Supplementary Online Appendix to "Repricing Avalanches"

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B The case of two-state productivity.

In this Online Appendix, we extend the baseline model to the case where productivity is not constant. Even though some portions are straightforward extension of the baseline model, we allow for some overlap with the main text for the sake of a self-contained exposition of the extended model.

B.1 Firms' problem with Markov-switching productivity a_t^i

We assume that each firm i produces output with a linear technology with labor being the single input,

$$y_t^i = a_t^i l_t^i, \tag{S.1}$$

and productivity at time t, $a_t^i \in \{a^L, a^H\}$, with $a^L \leq a^H$. By choosing the units in which we measure labor we may assume $\frac{a^L + a^H}{2} = 1$, so that the baseline model with $a^L = a^H = 1$ is a special case.

Firm i fixes a price P_t^i at time t and must satisfy demand, that is hire a labor force

$$l_t^i = \frac{(p_t^i)^{-\eta}}{a_t^i} Y_t. \tag{S.2}$$

The net income that firm i obtains from production when productivity is a^i, the firm sells output at price p^i, wages are W and final good output is Y is thus, measured in consumption-goods equivalent,

$$Z(p^i, a^i, w, Y) := p^i y^i - w l^i = ((p^i)^{1-\eta} - (w/a^i)(p^i)^{-\eta})Y \tag{S.3}$$

where $w = W/P$.

Firms are subject to a Calvo shock that is determined by a Poisson process with rate μ. When a Poisson time arrives, the firm is allowed to change prices at a zero-cost. We assume the Poisson shocks are independent across firms. At the time of a Calvo shock firms may also experience a productivity shock. The probability of a productivity change may depend on the current level of productivity, and we write $0 < \zeta^a < 1$ for the probability of *leaving* productivity a at time t, conditional on a Calvo shock at t. The proof of Proposition B.1 below, which characterizes the solution to intermediate-good firms' optimization problem, shows clearly that tying productivity shocks to Calvo shocks greatly simplifies the mathematics of the problem.

B.2 Value function in the stationary equilibrium

In a stationary equilibrium where the rate of inflation is π and $\bar{c} > 0$, if F denotes the stationary probability distribution of prices and productivities, and F_p its marginal with respect to prices, we must have (12), (13), (15), (16) and

$$\bar{N} = E^F\left[\frac{p^{-\eta}}{a}\bar{Y}\right]. \tag{S.4}$$

In a stationary equilibrium any firm that has productivity a and charges prices p has net real income from production, normalized by final-goods output $\bar{Y}$:

$$z(p,a) := \frac{Z(p,a,\bar{w},\bar{Y})}{\bar{Y}} = p^{1-\eta} - \frac{\bar{w}}{a}p^{-\eta}.$$

The function $z(\cdot,a)$ has a unique maximum at $\hat{p}_a := (\bar{w}/a)(\eta/(\eta-1))$ for each $a \in \{a^L, a^H\}$, and it is increasing for $p < \hat{p}_a$ and decreasing for $p > \hat{p}_a$. Furthermore $z(\hat{p}_a) > 0$.

Write $v(p,a) = V(p,a)/Y$, where $V(p,a)$ is the expected maximum discounted profits for a firm that charges p at time zero, and has productivity $a \in \{a^L, a^H\}$. Suppose that in a stationary equilibrium at some random time τ a Calvo shock arrives. Then if productivity stays the same the firm should choose $p_\tau = \arg\max v(p,a)$, whereas if productivity shifts to $b \neq a$, the firm should choose $p_\tau = \arg\max v(p,b)$. Suppose now that at a random time T' the firm decides to pay menu costs. Provided $\delta^+ \geq \delta^-$, if the firm chooses to increase price then it faces no restriction and would always choose $p_{T'} = \arg\max v(p,a)$, independent of the current price. On the other hand if it chooses to lower prices it would choose $p_{T'} = \arg\max_{\{p \leq p_{T'-}\}} v(p,a)$. Since $\delta^- > 0$ the optimal choice of $p_{T'}$, if it exists, is interior in this stationary equilibrium. Then if $b \neq a$,

$$\begin{aligned} v(p,a) = \sup_{p'_a, p'_b, T', p''_a \leq p_{T'-}} E_0\Bigg\{ &\int_0^{T' \wedge \tau} e^{-\rho t} z(p_t,a)dt + \mathbf{1}_{\tau < T'} e^{-\rho\tau}\left((1-\zeta^a)v(p'_a,a) + \zeta^a v(p'_b,b)\right) \\ &+ \mathbf{1}_{T' < \tau} e^{-\rho T'}(\max\{v(p'_a,a) - \delta^+, v(p''_a,a) - \delta^-\})\Bigg\}. \end{aligned} \tag{S.5}$$

A standard argument shows that we can restrict ourselves to feedback controls, that is stopping times T' such that $T' = 0$ if and only if $p_0 \in \mathcal{S}_a := \mathcal{S}_a^1 \cup \mathcal{S}_a^2$, where $\mathcal{S}_a^1 := \{p \in (0,\infty) : v(p,a) = \sup_{p'} v(p',a) - \delta^+\}$ and $\mathcal{S}_a^2 = \{p \in (0,\infty) : v(p,a) = \sup_{p' \leq p} v(p',a) - \delta^-\}$. Taking limits in the dynamic programming equation (S.5) we obtain:

$$z(p,a) - \rho v(p,a) - \pi p v'(p,a) + \mu[(1-\zeta^a)\sup_{p'_a} v(p'_a,a) + \zeta^a \sup_{p'_b} v(p'_b,b) - v(p,a)] \leq 0. \tag{S.6}$$

The left-hand side includes the profit at time zero and the expected drift of $e^{-\rho t}v(p_t,a)$ taking into consideration the fact that, if a Calvo shock occurs and productivity does not change, then the value would jump to $\sup_{p'} v(p', a)$, while if productivity changes the value would jump to $\sup_{p'} v(p', b)$, and the rate μ associated to the Poisson process of Calvo shocks. In addition, since stopping immediately is always a choice $v(p, a) \geq \max\{v(p'_a, a) - \delta^+, v(p''_a, a) - \delta^-\}$. Notice that if $p \notin \mathcal{S}_a$ then necessarily

$$z(p, a) - \rho v(p, a) - \pi p v'(p, a) + \mu[(1 - \zeta^a) \sup_{p'_a} v(p'_a, a) + \zeta^a \sup_{p'_b} v(p'_b, b) - v(p, a)] = 0, \quad \text{(S.7)}$$

so the function v satisfies the *quasi-variational inequality*: For each $a \in \{a^L, a^H\}$, if $b \neq a$,

$$\max\left\{ \begin{array}{c} z(p, a) - \rho v(p, a) - \pi p v'(p, a) + \mu[(1 - \zeta^a) \sup_{p'} v(p', a) + \zeta^a \sup_{p'} v(p', b) - v(p, a)]; \\ \max\{\sup_{p'} v(p', a) - \delta^+, \sup_{p'' \leq p} v(p'', a) - \delta^-\} - v(p, a) \end{array} \right\}$$
$$= 0. \quad \text{(S.8)}$$

We start by studying the differential equation implicit in the quasi-variational inequality.

Lemma 11. *Suppose that for each $a \in \{a^L, a^H\}$, if $b \neq a$, $R(\cdot, a)$ solves*

$$z(p, a) - \rho R(p, a) - \pi p R'(p, a) + \mu \left(\sup_{\{p'_a \geq \hat{p}_a, p'_b \geq \hat{p}_b\}} (1 - \zeta^a) R(p'_a, a) + \zeta^a R(p'_b, b) - R(p, a) \right)$$
$$= 0. \quad \text{(S.9)}$$

Then:

(a) *For each $\delta > 0$ there exists for each $a \in \{a^L, a^H\}$ at most one pair $(\underline{p}_a, p^*_a)$, with $\underline{p}_a \leq \hat{p}_a \leq p^*_a$ with $p^*_a > \underline{p}_a$ that solves,*

$$R'(\underline{p}_a, a) = R'(p^*_a, a) = 0 \quad \text{(S.10)}$$
$$R(\underline{p}_a, a) = R(p^*_a, a) - \delta^+. \quad \text{(S.11)}$$

(b) *If $\underline{p}_a$ (p^*_a) exists it is a global minimum of $R(p, a)$ in $(0, \hat{p}_a)$ (maximum in $(\hat{p}_a, \infty)$ resp.).*

(c) *Consequently, if p^*_a and p^*_b exist,*

$$R(p^*_\cdot, \cdot) = \frac{\mu \zeta^a z(p^*_b, b) + \mu \zeta^b z(p^*_a, a) + \rho z(p^*_\cdot, \cdot)}{\mu \rho (\zeta^a + \zeta^b) + \rho^2}, \quad \text{(S.12)}$$

and if $\underline{p}_a$ and p^*_a exist,

$$\delta^+ = R(p^*_a, a) - R(\underline{p}_a, a) = \frac{z(p^*_a, a) - z(\underline{p}_a, a)}{\mu + \rho}. \quad \text{(S.13)}$$

Proof. First note that if R solves (S.9) then:

$$z'(p,a) - (\pi + \mu + \rho)R'(p,a) - \pi p R''(p,a) = 0. \tag{S.14}$$

Thus whenever $R'(p,a) = 0$ and $p_a < \hat{p}_a$, $R''(p,a) > 0$. Hence there is at most one critical point in $(0, \hat{p}_a)$ and it is a global minimum in $(0, \hat{p}_a)$. Similarly for $p \in (\hat{p}_a, \infty)$ there is at most one critical point and it is a global maximum in $(\hat{p}_a, \infty)$. (c) is obvious. □

The next proposition characterizes the value and policy functions.

Proposition B.1. *Fix $w > 0$ and $\delta^+ > 0$. Then*

(a) There exist a unique function $R(p,a)$ and points $\underline{p}_a$, $\underline{p}_b$, p_a^* and p_b^* (which depend on (w, δ^+)) that satisfy equations (S.9)–(S.11).

(b) The following equation holds:

$$\frac{w}{a} = \frac{(\eta - 1)\varphi(p_a^*/\underline{p}_a, 1 + (\mu+\rho)/\pi - \eta)}{\eta\varphi(p_a^*/\underline{p}_a, (\mu+\rho)/\pi - \eta)}\underline{p}_a \tag{S.15}$$

with

$$\varphi(q,x) := (q^x - 1)/x.$$

(c) $v(p,a)$ given by:

$$v(p,a) = \begin{cases} R(p_a^*, a) - \delta^+ & \text{for } p < \underline{p}_a \\ R(p,a) & \text{for } p \in [\underline{p}_a, p_a^*] \\ \max\{R(p,a), R(p_a^*, a) - \delta^-\} & \text{for } p > p_a^* \end{cases}$$

is the value function.

(d) The optimal policy is: If $a^i_{t\text{-}} = a$ and $p_{t\text{-}} \in \mathcal{S}^1_a = \{p : v(p,a) = R(p_a^*, a) - \delta^+\}$, then pay the menu cost (δ^+) and set $p_t = p_a^*$, and if $p_{t\text{-}} \in \mathcal{S}^2_a = \{p : v(p,a) = R(p_a^*, a) - \delta^-\}$, then pay the menu cost (δ^-) and set $p_t = p_a^*$. Otherwise, unless you receive a Calvo shock, do nothing. If a Calvo shock arrives, move to $p^*_{a_t}$, where a_t is the productivity state at t.

Proof. If we write $S(p,a) = R'(p,a)$ and differentiate both sides of (S.9) we obtain,

$$S'(p,a) = \frac{z'(p,a)}{\pi p} - \left(\frac{\pi + \mu + \rho}{\pi p}\right) S(p,a). \tag{S.16}$$

Notice that the equations for $S(\cdot, a)$ do not involve $S(\cdot, b)$ for $b \neq a$ and can be solved separately.[27] Set

$$A(r) := \int_1^r -\frac{\pi + \mu + \rho}{\pi s} ds = -\frac{\pi + \mu + \rho}{\pi} \log(r).$$

[27] Separability requires that productivity changes are accompanied by Calvo shocks, as we assumed.

Then the general solution to (S.16) is:

$$S(\sigma,p,a)=\sigma e^{A(p)}+e^{A(p)}\int_{\hat{p}}^{p}e^{-A(r)}\frac{z'(r,a)}{\pi r}dr=p^{-\frac{\pi+\mu+\rho}{\pi}}\left[\sigma+\int_{\hat{p}}^{p}r^{\frac{\mu+\rho}{\pi}}\frac{z'(r,a)}{\pi}dr\right].$$

If

$$\Gamma(p,a):=\int_{\hat{p}}^{p}r^{\frac{\mu+\rho}{\pi}}\frac{z'(r,a)}{\pi}dr,$$

$S(\sigma,p,a)=0$ if and only if

$$\sigma+\Gamma(p,a)=0. \tag{S.17}$$

Since $\Gamma'(p,a):=p^{\frac{\mu+\rho}{\pi}}z'(p,a)/\pi$ is strictly positive (negative) for $p<\hat{p}_a$ ($p>\hat{p}_a$), $\Gamma(p,a)$ achieves a maximum of zero at $\hat{p}$. In addition, $\Gamma''(\hat{p},a):=\hat{p}^{\frac{\mu+\rho}{\pi}}z''(\hat{p},a)/\pi<0$. Thus there exists $\epsilon>0$ (which may depend on the parameters of the model) such that for $\sigma\in(0,\epsilon)$ there are exactly two solutions $p_1(\sigma)<\hat{p}<p_2(\sigma)$ to $S(\sigma,p,a)=0$ and $\lim_{\sigma\searrow 0}p_i(\sigma)=\hat{p}$, $i=1,2$. Moreover since

$$\frac{\partial p_i}{\partial\sigma}=-\frac{1}{\Gamma'(p_i,a)} \tag{S.18}$$

and $\Gamma'(p_i(\sigma),a)\neq 0$, unless $p_i=\hat{p}_a$, we can prolong the function $p_1(\sigma)$ until a $\sigma_{\max}$ such that $p_1(\sigma_{\max})=0$ ($\sigma_{\max}=\infty$ is not ruled out) and, since for any σ, $-1/\Gamma'(p_2(\sigma),a)$ is uniformly bounded above, we may also prolong $p_2(\sigma)$ until $\sigma_{\max}$. Furthermore if $\sigma'>\sigma$, $p_1(\sigma')<p_1(\sigma)<\hat{p}<p_2(\sigma)<p_2(\sigma')$.

If $\Psi(\sigma,a):=z(p_2(\sigma),a)-z(p_1(\sigma),a)$, then Ψ achieves a minimum of zero at $\sigma=0$. In addition, using equation (S.18), we obtain

$$\Psi'(\sigma,a)=\pi\left[p_1^{-\frac{\mu+\rho}{\pi}}-p_2^{-\frac{\mu+\rho}{\pi}}\right]>0,$$

provided $\sigma>0$. Since $z(0)=-\infty$, this delivers the existence and uniqueness of $\sigma(\delta^+,a)<\sigma_{\max}$ such that the value matching condition

$$R(\sigma,p_2(\sigma),a)-R(\sigma,p_1(\sigma),a)=\delta^+ \tag{S.19}$$

holds at $\sigma=\sigma(\delta^+,a)$ for any $\delta^+\geq 0$.

For each $a\in\{a^L,a^H\}$ set $p_a^*=p_2(\sigma(\delta^+,a))$ and $\underline{p}_a=p_1(\sigma(\delta^+,a))$ and set $R(\sigma(\delta^+,a),\cdot,a)$ using (S.9) and (S.12). In addition, (S.17) applied to $\underline{p}_a$ and p_a^*, implies

$$\int_{\underline{p}_a}^{p_a^*}r^{\frac{\mu+\rho}{\pi}}\frac{z'(r,a)}{\pi}dr=0,$$

which after integration yields (S.15).

Following the procedure in Øksendal and Sulem [54, Chapter 9],[28] one can in fact verify that since $R(\sigma(\delta^+, a), p, a)$ satisfies (S.9)–(S.11) then

$$v(p,a) = \begin{cases} R(\sigma(\delta^+, a), p, a) & \text{for } \underline{p}_a \leq p \leq p_a^* \\ R(\sigma(\delta^+, a), p_a^*, a) - \delta^+ & \text{for } p < \underline{p}_a \\ \max\{R(\sigma(\delta^+, a), p, a), R(\sigma(\delta^+, a), p_a^*, a) - \delta^-\} & \text{for } p > p_a^* \end{cases}$$

is the value function and the associated optimal policy is given by (d) in the statement of the Proposition. □

B.3 Stationary distribution

The characterization of the optimal policy in Proposition B.1 implies that if $i's$ productivity is a and $p_t^i \in (\underline{p}_a, p_a^*]$ then:

$$dp_t^i = -\pi p_t^i dt + \mu\left((1-\zeta^a)(p_a^* - p_t^i) + \zeta^a(p_b^* - p_t^i)\right)dt + \mathcal{M}_t,$$

where $b \neq a$ and $\mathcal{M}_t$ is a martingale.

In addition, if $h(p,t,a)$ denotes the time t density of prices for firms with productivity a, then h satisfies the *forward-equation*

$$\frac{\partial}{\partial t} h(p,t,a) = \frac{\partial}{\partial p}[\pi p h(p,t,a)] - \mu h(p,t,a). \tag{S.20}$$

Corollary 4. *(a) In a stationary equilibrium, the distribution of firm's prices conditional on productivity a has support in $(\underline{p}_a, p_a^*]$.*

(b) The stationary distribution of prices conditional on productivity a has density

$$f(p \mid a) = \frac{p^{\mu/\pi - 1}}{\underline{p}_a^{\mu/\pi} \varphi(p_a^*/\underline{p}_a, \mu/\pi)}. \tag{S.21}$$

(c) The marginal stationary density of prices is:

$$f_p(p) = \frac{\zeta^b}{\zeta^a + \zeta^b} f(p \mid a) + \frac{\zeta^a}{\zeta^a + \zeta^b} f(p \mid b).$$

(d) In the stationary equilibrium firms choose to pay menu costs only when increasing prices ($\lambda = \lambda^+$). The fraction of firms of productivity a that pay menu costs, λ_a, satisfies

$$\lambda_a = \frac{\mu}{(p_a^*/\underline{p}_a)^{\mu/\pi} - 1} \tag{S.22}$$

[28] A major difference between our set up and the one in [54] is the absence of a diffusion term, which allows for less smoothness of the candidate value function.

and consequently the fraction of firms that pay menu costs is

$$\lambda = \frac{\zeta^b}{\zeta^a+\zeta^b}\frac{\mu}{(p_a^*/\underline{p}_a)^{\mu/\pi}-1} + \frac{\zeta^a}{\zeta^a+\zeta^b}\frac{\mu}{(p_b^*/\underline{p}_b)^{\mu/\pi}-1}. \tag{S.23}$$

Proof. Equation (S.21) follows from (S.20) and Item (a). Moreover, $\lambda_a = f(\underline{p}_a|a)\pi\underline{p}_a = \pi/\varphi(p_a^*/\underline{p}_a, \mu/\pi) = \mu/((p_a^*/\underline{p}_a)^{\mu/\pi}-1)$. □

B.4 The equilibrium wage rate

In the previous section we showed that given w we can find $(\underline{p}_a, p_a^*, f(\cdot \mid a))_a$ that characterizes the behavior of the intermediate goods firms. However, a necessary condition for equilibrium is that the average price of intermediate goods must be one, that is, equation (13) must hold. In this section, we show that given the parameters of the model $(\eta, \rho, (\zeta^a)_a, \mu, \delta^+)$ and a rate of inflation π there is exactly one w such that (13) holds. In what follows, it is useful to introduce the notation

$$q_a := \frac{p_a^*}{\underline{p}_a} > 1 \quad \text{and} \quad \varsigma^a := \frac{\zeta^b}{\zeta^a+\zeta^b} \le 1, \quad b \neq a.$$

Thus q_a denotes the proportional increase in relative price when a firm with productivity a pays the menu cost, and ς^a the stationary fraction of firms with productivity a. With this notation, we may rewrite the zero-profit condition for the competitive firms, equation (13), as:

$$1 = \sum_{a=a^L,a^H} \int p^{1-\eta}\varsigma^a f(p \mid a)dp = \sum_{a=a^L,a^H} \frac{\varphi(q_a, \mu/\pi+1-\eta)}{\varphi(q_a,\mu/\pi)}\underline{p}_a^{1-\eta}\varsigma^a \tag{S.24}$$

and rewrite the value-matching condition (S.13) as:

$$(\rho+\mu)\delta^+ = \left((q_a^{1-\eta}-1) - \frac{w}{a\underline{p}_a}(q_a^{-\eta}-1)\right)\underline{p}_a^{1-\eta}, \quad \text{for } a = a^L, a^H. \tag{S.25}$$

The inverse of the “minimum mark-up” $\underline{p}_a/(w/a)$ satisfies (S.15), that is,

$$\frac{w}{a\underline{p}_a} = \frac{\eta-1}{\eta}\frac{\varphi(q_a, 1+(\mu+\rho)/\pi-\eta)}{\varphi(q_a,(\mu+\rho)/\pi-\eta)}, \quad \text{for } a = a^L, a^H. \tag{S.26}$$

Thus, equations (S.24)–(S.26) determine five unknowns $((\underline{p}_a, q_a)_a, w)$. By substituting (S.26) into (S.25), and substituting out $\underline{p}_a^{1-\eta}$ from (S.24), we obtain:

$$\frac{\eta-1}{(\mu+\rho)\delta^+} = \sum_{a=a^L,a^H} \frac{\varsigma^a\varphi(q_a,\mu/\pi+1-\eta)}{\varphi(q_a,\mu/\pi)\varphi(q_a,1-\eta)}\left(\frac{\varphi(q_a,(\rho+\mu)/\pi+1-\eta)}{\varphi(q_a,(\rho+\mu)/\pi-\eta)}\frac{\varphi(q_a,-\eta)}{\varphi(q_a,1-\eta)}-1\right)^{-1}. \tag{S.27}$$

Lemma 12. *There exists a unique positive vector* $((q_a, \underline{p}_a)_a, w) \in \mathbf{R}^5$*, with* $q_a > 1$ *for each* a*, that solves* (S.24)–(S.27)*. In addition* $q_{a^H} \leq q_{a^L}$*, with strict inequality if* $a^L < a^H$*. If* λ *given by* (S.23) *satisfies* $\lambda\delta^+ < 1$*, then there exists a unique positive vector* $(\bar{c}, \bar{N}, \bar{Y})$ *satisfying equations* (S.4) *and* (13)–(16)*.*

Proof. In B.5 we show that (S.27) has a unique solution pair (q_{a^L}, q_{a^H}) (and necessarily $q_{a^L} > q_{a^H}$). Then (S.24)–(S.25) determine a unique $\underline{p}_a$ and equilibrium wage rate w. Equation (S.23) determines λ and if $\lambda < 1/\delta^+$, (S.4) and (15) deliver $c > 0$ and $N > 0$ as linear functions of Y. The assumption that Inada conditions hold, $\partial^2 U(c, N)/\partial c^2 < 0$ and $\partial^2 U(c, N)/\partial c \partial N \leq 0$ guarantees that there exists exactly one level of Y that satisfies (16). □

There is however no guarantee that for the equilibrium wage rate w, $v(p, a) \geq 0$, for each p in the support of the stationary distribution for productivity level a, which should also be an equilibrium requirement. Since $\min_{p \in [\underline{p}_a, p^*_a]} v(p, a) = R(p^*_a, a) - \delta^+$, non-negativity of $v(p, a)$ for any p in the support of the stationary distribution for productivity level a is equivalent to:

$$\frac{\mu\zeta^a z(p^*_b, w/b) + \mu\zeta^b z(p^*_a, w/a) + \rho z(p^*_a, w/a)}{\mu\rho(\zeta^a + \zeta^b) + \rho^2} \geq \delta^+. \tag{S.28}$$

Hence we are left with two extra requirements not necessarily satisfied by the value function we constructed: (i) non-negativity in the relevant support and (ii) $\lambda\delta^+ < 1$. However we can show that (i) implies (ii):

Lemma 13. *If* $((q_a, \underline{p}_a)_a, w)$*, with* $q_a > 1$ *solves* (S.24)–(S.27) *and inequality* (S.28) *holds then* $\lambda\delta^+ < 1$.

Proof: See B.6.

In addition to the fixed exogenous parameters $(\eta, \rho, (\zeta^a)_a, \mu, \delta^+)$ the validity of (S.28) depends on the inflation parameter π and the endogenously determined real wage rate w. In general, if inflation and/or menu costs are very high (S.28) may fail. However,

Proposition B.2. *Let* $\bar{\delta}^+ = ((a^L)^{\eta-1}/\sum_{a'} \varsigma^{a'}(a')^{\eta-1})((\eta-1)^{\eta-1}/\eta^\eta)/(\rho+\mu)$*. Then there exists a* $\bar{\pi}$ *such that for any* $\pi \leq \bar{\pi}$ *and* $\delta^+ \leq \bar{\delta}^+$ *there exists an equilibrium* $((\underline{p}_a, p^*_a)_a, w, \lambda, c, N, Y)$*.*

Proof. Write $p^*_a(\delta^+, \pi, w)$ for the equilibrium for a fixed w established in Corollary 4, and $w(\delta^+, \pi)$ for the equilibrium wage rate established in Lemma 12. Since $p^*_a(\delta^+, \pi, w) \geq \hat{p}_a(w)$,

$$z(p^*_a(\delta^+, \pi, w), w/a) > \frac{1}{\eta}(p^*_a(\delta^+, \pi, w))^{1-\eta}.$$

Moreover, $\underline{p}_a < \hat{p}_a$. Hence $(p^*_a)^{1-\eta} = (q_a \underline{p}_a)^{1-\eta} > (q_a \hat{p}_a)^{1-\eta} = q_a^{1-\eta} a^{\eta-1}/(\sum_{a'} \varsigma^{a'}(a')^{\eta-1})$. In addition, as proved in B.7.2, $q_a(\delta^+, \pi, w(\delta^+, \pi))$ is an increasing function of δ^+. Thus

$$\begin{aligned}(p_a^*(\delta^+, \pi, w(\delta^+, \pi)))^{1-\eta} &> \frac{a^{\eta-1}}{\sum_{a'} \varsigma^{a'}(a')^{\eta-1}}(q_a(\delta^+, \pi, w(\delta^+, \pi)))^{1-\eta} \\ &> \frac{a^{\eta-1}}{\sum_{a'} \varsigma^{a'}(a')^{\eta-1}}(q_a(\bar{\delta}^+, \pi, w(\bar{\delta}^+, \pi)))^{1-\eta}.\end{aligned}$$

Since q_a is continuous and $\lim_{\pi \to 0} q_a(\bar{\delta}^+, \pi, w(\bar{\delta}^+, \pi)) = q_a(\bar{\delta}^+, 0, w(\bar{\delta}^+, 0)) < \eta/(\eta - 1)$ as we show in B.7.4, there exists $\bar{\pi} > 0$ such that,

$$z(p_a^*(\delta^+, \pi, w(\delta^+, \pi)), w(\delta^+, \pi)/a) > \rho\bar{\delta}^+$$

for each a, $\pi \in [0, \bar{\pi}]$ and $\delta^+ \in [0, \bar{\delta}^+]$. Hence, we obtain (S.28),

$$\frac{\mu\zeta^a z(p_b^*, w/b) + \mu\zeta^b z(p_a^*, w/a) + \rho z(p_a^*, w/a)}{\mu\rho(\zeta^a + \zeta^b) + \rho^2} \geq \delta^+,$$

for each $a \in \{a^L, a^H\}$, $\pi \in [0, \bar{\pi}]$ and $\delta^+ \in [0, \bar{\delta}^+]$.

B.5 Unique solution of Equation (S.27)

We define functions $A(q, \pi)$ and $B(q, \pi)$ as

$$\begin{aligned}A(q, \pi) &:= \frac{\varphi(q, (\rho + \mu)/\pi + 1 - \eta)}{\varphi(q, (\rho + \mu)/\pi - \eta)} \frac{\varphi(q, -\eta)}{\varphi(q, 1 - \eta)} \\ B(q, \pi) &:= \frac{\varphi(q, \mu/\pi + 1 - \eta)}{\varphi(q, \mu/\pi)\varphi(q, 1 - \eta)}\end{aligned}$$

and rewrite (S.27) as

$$1 = \frac{\delta^+(\rho + \mu)}{\eta - 1} \sum_a \varsigma^a \frac{B(q_a, \pi)}{A(q_a, \pi) - 1}. \tag{S.29}$$

The properties of functions A and B are derived in the main text: $\partial A/\partial q > 0$, $\lim_{q \to 1} A(q, \pi) = 1$, $\lim_{q \to \infty} A(q, \pi) < \infty$, $\partial B/\partial q < 0$, $\lim_{q \to 1} B(q, \pi) = \infty$, and $\lim_{q \to \infty} B(q, \pi) = 0$. These properties lead to that the right-hand side of (S.29) decreases monotonically from $+\infty$ to 0 as q_a increases from 1 to $+\infty$.

Equation (S.15) implies, for $a \in \{a^L, a^H\}$,

$$w = a\frac{\eta - 1}{\eta} \frac{\varphi(q_a, \frac{\rho+\mu}{\pi} + 1 - \eta)}{\varphi(q_a, \frac{\rho+\mu}{\pi} - \eta)} \underline{p}_a = aH(q_a, \pi) \tag{S.30}$$

where

$$H(q,\pi) := \frac{\eta-1}{\eta}\frac{\varphi(q,\frac{\rho+\mu}{\pi}+1-\eta)}{\varphi(q,\frac{\rho+\mu}{\pi}-\eta)}\left(\frac{\eta-1}{(\mu+\rho)\delta^+}\varphi(q,1-\eta)(A(q,\pi)-1)\right)^{1/(\eta-1)}.$$

Note that the second fraction in the expression for H is increasing in q, and so are $\varphi(q,1-\eta)$ and $A(q,\pi)$. Thus, $H(q_a,\pi)$ is an increasing function in q_a for each $a\in\{a^L,a^H\}$. Letting $q_{a^L}=q_{a^L}(q_{a^H})$ denote a function relating q_{a^H} and q_{a^L} determined by $a^H H(q_{a^H},\pi) = a^L H(q_{a^L},\pi)$, the function q_{a^L} is increasing in q_{a^H}. Furthermore, since $a^H > a^L$, this equation implies $q_{a^H} < q_{a^L}$. Thus, $q_{a^L}(q_{a^H}) > 1$ for $q_{a^H} > 1$. This leads to the unique existence of the solution for (S.29).

Finally, we verify that the solution to (S.24)–(S.27) generates inflation π. Note that, at any instant t, $P_t^{1-\eta}$ is increased by the price adjustments of firms hit by Calvo shocks and price adjustments of firms that pay menu costs. We compute these two effects on the growth rate $(dP_t^{1-\eta}/dt)/P_t^{1-\eta}$. The effect due to Calvo shocks is written as:

$$\sum_{a=a^L,a^H}\mu(1-\zeta^a)\varsigma^a\left((p_a^*)^{1-\eta}-\int p^{1-\eta}f(p|a)dp\right)+\mu\zeta^a\varsigma^a\left((p_b^*)^{1-\eta}-\int p^{1-\eta}f(p|a)dp\right).$$

By zero-profit condition (13) we have $1=\sum_a\varsigma^a\int p^{1-\eta}f(p|a)dp$. Also the stationary distribution implies $\zeta^a\varsigma^a=\zeta^b\varsigma^b$. Thus the above expression is reduced to

$$\begin{aligned}&\sum_{a=a^L,a^H}\mu(1-\zeta^a)\varsigma^a\left((p_a^*)^{1-\eta}-1\right)+\mu\zeta^a\varsigma^a\left((p_b^*)^{1-\eta}-1\right)\\&=\mu\sum_{a=a^L,a^H}\left((1-\zeta^a)\varsigma^a+\zeta^b\varsigma^b\right)\left((p_a^*)^{1-\eta}-1\right)=\mu\sum_{a=a^L,a^H}\varsigma^a\left((p_a^*)^{1-\eta}-1\right).\end{aligned}\tag{S.31}$$

The effect due to firms paying menu costs is

$$\sum_{a=a^L,a^H}\varsigma^a\lambda_a\left((p_a^*)^{1-\eta}-\underline{p}_a^{1-\eta}\right).\tag{S.32}$$

Using $\lambda_a=\mu/(q_a^{\mu/\pi}-1)$ from (S.22), we have

$$\left((p_a^*)^{1-\eta}-1\right)+\frac{\lambda_a}{\mu}\left((p_a^*)^{1-\eta}-\underline{p}_a^{1-\eta}\right)=\frac{q_a^{1-\eta+\mu/\pi}-1}{q_a^{\mu/\pi}-1}\underline{p}_a^{1-\eta}-1.$$

Hence, by summing (S.31) and (S.32) and using (13), we obtain:

$$\frac{dP_t^{1-\eta}/dt}{P_t^{1-\eta}}=\mu\sum_{a=a^L,a^H}\varsigma^a\left(\frac{q_a^{1-\eta+\mu/\pi}-1}{q_a^{\mu/\pi}-1}\underline{p}_a^{1-\eta}-1\right)=\pi(1-\eta),\tag{S.33}$$

where we used $\frac{q_a^{1-\eta+\mu/\pi}-1}{q_a^{\mu/\pi}-1}\underline{p}_a^{1-\eta}=\frac{1-\eta+\mu/\pi}{\mu/\pi}\frac{\varphi(q_a,1-\eta+\mu/\pi)}{\varphi(q_a,\mu/\pi)}\underline{p}_a^{1-\eta}=\frac{1-\eta+\mu/\pi}{\mu/\pi}$. Thus we obtain the desired result $(dP_t/dt)/P_t=\pi$.

B.6 Non-negative average $v(p)$ implies $\lambda\delta^+ < 1$

Proof of Lemma 13. Let $(\Omega, \mathcal{F}, Q)$ be the probability space that defines the Poisson processes and the history of productivities. Then,

$$v(p,a) = E^Q\left[\int_0^\infty e^{-\rho t}\left(z(p_t(\omega), a_t(\omega), t) - \delta^+ \mathbf{1}_{\{t: p_t(\omega) = \underline{p}_{a_t(\omega)}\}}\right) dt \mid p_0 = p, a_0 = a\right].$$

Averaging over the stationary distribution of prices and productivities F, we obtain:

$$E^F v(p,a) = E^F E^Q\left[\int_0^\infty e^{-\rho t}\left(z(p_t(\omega), a_t(\omega)) - \delta^+ \mathbf{1}_{\{t: p_t(\omega) = \underline{p}_{a_t(\omega)}\}}\right) dt \mid p_0 = p, a_0 = a\right].$$

Since $z(p_t, a_t) - \delta^+$ is bounded below by $\min\{z(p^*, a_t) - \delta^+; z(\underline{p}, a_t) - \delta^+\}$, we can apply Tonelli's theorem to obtain

$$\begin{aligned} E^F v(p,a) &= E^Q \int_0^\infty E^F\left[e^{-\rho t}\left(z(p_t(\omega), a_t(\omega)) - \delta^+ \mathbf{1}_{\{t: p_t(\omega) = \underline{p}_{a_t(\omega)}\}}\right)\right] dt \\ &= \frac{1}{\rho}\left(E^F z(p,a) - \lambda\delta^+\right). \end{aligned}$$

Since w is necessarily positive, (S.24) implies that $E^F z(p,a) < 1$. Thus if $E^F v(p,a) > 0$,

$$0 < E^F \rho v(p,a) = E^F z(p,a) - \lambda\delta^+ < 1 - \lambda\delta^+.$$

B.7 Function $q_a(\delta^+, \pi)$

B.7.1 Continuity of $q_a(\delta^+, \pi)$ at $\delta^+ = 0$

When $\delta^+ = 0$, it is optimal for firms to adjust price immediately at any time regardless of π. This implies $q_a = 1$ and $p_a^* = \hat{p}_a$. With all the firms pricing at $\hat{p}_a$, price aggregation implies $\sum_a \varsigma^a \hat{p}_a^{1-\eta} = 1$. Hence, $w = (\sum_a \varsigma^a a^{\eta-1})^{1/(\eta-1)}(\eta-1)/\eta$ and $p_a^* = \hat{p}_a = (w/a)\eta/(\eta-1) = (\sum_{a'} \varsigma^{a'} a'^{\eta-1})^{1/(\eta-1)}/a$ hold.

As δ^+ decreases to 0, the right-hand side of (S.29) decreases to 0 for fixed $(q_a)_a$. This implies that the solution q_a of (S.29) decreases as δ^+ decreases, since $B(q,\pi)/(A(q,\pi)-1)$ is a decreasing function in q. Moreover, the solution q_a converges to 1 as $\delta^+ \to 0$, since $\lim_{q\to 1} A(q,\pi) = 1$. Hence, $q_a(\delta^+, \pi)$ is continuous in $\delta^+ \geq 0$ and $q_a(0,\pi) = 1$ for any π.

B.7.2 $dq_a/d\delta^+ > 0$

By taking derivative of (S.29) with respect to $(\delta^+, (q_a)_a)$ and rearranging, we have

$$\begin{aligned} 0 = &\left(\sum_a \varsigma^a \frac{B(q_a,\pi)}{A(q_a,\pi)-1}\right)\frac{d\delta^+}{\delta^+} \\ &+ \sum_a \frac{\varsigma^a}{(A(q_a,\pi)-1)^2}\left(\frac{\partial B(q_a,\pi)}{\partial q_a}(A(q_a,\pi)-1) - \frac{\partial A(q_a,\pi)}{\partial q_a}B(q_a,\pi)\right) dq_a. \end{aligned}$$

We have $\partial A/\partial q_a > 0$ and $\partial B/\partial q_a < 0$. Moreover, since (S.30) and $aH(q_a,\pi) = bH(q_b,\pi)$ are independent of δ^+, we have that q_{a^L} and q_{a^H} comove when δ^+ varies. Thus we obtain $dq_a/d\delta^+ > 0$.

B.7.3 $d(w/p_a^*)/d\pi < 0$ and $d(w/p_a^*)/d\delta^+ < 0$

Finally, from (S.15) we have

$$\frac{w}{p_a^*} = \frac{(\eta-1)a}{\eta}\frac{\varphi(q_a, 1-\eta+\frac{\rho+\mu}{\pi})}{q_a\varphi(q_a, -\eta+\frac{\rho+\mu}{\pi})} = \frac{(\eta-1)a}{\eta}\frac{\varphi(q_a, \eta-1-\frac{\rho+\mu}{\pi})}{\varphi(q_a, \eta-\frac{\rho+\mu}{\pi})}. \tag{S.34}$$

From (S.34) we may write $w/p_a^* = (w/p_a^*)(q_a,\pi)$ and Lemma 4(c) guarantees that

$$\frac{\partial(w/p_a^*)}{\partial q_a} < 0.$$

Furthermore, (69) implies that

$$\frac{\partial(w/p_a^*)}{\partial \pi} < 0.$$

Since $dq_a/d\pi > 0$, the claim $d(w/p_a^*)/d\pi < 0$ follows. Finally, from $\partial(w/p_a^*)/\partial q_a < 0$ and $dq_a/d\delta^+ > 0$, we obtain that $d(w/p_a^*)/d\delta^+ < 0$.

B.7.4 Continuity of $q_a(\delta^+,\pi)$ at $\pi = 0$

When there is no inflation, the relative price does not move, unless the firm chooses to change it. Once a firm adjusts its price, it would opt to change the price again only when it receives a productivity shock. Thus, in any productivity state a, the relative price of a firm is

$$p_a^*(0) = \hat{p}_a(0) = \frac{\eta}{\eta-1}\frac{w(0)}{a}.$$

Hence, the zero-profit condition in the competitive sector (13) implies

$$1 = E^{F_p}[p^{1-\eta}] = \sum_a \varsigma^a (p_a^*)^{1-\eta} = \left(\frac{w(0)}{\tilde{a}}\frac{\eta}{\eta-1}\right)^{1-\eta}$$

where we define $\tilde{a} := (\sum_a \varsigma^a a^{\eta-1})^{1/(\eta-1)}$.

The firm pays menu-costs and reprices if $p < \underline{p}_a(0)$ where $\underline{p}_a(0)$ is determined by the value matching condition (S.25). Using the results above, (S.25) is modified as

$$\frac{(\rho+\mu)\delta^+}{\eta-1}\left(\frac{a}{\tilde{a}}\right)^{1-\eta} = \varphi(q_a(0),\eta) - \varphi(q_a(0),\eta-1). \tag{S.35}$$

Equation (S.35) shows that $q_{a^L}(0) > q_{a^H}(0) > 1$, since function $h(q) := ((\eta-1)/\eta)q^\eta - q^{\eta-1} + 1/\eta - (\rho+\mu)\delta^+(a/\tilde{a})^{1-\eta}$ is increasing in $q \geq 1$ toward infinity and $h(1) < 0$. Also we obtain $q(0) < \eta/(\eta-1)$, since we have $h(\eta/(\eta-1)) = 1/\eta - (\rho+\mu)\delta^+(a/\tilde{a})^{1-\eta} \geq 1/\eta - (\rho+\mu)\bar{\delta}^+(a^L/\tilde{a})^{1-\eta} > 0$ under the premise of Proposition B.2.

Now we show that $\lim_{\pi\to 0} q_a(\pi) = q_a(0)$. First, notice that for any fixed $q > 1$, using l'Hôpital's rule:

$$\lim_{y\to\infty} \frac{q^x \varphi(q,y)}{\varphi(q,x+y)} = \lim_{y\to\infty} \frac{q^x(q^y-1)}{q^{x+y}-1} \lim_{y\to\infty} \frac{x+y}{y} = \lim_{y\to\infty} \frac{q^x q^y \log q}{q^{x+y}\log q} = 1.$$

In addition, Lemma 4(c) states that, for $q > 1$, if $x > 0$ then $q^x\varphi(q,y)/\varphi(q,x+y)$ increases with q, and if $x < 0$, $q^x\varphi(q,y)/\varphi(q,x+y)$ decreases with q.

Let $\bar{q}$ be an upper bound of $q_{a^L}(\pi)$ in $\pi \in (0,\bar{\pi})$. Thus for $\pi \in (0,\bar{\pi})$, since $\bar{q} > q_{a^L}(\pi) > q_{a^H}(\pi) > 1$,

$$\lim_{q\to 1} \frac{\varphi(q, 1-\eta+\mu/\pi)}{q^{1-\eta}\varphi(q,\mu/\pi)} \leq \frac{\varphi(q_a(\pi), 1-\eta+\mu/\pi)}{q_a(\pi)^{1-\eta}\varphi(q_a(\pi),\mu/\pi)} < \frac{\varphi(\bar{q}, 1-\eta+\mu/\pi)}{\bar{q}^{1-\eta}\varphi(\bar{q},\mu/\pi)}.$$

By l'Hôpital's rule, the leftmost side is equal to 1 for any π. Thus, taking the limit $\pi \to 0$ for all the terms, we obtain

$$\lim_{\pi\to 0} \frac{\varphi(q_a(\pi), 1-\eta+\mu/\pi)}{q_a(\pi)^{1-\eta}\varphi(q_a(\pi),\mu/\pi)} = 1$$

and if $q_a^o := \lim_{\pi\to 0} q_a(\pi)$ then

$$\lim_{\pi\to 0} \frac{\varphi(q_a(\pi), 1-\eta+\mu/\pi)}{\varphi(q_a(\pi),\mu/\pi)} = (q_a^o)^{1-\eta}.$$

Similarly,

$$\lim_{\pi\to 0} \frac{\varphi(q_a(\pi), 1-\eta+\frac{\mu+\rho}{\pi})}{\varphi(q_a(\pi), -\eta+\frac{\mu+\rho}{\pi})} = q_a^o.$$

Applying these limits to (S.24) and (S.26), we obtain $\lim_{\pi\to 0} w(\pi) = \tilde{a}(\eta-1)/\eta$ and $\lim_{\pi\to 0} \underline{p}_a(\pi) = \lim_{\pi\to 0} w(\pi)\eta/((\eta-1)aq_a^o)$. Substituting these into (S.25) yields

$$\frac{(\rho+\mu)\delta}{\eta-1}\left(\frac{a}{\tilde{a}}\right)^{1-\eta} = \varphi(q_a^o,\eta) - \varphi(q_a^o,\eta-1).$$

This equation is equivalent to (S.35). Hence, $q_a(\pi)$ is continuous at $\pi = 0$.

B.8 θ in the extended model

In this section we derive the parameter θ, which measures the complementarity of price-adjustments across firms, in the extended model.

Given productivity level a and $\nu \geq 0$, define $m(\nu, a)$ as the measure of firms with productivity a that charge (log relative) prices between $\log \underline{p}_a$ and $\log \underline{p}_a + \nu$. It is convenient to define a relative log price

$$s_t^i := \frac{\log p_t^i - \log \underline{p}_{a^i}}{\log p^*_{a^i} - \log \underline{p}_{a^i}} \in [0, 1],$$

where a^i is the productivity of firm i at t. Using (S.21) one obtains the stationary density of s,

$$\begin{aligned} g(s|a) &= \frac{\log p_a^* - \log \underline{p}_a}{\varphi(p_a^*/\underline{p}_a, \mu/\pi)} \left(\frac{p_a^*}{\underline{p}_a}\right)^{\frac{\mu s}{\pi}}, \\ g_s(s) &= \sum_a \varsigma^a g(s|a). \end{aligned} \tag{S.36}$$

Then we have

$$m(\nu, a) = \varsigma^a \int_0^{\nu/\log q_a} g(s|a) ds.$$

Suppose we perform the following thought experiment. For any a, all firms with productivity a and $s \leq \nu/\log q_a$ reprice (set $p = p_a^*$). This increases the price level P to a new level P' that can be computed using the zero-profit condition (S.24), that is:

$$\frac{P'}{P} = \left(\sum_a \left[(p_a^*)^{1-\eta} m(\nu, a) + \varsigma^a \int_{\underline{p}_a e^\nu}^{p_a^*} p^{1-\eta} f(p|a) dp\right]\right)^{1/(1-\eta)} > 1$$

and thus a firm of productivity a that charged $p > \underline{p}_a e^\nu$ in the original stationary equilibrium now charges $(P/P')p$. Let

$$\nu'(\nu, a) = \nu + \log(P'/P).$$

If firms expect that the equilibrium stationary dynamics prevails, all firms of productivity a with original prices p such that $\log \underline{p}_a + \nu < \log p \leq \log \underline{p}_a + \nu'(\nu, a)$ would then raise their price to p_a^*. Set

$$m'(\nu, a) := \varsigma^a \int_{\nu/\log q_a}^{\nu'(\nu,a)/\log q_a} g(s|a) ds,$$

the measure of productivity a firms that reprice as a result of the first round of repricing.

We define the complementarity coefficient θ as a limit of the ratio of the measure of firms m' induced to raise prices to the measure of firms m that originally raise prices as $m \to 0$.

$$\theta := \lim_{\nu\to 0} \frac{\sum_a m'(\nu, a)}{\sum_a m(\nu, a)} = \left. \frac{\sum_a dm'(\nu, a)/d\nu}{\sum_a dm(\nu, a)/d\nu} \right|_{\nu=0}$$

In B.9.1 we show that

$$\theta = \left. \frac{d \log(P'/P)}{d\nu} \right|_{\nu=0} = \sum_a \varsigma^a \frac{\varphi(q_a, 1-\eta)}{\varphi(q_a, \mu/\pi)} \underline{p}_a^{1-\eta}. \tag{S.37}$$

Furthermore in B.9.1 we show that

$$\pi = \frac{\mu}{1-\theta} \sum_a \varsigma^a \varphi(p_a^*, 1-\eta), \tag{S.38}$$

which relates inflation to the multiplier effect $1/(1-\theta)$.

It follows from (S.24) that $\theta < 1$, since $\varphi(q, x)$ is increasing on x if $q > 1$ by Lemma 4(b). θ is a function of other parameters in the model, in particular of π. Since $\varphi(p_a^*, 1-\eta) < 1/(\eta-1)$, (S.38) implies $\lim_{\pi\to\infty} \theta(\pi) \to 1$ and

$$\frac{1}{1-\theta} \geq \frac{\eta-1}{\mu}\pi.$$

In a similar manner, we can also define the impact of a Calvo-shock on a firm which is in state s on price-adjustments by other firms. Suppose a small measure m_c of firms around s receive a Calvo shock. These firms would adjust prices and the effect on the aggregate log prices equals a constant $\nu_0(s)$. We can now proceed as before and calculate $m(\nu_0(s), a)$, the measure of firms with productivity a that charge prices between $\log \underline{p}_a$ and $\log \underline{p}_a + \nu_0(s)$. We define $\theta_0(s)$ as the limit as $m_c \to 0$ of the ratio of the measure of firms that are induced to change prices as a direct effect of the repricing by the Calvo-hit firms with s to the initial measure m_c, that is:

$$\theta_0(s) := \lim_{m_c\to 0} \frac{\sum_a m(\nu_0(s), a)}{m_c}. \tag{S.39}$$

We also define θ_0 as the average of $\theta_0(s)$ over s, that is,

$$\theta_0 := \int_0^1 \theta_0(s) g_s(s) ds. \tag{S.40}$$

In B.9.1 we provide an expression for $\theta_0(s)$ and θ_0 (see (S.43) and (S.44)) and derive a relation,

$$\lambda = \frac{\mu\theta_0}{1-\theta}. \tag{S.41}$$

Note that (S.41) coincides with (40), the expression for λ when $a^L = a^H$. Equation (S.41) shows that the measure of firms that pay menu costs at each instant is a multiple of the rate of Calvo shock μ, the direct effect of the Calvo-hit firms on the firms at the extensive margin θ_0, and the multiplier effect $1/(1-\theta)$.

B.9 Derivation of θ and θ_0 in Section B.8

B.9.1 Derivation of (S.37), (S.38), and (S.41)

Taking derivatives of $m', \nu', m, P'/P$ we obtain

$$\begin{aligned}
\frac{dm'(\nu,a)}{d\nu} &= \varsigma^a \left(g(\nu'(\nu,a)/\log q_a \mid a) \frac{d\nu'(\nu,a)}{d\nu} \frac{1}{\log q_a} - g(\nu/\log q_a \mid a) \frac{1}{\log q_a} \right) \\
\frac{d\nu'(\nu,a)}{d\nu} &= 1 + \frac{d\log(P'/P)}{d\nu} \\
\frac{dm(\nu,a)}{d\nu} &= \varsigma^a g(\nu/\log q_a \mid a) \frac{1}{\log q_a} = \varsigma^a \frac{q_a^{(\mu/\pi)\nu/\log q_a}}{\varphi(q_a,\mu/\pi)} \\
\frac{d\log(P'/P)}{d\nu} &= \frac{1}{1-\eta} \left(\frac{P'}{P}\right)^{\eta-1} \left(\sum_a \left[(p_a^*)^{1-\eta} \frac{dm(\nu,a)}{d\nu} - \varsigma^a (\underline{p}_a e^\nu)^{1-\eta} f(\underline{p}_a e^\nu \mid a) \underline{p}_a e^\nu \right] \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\left.\frac{d\log(P'/P)}{d\nu}\right|_{\nu=0} &= \frac{1}{1-\eta} \sum_a \varsigma^a \frac{(p_a^*)^{1-\eta} - (\underline{p}_a)^{1-\eta}}{\varphi(q_a,\mu/\pi)} = \sum_a \varsigma^a \frac{\varphi(q_a, 1-\eta)}{\varphi(q_a,\mu/\pi)} \underline{p}_a^{1-\eta} \\
\left.\frac{dm'(\nu,a)}{d\nu}\right|_{\nu=0} &= \frac{\varsigma^a}{\varphi(q_a,\mu/\pi)} \left.\frac{d\log(P'/P)}{d\nu}\right|_{\nu=0} \\
\left.\frac{dm(\nu,a)}{d\nu}\right|_{\nu=0} &= \frac{\varsigma^a}{\varphi(q_a,\mu/\pi)}.
\end{aligned} \tag{S.42}$$

Substituting these results into θ, we obtain (S.37).

Also, from (S.33) we obtain

$$\mu \sum_{a=a^L,a^H} \varsigma^a \left(\left((p_a^*)^{1-\eta} - 1\right) + \frac{(p_a^*)^{1-\eta} - \underline{p}_a^{1-\eta}}{q_a^{\mu/\pi} - 1} \right) = \pi(1-\eta),$$

which implies

$$\left(1 - \sum_a \varsigma^a \frac{\varphi(q_a, 1-\eta)}{\varphi(q_a,\mu/\pi)} \underline{p}_a^{1-\eta} \right) \pi = \mu \sum_a \varsigma^a \varphi(p_a^*, 1-\eta).$$

Combining this with (S.37), we obtain (S.38).

In order to derive (S.41), we first characterize $\theta_0(s)$. Let m_c denote a measure of firms who receive Calvo shocks with s and $\nu_0(s)$ the increase in $\log P$ caused by the repricing of the m_c firms. By the definition of $\nu_0(s)$ and the zero-profit condition (S.24), we have

$$e^{\nu_0(s)} = \left(\sum_a \varsigma^a \left[\int_{\underline{p}_a}^{p_a^*} p^{1-\eta} f(p \mid a) dp + m_c \left((p_a^*)^{1-\eta} - (\underline{p}_a q_a^s)^{1-\eta}\right)\right]\right)^{1/(1-\eta)}$$
$$= \left(1 + m_c \sum_a \varsigma^a ((p_a^*)^{1-\eta} - (\underline{p}_a q_a^s)^{1-\eta})\right)^{1/(1-\eta)}.$$

Hence,

$$\left.\frac{d\nu_0(s)}{dm_c}\right|_{m_c=0} = \sum_a \varsigma^a \frac{(p_a^*)^{1-\eta} - (\underline{p}_a q_a^s)^{1-\eta}}{1-\eta}.$$

Thus, combined with (S.42), this yields

$$\theta_0(s) = \lim_{m_c \to 0} \frac{\sum_a m(\nu_0(s), a)/\nu_0(s)}{m_c/\nu_0(s)} = \sum_{a'} \frac{\varsigma^{a'}}{\varphi(q_{a'}, \mu/\pi)} \sum_a \varsigma^a \frac{(p_a^*)^{1-\eta} - (\underline{p}_a q_a^s)^{1-\eta}}{1-\eta}$$
$$= \frac{\lambda}{\pi} \sum_a \varsigma^a \frac{(p_a^*)^{1-\eta} - (\underline{p}_a q_a^s)^{1-\eta}}{1-\eta}, \tag{S.43}$$

where the last equality holds by (S.22) and (S.23). Also, using $E^{F_p}[p^{1-\eta}] = 1$ we obtain

$$\theta_0 = E^{G_s}[\theta_0(s)] = \sum_a \frac{\varsigma^a}{\varphi(q_a, \mu/\pi)} \sum_{a'} \varsigma^{a'} \varphi(p_{a'}^*, 1-\eta). \tag{S.44}$$

Using (S.22) and (S.38) with the above equation, we obtain (S.41) as $\lambda = \sum_a \varsigma^a \lambda_a = \theta_0 \mu/(1-\theta)$.

B.9.2 As $\pi \to \infty$, $\log q_a \to \infty$ and $p_a^* \to \infty$ for each a

First, we show that $\log q_a(\pi) \to \infty$ as $\pi \to \infty$. Suppose to the contrary that there exists a sequence $\pi_n \to \infty$ such that $\log q_{a,n} = q_a(\pi_n)$ is bounded above. Then $A(q_{a,n}, \pi_n)$ converges to 1 as $\pi_n \to \infty$. Hence, (S.29) implies that $B(q_{a,n}, \pi_n)$ for $a \in \{a^L, a^H\}$ must converge to 0 as $\pi_n \to \infty$. The numerator of B, $\varphi(q, 1-\eta+\mu/\pi)$, is strictly positive and increasing in q for $q > 1$. Therefore, the denominator must tend to infinity. However, $\varphi(q, 1-\eta)$ is bounded. Thus, $\varphi(q_{a,n}, \mu/\pi_n)$ must tend to infinity. For a fixed $\log q$, we have $\lim_{\pi \to \infty}(e^{\mu(\log q)/\pi} - 1)/(\mu/\pi) = \lim_{\pi \to \infty}(e^{\mu(\log q)/\pi} \mu(\log q)/(-\pi^2))/(\mu/(-\pi^2)) = \log q$. Since $\log q_{a,n}$ is bounded by our hypothesis, this contradicts the divergence of $\varphi(q_{a,n}, \mu/\pi_n)$. Hence, $\log q_a(\pi) \to \infty$ as $\pi \to \infty$.

From (S.24), we have

$$1 = \sum_a \frac{\varphi(q_a, \eta - 1 - \mu/\pi)}{\varphi(q_a, -\mu/\pi)} (p_a^*)^{1-\eta} \varsigma^a.$$

Since $\lim_{\pi\to\infty} \log q_a = \infty$, the first fraction $\to \infty$ as $\pi \to \infty$ for each a. Hence, $p_a^* \to \infty$ as $\pi \to \infty$ for each a.

B.10 Complementarity of repricing at the extensive margin and derivation of the first two moments of L.

As shown in the baseline model, an economy with finite n firms converges to the continuum economy in the steady state as $n \to \infty$. In this section, we focus on establishing that the repricing avalanche has the same fluctuation properties as in the baseline model, in particular the key Proposition 7(d).

Suppose the aggregate good price is $P_{t\text{-}}$ and that a single firm i changes its (relative) price to $p^*_{a^i_t}$ at t. Then $\Delta \log P^i_t := \log p^*_{a^i_t} - \log p^i_{t\text{-}}$. We consider the case of a price increase, whereas the case of price cut can be analyzed similarly. To compute the impact of firm i's price-change on the final-good price P_t, recall that

$$\Delta \log P_t = \frac{(p^i_{t\text{-}})^{1-\eta}}{n} \frac{e^{(1-\eta)\Delta \log P^i_t} - 1}{1-\eta} - \epsilon_P(s^i_{t\text{-}}), \tag{S.45}$$

where $\epsilon_P(s^i_{t\text{-}}) = O(n^{-2})$.

To analyze the avalanche initiated by a Calvo shock it is again helpful to work with the normalized price $s^i_{t\text{-}} = (\log p^i_{t\text{-}} - \log \underline{p}_{a^i_t})/\log q_{a^i_t}$ whose stationary distribution is (S.36). Fix a time t that, to simplify notation, we temporarily omit, and choose s^1 using the distribution $g(s)$ and assume that firm 1 receives a Calvo shock. In addition, we draw independently $n-1$ other firms and thus $(s^\ell)_{\ell=2}^n$ is distributed as $g^{n-1}(s)$. Equation (S.45) implies that s^j for $j > 1$ decreases by

$$\epsilon_0^{a^j} = \frac{1}{\log p^*_{a^j} - \log \underline{p}_{a^j}} \left[\frac{1}{n} \frac{(p^*_{a^1})^{1-\eta} - (p^1)^{1-\eta}}{1-\eta} - \epsilon_P(s^1) \right]. \tag{S.46}$$

Set $m_0^a = \# M_0^a$, where $M_0^a := \{j > 1 : a^j = a \text{ and } s^j \le \epsilon_0^a\}$ is the set of firms in state a that choose to reprice because firm 1 repriced. Write $G(\cdot|a)$ for the cumulative distribution with density $g(\cdot|a)$. Then, m_0^a conditional on s^1 follows a binomial distribution with population $n-1$ and probability $\kappa_0^a := \varsigma^a G(\epsilon_0^a|a)$.

Notice that s^ℓ for $\ell \notin \cup_a M_0^a$, $\ell > 1$, and $a^\ell = a'$ decreases by $\epsilon_0^{a'} + \sum_{j\in\cup_a M_0^a} \tilde{\epsilon}_1^{j,a'}$, with

$$\tilde{\epsilon}_1^{j,a'} := \epsilon_1^{a^j,a'} - \epsilon'_P(s^j)$$

where

$$\epsilon_1^{a^j,a'} := \frac{1}{n} \frac{\underline{p}_{a^j}^{1-\eta}}{\log p^*_{a'} - \log \underline{p}_{a'}} \frac{(p^*_{a^j}/\underline{p}_{a^j})^{1-\eta} - 1}{1-\eta} \tag{S.47}$$

$$\epsilon'_P(s^j) := \frac{1}{\log p^*_{a'} - \log \underline{p}_{a'}} \left[\frac{\underline{p}_{a^j}^{1-\eta}}{n} \frac{1 - (p^*_{a^j}/\underline{p}_{a^j})^{(1-\eta)s^j}}{1-\eta} + \epsilon_P(s^j) \right] > 0.$$

Since $0 < s^j < \epsilon_0^{a^j} = O(n^{-1})$, the first term in the square brackets is also of $O(n^{-2})$.

The set,

$$M_1^a := \{\ell > 1 : a^\ell = a \text{ and } s^\ell \leq \epsilon_0^a + \sum_{j \in \cup_{a'} M_0^{a'}} \tilde{\epsilon}_1^{j,a}\} \setminus M_0^a,$$

is the set of firms in state a that reprice because the Calvo shock led firm 1 and firms in $\cup_a M_0^a$ to reprice. Random variable $m_1^a = \# M_1^a$ is distributed according to a binomial distribution with population $n - 1 - \sum_{a'} m_0^{a'}$ and probability:

$$\kappa_1^a := \varsigma^a \left(G\left(\epsilon_0^a + \sum_{j \in \cup_{a'} M_0^{a'}} \tilde{\epsilon}_1^{j,a} \mid a \right) - G\left(\epsilon_0^a \mid a\right) \right).$$

In turn, the price changes of the firms in $\cup_{a'} M_1^{a'}$ cause an additional set of firms to change prices and we write,

$$M_2^a := \{\ell > 1 : a^\ell = a \text{ and } s^\ell \leq \epsilon_0^a + \sum_{j \in \cup_{a'} (M_0^{a'} \cup M_1^{a'})} \tilde{\epsilon}_1^{j,a}\} \setminus (M_0^a \cup M_1^a),$$

for the set of firms that react to the price changes by the firms in $\cup_{a'} M_1^{a'}$, and $m_2^a = \# M_2^a$ and so forth.

Conditional on $((m_k^{a'})_{a'})_{k=0}^u$, m_{u+1}^a follows a binomial distribution with population $n - 1 - \sum_{k=0}^u \sum_{a'} m_k^{a'}$ and probability κ_u^a where,

$$\kappa_u^a := \varsigma^a \left(G\left(\epsilon_0^a + \sum_{j \in \bigcup_{k=0}^u \cup_{a'} M_k^{a'}} \tilde{\epsilon}_1^{j,a} \mid a \right) - G\left(\epsilon_0^a + \sum_{j \in \bigcup_{k=0}^{u-1} \cup_{a'} M_k^{a'}} \tilde{\epsilon}_1^{j,a} \mid a \right) \right). \tag{S.48}$$

Write $m_u := \sum_a m_u^a$. The total size of the avalanche initiated by a Calvo shock at s^1 is given by,

$$L^n = \sum_{u=0}^{\infty} m_u.$$

Notice that if $m_U = 0$ then (S.48) implies that $m_u^a = 0$ for each a and $u > U$, that is the avalanche stops whenever $m_U = 0$.

It is well known that the distribution of properly normalized binomials converge to the distribution of a Poisson as the number of observations $n \to \infty$. The asymptotic mean of m_0 conditional on s^1, $\lim_{n\to\infty}(n-1)\sum_a \varsigma^a G(\epsilon_0^a|a)$, is derived as, using (S.23), (S.36), and (S.46),

$$\lim_{n\to\infty}(n-1)\sum_a \varsigma^a g(0|a)\epsilon_0^a = \frac{\lambda}{\pi}\frac{(p_{a^1}^*)^{1-\eta}-(p^1)^{1-\eta}}{1-\eta}.$$

The right-hand side coincides with $\theta_0(s^1)$ defined in (S.39) and derived in (S.43). Hence, m_0 conditional on s^1 asymptotically follows a Poisson distribution with mean $\theta_0(s^1)$. Using the zero-profit condition (13) and the formula (S.44) for θ_0, we obtain

$$\lim_{n\to\infty} E^{F_p}[m_0] = E^{G_s}[\theta_0(s^1)] = \frac{\lambda}{\pi}\sum_{a'}\varsigma^{a'}\varphi(p_{a'}^*, 1-\eta) = \theta_0,$$

that is, the asymptotic unconditional mean of m_0 equals θ_0.

Conditional on $((m_k^a)_a)_{k=0}^u$, the asymptotic mean of the binomial m_{u+1} is obtained by using (S.36), (S.47), and (S.48) as,

$$\lim_{n\to\infty} E[m_{u+1} \mid ((m_k^a)_a)_{k=0}^u] = \lim_{n\to\infty}(n-1-\sum_{k=0}^u m_k)\sum_a \kappa_u^a = \frac{\lambda}{\pi}\sum_a \underline{p}_a^{1-\eta}\varphi(q_a, 1-\eta)m_u^a.$$

Thus, as $n\to\infty$, m_{u+1} conditional on $(m_u^a)_a$ asymptotically follows a sum of m_u^a-fold convolution of a Poisson distribution with mean $\theta_a := (\lambda/\pi)\underline{p}_a^{1-\eta}\varphi(q_a, 1-\eta)$.

Conditional on m_u, m_u^a follows a binomial distribution with population m_u and probability $\kappa_{u-1}^a/\sum_{a'}\kappa_{u-1}^{a'}$.[29] The probability of the binomial converges as $n\to\infty$ to $\hat{\omega}_a := \varsigma^a\lambda_a/\lambda$. The mean of m_u^a conditional on m_u thus satisfies

$$\lim_{n\to\infty} E[m_u^a \mid m_u] = \hat{\omega}_a m_u.$$

We also note that $\sum_a \hat{\omega}_a\theta_a = \theta$ holds, where θ is the complementarity measure defined in (S.37). Hence, by the law of iterated expectation, we obtain

$$\lim_{n\to\infty} E[m_{u+1} \mid m_u] = \lim_{n\to\infty} E[E[m_{u+1} \mid ((m_k^a)_a)_{k=0}^u, m_u] \mid m_u] = \theta m_u.$$

Note that m_{u+1} conditional on m_u is an m_u-fold convolution of a mixed Poisson distribution in which the mean of a Poisson distribution θ_a is randomly drawn with probability $\hat{\omega}_a$

[29] In case where the number of productivity states is greater than two, the following argument still holds with a multinomial distribution replacing the binomial distribution.

for $a = a^L, a^H$. The mixed Poisson distribution has mean θ and variance $\theta + \sigma^2_{\theta_a}$ where $\sigma^2_{\theta_a}$ denotes the variance of θ_a.[30]

Hence, in the limit as $n \to \infty$, m_{u+1} follows a branching process in which the number of children per parent follows the mixed Poisson distribution. The total progeny of the branching process, L^n, represents the stochastic size of repricing avalanche. We let L denote the limit of L^n as $n \to \infty$. Writing $V^F(x)$ for the variance of the random variable x under the stationary distribution F, we can prove that:

Proposition B.3. *The unconditional mean and variance of L, the size of an avalanche initiated by a Calvo shock, is, respectively, $\theta_0/(1-\theta)$ and*

$$\frac{\theta_0(1+\sigma^2_{\theta_a})}{(1-\theta)^3} + \frac{\sigma^2_{\theta_0}}{(1-\theta)^2}, \quad \textit{where}$$

$$\sigma^2_{\theta_a} = \frac{\lambda}{\pi}\left(\sum_a \varsigma^a \frac{(\underline{p}_a^{1-\eta}\varphi(q_a, 1-\eta))^2}{\varphi(q_a, \mu/\pi)}\right) - \theta^2,$$

$$\sigma^2_{\theta_0} = \left(\frac{\lambda/\pi}{1-\eta}\right)^2 V^{G_s}\big(\sum_a \varsigma^a \underline{p}_a^{1-\eta} q_a^{s(1-\eta)}\big).$$

Proof: See B.11.

B.11 Proof of Proposition B.3

First we extend Lemma 10 to the two-productivity case where we show that m_u follows a branching process with a mixed Poisson distribution with mean θ and variance $\theta + \sigma^2_{\theta_a}$.

The proof proceeds similarly to that of Lemma 10. The probability generating function $\Psi(z)$ of the sum L of a branching process with initial value 1 is a fixed point of a functional equation $\Psi(z) = z\Phi(\Psi(z))$, where Φ is a probability generating function of the mixed Poisson distribution. Thus, the mean and variance of the mixed Poisson correspond to $\Phi'(1) = \theta$ and $\Phi''(1) + \Phi'(1) - (\Phi'(1))^2 = \theta + \sigma^2_{\theta_a}$, respectively. Hence, $\Phi''(1) = \theta^2 + \sigma^2_{\theta_a}$.

Using the functional equation, we obtain $\Psi'(z) = \Phi(\Psi(z)) + z\Phi'(\Psi(z))\Psi'(z)$ and $\Psi''(z) = 2\Phi'(\Psi(z))\Psi'(z) + z\Phi''(\Psi(z))(\Psi'(z))^2 + z\Phi'(\Psi(z))\Psi''(z)$. Evaluating at $z = 1$, we obtain

$$E[L \mid m_0 = 1] = \Psi'(1) = \frac{1}{1-\theta}.$$

For $\Psi''(1)$, we have

$$\Psi''(1) = \frac{1}{1-\theta}\left(\frac{2\theta}{1-\theta} + \frac{\Phi''(1)}{(1-\theta)^2}\right) = \frac{2\theta}{(1-\theta)^2} + \frac{\theta^2+\sigma^2_{\theta_a}}{(1-\theta)^3}.$$

[30] Letting X denote the mixed Poisson, its variance is decomposed as $\mathrm{V}(E[X \mid a]) + E[\mathrm{V}(X \mid a)]$. Since X conditional on a follows a Poisson distribution with mean θ_a, we have $E[X \mid a] = V(X \mid a) = \theta_a$. Hence the variance of X is $\sigma^2_{\theta_a} + E[\theta_a]$. Since $E[\theta_a] = \sum_a \hat{\omega}\theta_a = \theta$, we obtain the result.

Hence, the variance of L conditional on $m_0 = 1$ is

$$V(L \mid m_0 = 1) = \Psi''(1) + \Psi'(1) - (\Psi'(1))^2 = \frac{2\theta - \theta^2 + \sigma^2_{\theta_a}}{(1-\theta)^3} + \frac{1}{1-\theta} - \frac{1}{(1-\theta)^2}$$
$$= \frac{\theta + \sigma^2_{\theta_a}}{(1-\theta)^3}.$$

Next, using (S.41) and the above result, we obtain

$$E[L] = E^{F_p}[E[L \mid m_0]] = E^{F_p}[m_0 E[L \mid m_0 = 1]] = E^{F_p}\left[\frac{m_0}{1-\theta}\right] = \frac{\theta_0}{1-\theta},$$

which is equal to λ/μ. Denoting $\sigma^2_{\theta_0} := V^{G_s}(\theta_0(s))$, we have

$$V(m_0) = E^{G_s}[V(m_0 \mid s)] + V^{G_s}(E[m_0 \mid s]) = E^{G_s}[\theta_0(s)] + V^{G_s}(\theta_0(s)) = \theta_0 + \sigma^2_{\theta_0}.$$

Using this, we obtain

$$\begin{aligned}
V(L) &= E^{G_s}\left[V(L \mid m_0)\right] + V^{G_s}\left(E\left[L \mid m_0\right]\right) \\
&= E^{G_s}[m_0 V(L \mid m_0 = 1)] + V^{G_s}(m_0 E[L \mid m_0 = 1]) \\
&= E^{G_s}[m_0] V(L \mid m_0 = 1) + V^{G_s}(m_0) E[L \mid m_0 = 1]^2 \\
&= \frac{\theta_0(\theta + \sigma^2_{\theta_a})}{(1-\theta)^3} + \frac{\theta_0 + \sigma^2_{\theta_0}}{(1-\theta)^2} = \frac{\theta_0(1 + \sigma^2_{\theta_a})}{(1-\theta)^3} + \frac{\sigma^2_{\theta_0}}{(1-\theta)^2}.
\end{aligned}$$

Finally,

$$\sigma^2_{\theta_a} = \sum_a \hat{\omega}_a \theta_a^2 - (\sum_a \hat{\omega}_a \theta_a)^2 = \frac{\lambda}{\pi}\left(\sum_a \varsigma^a \frac{(\underline{p}_a^{1-\eta} \varphi(q_a, 1-\eta))^2}{\varphi(q_a, \mu/\pi)}\right) - \theta^2.$$

Also, using (S.43), we obtain

$$\sigma^2_{\theta_0} = \left(\frac{\lambda/\pi}{1-\eta}\right)^2 V^{G_s}(\sum_a \varsigma^a((p_a^*)^{1-\eta} - \underline{p}_a^{1-\eta} q_a^{s(1-\eta)})) = \left(\frac{\lambda/\pi}{1-\eta}\right)^2 V^{G_s}(\sum_a \varsigma^a \underline{p}_a^{1-\eta} q_a^{s(1-\eta)}).$$

Remark: If $a^L = a^H$, then $\sigma^2_{\theta_a} = 0$, and $\sigma^2_{\theta_0}$ is reduced to $\sigma^2_{\theta_0}$ defined in Proposition 7(d). Hence the formula for the variance of L shown in Proposition B.3 is a generalization of that in Proposition 7(d).

B.12 λ converges to a non-zero constant when $\pi \to \infty$

Function $\varphi(q,x) = (q^x - 1)/x$ for fixed $q > 1$ is analytic in region $x < 0$, and so is $\log \varphi(q,x)$. Thus, a Taylor series expansion of $\log \varphi(q,x)$ around $x = -\eta$ yields

$$\log \varphi(q, -\eta + u) - \log \varphi(q, -\eta) = u \frac{\varphi_x}{\varphi}(q, -\eta) + \frac{\partial}{\partial x} \frac{\varphi_x}{\varphi}(q,x)\bigg|_{x=x_1} \frac{u^2}{2}$$

for $|u| < 1$ where $x_1 \in [-\eta, -\eta + u]$. Similar expansion around $x = 1 - \eta$ yields, for some $x_2 \in [1-\eta, 1-\eta+u]$,

$$\log \varphi(q, 1 - \eta + u) - \log \varphi(q, 1 - \eta) = u \frac{\varphi_x}{\varphi}(q, 1-\eta) + \frac{\partial}{\partial x} \frac{\varphi_x}{\varphi}(q,x)\bigg|_{x=x_2} \frac{u^2}{2}.$$

Note that, from Lemma 4(b), the coefficients of u^2 terms in the above two equations are uniformly bounded for $q \in [1, \infty]$. Moreover, Lemma 4(b) gives

$$\frac{\varphi_x}{\varphi}(q, 1-\eta) - \frac{\varphi_x}{\varphi}(q, -\eta) = \int_{-\eta}^{1-\eta} \frac{1}{x^2} \left(1 - \frac{q^x (\log q^x)^2}{(q^x - 1)^2}\right) dx. \tag{S.49}$$

Using again the notation $u = (\rho + \mu)/\pi$, we obtain the first-order Taylor expansion of $\log A$ around $u = 0$ as

$$\begin{aligned} \log A(q,\pi) &= \log \varphi(q, 1-\eta+u) - \log \varphi(q, 1-\eta) - \log \varphi(q, -\eta+u) + \log \varphi(q, -\eta) \\ &= \left(\frac{\varphi_x}{\varphi}(q, 1-\eta) - \frac{\varphi_x}{\varphi}(q, -\eta)\right) u + O(u^2). \end{aligned}$$

Combining with (S.49), we obtain

$$\pi \log A(q,\pi) = (\rho + \mu) \int_{-\eta}^{1-\eta} \frac{1}{x^2} \left(1 - \frac{q^x (\log q^x)^2}{(q^x-1)^2}\right) dx + O(u).$$

From B.9.2, $\lim_{\pi\to\infty} q_a(\pi) = \infty$. Thus, $\lim_{\pi\to\infty} q_a^x(\pi)(\log q_a^x(\pi))^2/(q_a^x(\pi)-1)^2 = 0$ for $x < 0$. Also note $\int_{-\eta}^{1-\eta} 1/x^2 dx = 1/(\eta(\eta-1))$. Thus, $\lim_{\pi\to\infty} \pi \log A(q_a(\pi), \pi) = (\rho+\mu)/(\eta(\eta-1))$. This implies $\lim_{\pi\to\infty} A(q_a(\pi),\pi) = 1$. Hence, $\lim_{\pi\to\infty}(A(q_a(\pi),\pi) - 1)/\log A(q_a(\pi),\pi) = 1$ by l'Hôpital's rule. Combining with the above result yields

$$\pi(A(q_a(\pi),\pi) - 1) \to \frac{\rho + \mu}{\eta(\eta - 1)} \quad \text{as } \pi \to \infty. \tag{S.50}$$

Moreover, a Taylor series expansion of $\varphi(q, 1-\eta+\mu/\pi)$ around $x = 1-\eta$ yields, for some $x_3 \in [1-\eta, 1-\eta+\mu/\pi]$,

$$\varphi(q, 1-\eta+\mu/\pi) = \varphi(q, 1-\eta) + \frac{1}{x_3^2}(x_3^2 q^{x_3} \log(q) - (q^{x_3} - 1))\frac{\mu}{\pi}.$$

Since $x_3 < 0$ for sufficiently large π such that $1 - \eta + \mu/\pi < 0$, the final term tends to 0 as $q \to \infty$. Thus we have,

$$B(q_a(\pi), \pi) = \frac{\varphi(q_a(\pi), 1 - \eta + \mu/\pi)}{\varphi(q_a(\pi), \mu/\pi)\varphi(q_a(\pi), 1 - \eta)} \sim \frac{1}{\varphi(q_a(\pi), \mu/\pi)} \quad \text{as } \pi \to \infty. \tag{S.51}$$

Applying (S.50) and (S.51) to (S.29) and using (S.23), we obtain

$$1 = \frac{\delta^+(\rho + \mu)}{\eta - 1} \sum_a \varsigma^a \frac{B(q_a(\pi), \pi)}{A(q_a(\pi), \pi) - 1} \sim \delta^+ \eta \lambda \quad \text{as } \pi \to \infty.$$

Hence, λ converges to $1/(\delta^+ \eta) > 0$ as $\pi \to \infty$.