Appendix A Proofs

Proof of Lemma 1. By definition,

$$\begin{aligned} EV^m &= \log \frac{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_0}, v(p_{t_0}, I_{t_0}; x_{t_1}); x_{t_1}\right)} \\ &= \log \frac{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_0}, v(p_{t_0}, I_{t_0}; x_{t_1}); x_{t_1}\right)} \frac{e\left(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)} \\ &= \log \frac{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{I_{t_0}} \frac{I_{t_1}}{e\left(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}. \end{aligned}$$

To finish, rewrite

$$\log \frac{e\left(p_{t_0}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p_{t_1}, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1}\right)} = -\int_{t_0}^{t_1} \frac{\partial \log e(p, v(p_{t_1}, I_{t_1}; x_{t_1}); x_{t_1})}{\partial \log p} \frac{d \log p}{dt} dt,$$

and use the Shephard's lemma to express the price elasticity of the expenditure function in terms of budget shares. If the path of prices between t_0 and t_1 is not differentiable, then construct a new a modified path of prices that is differentiable, and apply the integral to this modified path. Since the integral is path independent, it only depends on p_{t_0} and p_{t_1}. Therefore any path that connects p_{t_0} and p_{t_1} gives the same integral. □

Proof of Proposition 1. If the path of prices is continuously differentiable, we can combine Lemma 1 with the definition of real consumption. □

Proof of Corollary 1. From Proposition 1, given any change in prices, income, and preferences, $EV^m - \Delta \log Y = 0$ if, and only if, $b = b^{ev}$, which is the case if, and only if, budget shares do not depend on the level of utility and x – that is preferences are homothetic and stable. □

Proof of Lemma 2. Differentiate real consumption

$$\Delta \log Y = \int_{t_0}^{t_1} d \log I(t) - \sum_{i \in N} b_i(p(t), u(t); x(t)) \frac{d \log p_i}{dt} dt$$

twice with respect to t_1 and evaluate the derivative at $t_1 = t_0$. This yields the desired expression. □

Proof of Proposition 2. By Lemma 1:

$$EV = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} \frac{\partial \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p} \frac{d \log p}{d \log t} d \log t$$

Differentiate EV twice with respect to t_1 and evaluate the derivative at $t_1 = t_0$

$$\begin{aligned}
\frac{dEV}{dt_1} &= \frac{d \log I}{dt} - \sum_{i \in N} \frac{\partial \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p_i} \frac{d \log p_i}{d \log t} - \\
&\quad - \int_{t_0}^{t_1} \sum_{i \in N} d \log v \frac{\partial^2 \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log u \partial \log p_i} d \log p_i \\
&\quad - \int_{t_0}^{t_1} \sum_{i \in N} d \log x \frac{\partial^2 \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log x \partial \log p_i} d \log p_i \\
\frac{d^2 EV}{dt_1^2} &= - \sum_{i \in N} b_i \frac{d^2 \log p}{d \log t^2} - \sum_{i \in N} \sum_{j \in N} \frac{\partial^2 \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p_i \partial \log p_j} d \log p_i d \log p_j \\
&\quad - 2 \sum_{i \in N} d \log v \frac{\partial^2 \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p_i \partial \log u} d \log p_i - 2 \sum_{i \in N} d \log x' \frac{\partial \log e(p, v(p_{t_1}, x_{t_1}), x_{t_1})}{\partial \log p_i \partial \log x} d \log p_i \\
&= - \sum_{i \in N} \sum_{j \in N} \frac{\partial b_i}{\partial \log p_i} d \log p_i d \log p_j - \sum_{i \in N} b_i \frac{d^2 \log p}{d \log t^2} \\
&\quad - 2 \sum_{i \in N} d \log v \frac{\partial b_i}{\partial \log u} d \log p_i - 2 \sum_{i \in N} d \log x' \frac{\partial b_i}{\partial \log x} d \log p_i \\
&= - \sum_{i \in N} \left[\sum_{j \in N} \frac{\partial b_i}{\partial \log p_j} d \log p_j + d \log v \frac{\partial b_i}{\partial \log u} + d \log x' \frac{\partial b_i}{\partial \log x} \right] d \log p_i - \sum_{i \in N} b_i \frac{d^2 \log p}{d \log t^2} \\
&\quad - \sum_{i \in N} d \log v \frac{\partial b_i}{\partial \log u} d \log p_i - \sum_{i \in N} d \log x' \frac{\partial b_i}{\partial \log x} d \log p_i \\
&= - \sum_{i \in N} db_i d \log p_i - \sum_{i \in N} b_i \frac{d^2 \log p}{d \log t^2} - \sum_{i \in N} d \log v \frac{\partial b_i}{\partial \log u} d \log p_i - \sum_{i \in N} d \log x' \frac{\partial b_i}{\partial \log x} d \log p_i
\end{aligned}$$

By Lemma 2, the first two terms are equal to the second-order expansion of $\Delta \log Y$, and the remaining terms are the bias. □

Proof of Proposition 3. By Lemma 2, we have

$$\Delta \log Y \approx \Delta \log I - \sum_i b_i \Delta \log p_i - \frac{1}{2} \sum_i \Delta b_i \Delta \log p_i.$$

Substitute (9) in place of Δb to get the desired expression. For the bias, note that Proposi-

tion 1 implies that

$$EV - \Delta \log Y \approx -\frac{1}{2} \sum_i \left[\Delta b_i - \sum_j \frac{\partial b_i^H}{\partial \log p_j} \Delta \log p_j \right] \Delta \log p_i$$

where b^H is the Hicksian budget share (holding fixed utility and demand shifters). Using (9) in place of Δb above and the fact that $\frac{\partial b_i^H}{\partial \log p_i} = (1 - \theta_0) b_i (1 - b_i)$ for $i = j$ and $\frac{\partial b_i^H}{\partial \log p_j} = \theta_0 b_i b_j$ for $i \neq j$, yields the following

$$\Delta \log EV - \Delta \log Y \approx -\frac{1}{2} \sum_{i \in N} \left[(\varepsilon_i - 1) b_i \left(d \log I - \sum_{j \in N} b_j \Delta \log p_j \right) + b_i \Delta \log x_i \right] \Delta \log p_i,$$

which can be rearranged to give the desired expression. □

Proof of Lemma 3. Setting nominal GDP to be the numeraire, we can write

$$\begin{aligned}
\Delta \log Y &= -\int_{t_0}^{t_1} b' d \log p \\
&= -\int_{t_0}^{t_1} b' \left[-\Psi d \log A + \Psi^F d \log \Lambda \right] \\
&= \int_{t_0}^{t_1} b' \Psi d \log A - \int_{t_0}^{t_1} b' \Psi^F d \log \Lambda \\
&= \int_{t_0}^{t_1} \lambda' d \log A - \int_{t_0}^{t_1} \Lambda d \log \Lambda \\
&= \int_{t_0}^{t_1} \lambda' d \log A
\end{aligned}$$

where the second line uses Proposition 7, and we use the fact that Using $\lambda' = b' \Psi$, $\Lambda' = b' \Psi^F$, and $b' \Psi^F d \log \Lambda = \Lambda' d \log \Lambda = 0$ because the factor shares always sum to one: $\sum_{f \in F} \Lambda_f = 1$. □

Proof of Proposition 4. Similarly to Lemma 1, EV^M can be expressed as

$$\begin{aligned}
EV^M &= \log \frac{e\left(p(A_{t_0}, x_{t_1}), v(p(A_{t_1}, x_{t_1}), I(A_{t_1}, x_{t_1}); x_{t_1}); x_{t_1}\right)}{e\left(p(A_{t_0}, x_{t_1}), v(p(A_{t_0}, x_{t_1}), I(A_{t_0}, x_{t_1}); x_{t_1}); x_{t_1}\right)} \\
&= \Delta \log I - \log \frac{e\left(p(A_{t_1}, x_{t_1}), v_{t_1}; x_{t_1}); x_{t_1}\right)}{e\left(p(A_{t_0}, x_{t_1}), v_{t_1}; x_{t_1}); x_{t_1}\right)}
\end{aligned}$$

where $v_{t_1} \equiv v(p(A_{t_1}, x_{t_1}), I(A_{t_1}, x_{t_1}); x_{t_1})$.

Define a hypothetical economy with fictional households that have stable homoth-

etic preferences defined by the expenditure function $e^{ev}(p,u) = e(p, v_{t_1}; x_{t_1}) \frac{u}{v_{t_1}}$. Budget shares of this fictional consumer are $b_i^{ev}(p) \equiv \frac{\partial e^{ev}(p,u)}{\partial p_i} = \frac{\partial e\left(p, v_{t_1}; x_{t_1}\right)}{\partial p_i}$. Given any technology vector, in this hypothetical economy we denote the Leontief inverse matrix by Ψ^{ev} and sales shares by λ^{ev}. Prices in this hypothetical economy are $p_t^{ev} \equiv p(A_t, x_{t_1})$, and changes in prices satisfy

$$d \log p^{ev} = -\Psi^{ev} d \log A + \Psi^{evF} d \log \Lambda^{ev}. \tag{24}$$

Re-write EV^M (given the constant income normalization) as

$$EV^M = -\int_{t_0}^{t_1} \sum_{i \in N+F} \frac{\partial \log e(p, v_{t_1})}{\partial \log p_i} d \log p_i^{ev} = -\int_{t_0}^{t_1} \sum_{i \in N+F} b_i^{ev} d \log p_i^{ev}.$$

Following the same steps as in Lemma 3, we obtain

$$EV^M = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i^{ev} d \log A_i.$$

□

As mentioned earlier, in general, macro and micro welfare changes are not the same when preferences are unstable. However, when the PPF is linear, the two coincide.

Proposition 10 (Macro vs. Micro Welfare)**.** *When preferences are unstable, macro and micro welfare changes are equal ($EV^m = EV^M$) if factor income shares are constant (as in a one factor economy).*

Proof of Proposition 10. If factor income shares, Λ, are constant, then by Proposition 7, changes in prices in response to changes in A and x are given by the following differential equation:

$$d \log p = -\Psi d \log A.$$

Furthermore, note that changes in Ψ are determined by changes in Ω since $\Psi = (I - \Omega)^{-1}$. Since every $i \in N$ has constant returns to scale, changes in Ω_{ij} depend only on changes in relative prices for every $i \in N$. This means that changes in Ω only depend on changes in relative prices, therefore changes in Ψ depend only on changes in relative prices. Since x does not appear in any of these expressions, this means that prices and incomes $p(A, x)$ and $I(A, x)$, relative to the numeraire, do not depend on x. Thus, $p(A_t, x_{t_1})/I(A_t, x_{t_1}) = p(A_t, x_{t_0})/I(A_t, x_{t_0})$. This implies that $EV^m(p_{t_0}, I_{t_0}, p_{t_1}, I_{t_1}; x_{t_1}) = EV^M(A_{t_0}, A_{t_1}; x_{t_1})$ from the definitions. □

Proof of Lemma 4. Differentiate real GDP,

$$\Delta \log Y = \int_{t_0}^{t_1} \sum_{i \in N} \lambda_i(A(t); x(t)) \frac{d \log A_i}{dt} dt,$$

twice with respect to t_1 and evaluate the derivative at $t_1 = t_0$. This yields the desired expression. □

Proof of Proposition 5. Following similar steps as in the proof of Proposition 3,

$$EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[\Delta \lambda_i - \sum_{j \in N} \frac{\partial \lambda_i^{ev}}{\partial \log A_j} \Delta \log A_j \right] \Delta \log A_i.$$

The term in square brackets is the change in sales shares due to changes in utility and demand shifters. This expression can be written as

$$EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{25}$$

□

Proof of Proposition 6. Normalize nominal GDP to one. Applying Proposition 7 to a one-factor model yields

$$d \log p = -\Psi d \log A,$$

so that relative prices do not respond to changes in demand or income.

To solve for $\Delta \log Y$, use Lemma 4 in combination with the expression for $d \log p$ and $d\lambda$ in Proposition 7 in the case of one factor. To solve for EV^M, by Proposition 10, $EV^M = EV^m$. Solve for $EV^m - \Delta \log Y$ by plugging the expression for $d \log p$ into Proposition 2 and noting that $b' = \Omega^{(0)}$. □

Proof of Proposition 7. We normalize nominal GDP to be the numeraire. Then Shephard's lemma implies that, for each $i \in N$

$$d \log p_i = -d \log A_i + \sum_j \Omega_{ij} d \log p_j.$$

Furthermore, for $i \in F$

$$d \log p_i = -d \log A_i + d \log \Lambda_i.$$

Combining these yields the desired expression for changes in prices

$$d\log p = -\Psi d\log A + \Psi^F d\log\Lambda.$$

To get changes in sales shares, note that

$$\begin{aligned}\lambda &= b'\Psi\\ d\lambda &= d(b'\Psi)\\ &= b'\Psi d\Omega\Psi + db'\Psi\end{aligned}$$

$$\begin{aligned}\Omega_{ij}d\log\Omega_{ij} &= (1-\theta_i)\Omega_{ij}(d\log p_j - \sum_k \Omega_{ik}d\log p_k)\\ d\Omega_{ij} &= (1-\theta_i)Cov_{\Omega^{(i)}}(d\log p, I_{(j)})\\ \sum_j d\Omega_{ij}\Psi_{jk} &= (1-\theta_i)Cov_{\Omega^{(i)}}(d\log p, I_{(j)})\Psi_{jk}\\ &= (1-\theta_i)\sum_j Cov_{\Omega^{(i)}}(d\log p, \Psi_{jk}I_{(j)})\\ [d\Omega\Psi]_{ik} &= (1-\theta_i)Cov_{\Omega^{(i)}}(d\log p, \Psi_{(k)})\end{aligned}$$

Meanwhile

$$\begin{aligned}d\log b_i &= (1-\theta_0)\left(d\log p_i - \sum_i b_i d\log p_i\right) + (\varepsilon_i - 1)d\log\Upsilon + d\log x_i\\ &= (1-\theta_0)Cov_{\Omega^{(0)}}\left(d\log p, I_{(i)}\right) + Cov_{\Omega^{(0)}}\left(\varepsilon, I_{(i)}\right)d\log\Upsilon + Cov_{\Omega^{(0)}}\left(d\log x, I_{(i)}\right)\\ \sum_i db_i\Psi_{ik} &= Cov_{\Omega^{(0)}}\left((1-\theta_0)d\log p + \varepsilon d\log\Upsilon + d\log x, \Psi_{(k)}\right)\end{aligned}$$

Hence,

$$d\lambda' = \lambda' d\Omega\Psi + db'\Psi$$

can be written as

$$d\lambda_k = \sum_i \lambda_i(1-\theta_i)Cov_{\Omega^{(i)}}(d\log p, \Psi_{(k)}) + Cov_{\Omega^{(0)}}\left(\varepsilon, \Psi_{(k)}\right)d\log\Upsilon + Cov_{\Omega^{(0)}}\left(d\log x, \Psi_{(k)}\right).$$

□

Proof of Proposition 8. Start by setting nominal GDP to be the numeraire. To model the

industry-structure, for each industry I, add two new CES aggregators. One buys the good for the household and one buys the good for firms. Let the price of the household aggregator be given by p_I^c and the price of the non-household aggregator be p_I^f and let p_I be the price of the original industry. Let firm $i's$ share of industry I from household expenditures be b_{iI}. Let the expenditure share of other firms on firm i be s_{iI}. We have

$$\sum_{i \in I} b_{iI} = 1$$
$$\sum_{i \in I} s_{iI} = 1.$$

Let λ_I^c and λ_I^f be sales of industry I to households and firms. Then we have

$$d\lambda_I = d\lambda_I^c + d\lambda_I^f.$$

The sales of an individual firm i in industry I is given by

$$\begin{aligned}
\lambda_i &= b_{iI}\lambda_I^c + s_{iI}\lambda_I^f \\
d\lambda_i &= db_{iI}\lambda_I^c + b_{iI}d\lambda_I^c + ds_{iI}\lambda_I^f + s_{iI}d\lambda_I^f \\
db_{iI} &= Cov_{b_I}(d\log x_{(iI)} + (1-\sigma_I)d\log A_{(iI)}, I_{(i)}) \\
ds_{iI} &= Cov_{b_I}((1-\sigma_I)d\log A_{(iI)}, I_{(i)}).
\end{aligned}$$

The gap between macro welfare and real GDP, $EV^M - \Delta \log Y$, is approximately given by

$$\frac{1}{2} d\log x \frac{\partial \lambda}{\partial \log x} d\log A = \frac{1}{2} \sum_{i \in N} \left[\sum_{j \in N} d\log x_j \frac{\partial \lambda_i}{\partial \log x_j} \right] d\log A_i,$$

where the sums can be re-written as

$$\begin{aligned}
\sum_{i \in N} \left[\sum_{j \in N} d\log x_j \frac{\partial \lambda_i}{\partial \log x_j} \right] d\log A_i = \sum_{i \in N} \Bigg[& d\log x \frac{\partial b_{iI}}{\partial \log x} \lambda_I^c d\log A_i + b_{iI} d\log x \frac{\partial \lambda_I^c}{\partial \log x} d\log A_i \\
& + d\log x \frac{\partial s_{iI}}{\partial \log x} \lambda_I^f d\log A_i + s_{iI} d\log x \frac{\partial \lambda_I^f}{\partial \log x} d\log A_i \Bigg].
\end{aligned}$$

The individual terms of this expression are:

$$\begin{aligned}\sum_{i\in N}\left[d\log x\frac{\partial b_{iI}}{\partial\log x}\lambda_I^c d\log A_i\right] &= \sum_{i\in N} Cov_{b_I}(d\log x_{(iI)}, I_{(i)})\lambda_I^c d\log A_i \\ &= Cov_{b_I}(d\log x_{(iI)}, \sum_{i\in N} I_{(i)} d\log A_i)\lambda_I^c \\ &= Cov_{b_I}(d\log x_{(iI)}, d\log A_{iI})\lambda_I^c;\end{aligned}$$

$$\sum_{i\in N}\left[b_{iI}d\log x\frac{\partial\lambda_I^c}{\partial\log x}d\log A_i\right] = \mathbb{E}_{b_I}\left(d\log A_{iI}\right)d\log x\frac{\partial\lambda_I^c}{\partial\log x};$$

$$\sum_{i\in N} d\log x\frac{\partial s_{iI}}{\partial\log x}\lambda_I^f d\log A_i = 0;$$

and

$$\sum_i s_{iI}d\log x\frac{\partial\lambda_I^f}{\partial\log x}d\log A_i = \mathbb{E}_{s_I}\left(d\log A_{iI}\right)d\log x\frac{\partial\lambda_I^f}{\partial\log x}.$$

Of the four terms, two depend on changes on industry-level sales shares (the sum of which is denoted by Θ in the proposition), one of them is zero, and the remaining one (the first term) is the within-industry covariance of supply and demand shocks in the proposition. □

Proof of Proposition 9. Consider a household with preferences given by

$$C = \left(\int_0^{x^*} c(x)^{\frac{\sigma-1}{\sigma}}dx\right)^{\frac{\sigma}{\sigma-1}}.$$

Note that

$$\begin{aligned}\lambda^{ev}(x,t,t_1) &= \frac{p(x,t)^{1-\sigma}}{\left(\int_0^{x^*(t_1)} p(x,t)^{1-\sigma}dx\right)} \\ \lambda(x,t) &= \frac{p(x,t)^{1-\sigma}}{\left(\int_0^{x^*(t)} p(x)^{1-\sigma}dx\right)}\end{aligned}$$

Hence

$$EV = \frac{I}{\left(\int_0^{x^*(t_1)} p(x)^{1-\sigma}dx\right)^{\frac{1}{1-\sigma}}}.$$

Next

$$\Delta \log EV = \Delta \log I - \int_{t_0}^{t_1} \int_0^{x^*(t_1)} \lambda^{ev}(x,t,t_1) \frac{d \log p(x,t)}{dt} dx dt.$$

Without loss of generality, let's normalize changes in nominal income to zero. Let $\partial_i \lambda^{ev}$ refer to the partial derivative of λ^{ev} with respect to its ith argument. Differentiating and evaluating at the initial point, we get

$$\begin{aligned}
\frac{d \log EV}{dt_1} &= - \int_{t_0}^{t_1} \int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x,t,t_1) \frac{d \log p(x,t)}{dt} dx dt \\
&\quad - \int_{t_0}^{t_1} \lambda^{ev}(x^*(t_1),t,t_1) \frac{d \log p(x^*(t_1),t)}{dt} \frac{dx^*}{dt_1} dt - \int_0^{x^*(t_1)} \lambda^{ev}(x,t_1,t_1) \frac{d \log p(x,t_1)}{dt_1} dx \\
\frac{d^2 \log EV}{dt_1^2} &= - \int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x,t_1,t_1) \frac{d \log p(x,t_1)}{dt_1} dx - \lambda^{ev}(x^*(t_1),t_1,t_1) \frac{d \log p(x^*(t_1),t_1)}{dt} \frac{dx^*}{dt_1} \\
&\quad - \lambda^{ev}(x^*(t_1),t_1,t_1) \frac{dx^*}{dt_1} \frac{d \log p(x^*,t_1)}{dt_1} - \int_0^{x^*(t_1)} \frac{d\lambda^{ev}(x,t_1,t_1)}{dt_1} \frac{d \log p(x,t_1)}{dt_1} dx \\
&\quad - \int_0^{x^*(t_1)} \lambda^{ev}(x,t_1,t_1) \frac{d^2 \log p(x,t_1)}{dt_1^2} dx
\end{aligned}$$

Evaluating at the initial point this simplifies to

$$\begin{aligned}
\frac{d \log EV}{dt_1} &= - \int_0^{x^*} \lambda(x) \frac{d \log p(x,t)}{dt} dx \\
\frac{d^2 \log EV}{dt_1^2} &= - \int_0^{x^*(t_1)} \partial_3 \lambda^{ev}(x,t_1,t_1) \frac{d \log p(x,t_1)}{dt_1} dx - \lambda^{ev}(x^*(t_1),t_1,t_1) \frac{d \log p(x^*(t_1),t_1)}{dt} \frac{dx^*}{dt_1} \\
&\quad - \lambda^{ev}(x^*(t_1),t_1,t_1) \frac{dx^*}{dt_1} \frac{d \log p(x^*,t_1)}{dt_1} - \int_0^{x^*(t_1)} \frac{d\lambda^{ev}(x,t_1,t_1)}{dt_1} \frac{d \log p(x,t_1)}{dt_1} dx \\
&\quad - \int_0^{x^*(t_1)} \lambda^{ev}(x,t_1,t_1) \frac{d^2 \log p(x,t_1)}{dt_1^2} dx
\end{aligned}$$

We note that

$$\lambda^{ev}(x,t,t_1) = \frac{p(x,t)^{1-\sigma}}{\left(\int_0^{x^*(t_1)} p(x,t)^{1-\sigma} dx \right)}$$

$$\frac{\partial \log \lambda^{ev}(x,t,t_1)}{\partial t} = (1-\sigma) \left(\frac{d \log p(x,t)}{dt} - \int_0^{x^*(t_1)} \lambda(x,t) \frac{d \log p(x,t)}{dt} dx \right).$$

$$\partial_3 \log \lambda^{ev}(x,t,t_1) = \frac{\partial \log \lambda^{ev}(x,t,t_1)}{\partial t_1} = \left(-\lambda(x^*,t) \frac{dx^*}{dt_1} \right).$$

Meanwhile, real consumption changes are given by

$$\begin{aligned}
\log Y &= -\int_{t_0}^{t_1}\int_0^{x^*(t)} \lambda(x,t)\frac{d\log p}{dt}dxdt \\
\frac{d\log Y}{dt_1} &= -\int_0^{x^*(t_1)} \lambda(x,t_1)\frac{d\log p}{dt_1}dx \\
\frac{d^2\log Y}{dt_1^2} &= -\lambda(x^*(t_1),t_1)\frac{dx^*}{dt_1}\frac{d\log p}{dt_1} \\
&\quad -\int_0^{x^*(t_1)} \frac{d\lambda(x,t_1)}{dt_1}\frac{d\log p}{dt_1}dx - \int_0^{x^*(t_1)} \lambda(x,t_1)\frac{d^2\log p}{dt_1^2}dx
\end{aligned}$$

where

$$\frac{d\log\lambda(x,t_1)}{dt_1} = (1-\sigma)\left(\frac{d\log p(x,t)}{dt} - \int_0^{x^*(t_1)}\lambda(x,t)\frac{d\log p(x,t)}{dt}dx\right) - \lambda(x^*,t)\frac{dx^*}{dt}.$$

Hence

$$\begin{aligned}
\frac{d\log EV}{dt_1} &= \frac{d\log Y}{dt_1} \\
\frac{d^2\log EV}{dt_1^2} &= \frac{d^2\log Y}{dt_1^2} - \int_0^{x^*(t_1)} \partial_3\lambda^{ev}(x,t_1,t_1)\frac{d\log p(x,t_1)}{dt_1}dx - \lambda^{ev}(x^*(t_1),t_1,t_1)\frac{d\log p(x^*(t_1),t_1)}{dt}\frac{dx^*}{dt_1} \\
&= \frac{d^2\log Y}{dt_1^2} + \lambda(x^*)\frac{dx^*}{dt_1}\int_0^{x^*(t_1)}\lambda(x)\frac{d\log p(x,t_1)}{dt_1}dx - \lambda(x^*)\frac{d\log p(x^*(t_1),t_1)}{dt}\frac{dx^*}{dt_1} \\
&= \frac{d^2\log Y}{dt_1^2} + \lambda(x^*)\frac{dx^*}{dt_1}\left[\int_0^{x^*(t_1)}\lambda(x)\frac{d\log p(x,t_1)}{dt_1}dx - \frac{d\log p(x^*(t_1),t_1)}{dt}\right] \\
&= \frac{d^2\log Y}{dt_1^2} + \lambda(x^*)\frac{dx^*}{dt_1}\left[\mathbb{E}_\lambda\left[\frac{d\log p}{dt}\right] - \frac{d\log p(x^*)}{dt}\right].
\end{aligned}$$

□

Appendix B Extension to Other Welfare Measures

Our baseline measure of welfare changes is equivalent variation under final preferences. Alternatively, we could measure changes in welfare using compensating (instead of equivalent) variation, or by using initial (rather than final) preferences. We focus on equivalent variation with final preferences since it uses indifference curves in the final allocation to make welfare comparisons (that is, preferences "today" for growth-accounting purposes). In this appendix, we show that our methods generalize to the other welfare measures.

Note that if preferences are homothetic, then equivalent and compensating variation are equal, and if preferences are stable, then initial and final preferences are equal.

B.1 Micro welfare changes

We consider four alternative measures of micro welfare changes. For each measure, we present expression for global welfare changes and the approximate gap with real consumption.

The *compensating variation* with *initial preferences* is $CV^m(p_{t_0}, I_{t_0}, p_{t_1}, I_{t_1}; x_{t_0}) = \phi$, where ϕ solves

$$v(p_{t_1}, e^{-\phi} I_{t_1}; x_{t_0}) = v(p_{t_0}, I_{t_0}; x_{t_0}). \tag{26}$$

The analog to (7) in Lemma 1 is

$$CV^m = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} b_i^{cv} d \log p_i, \tag{27}$$

where $b_i^{cv}(p) \equiv b_i(p, v(p_{t_0}, I_{t_0}; x_{t_0}); x_{t_0})$.

Whereas EV^m weights price changes by hypothetical budget shares evaluated at current prices for fixed *final preferences and final utility*, CV^m uses budget shares evaluated at current prices for fixed *initial preferences and initial utility*. An alternative way of calculating CV^m is to reverse the flow of time (the final period corresponds to the initial period), calculate the baseline EV measure under this alternative timeline, and then set $CV^m = -EV^m$.

Since CV^m is based on budget shares at fixed initial preferences and initial utility, to a second-order approximation

$$\begin{aligned}
\Delta \log CV^m &\approx \Delta \log I - b' \Delta \log p - \frac{1}{2} \sum_{i \in N} \left[\Delta \log p' \frac{\partial b_i}{\partial \log p} \right] \Delta \log p \\
&\approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial b_i}{\partial \log x} + \Delta \log v \frac{\partial b_i}{\partial \log u} \right] \Delta \log p.
\end{aligned}$$

This implies that up to a second order approximation,

$$0.5 \left(EV^m + CV^m \right) \approx \Delta \log Y.$$

That is, locally (but not globally) changes in real consumption equal a simple average of equivalent variation under final preferences and compensating variation under initial preferences.

Alternatively, we can measure the change in welfare using the *micro equivalent variation* with *initial preferences*, $EV^m(p_{t_0}, I_{t_0}, p_{t_1}, I_{t_1}; x_{t_0}) = \phi$ where ϕ solves

$$v(p_{t_1}, I_{t_1}; x_{t_0}) = v(p_{t_0}, e^{\phi} I_{t_0}; x_{t_0}). \tag{28}$$

Globally, changes in welfare are

$$EV^m = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} b_i^{ev} d \log p_i, \tag{29}$$

where $b_i^{ev}(p) \equiv b_i(p, v(p_{t_1}, I_{t_1}; x_{t_0}); x_{t_0})$. The gap between changes in welfare and real consumption is, up to a first order approximation,

$$\Delta \log EV^m - \Delta \log Y \approx \frac{1}{2} \sum_{i \in N} \left[-\Delta \log x' \frac{\partial b_i}{\partial \log x} + \Delta \log v \frac{\partial b_i}{\partial \log u} \right] \Delta \log p.$$

Finally, the change in welfare measured using the *micro compensating variation* with *final preferences* is $CV^m(p_{t_0}, I_{t_0}, p_{t_1}, I_{t_1}; x_{t_1}) = \phi$ where ϕ solves

$$v(p_{t_1}, e^{-\phi} I_{t_1}; x_{t_1}) = v(p_{t_0}, I_{t_0}; x_{t_1}). \tag{30}$$

Globally, changes in welfare are given by

$$CV^m = \Delta \log I - \int_{t_0}^{t_1} \sum_{i \in N} b_i^{cv} d \log p_i, \tag{31}$$

where $b_i^{cv}(p) \equiv b_i(p, v(p_{t_0}, I_{t_0}; x_{t_0}); x_{t_1})$. The gap between changes in welfare and real consumption is, up to a first order approximation,

$$\Delta \log CV^m - \Delta \log Y \approx \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial b_i}{\partial \log x} - \Delta \log v \frac{\partial b_i}{\partial \log u} \right] \Delta \log p.$$

Note for EV with initial preferences or CV with final preferences, we must be able to separate demand instability from income effects. For this reason, to compute welfare changes, the elasticities of substitution are not sufficient — we must also know income elasticities or the demand shocks.

Finally, we note that real consumption can be interpreted as representing an alternative measure of welfare, defined as the sum of instantaneous welfare changes using current preferences at each point in time. In particular, real consumption can be written

as

$$\Delta \log Y = \int_{t_0}^{t_1} \frac{\partial EV^m(p(t), I(t), p(t), I(t); x(t))}{\partial p_{t_1}} dp(t) + \frac{\partial EV^m(p(t), I(t), p(t), I(t); x(t))}{\partial I_{t_1}} dI(t),$$

where, at every $t \in [t_0, t_1]$, the integrand is the instantaneous welfare change in response to changes in prices and income, dp and dI, measured using equivalent variation with preferences $x(t)$. In contrast to our welfare measures, this measure does not represent the welfare change over a single preference ordering, and is path dependent (it does not only depend only on initial and final income and prices).

B.2 Macro welfare changes

For each alternative micro welfare measure there is a corresponding macro welfare measure. For example, the *macro compensating variation* with *initial preferences* is $CV^M(A_{t_0}, A_{t_1}; x_{t_0}) = \phi$, where ϕ solves

$$v(p(A_{t_0}, x_{t_0}), I(A_{t_0}, x_{t_0}); x_{t_0}) = v(p(A_{t_1}, x_{t_0}), e^{-\phi} I(A_{t_1}, x_{t_0}); x_{t_0}).$$

In words, CV^M is the proportional change in income necessary to make a consumer with preferences $\succeq_{x_{t_0}}$ indifferent between the budget set that prevails under the initial PPF A_{t_0} and the prices associated with the new PPF A_{t_1}.

Equation (13) in Proposition 4 applies using $\lambda^{cv}(A)$, the sales shares in a fictional economy with the PPF A but where consumers have stable homothetic preferences represented by the expenditure function $e^{cv}(p, u) = e(p, v_{t_0}, x_{t_0}) \frac{u}{v_{t_0}}$ where $v_{t_0} = v(p_{t_0}, I_{t_0}; x_{t_0})$. Growth accounting for welfare is based on hypothetical sales shares evaluated at current technology but for fixed initial preferences and initial utility. The only information on preferences we need to know is elasticities of substitution at the final allocation. As discussed above, CV^M is equal to $-EV^M$ if we reverse the flow of time.

The gap between changes in welfare and real GDP is, to a second-order approximation (the analog of equation 15 in Proposition 5) is

$$CV^M \approx \Delta \log Y - \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{32}$$

We can also define *macro equivalent variation* with *initial preferences*, $EV^M(A_{t_0}, A_{t_1}; x_{t_0}) = \phi$, where ϕ solves

$$v(p(A_{t_0}, x_{t_0}), e^{\phi} I(A_{t_0}, x_{t_0}); x_{t_0}) = v(p(A_{t_1}, x_{t_0}), I(A_{t_1}, x_{t_0}); x_{t_0}).$$

Growth accounting for welfare is based on hypothetical sales shares evaluated at current technology for fixed initial preferences and final utility. In contrast to our previous measures, in order to implement this measure we must know initial demand shifters or income effects. The gap between changes in welfare and real GDP is

$$EV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[-\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} + \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{33}$$

Finally, define *macro compensating variation* with *final preferences*, $CV^M(A_{t_0}, A_{t_1}; x_{t_1}) = \phi$, where ϕ solves

$$v(p(A_{t_0}, x_{t_1}), I(A_{t_0}, x_{t_1}); x_{t_0}) = v(p(A_{t_1}, x_{t_1}), e^{-\phi} I(A_{t_1}, x_{t_1}); x_{t_1}).$$

Growth accounting for welfare is based on hypothetical sales shares evaluated at current technology for fixed final preferences and initial utility, which requires information on demand shifters or income effects. The gap between changes in welfare and real GDP is

$$CV^M \approx \Delta \log Y + \frac{1}{2} \sum_{i \in N} \left[\Delta \log x' \frac{\partial \lambda_i}{\partial \log x} - \Delta \log A' \frac{\partial \log v}{\partial \log A} \frac{\partial \lambda_i}{\partial \log v} \right] \Delta \log A_i. \tag{34}$$

Appendix C Non-homothetic CES preferences

This appendix provides a derivation of the log-linearized expression (9). Changes in Marshallian budget share are given by

$$\begin{aligned} d \log b_i^M &= d \log p_i - d \log I + \sum_j \varepsilon_{ij}^M d \log p_j + \varepsilon_i^w d \log I + d \log x_i, \\ &= d \log p_i - d \log I + \sum_j \left(\varepsilon_{ij}^H - \varepsilon_i^w b_j \right) d \log p_j + \varepsilon_i^w d \log I + d \log x_i, \end{aligned}$$

where ε^H and ε^M are the Hicksian and Marshallian price elasticities, $\varepsilon^w$are the income elasticities, and $d \log x_i$ is a residual that captures changes in shares not attributed to changes in prices or income. The third line is an application of Slutsky's equation. When preferences are non-homothetic CES, then the Hicksian demand curve can be written as

$$c_i = \gamma_i \left(\frac{p_i}{\sum_j p_j c_j} \right)^{-\theta_0} u^{\xi_i},$$

where γ_i and ξ_i are some parameters. The Hicksian price elasticity for $j \neq i$ is

$$\frac{\partial \log c_i}{\partial \log p_j} = \varepsilon_{ij}^H = \theta_0 \frac{p_j c_j}{I} = \theta_0 b_j.$$

Using this fact and the identity $\varepsilon_{ii}^H = -\sum_{j \neq i} \varepsilon_{ij}^H$, we can rewrite changes in budget shares as

$$\begin{aligned}
d \log b_i^M &= \sum_j \left(\varepsilon_{ij}^H - \varepsilon_i^w b_j \right) d \log p_j + d \log p_i + (\varepsilon_i^w - 1) \, d \log I + d \log x \\
&= \left(1 - \sum_{j \neq i} \varepsilon_{ij}^H \right) d \log \frac{p_i}{I} + \sum_{j \neq i} \varepsilon_{ij}^H d \log \frac{p_j}{I} + \varepsilon_i^w \left[d \log I - \sum_j b_j d \log p_j \right] + d \log x_i \\
&= \left(1 - \sum_{j \neq i} \theta_0 b_j \right) d \log \frac{p_i}{I} + \sum_{j \neq i} \theta_0 b_j d \log \frac{p_j}{I} + \varepsilon_i^w \left[d \log I - \sum_j b_j d \log p_j \right] + d \log x_i \\
&= (1 - \theta_0 (1 - b_i)) \, d \log \frac{p_i}{I} + \sum_{j \neq i} \theta_0 b_j d \log \frac{p_j}{I} + \varepsilon_i^w \left[d \log I - \sum_j b_j d \log p_j \right] + d \log x_i \\
&= (1 - \theta_0) \left[d \log p_i - \sum_j b_j d \log p_j \right] + (\varepsilon_i^w - 1) \left[d \log I - \sum_j b_j d \log p_j \right] + d \log x_i.
\end{aligned}$$

Appendix D The Role of Elasticities of Substitution for Changes in Welfare-Based Productivity

In this appendix, we show why the results in Table 1 are differentially sensitive to changes in different elasticities of substitution. Combine Propositions 6 and 10 to obtain the following second-order approximation:

$$\Delta \log TFP^{\text{welfare}} \approx \sum_i \lambda_i \Delta \log A_i + \frac{1}{2} \sum_{j \in \{0\} + \mathcal{N}} (\theta_j - 1) \lambda_j Var_{\Omega^{(j)}} \left(\sum_{k \in \mathcal{N}} \Psi_{(k)} \Delta \log A_i \right). \quad (35)$$

The second term is half the sum of changes in Domar weights due to substitution effects (i.e. changes in welfare-relevant sales shares) times the change in productivities. Note that changes in these welfare-relevant sales shares are linear in the microeconomic elasticities of substitution. The importance of some elasticity θ depends on

$$\sum_j \lambda_j Var_{\Omega^{(j)}} \left(\sum_{k \in \mathcal{N}} \Psi_{(k)} \Delta \log A_i \right),$$

where the index j sums over all CES nests whose elasticity of substitution is equal to θ (i.e. all j such that $\theta_j = \theta$). Therefore, elasticities of substitution are relatively more potent if: (1) they control substitution over many nests with high sales shares, or (2) if the nests corresponding to those elasticities are heterogeneously exposed to the productivity shocks.

We compute the coefficients in (35) for our model's various elasticities using the IO table at the end of the sample. The coefficient on $(\theta_0 - 1)$, the elasticity of substitution between agriculture, manufacturing, and services in consumption is only 0.01. This explains why the results in Table 1 are not very sensitive to this elasticity. On the other hand, the coefficient on $(\theta_1 - 1)$, the elasticity across disaggregated consumption goods, is much higher at 0.21. The coefficient on $(\theta_2 - 1)$, the elasticity between materials and value-added bundles is 0.07. Finally, the coefficient on $(\theta_3 - 1)$, the elasticity between disaggregated categories of materials is 0.25. This underscores the fact that elasticities of substitution are more important if they control substitution in CES nests which are very heterogeneously exposed to productivity shocks — that is, nests that have more disaggregated inputs.

According to equation (35), setting $\theta_1 = \theta_2 = \theta_3 = 1$ (which is similar to abstracting from heterogeneity within the three broader sectors and heterogeneity within intermediate inputs), then θ_0 is the only parameter that can generate complementarities in the model. This may help understand why more aggregated models of structural transformation (e.g. Buera et al., 2015 and Alder et al., 2019) require low values of θ_0 to account for the extent of sectoral reallocation in the data.

Appendix E Non-CES

In this appendix, we generalize Proposition 7 beyond CES functional forms. To do this, for each producer k with cost function $\mathbf{C}_k$, we define the Allen-Uzawa elasticity of substitution between inputs x and y as

$$\theta_k(x,y) = \frac{\mathbf{C}_k d^2\mathbf{C}_k/(dp_x dp_y)}{(d\mathbf{C}_k/dp_x)(d\mathbf{C}_k/dp_y)} = \frac{\epsilon_k(x,y)}{\Omega_{ky}},$$

where $\epsilon_k(x,y)$ is the elasticity of the demand by producer k for input x with respect to the price p_y of input y, and Ω_{ky} is the expenditure share in cost of input y. For the household $k = 0$, we use the household's expenditure function in place of the cost function (where the Allen-Uzawa elasticities are disciplined by Hicksian cross-price elasticities and ex-

penditure shares).

Following Baqaee and Farhi (2019c), define the *input-output substitution operator* for producer k as

$$\Phi_k(\Psi_{(i)}, \Psi_{(j)}) = -\sum_{1\leq x,y\leq N+1+F} \Omega_{kx}[\delta_{xy} + \Omega_{ky}(\theta_k(x,y)-1)]\Psi_{xi}\Psi_{yj}, \tag{36}$$

$$= \frac{1}{2}E_{\Omega^{(k)}}\left((\theta_k(x,y)-1)(\Psi_i(x)-\Psi_i(y))(\Psi_j(x)-\Psi_j(y))\right), \tag{37}$$

where δ_{xy} is the Kronecker delta, $\Psi_i(x) = \Psi_{xi}$ and $\Psi_j(x) = \Psi_{xj}$, and the expectation on the second line is over x and y. The second line can be obtained from the first using the symmetry of Allen-Uzawa elasticities of substitution and the homogeneity identity.

Then, Proposition 7 generalizes as follows:

Proposition 11. *Consider some perturbation in final demand $d\log x$ and technology $d\log A$. Then changes in prices of goods and factors are*

$$d\log p_i = -\sum_{j\in N}\Psi_{ij}d\log A_j + \sum_{f\in F}\Psi^F_{if}d\log\lambda_f. \tag{38}$$

Changes in sales shares for goods and factors are

$$\lambda_i d\log\lambda_i = \sum_{j\in\{0\}+N}\lambda_j\Phi_j\left(-d\log p, \Psi_{(i)}\right) \tag{39}$$

$$+ Cov_{\Omega^{(0)}}\left(d\log x, \Psi_{(i)}\right) + Cov_{\Omega^{(0)}}(\varepsilon, \Psi_{(i)})\left(\sum_{k\in N}\lambda_k d\log A_k\right).$$

Since Φ_j shares many of the same properties as a covariance (it is bilinear and symmetric in its arguments, and is equal to zero whenever one of the arguments is a constant), the intuition for Proposition 11 is very similar to that of Proposition 7. Computing the equilibrium response in Proposition 11 requires solving a linear system exactly as in Proposition 7.

Appendix F Heterogeneous Agents

Consider an economy with a set of agents indexed by h, where each agent has stable and homothetic preferences. This means that real consumption for agent h, $\Delta\log Y_h$, is equal

to EV_h. The social welfare function is given by

$$W = \sum_h \frac{\bar{p_h}\bar{Y_h}}{\sum_j \bar{p_j}\bar{Y_j}} \left[\frac{Y_h}{\bar{Y_h}}\right],$$

where bars denote values at the initial equilibrium. We show that to a second order approximation

$$\Delta \log W \approx \Delta \log Y - \frac{1}{2} Cov_\chi(\mathbb{E}_{b_h}[\Delta \log p], \Delta \log Y_h),$$

where χ is the vector of initial expenditure shares for each agent as a share of aggregate spending, and the covariance is applied across individuals.

To see this,

$$\begin{aligned}
dW &= \sum_h \frac{\bar{p_h}Y_h}{\sum_j \bar{p_j}\bar{Y_j}} d\log Y_h, \\
d\log W &= \frac{1}{W}\sum_h \frac{\bar{p_h}Y_h}{\sum_j \bar{p_j}\bar{Y_j}} d\log Y_h, \\
d^2 \log W &= \frac{1}{W}\sum_h \frac{\bar{p_h}Y_h}{\sum_j \bar{p_j}\bar{Y_j}} d^2\log Y_h + \frac{1}{W}\sum_h \frac{\bar{p_h}Y_h}{\sum_j \bar{p_j}\bar{Y_j}} \left(d\log Y_h\right)^2 - d\log W^2, \\
&= \sum_h \chi_h d^2 \log Y_h + \sum_h \chi_h \left(d\log Y_h\right)^2 - d\log Y^2,
\end{aligned}$$

where the last line uses the fact that $d\log W = d\log Y$ at the initial point.

Next consider the change in real GDP:

$$\log Y = \int_{t_0}^{t_1} \sum_i \frac{p_i(t)q_i(t)}{\sum_j p_j(t)q_j(t)} d\log q_i(t),$$

and

$$p_i q_i = \sum_h \chi_h b_{hi} GDP,$$

where b_{hi} is the budget share of agent h on good i and GDP is nominal GDP. We have

$$\begin{aligned}
d\log q_i &= \sum_h \frac{q_{hi}}{q_i} d\log q_{hi} \\
&= \sum_h \frac{\chi_h b_{hi}}{\sum_g \chi_g b_{gi}} d\log q_{hi} \\
\log Y &= \int^{t_1} \sum_h \chi_h \sum_i b_{hi} d\log q_{hi} \\
&= \int^{t_1} \sum_h \chi_h d\log Y_h. \\
d\log Y &= \chi \cdot d\log Y_{(h)}.
\end{aligned}$$

Therefore, to a second order

$$\begin{aligned}
d^2\log Y &= d\chi \cdot d\log Y_{(h)} + \chi \cdot d^2\log Y_{(h)} \\
&= d\chi \cdot d\log Y_{(h)} + d^2\log W - \sum_h \chi_h \left(d\log Y_h\right)^2 + d\log Y^2.
\end{aligned}$$

Using the fact that

$$d\chi_h = \chi_h d\log\chi_h = \chi_h \left[\mathbb{E}_{b_h}[d\log p] + d\log Y_h - \sum_j \chi_j \left[\mathbb{E}_{b_j}[d\log p] + d\log Y_j\right]\right],$$

we have

$$\begin{aligned}
d^2\log Y &= \sum_h \chi_h \left[\mathbb{E}_{b_h}[d\log p] + d\log Y_h - \sum_j \chi_j \left[\mathbb{E}_{b_j}[d\log p] + d\log Y_j\right]\right] d\log Y_h \\
&\quad + d^2\log W - \sum_h \chi_h \left(d\log Y_h\right)^2 + d\log Y^2 \\
&= \sum_h \chi_h \mathbb{E}_{b_h}[d\log p] d\log Y_h + \sum_h \chi_h d\log Y_h d\log Y_h - \sum_h \chi_h d\log Y_h \sum_j \chi_j \left[\mathbb{E}_{b_j}[d\log p] + d\log Y_j\right] \\
&\quad + d^2\log W - \sum_h \chi_h \left(d\log Y_h\right)^2 + d\log Y^2 \\
&= Cov_\chi(\mathbb{E}_{b_h}[d\log p], d\log Y_{(h)}) + d^2\log W.
\end{aligned}$$

This implies that

$$d^2\log W = d^2\log Y - Cov_\chi(\mathbb{E}_{b_h}[d\log p], d\log Y_{(h)}).$$

To a second order approximation changes in the social welfare function are given by

$$d\log W + \frac{1}{2}d^2\log W = d\log Y + \frac{1}{2}d^2\log Y - \frac{1}{2}Cov_{\chi}(\mathbb{E}_{b_h}[d\log p], d\log Y_{(h)}).$$