuitively, the non-zero probability of hitting the constraint in the second period affects the Nash surplus in the first period. Second, it is straightforward to verify that if period-1 wages were unconstrained (i.e.,  $w_1 \geq w_m$ ), both  $J_1$  and  $W_1$  would be identical to their respective values under unconstrained Nash bargaining, even if wage adjustment is still constrained in period 2 ( $w_2 \geq w_m$ ). This result reflects the argument in [Pissarides \(2009\)](#), who shows that, as long as the wage of new hires is determined by the unconstrained Nash rule, wage determination in subsequent periods for a continuing match is irrelevant for job creation.

This result does not hold when the constraint on wage bargaining is in place in *both* periods. That is, the requirement that  $w_1 \geq w_m$  implies that at time 0 the firm anticipates the existence of states of the world at  $t = 1$  in which it will not be possible to reduce  $w_1$  sufficiently conditional on the increase in the variance of  $Z_2$ . As a result, both  $E_1(J_2)$  and  $\bar{y}_0 - \bar{w}_0$  fall when  $var(Z_2)$  rises, reducing job creation.

To conclude, notice that we derive the results of this section using a model where the response of expected profits to an increase in the variance of  $Z_t$  is independent of the state of the economy at time  $t - 1$ . By contrast, when the state of the economy affects the future path of endogenous variables, expected profits may fall by a larger amount when the  $t - 1$  level of productivity is low. In the general equilibrium model presented in the next section, a low value of  $Z_{t-1}$  increases the likelihood of a productivity draw  $Z_t < Z^m$  for which the surplus share of the firm is lower.

## Second-Order Stochastic Dominance

An alternative way to prove that uncertainty lowers expected profits in our model is to show that the distribution of the random variable  $Z_2$  with range  $\bar{\sigma}$  (denoted by  $Z_{\bar{\sigma}}$  below) second-order stochastically dominates the distribution of the random variable  $Z_2$  with range  $\sigma > \bar{\sigma}$  (denoted by  $Z_{\sigma}$  below), i.e., the expected profits implied the distribution  $Z_{\bar{\sigma}}$  are higher relative to  $Z_{\sigma}$ . Since the profit function—which is equal to  $y_2(Z_2) - w_m$  for  $Z \leq Z^m$  and to  $y_2(Z_2)/2$  for  $Z > Z^m$ —is concave and weakly increasing and  $E(Z_{\bar{\sigma}}) = E(Z_{\sigma})$ , second-order stochastic dominance

of  $Z_{\bar{\sigma}}$  relative to  $Z_{\sigma}$  can be proven by showing that  $\int_{Z^a}^t G(Z)dZ \geq \int_{Z^a}^t F(Z)dZ \forall t > Z^a$ , where  $Z_{\sigma} \sim G(Z)$  and  $Z_{\bar{\sigma}} \sim F(Z)$ . We provide only the main steps of the proof, since the conditions for  $\int_{Z^a}^t G(Z)dZ \geq \int_{Z^a}^t F(Z)dZ \forall t > Z^a$  are always satisfied when  $G(Z)$  and  $F(Z)$  intersect only once and  $F(Z) < G(Z)$  for  $Z$  smaller than the productivity value corresponding to the intersection point, as in the case we are considering.

First, notice that the distribution functions  $G(Z)$  and  $F(Z)$  are weakly increasing, and  $G(Z) > F(Z) = 0$  in the interval  $[Z^a, \bar{Z}^a]$ . Then, since  $G(E\{Z\}) = F(E\{Z\}) = 1/2$  we obtain that

$$G(Z) > F(Z) \forall Z \in [Z^a, E\{Z\}].$$

This in turn implies:

$$\int_{Z^a}^t G(Z) > \int_{Z^a}^t F(Z)dZ \forall t \in [Z^a, E\{Z\}]. \quad (\text{A-25})$$

Since it holds that:

$$G(E\{Z\} - \varepsilon) - F(E\{Z\} - \varepsilon) = F(E\{Z\} + \varepsilon) - G(E\{Z\} + \varepsilon) \forall \varepsilon \geq 0,$$

we obtain

$$\int_{E\{Z\}-Z^a}^0 G(E\{Z\} - \varepsilon) - F(E\{Z\} - \varepsilon)d\varepsilon + \int_0^{E\{Z\}-Z^a} G(E\{Z\} + \varepsilon) - F(E\{Z\} + \varepsilon)d\varepsilon = 0.$$

The previous result can be rewritten as

$$\begin{aligned} & \int_{Z^a}^{E\{Z\}} G(Z) - F(Z)dZ + \int_{E\{Z\}}^{Z^b} G(Z) - F(Z)dZ \\ &= \int_{Z^a}^{Z^b} G(Z) - F(Z)dZ = 0. \end{aligned} \quad (\text{A-26})$$

Since  $G(Z)$  and  $F(Z)$  are monotonically increasing, and also  $\int_{Z^a}^t G(Z)dZ$  and  $\int_{Z^a}^t F(Z)dZ$  are monotonically increasing, the results in equation (A-25) and (A-26) imply that  $\int_{Z^a}^t G(Z)dZ \geq \int_{Z^a}^t F(Z)dZ \forall t > Z^a$ .

The proof above highlights the conditions under which an increase in the variance may *increase* the expected second-period profits. Under the assumptions in the main text, the profit function is concave. Thus, verifying that the random variable  $Z_{\bar{\sigma}}$  second-order stochastically dominates the random variable  $Z_{\sigma}$  implies that the profit function is decreasing in the variance of  $Z$ . If the profit

function is not concave in  $Z$ , the condition  $\int_{Z^a}^t G(Z)dZ \geq \int_{Z^a}^t F(Z)dZ$  is not sufficient to ensure that  $E_1(J_2|Z_{\bar{\sigma}}) > E_1(J_2|Z_{\sigma})$ , and there may exist parameterizations such that firms prefer higher variance in  $Z$ . One example is provided by a profit function equal to  $y_2(Z_2)/2$  for  $Z > Z^m$ , and equal to zero for  $Z \leq Z^m$ , as discussed above. In this case, an increase in the variance of  $Z$  will lower profits only if condition (A-30) is verified. It is straightforward to introduce constraints in wage setting that result in non-concave profit functions, thus allowing for the possibility of a rise in profits as the variance increases. For example, bargaining may be constrained by a higher value of  $w_{m_1}$  for  $Z^{m_2} < Z < Z^{m_1}$ , and a lower value  $w_{m_2} < w_{m_1}$  for  $Z < Z^{m_2}$ . Alternatively the firm may be constrained to pay a fixed  $w_{m_1}$  for  $Z^{m_2} < Z < Z^{m_1}$ , and pay the Nash wage  $w = y(Z)/2$  for  $Z < Z^{m_2}$ , lowering the wage under the threat of bankruptcy.

## Proofs

### *Proposition 1*

*Part 1: With flexible wages, an increase in the variance of  $Z_2$  holding constant  $E(Z_2)$  does not affect period-2 expected profits.*

With fully flexible wages, our assumptions about the distribution of  $Z_2$ ,  $y(Z_2)$ , and Nash bargaining imply:

$$E_1(J_2) = \int_{Z^a}^{Z^b} Z_2 dF(Z_2) = E_1(Z_2).$$

Given that the increase in uncertainty for  $Z_2$  holds  $E_1(Z_2)$  constant,  $E_1(J_2)$  is also constant.

*Part 2: With fully rigid wages, an increase in the variance of  $Z_2$  holding constant  $E(Z_2)$  does not affect period-2 expected profits.*

When wages are completely rigid and do not respond to productivity, the wage constraint is binding over the entire support of  $Z$ . This implies that a constant wage (set at any feasible level) delivers an expected period-2 profit that depends exclusively on the expected level of productivity:  $E_1(J_2) = \int_{Z^a}^{Z^b} [y(Z_2) - w_{const}] dF(Z_2)$ . Since the increase in uncertainty for  $Z_2$  holds  $E_1(Z_2)$  and  $E_1[y(Z_2)]$  constant,  $E_1(J_2)$  is also constant. The elasticity of profits with respect to  $Z$  is constant, regardless of the wage choice. The realized and expected level of profits are instead a function of the wage choice  $w_{const}$ :

$$E_1(J_2) = 2E(Z_2) - w_{const}.$$

*Part 3: An increase in the variance of  $Z_2$  holding constant  $E(Z_2)$  lowers period-2 expected*

profits if wage bargaining is constrained by a lower bound  $w_m$  for wages.

Under constrained-Nash bargaining, the expected period-2 profit can be written as:

$$E_1(J_2) = \int_{Z^a}^{Z^m} [y_2(Z_2) - w_m] dF(Z_2) + \int_{Z^m}^{Z^b} \frac{y_2(Z_2)}{2} dF(Z_2) \quad (\text{A-27})$$

$$= \int_{Z^a}^{Z^m} [y_2(Z_2) - w_m] dF(Z_2) + \int_{Z^m}^{Z^b} Z_2 dF(Z_2) \quad (\text{A-28})$$

$$= \int_{Z^a}^{Z^m} 2Z_2 dF(Z_2) - \int_{Z^a}^{Z^m} w_m dF(Z_2) + \int_{Z^m}^{Z^b} Z_2 dF(Z_2). \quad (\text{A-29})$$

Define  $\sigma = Z^b - Z^a$  and  $\bar{\sigma} = \bar{Z}^b - \bar{Z}^a$ , where  $\bar{Z}^a$  and  $\bar{Z}^b$  are two reference points on the support of  $Z$  such that  $Z^a < \bar{Z}^a < \bar{Z}^b < Z^b$  and  $Z^a - \bar{Z}^a = \bar{Z}^b - Z^b$ . This implies that  $\Delta\sigma = \sigma - \bar{\sigma}$  defines an increase in variance which keeps the expected value of  $Z$  constant. Then we can rewrite the extremes of the support as a function of the dispersion measure:

$$\begin{aligned} Z^a &= \bar{Z}^a - \frac{1}{2}(\sigma - \bar{\sigma}), \\ Z^b &= \bar{Z}^b + \frac{1}{2}(\sigma - \bar{\sigma}). \end{aligned}$$

Note that for a uniform distribution, the range  $\sigma$  is related to the variance of the distribution by the relationship  $Var(Z) = 1/12\sigma^2$ . The introduction of the variable  $\Delta\sigma$  lets us evaluate the derivative of expected profits with respect to a simultaneous change in both  $Z^a$  and  $Z^b$  as a function of the range  $\sigma$ .

Consider the second integral in equation (A-28). Evaluating the integral we obtain:

$$\begin{aligned} E_1(J_2) &= \int_{Z^m}^{Z^b} Z \frac{1}{Z^b - Z^a} dZ \\ &= \frac{1}{\sigma} \frac{1}{2} \left[ \left( \bar{Z}^b + \frac{1}{2}\Delta\sigma \right)^2 - Z^{m^2} \right] \\ &= \frac{1}{2} \bar{Z}^{b^2} \sigma^{-1} + \frac{1}{8} \sigma + \frac{1}{8} \bar{\sigma}^2 \sigma^{-1} - \frac{1}{4} \bar{\sigma} + \frac{1}{2} \bar{Z}^b - \frac{1}{2} \bar{Z}^b \bar{\sigma} \sigma^{-1} - \frac{1}{2} Z^{m^2} \sigma^{-1}. \end{aligned}$$

where we have omitted the time-2 subscript on  $Z$  to save on notation. To assess the marginal

impact of an increase in the dispersion  $\sigma$ , evaluate the derivative of the latter expression:

$$\begin{aligned}
\frac{\partial E_1(J_2)}{\partial \sigma} &= -\frac{1}{2}\bar{Z}^b \sigma^{-2} + \frac{1}{8} - \frac{1}{8}\bar{\sigma}^2 \sigma^{-2} + \frac{1}{2}\bar{Z}^b \bar{\sigma} \sigma^{-2} + \frac{1}{2}Z^m \sigma^{-2} \\
&= -\frac{1}{2}\frac{1}{\sigma^2} \left[ \frac{1}{4}(\bar{Z}^b + \bar{Z}^a)^2 - Z^m \right] + \frac{1}{8} \\
&= -\frac{1}{2}\frac{1}{\sigma^2} \left[ E\{Z\}^2 - Z^m \right] + \frac{1}{8},
\end{aligned} \tag{A-30}$$

Equation (A-30) implies that  $\partial E_1(J_2)/\partial \sigma < 0$  if  $E\{Z\}^2 - Z^m > \sigma^2/4$ .

Equation (A-30) has a useful interpretation. Assume that the value of the first integral in equation (A-28) is always equal to zero. This implies that when the bargained wage violates the constraint  $w_t \geq w_m$  (or, equivalently,  $Z < Z^m$ ) all the surplus is assigned to the worker. Then equation (A-30) is equal to  $\partial E_1(J_2)/\partial \sigma$ . In this case, a sufficient condition for  $E_1(J_2)$  to fall when the dispersion of  $Z$  increases is that  $Z^m$  is sufficiently below the average productivity level (and, correspondingly,  $w_m$  sufficiently below the Nash-bargained wage for the average productivity draw,  $w[E(Z_2)]$ ). Assume instead that  $[E\{Z\}^2 - Z^m] < \frac{\sigma^2}{4}$  and  $\partial E_1(J_2)/\partial \sigma > 0$ , as would be the case if  $Z^m = E\{Z\}$ . Then, a larger variance in  $Z$  leaves unchanged both the probability of outcomes where  $Z < Z^m$  and profits equal zero, and the conditional expectation  $E\{Z|Z < Z^m\} = 0$ . On the contrary, the expected value of profits conditional on  $Z > Z^m$  increases, since the probability of the event  $Z > Z^m$  is unchanged (equal to 1/2), while the average productivity outcome conditional on  $Z > Z^m$  has increased. This in turn will raise the unconditional expectation of profits. This example shows that there exist wage schedules such that expected profits may increase as uncertainty rises. For the case illustrated in the main text, an increase in uncertainty always lowers profits, as we now demonstrate.

To find the conditions such that  $\partial E_1(J_2)/\partial \sigma < 0$ , evaluate the derivative of the first integral in

equation (A-28), obtaining:

$$\begin{aligned}
& \int_{Z^a}^{Z^m} [y_2(Z_2) - w_m] dF(Z_2) \\
&= \int_{Z^a}^{Z^m} 2Z \frac{1}{Z^b - Z^a} dZ - \int_{Z^a}^{Z^m} w_m \frac{1}{Z^b - Z^a} dZ \\
&= \frac{1}{\sigma} \left[ Z^{m^2} - \left( \bar{Z}^a - \frac{1}{2} \Delta\sigma \right)^2 \right] + \\
&\quad - \frac{1}{\sigma} w_m \left[ Z^m - \left( \bar{Z}^a - \frac{1}{2} \Delta\sigma \right) \right] \\
&= \sigma^{-1} Z^{m^2} - \sigma^{-1} \bar{Z}^{a^2} + \bar{Z}^a - \bar{Z}^a \bar{\sigma} \sigma^{-1} - \frac{1}{4} \sigma - \frac{1}{4} \sigma^{-1} \bar{\sigma}^2 + \frac{1}{2} \bar{\sigma} \\
&\quad - w_m \sigma^{-1} Z^m + w_m \sigma^{-1} \bar{Z}^a - w_m \frac{1}{2} + w_m \frac{1}{2} \sigma^{-1} \bar{\sigma}.
\end{aligned}$$

Consider the derivative of the latter expression:

$$\frac{\partial \left[ \int_{Z^a}^{Z^m} [y_2(Z_2) - w_m] dF(Z_2) \right]}{\partial \sigma} = -Z^{m^2} \sigma^{-2} + \bar{Z}^{a^2} \sigma^{-2} - \frac{1}{4} + \frac{1}{4} \bar{\sigma}^2 \sigma^{-2} \\
+ \bar{Z}^a \bar{\sigma} \sigma^{-2} + w_m Z^m \sigma^{-2} - w_m \bar{Z}^a \sigma^{-2} - \frac{1}{2} w_m \bar{\sigma} \sigma^{-2} \quad (\text{A-31})$$

$$= \frac{1}{\sigma^2} \left[ E \{ Z \}^2 - Z^{m^2} \right] - \frac{1}{4} + w_m \frac{1}{\sigma^2} \left[ Z_m - \bar{Z}^a - \frac{1}{2} \bar{\sigma} \right]. \quad (\text{A-32})$$

Since the definition of  $Z^m$  and the properties of the uniform distribution imply that  $w_m = Z^m$ , we can combine equation (A-30) and (A-32) to obtain:

$$\begin{aligned}
\frac{\partial E_1(J_2)}{\partial \sigma} &= \frac{1}{2} \frac{1}{\sigma^2} \left[ E \{ Z \}^2 - Z^{m^2} \right] - Z^m \frac{1}{\sigma^2} [E \{ Z \} - Z^m] - \frac{1}{8} \\
&= [E \{ Z \} - Z^m]^2 - \frac{\sigma^2}{4}.
\end{aligned}$$

Rearranging terms, we obtain that the condition for  $\partial E_1(J_2)/\partial \sigma < 0$  is:

$$[E \{ Z \} - Z^m] < \frac{\sigma}{2}. \quad (\text{A-33})$$

Using equation (A-33) we obtain the following three results:

1. The condition in (A-33) for  $\partial E_1(J_2)/\partial \sigma$  to be less than zero can be rewritten as  $Z^m > Z^a$ ,

which for the assumed distribution of  $Z$  is always verified since  $Z^a = \min(Z)$ . Therefore, an increase in uncertainty always lowers expected profits.

2. The absolute value of the derivative  $\partial E_1(J_2)/\partial\sigma$  is increasing in  $\sigma$ . Therefore, for given  $Z^m$ , the impact of an increase in uncertainty on expected profits is stronger when  $\sigma$  is larger, and the implied risk premium rises with uncertainty.
3. For values of  $E\{Z\}$  such that  $E\{Z\} > Z^m$ , the absolute value of the derivative  $\partial E_1(J_2)/\partial\sigma$  increases as  $Z^m \rightarrow E\{Z\}$ . This implies that as the value of  $Z$  for which the wage setting constraint becomes binding gets closer to the average productivity, the impact of a marginal increase in uncertainty becomes larger. ■

In the context of our simple model, result (3) above describes the implications of increasing  $Z^m$  to a value closer to a given average productivity  $E\{Z\}$ . As  $Z^m$  increases, the probability of a draw  $Z_2 < Z^m$  increases, and a given change in uncertainty has a larger impact on profits.

**Proposition 2**

*Under constrained Nash bargaining the total stream of expected profits  $\bar{y}_0 - \bar{w}_0$  falls when a mean-preserving spread of  $Z_2$  lowers period-2 expected profits.*

Pissarides (2009) shows that as long as the initial wage is not constrained, wage rigidity for continuing matches is irrelevant for job creation. This result would hold true also in our simple model if wages could be set freely in period 1. We can rewrite the first-period Nash wage as:

$$\begin{aligned} w_1 &= \frac{y_1(Z_1)}{2} + \frac{1}{2} [E_1(J_2) - E_1(W_2)] \\ &= Z + \frac{1}{2} [E_1(J_2) - E_1(W_2)], \end{aligned} \tag{A-34}$$

where  $W_2$  is the period-2 value of the match to the worker. Note that  $y(Z_1)/2$  is the wage that would be negotiated under unconstrained Nash bargaining in all period, when equal bargaining weight in period 2 implies  $E_1(\Pi_2) - E_1(V_2) = 0$ .

Assume that wages can be set freely in period 1, yet remaining constrained by  $w_m$  in period 2. Regardless of the level of the constraint, period-2 profits do not depend on the realization  $Z_1$ , and so is  $w_1$ . To see this, recall that vacancy posting depends on the expected total profit from the

match:

$$E_0(J_1) = \int_{Z^a}^{Z^b} [y_1(Z_1) - w_1] dF(Z_1) + E_0(J_2).$$

Using the wage solution, we obtain:

$$\begin{aligned} E_0(J_1) &= \int_{Z^a}^{Z^b} \left[ y_1(Z_1) - \frac{y_1(Z_1)}{2} - \frac{1}{2} [E_1(J_2) - E_1(W_2)] \right] dF(Z_1) + E_0(J_2) \\ &= E(Z_1) - \frac{1}{2} [E_1(J_2) - E_1(W_2)] + E_0(J_2), \end{aligned}$$

where the term  $E(Z_1)$  is the period-1 profit obtained if wage bargaining were unconstrained at all times. Observe then that  $E_0(J_2)$  is a function of  $w_m$ , but the total surplus is not, given the fact that  $t = 2$  is the terminal period. Then  $E_1(J_2) + E_1(W_2) = G$  where  $G$  is a constant. This then implies that

$$\begin{aligned} E_0(J_1) &= E(Z_1) - \left[ \frac{2E_0(J_2) - G}{2} \right] + E_0(J_2) \\ &= E(Z_1) + G/2 \end{aligned}$$

which is also constant. Therefore, the total profit from the match is independent from  $E_0(J_2)$  and  $w_m$ , implying the wage bound at  $t = 2$  leaves the ex-ante incentive to post vacancies unchanged.

Finally, note that this result is only obtained because unconstrained Nash bargaining allows for a *lower* wage—compared to the unconstrained scenario—in period 1, given that a lower wage limit  $w_m$  exists in period 2 and splits the surplus so that  $E_1(J_2) - E_1(W_2) < 0$ . When the wage cannot adjust freely in period 1 (as in our model),  $E_0(J_1)$  may fall whenever  $E_0(J_2)$  falls and the wage cannot be lowered all the way to its Nash value defined in equation (A-34).

## H Empirical Analysis: Additional Evidence

### Data Description

We use the following series:

- Real Gross Domestic Product (FRED: GDPC1).

- Quarterly average of monthly, seasonally adjusted civilian unemployment rate (BLS series id LNS14000000).
- Consumer price index excluding food and energy: Quarterly average of monthly, seasonally adjusted consumer price index for all urban consumers, all items less food and energy (BLS series *CUUR0000SA0L1E*).
- Quarterly average of the effective federal funds rate.

### Interacted-VAR

We provide additional evidence about the state-dependent effects of uncertainty shocks by estimating a non-linear Interacted VAR (I-VAR):

$$\mathbf{Y}_t = a + \sum_{j=1}^p B_j \mathbf{Y}_{t-j} + \sum_{j=1}^p C_j (UNC_{t-j} \times BC_{t-j}) + \mathbf{u}_t,$$

where  $\mathbf{Y}_t$  is a vector of endogenous variables, and  $\mathbf{u}_t$  is a vector of reduced-form shocks. The I-VAR includes the interaction term  $UNC_{t-j} \times BC_{t-j}$ , where  $UNC_t$  corresponds to the uncertainty measure and  $BC_t$  is variable that captures the state of the business cycle. The interaction allows uncertainty to have different effects depending on the state of the business cycle. The I-VAR provides a parsimonious specification to model the interaction between business cycle dynamics and uncertainty. Since both interaction variables evolve endogenously, I-VAR impulse responses account for the endogenous evolution of the state of the economy, including its impact on uncertainty.<sup>4</sup>

We focus on U.S. quarterly time series data for the period 1960:Q3-2015:Q1. The vector  $\mathbf{Y}_t$  includes: a measure of uncertainty; real GDP; the unemployment rate; the 4-quarter difference of the log-consumer price index excluding food and energy; the federal funds rate.<sup>5</sup> We compute log deviations of real quarterly GDP from real potential Gross Domestic Product (FRED: GDPPOT), since the interaction variable needs to be stationary.

As in the main text, we measure uncertainty using the median estimates from a backward-smoothing version of the particle filter on the stochastic volatility model. Following the previous literature, we identify uncertainty shocks using a Cholesky decomposition (Basu and Bundick, 2017 and Leduc and Liu, 2016). The uncertainty variable is ordered first, so that unexpected increases

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<sup>4</sup>See Pellegrino (2017) for an exhaustive discussion of the advantages of I-VARs relative to other non-linear VARs.

<sup>5</sup>Output, hours and hourly compensation are measured as deviations from trend.

in uncertainty have contemporaneous effects on real variables, while real variables do not have a contemporaneous impact on uncertainty.

We include three lags of each variable. We measure the variable describing the business cycle state,  $BC_{t-j}$ , using lagged values of output (the results are robust to using lags of the unemployment rate). We identify the recessionary states as the quarters corresponding to the bottom 15% of the output distribution. This criterion implies that the recessionary periods are in line with NBER recession dates. In these episodes, output is on average 3.15% below trend.

The top row in Figure A.5 presents generalized impulse responses following a one-standard deviation increase in uncertainty. Continuous lines identify the estimated response in the recessionary state, while dashed lines correspond to the non-recessionary state. In the recessionary state, the increase in uncertainty lowers output between 0.25 and 0.8 percentage points on average, while the corresponding increase in unemployment ranges from 0.15 to 0.45 percentage points. In the non-recessionary state, the output response is statistically not significant, while the unemployment response is muted but more persistent across all specifications. The change in unemployment during recessionary periods appears to be driven by a fall in the job finding rate. Finally, hours per worker fall initially both in recessionary and non-recessionary states, although the fall is not always statistically significant. The bottom row in Figure A.5 shows the difference between the response in the recessionary and non-recessionary states for output and employment is significant at the 95% confidence level.

To summarize, the I-VAR results confirm the local-projections analysis. The impact of uncertainty shocks varies over the business cycle, consistent with the results from the model simulations. To provide additional robustness, Figure A.6 re-estimate the local projections discussed in the main paper using the same output measure of the I-VAR (log deviations of real quarterly GDP from its potential level). The results remain very similar.

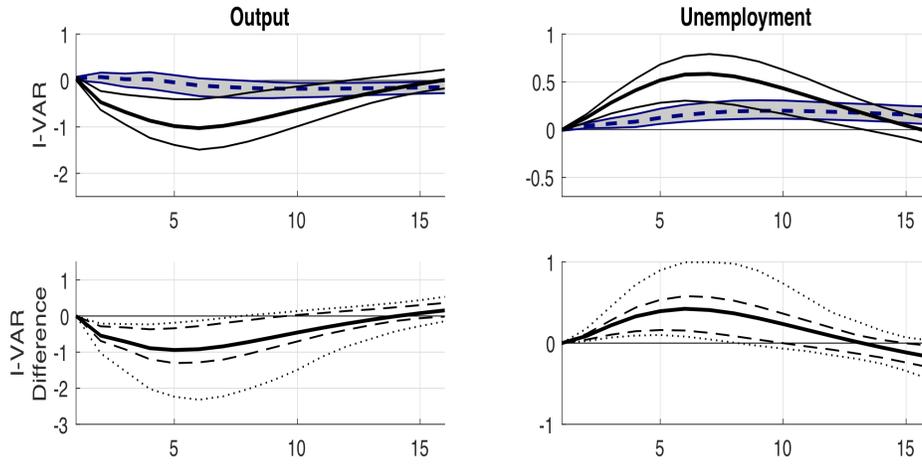


Figure A.6. I-VAR, one-standard deviation uncertainty shock. *First row*: impulse responses in the recessionary state (solid line) and non recessionary state (continuous lines); *Second row*: difference between responses in the recessionary and non-recessionary state. Confidence bands at 90% significance level.

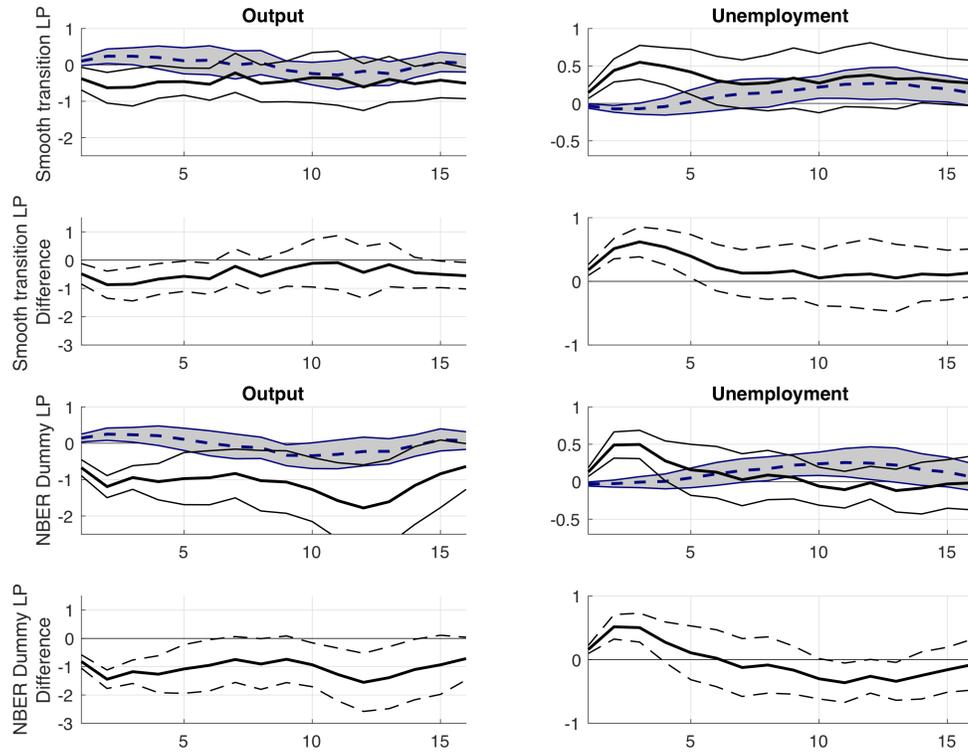


Figure A.6. Local projections (LP) following an increase in TFP uncertainty, real economic activity measured by log deviations of real GDP from its potential level. *First row*: smooth-transition LP, recessionary state (solid line) and non recessionary state (continuous lines); *Second row*: smooth transition LP, difference between responses in the recessionary and non-recessionary state; *Third row*: NBER-dates LP, recessionary state (solid line) and non recessionary state (continuous lines); *Fourth row*: NBER-dates LP, difference between responses in the recessionary and non-recessionary state. Confidence bands at 90% significance level.

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