

Appendix

A The Supply of Loanable Shares

In this section we provide the micro-foundations for the supply curve $f_t = l(y_t)$.

To start, we extend the model by introducing competitive firms that we label “brokers.” Brokers facilitate share lending operations. A broker that facilitates the lending of S_t shares obtains revenue equal to $f_t S_t P_t$, where f_t is the market lending rate and P_t is the value of the stock. The broker only retains a fraction $(1 - \tau)$ of that income, with the remaining income being rebated to (long-position) stock investors as the outcome of a bargaining game to obtain their permission to lend out their shares.

Facilitating the share lending also requires a resource (“human capital”) that is owned by households. With these assumptions, a firm’s problem is

$$\max_{S_t \geq 0} (1 - \tau) f_t S_t P_t - q_t H(S_t), \quad (48)$$

where q_t is the prevailing market compensation (“wage”) per unit of resource employed and $H(S_t)$ are the resource units that are required to process S_t shares. (We assume $H'(S_t) \geq 0$ and $H''(S_t) \geq 0$.) Solving the broker’s optimization problem leads to the resource demand function

$$H'(S_t) = \frac{(1 - \tau) f_t P_t}{q_t}. \quad (49)$$

To model the supply of the household-controlled resource, we assume that each household incurs a disutility equal to $\frac{1}{\rho + \pi} d\left(\frac{h_t^i}{\chi_t^i}\right)$ from providing h_t^i units of human capital where $d(\cdot) \geq 0$, $d''(\cdot) \geq 0$ and $\chi_t^i \equiv \frac{W_t^i}{\bar{W}}$. The term χ_t^i ensures that households with higher relative wealth, $\frac{W_t^i}{\bar{W}}$, are de-facto endowed with a better ability to provide efficiency units of human

capital (say due to their higher education). But fundamentally, the assumption that wealthier households obtain a higher fraction of the human-capital compensation is for technical reasons, since this assumption safeguards that a household's value function continues to be logarithmic in wealth. (Otherwise, one would have to introduce the present value of human capital compensation as a separate state variable in each household's problem, which would undermine the tractability of the model.)

Given logarithmic utilities, a household's supply decision is characterized by the first-order condition

$$\frac{1}{\rho + \pi} \frac{1}{\chi_t^i} d' \left(\frac{h_t^i}{\chi_t^i} \right) = \frac{1}{(\rho + \pi) W_t^i} \times q_t, \quad (50)$$

where the right-hand side is the marginal disutility of labor, whereas the right hand side is the marginal utility of consumption, $\frac{1}{(\rho + \pi) W_t^i}$, times the wage q_t . Using the definition of χ_t^i and simplifying gives

$$h_t^i = d'^{-1} \left(\frac{\bar{H}}{W_t} q_t \right) \chi_t^i, \quad (51)$$

where $d'^{-1}(\cdot)$ is the inverse function of $d'(\cdot)$. Aggregating (51) across all households j (across all cohorts and agent types) and noting that $\int_j x_t^j dj = 1$ leads to $H_t \equiv \int_j h_t^j dj = \bar{H} d'^{-1} \left(\bar{H} \frac{q_t}{W_t} \right)$ or equivalently

$$q_t = \frac{W_t}{\bar{H}} d' \left(\frac{H_t}{\bar{H}} \right). \quad (52)$$

Substituting (52) into (49), noting that in equilibrium $P_t = W_t$ and also that the supply H_t equals the demand $H(S_t)$ leads to

$$f_t = \frac{1}{(1 - \tau) \bar{H}} H'(S_t) d' \left(\frac{H(S_t)}{\bar{H}} \right). \quad (53)$$

The right-hand side of (53) is non-decreasing in S_t (given the assumed monotonicity and convexity of $d(\cdot)$ and $H(\cdot)$). In turn, the equilibrium short interest equals $S_t = \frac{y_t}{1-y_t}$.¹⁷ Accordingly, S_t is a monotone function of y_t and therefore f_t can be expressed as a non-decreasing function of the equilibrium value of y_t . We write

$$f_t = l(y_t) \text{ with } l'(\cdot) \geq 0.$$

Assuming that $H(S_t) = \phi S_t$ for some constant $\phi > 0$, the broker makes no profits and therefore $(1 - \tau) f_t S_t P_t = H_t q_t$. In turn, combining (51) with (52) implies

$$\frac{h_t^i q_t}{H_t q_t} = \frac{W_t^i}{W_t}. \quad (54)$$

Accordingly, the wealth evolution of household i is

$$\begin{aligned} \frac{dW_t^i}{W_t^i} &= \left((r_t + \pi + w_{t,s}^i (\mu_t - r_t + s_{t,s}^i)) - \frac{c_t^i}{W_t^i} + \frac{h_t^i q_t}{W_t^i} \right) dt + w_t^i \sigma_t dB_t \\ &= \left((r_t + \pi + w_{t,s}^i (\mu_t - r_t + s_{t,s}^i)) - \frac{c_t^i}{W_t^i} + \frac{(1 - \tau) f_t S_t P_t}{W_t} \right) dt + w_t^i \sigma_t dB_t \\ &= \left((r_t + \pi + w_{t,s}^i (\mu_t - r_t + s_{t,s}^i)) - \frac{c_t^i}{W_t^i} + (1 - \tau) f_t \frac{y_t}{1 - y_t} \right) dt + w_t^i \sigma_t dB_t, \end{aligned} \quad (55)$$

where the second line follows from combining (54) with $(1 - \tau) f_t S_t P_t = H_t q_t$ and the last line follows from $P_t = W_t$ and $S_t = \frac{y_t}{1-y_t}$.

We note that, if we were to assume a non-linear $H(S_t)$, then we would additionally have to assume the broker profits to be distributed to the households in proportion to their wealth.

¹⁷The amount of shorted shares S_t is equal to $S_t = \omega_t^R |w_t^R|$. The definition $y_t = \frac{\omega_t^R |w_t^R|}{\omega_t^I w_t^I}$ along with the stock market clearing requirement $\omega_t^I w_t^I - \omega_t^R |w_t^R| = 1$ implies that

$$S_t = \omega_t^R |w_t^R| = \frac{y_t}{1 - y_t}.$$

B The Price-Dividend Ratio of a Small Stock

This section provides the details of the entry-and-exit process for the model of section 5.2 and proves Proposition 6.

We assume that the fraction of investors that pay attention to stock 1, which we denote by $\widehat{\omega}_t$, is small: $\widehat{\omega}_t = \omega_t^R + \omega_t^I \approx 0$. Further, in the interest of tractability, we allow for entry and exit into the market for stock 1 to not be fully driven by entry and exit into the economy (via births and deaths). Specifically, letting W_t^i denote the (aggregate) wealth of type- i investors (where $i \in \{I, R\}$), we assume

$$dW_t^i = dW_t^{i,\text{part}} + \delta (\nu_i (W_t^I + W_t^R) - W_t^i) dt + \omega_t^i (dL_t - dN_t), \quad (56)$$

where $dW_t^{i,\text{part}}$ is the wealth growth of an investor of type $i \in \{I, R\}$ who is already participating in the market¹⁸ and the last two terms on the right-hand side of equation (56) capture entry and exit into market 1, i.e., how investors from the broad economy gain and lose interest in stock 1. The term $\delta (\nu_i (W_t^I + W_t^R) - W_t^i) dt$ reflects that investors enter and exit the market for stock 1 at a rate δ per unit of time dt for exogenous reasons. Similar to the baseline model, this term affects the composition, but not the sum of $W_t^I + W_t^R$, since $\sum_{i \in \{I, R\}} \delta (\nu_i (W_t^I + W_t^R) - W_t^i) = 0$. The third term on the right-hand side of (56) affects the sum of $W_t^I + W_t^R$, but not the shares $\omega_t^i = \frac{W_t^i}{W_t^I + W_t^R}$. Specifically, we assume that dL_t and dN_t are two singular, increasing processes that “control” $W_t^I + W_t^R$ so that the ratio of stock market capitalization in market 1 to the total wealth of investors participating in market one, $\widetilde{m}_t = \frac{M_{1,t}}{W_t^I + W_t^R}$ stays constant across time ($\widetilde{m}_t = \widetilde{m}$). This assumption ensures that in

¹⁸For completeness, $dW_t^{i,\text{part}} = W_t^{i,\text{part}} \mu_W^i dt + W_t^{i,\text{part}} \vec{\sigma}_W^i dW_t$, where

$$\mu_W^i = r_t + \pi + n_t + \vec{w}_{t,s}^i \left(\vec{\mu}_t - r_t \mathbf{1}_{\{2 \times 1\}} + \lambda_{t,s}^i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) - \frac{c_{t,s}^i}{W_{t,s}^i}$$

and $\vec{\sigma}_W^i = \vec{w}^{i'} \sigma_t$.

the absence of any differences of opinion ($\eta = 0$), the excess return, the price-dividend ratio, and the volatility of stock 1 would all be constant. Thus, we can attribute any fluctuations in the price-dividend ratios exclusively to the forces we wish to highlight — namely, the differences in opinion and the shorting frictions — while eliminating the fluctuations that would emerge from limited participation in the market for stock 1. In addition, these assumptions on dL_t and dN_t are convenient from a technical standpoint, since they allow us to express the price-dividend ratio for stock 1 as a function of $\tilde{\omega}_t^R$, rather than a function of three state variables $(\tilde{\omega}_t^R, \tilde{m}_t, z_t)$, which would require the solution of a non-linear, three-dimensional partial differential equation.

Using the market clearing condition $\sum_{i \in \{I, R\}} \tilde{\omega}_t^i w_t^{i,1} = \tilde{m}$, and applying Ito's Lemma to $\tilde{\omega}_t^i = \frac{W_t^i}{W_t^I + W_t^R}$ leads to

$$d\tilde{\omega}_t^i = \mu_t^i dt + \sigma_t^i dB_{1,t}, \quad (57)$$

where

$$\begin{aligned} \mu_t^i &= \tilde{\omega}_t^i [(w_{1,t}^i - \tilde{m}) \sigma_{1,t} (\kappa_t - \sigma_{1,t} \tilde{m}) + w_{1,t}^i f_t + \tilde{n}_t] + \delta (\nu^i - \tilde{\omega}_t^i), \\ \sigma_t^i &= \tilde{\omega}_t^i (w_{1,t}^i - \tilde{m}) \sigma_{1,t}, \end{aligned}$$

and¹⁹

$$\tilde{n}_t \equiv - \sum_{i \in \{I, R\}} w_{1,t}^i \tilde{\omega}_t^i \lambda_{t,s}^i = \frac{y_t \tilde{m}}{1 - y_t} f_t (1 - \tau).$$

¹⁹Using $\sum_{i \in \{I, R\}} w_{t,s}^i \tilde{\omega}_t^i = \tilde{m}_t$, the definition $y_t = - \frac{(w_{t,s}^R \tilde{\omega}_t^R) 1_{\{w_{t,s}^R < 0\}}}{w_{t,s}^I \tilde{\omega}_t^I}$ and the definition of $\lambda_{t,s}^i$ leads to

$$- \sum_{i \in \{I, R\}} w_{t,s}^i \tilde{\omega}_t^i \lambda_{t,s}^i = \frac{y_t \tilde{m}}{1 - y_t} f_t (1 - \tau).$$

Since $\frac{\phi_1}{\phi_2} \approx 0$, the aggregate endowment follows a geometric Brownian motion in the limit, and the interest rate is constant $r_t = r$. Accordingly, price of a stock of type 1 follows the dynamics

$$\frac{dP_{1,t,s} + D_{1,t,s}dt}{P_{1,t,s}} = (r + \kappa_{1,t}\sigma_{1,t})dt + \sigma_t dB_{1,t}. \quad (58)$$

Applying Ito's Lemma to the product $P_{t,s} = p(\omega_t^R) D_{1,t,s}$ also implies that

$$\frac{dP_{1,t,s}}{P_{1,t,s}} = \frac{dp_t}{p_t} + \frac{dD_{1,t,s}}{D_{1,t,s}} + \frac{p'(\omega_t^R)}{p(\omega_t^R)} \sigma_t^R \sigma_D dt. \quad (59)$$

Combining (58) with (59) and using $\sigma_{1,t} = \frac{p'(\omega_t^R)}{p(\omega_t^R)} \sigma_t^R + \sigma_D$ and Ito's Lemma to compute the drift of $\frac{dp_t}{p_t}$ leads to

$$\frac{1}{2} \frac{\partial^2 p}{(\partial \omega^R)^2} (\sigma_t^R)^2 + \frac{\partial p}{\partial \omega^R} (\mu_t^R + \sigma_t^R \sigma_D) - p \times (r + \delta_1 + \kappa_{1,t}\sigma_{1,t}) + 1 = 0, \quad (60)$$

which in turn leads to (44) after substituting $\sigma_{1,t} = \frac{p'(\omega_t^R)}{p(\omega_t^R)} \sigma_t^R + \sigma_D$.

C Proofs

Proof of Proposition 1. Fix a value η and consider a sequence of parameters such that $\sigma_D \rightarrow 0$ and along that sequence β is set according to

$$\beta = \sigma_D (\eta - \xi \sigma_D), \quad (61)$$

for some $\xi > 1$. Note that since $\sigma_D \rightarrow 0$, $\eta - \xi \sigma_D$ is positive for sufficiently small σ_D , and hence β is positive.

We show next that as σ_D gets close to zero, Assumption 2 is satisfied.

Re-arranging (61) gives

$$\frac{\eta}{\frac{\beta}{\sigma_D}} = \frac{1}{1 - \xi \frac{\sigma_D}{\eta}}. \quad (62)$$

For sufficiently small σ_D we obtain

$$1 + \tau > \frac{1}{1 - \xi \frac{\sigma_D}{\eta}} > 1. \quad (63)$$

Combining (62) and (63) yields (19).

Turning to (20), we note that the definition of $\omega^{(1)}$ along with (61) implies

$$\omega^{(1)} = 1 - \frac{\sigma_D}{\xi \sigma_D} = \frac{\xi - 1}{\xi} > 0,$$

while also

$$\lim_{\sigma_D \rightarrow 0} \frac{\sigma_D}{(1 + \tau) \frac{\beta}{\sigma_D} - \eta} = \lim_{\sigma_D \rightarrow 0} \frac{\sigma_D}{(1 + \tau) (\eta - \xi \sigma_D) - \eta} = 0.$$

Therefore, for sufficiently small σ_D , the left-hand side of (20) converges to $\frac{\xi-1}{\xi} > 0$, while

the right-hand side converges to zero, and therefore the inequality holds.

We conclude the proof by showing that $F(\omega)$ has a unique root in the interval $(0, 1)$. To this end, it is useful to introduce the definitions

$$A(\omega) \equiv \tau \frac{\omega}{\sigma_D} \beta, \quad (64)$$

$$B(\omega) \equiv \sigma_D - \omega \left((1 + \tau) \frac{\beta}{\sigma_D} - \eta \right), \quad (65)$$

$$C(\omega) \equiv \frac{\omega}{1 - \omega} \left(\sigma_D + (1 - \omega) \left(\frac{\beta}{\sigma_D} - \eta \right) \right). \quad (66)$$

With these definitions, $F(\omega)$ can be written as $F(\omega) = B^2(\omega) - 4A(\omega)C(\omega)$. We start by observing that $C(\omega^{(1)}) = 0$ for any parametric choice (since the definition of $\omega^{(1)}$ in equation (17) implies $\sigma_D + (1 - \omega^{(1)}) \left(\frac{\beta}{\sigma_D} - \eta \right) = 0$). Also the inequality (20) implies that $B(\omega^{(1)}) \neq 0$, which implies that $B^2(\omega^{(1)}) > 0$. Accordingly, $F(\omega^{(1)}) > 0$. Also $B(1) < \infty$, while $C(1) = \infty$. By continuity, there exists at least one value $\omega^{(2)} \in (\omega^{(1)}, 1)$ such that $F(\omega^{(2)}) = 0$.

To show that this value is unique, consider any value $\omega^{(2)} \in (\omega^{(1)}, 1)$ such that $F(\omega^{(2)}) = 0$. We next show that $F'(\omega^{(1)}) < 0$.

To this end, note that

$$\begin{aligned} F'(\omega) &= 2B(\omega)B'(\omega) - 4[A'(\omega)C(\omega) + A(\omega)C'(\omega)] \\ &= 2B^2(\omega) \frac{B'(\omega)}{B(\omega)} - 4A(\omega)C(\omega) \left(\frac{A'(\omega)}{A(\omega)} + \frac{C'(\omega)}{C(\omega)} \right). \end{aligned}$$

Since $\omega^{(2)}$ is a root of $F(\omega)$ it follows that $B^2(\omega^{(2)}) = 4A(\omega^{(2)})C(\omega^{(2)})$. Therefore,

$$F'(\omega^{(2)}) = B^2(\omega^{(2)}) \left(2 \frac{B'(\omega^{(2)})}{B(\omega^{(2)})} - \frac{A'(\omega^{(2)})}{A(\omega^{(2)})} - \frac{C'(\omega^{(2)})}{C(\omega^{(2)})} \right). \quad (67)$$

The sign of $F'(\omega^{(2)})$ is the same as the sign of the expression inside the parentheses in

equation (67). We have

$$\frac{A'(\omega^{(2)})}{A(\omega^{(2)})} = \frac{1}{\omega^{(2)}}, \quad \frac{B'(\omega^{(2)})}{B(\omega^{(2)})} = -\frac{\left((1+\tau)\frac{\beta}{\sigma_D} - \eta\right)}{\sigma_D - \omega^{(2)}\left((1+\tau)\frac{\beta}{\sigma_D} - \eta\right)}$$

and

$$\frac{C'(\omega^{(2)})}{C(\omega^{(2)})} = \frac{1}{\omega^{(2)}(1-\omega^{(2)})} + \frac{\eta - \frac{\beta}{\sigma_D}}{\sigma_D + (1-\omega^{(2)})\left(\frac{\beta}{\sigma_D} - \eta\right)}.$$

Combining terms gives

$$\begin{aligned} & 2\frac{B'(\omega^{(2)})}{B(\omega^{(2)})} - \frac{A'(\omega^{(2)})}{A(\omega^{(2)})} - \frac{C'(\omega^{(2)})}{C(\omega^{(2)})} = \\ & -\frac{2\left((1+\tau)\frac{\beta}{\sigma_D} - \eta\right)}{\sigma_D - \omega^{(2)}\left((1+\tau)\frac{\beta}{\sigma_D} - \eta\right)} - \frac{1}{\omega^{(2)}} - \frac{1}{\omega^{(2)}(1-\omega^{(2)})} - \frac{\eta - \frac{\beta}{\sigma_D}}{\sigma_D + (1-\omega^{(2)})\left(\frac{\beta}{\sigma_D} - \eta\right)}. \end{aligned} \quad (68)$$

For future reference, we note that using $\omega^{(2)} > \omega^{(1)}$ along with (19) and the definition of $\omega^{(1)}$ implies that

$$\sigma_D + (1-\omega^{(2)})\left(\frac{\beta}{\sigma_D} - \eta\right) > \sigma_D + (1-\omega^{(1)})\left(\frac{\beta}{\sigma_D} - \eta\right) = 0. \quad (69)$$

Using (61) we can write the right-hand side of (68) as

$$-\frac{2\left((1+\tau)(\eta - \xi\sigma_D) - \eta\right)}{\sigma_D - \omega^{(2)}\left((1+\tau)(\eta - \xi\sigma_D) - \eta\right)} - \frac{1}{\omega^{(2)}} - \frac{1}{\omega^{(2)}(1-\omega^{(2)})} - \frac{\xi}{1-\xi(1-\omega^{(2)})}. \quad (70)$$

Taking the limit as σ_D approaches zero, the expression (70) converges to

$$-\frac{1}{1 - \omega^{(2)}} - \frac{\xi}{1 - \xi(1 - \omega^{(2)})} < 0,$$

where the inequality follows from (69) along with (61).²⁰

The fact that the derivative $F'(\omega^{(2)}) < 0$ for any root of the equation $F(\omega^{(2)}) = 0$ in the interval $(\omega^{(1)}, 1)$ implies that the root $\omega^{(2)}$ must be unique. ■

Proof of Proposition 2. In preparation for the proof of Proposition 2, we state and prove an auxiliary result.

Lemma 2 *The following statements hold for the quadratic equation (22).*

1. $\omega^{(1)} < \omega^{(2)}$ and the discriminant of (22) is non-negative for all $\omega_t^R \leq \omega^{(2)}$.
2. When $\omega^{(1)} \leq \omega_t^R \leq \omega^{(2)}$, the two roots of the equation are both in the interval $[0, 1)$.
3. For $\omega_t^R \in [0, \omega^{(1)})$, only the larger root of (22) is in the interval $(0, 1)$.
4. If y is a root of (22), then $(1 - \omega_t^R)\eta - \sigma_D - \frac{1 - \omega_t^R}{\sigma_D}\beta(1 - \tau y) > 0$.

Proof of Lemma 2. We start with part 1. Using the definitions (64)–(66), equation (22) can be written in the familiar form

$$A(\omega_t^R)y^2 + B(\omega_t^R)y + C(\omega_t^R) = 0,$$

and the discriminant of this quadratic equation is given by $F(\omega_t^R)$, as defined in equation (18).

For $\omega_t^R \leq \omega^{(1)}$, $C(\omega_t^R) < 0$ and the discriminant, $B^2(\omega_t^R) - 4A(\omega_t^R)C(\omega_t^R)$, is positive. The assumption that $\omega^{(2)}$ is the unique root of $F(\omega)$ along with the facts that $F(\omega^{(1)}) =$

²⁰Equation (61) implies that $\frac{\beta}{\sigma_D} = \eta - \xi\sigma_D$, and therefore $\sigma_D + (1 - \omega^{(2)})\left(\frac{\beta}{\sigma_D} - \eta\right) = \sigma_D [1 - (1 - \omega^{(2)})\xi] < 0$, where the inequality follows from (69).

$B^2(\omega^{(1)}) > 0$ and $F(1) = -\infty$ imply that $\omega^{(1)} < \omega^{(2)}$.²¹ The uniqueness of the root $\omega^{(2)}$ also implies that $F(\omega_t^R) = B^2(\omega_t^R) - 4A(\omega_t^R)C(\omega_t^R) \geq 0$ for all $\omega_t^R \leq \omega^{(2)}$.

We now turn to part 2. To economize on notation we write A rather $A(\omega_t^R)$ and similarly for B and C . Fix a given ω_t^R and let $g(y) = Ay^2 + By + C$. We have that $g(1) = A + B + C = \frac{\sigma_D}{1 - \omega_t^R} > 0$ and $g'(1) = 2A + B = \sigma_D + \omega_t^R \left(\eta^I - (1 - \tau) \frac{\beta}{\sigma_D} \right) > 0$, where the inequality follows from (19). Since $A > 0$, it follows that all roots of $g(y)$ must be smaller than one. Also, the fact that $\omega_t^R \geq \omega^{(1)}$ implies that $g(0) = C > 0$, while assumptions (19) and (20) together with the fact that $\omega_t^R \geq \omega^{(1)}$ imply that $g'(0) = B < 0$.

The facts that i) $g(y)$ is a convex, quadratic function of y , ii) $g(1) > 0, g(0) > 0$ and $g'(1) > 0, g'(0) < 0$ and iii) $B^2 - 4AC > 0$ for $\omega_t^R \in [\omega^{(1)}, \omega^{(2)})$ imply that there are two roots in $(0, 1)$.

For part 3, we note that, when $\omega_t^R < \omega^{(1)}$, $g(0) = C < 0$, while $g(1) = A + B + C = \frac{\sigma_D}{1 - \omega_t^R} > 0$. Therefore there exists one and only one root in $(0, 1)$.

Finally, let $y \in (0, 1)$ denote a root of the quadratic equation (22). Accordingly,

$$\begin{aligned} (1 - \omega_t^R) \eta - \sigma_D - (1 - \omega_t^R) \frac{\beta}{\sigma_D} (1 - \tau y) &= \frac{1 - \omega_t^R}{\omega_t^R} y \left(\sigma_D + \omega_t^R \eta^I - \omega_t^R \frac{\beta}{\sigma_D} (1 - \tau y) \right) \\ &= \frac{1 - \omega_t^R}{\omega_t^R} y \left[\sigma_D + \omega_t^R \left(\eta - \frac{\beta}{\sigma_D} \right) + \omega_t^R \frac{\beta}{\sigma_D} \tau y \right] \\ &> 0 \end{aligned}$$

where the last inequality follows from (19). This proves property 4. ■

We now continue with the proof of the proposition. We provide expressions for r_t and κ_t that apply in any equilibrium where $w_t^R \neq 0$. Since $\sum_i \omega_t^i = 1$, it follows that $\sum_i \sigma_t^i = 0$ and $\sum_i \mu_t^i = 0$. Using (15) and $\sum_i \sigma_t^i = 0$ implies that $\sum_i \omega_t^i w_t^i = 1$. Combining $\sum_i \omega_t^i w_t^i = 1$ with

²¹Assumption (20) implies that $B(\omega^{(1)}) \neq 0$ and therefore $B^2(\omega^{(1)}) > 0$.

(12) along with the definition $y_t = \frac{\overline{W}_t^-}{\overline{W}_t^+}$ gives

$$\kappa_t + (1 - \omega^R) \eta + \left(\omega_t^R \frac{1}{\sigma_D} \beta + (1 - \omega_t^R) \tau y_t \frac{1}{\sigma_D} \beta \right) 1_{\{\omega_t^R < 0\}} = \sigma^D. \quad (71)$$

Similarly, using (16) along with $\sum_i \mu_t^i = 0$ and $\sum_i \omega_t^i (n_t + w_t^i s_t^i) = 0$ gives (24).

We next describe the equilibria for the three subintervals of ω_t^R described in the statement of the proposition.

- i) In this case, $\omega_t^R > \omega^{(2)}$. The equilibrium prescribes non-negative portfolios for both investors. If $\omega_t^R > 1 - \frac{\sigma_D}{\eta}$, equation (71) implies that $\kappa_t > 0$ and (12) implies that both investors hold positive portfolios and the shorting market is inactive. If $\omega_t^R \in [\omega^{(1)}, 1 - \frac{\sigma_D}{\eta})$, then there is an equilibrium that involves no shorting and a zero portfolio for investor R . Under this supposition the market clearing requirement becomes $(1 - \omega_t^R) w_t^I = 1$, which together with $y_t = 0$ leads to (21). To confirm that this is indeed an equilibrium, note that

$$\begin{aligned} \kappa_t + \frac{\beta}{\sigma_D} &= \frac{\sigma_D}{1 - \omega_t^R} - \eta + \frac{\beta}{\sigma_D} \\ &> \frac{\sigma_D}{1 - \omega^{(1)}} - \eta + \frac{\beta}{\sigma_D} \\ &= 0. \end{aligned} \quad (72)$$

The first line follows from (21), the second line follows from $\omega_t^R > \omega^{(1)}$ and the third line follows from the definition of $\omega^{(1)}$. Since $\kappa_t + \frac{\beta}{\sigma_D} > 0$, investor R does not choose a negative portfolio. And since $\kappa_t < 0$ for $\omega_t^R \in [\omega^{(1)}, 1 - \frac{\sigma_D}{\eta})$, the investor chooses a zero portfolio.

- ii) In this case, $\omega^{(1)} < \omega_t^R < \omega^{(2)}$. Since $\omega_t^R > \omega^{(1)}$, equation (72) implies that the no-shorting equilibrium continues to be an equilibrium. However, we have two more

equilibria. To compute them, we guess (and verify shortly) that $w_t^R < 0$. Using (12) and (71) gives

$$\begin{aligned} y_t &= \frac{\overline{W}_t^-}{\overline{W}_t^+} = \frac{-\omega_t^R w_{t,s}^R}{(1 - \omega_t^R) w_{t,s}^I} = \frac{\omega_t^R}{1 - \omega_t^R} \frac{-\left[\kappa_t + \frac{1}{\sigma_D} \beta\right]}{\kappa_t + \eta_t + \frac{1}{\sigma_D} \beta \tau y_t} \\ &= \frac{\omega_t^R}{1 - \omega_t^R} \frac{(1 - \omega_t^R) \eta - \sigma_D - \frac{1 - \omega_t^R}{\sigma_D} \beta (1 - \tau y_t)}{\sigma_D + \omega_t^R \eta - \frac{\omega_t^R}{\sigma_D} \beta (1 - \tau y_t)}. \end{aligned}$$

Re-arranging leads to (22). Statement 1 of Lemma 2 implies that when $\omega_t \in (\omega^{(1)}, \omega^{(2)})$, then (22) has two roots between $(0, 1)$. Under the supposition that $w_t^R < 0$, (71) leads to (23). In turn

$$\begin{aligned} \kappa_t^\pm + \frac{\beta}{\sigma_D} &= \sigma_D - (1 - \omega_t^R) \eta - \frac{\omega_t^R}{\sigma_D} \beta \left[1 + \tau y_t^\pm \frac{1 - \omega_t^R}{\omega_t^R} \right] + \frac{\beta}{\sigma_D} \\ &= \sigma_D - (1 - \omega_t^R) \left(\eta + \frac{\beta}{\sigma_D} (1 - \tau y_t^\pm) \right) < 0, \end{aligned} \quad (73)$$

where the last inequality follows from statement 4 of Lemma 2. Combining this observation with (12) confirms that $w_t^R < 0$. Note that in the second and third equilibria we have that

$$\kappa_t^\pm + \eta_t + \frac{1}{\sigma_D} \beta \tau y_t^\pm = \sigma_D + \omega_t^R \eta - \frac{\beta \omega_t^R}{\sigma_D} (1 - \tau y_t^\pm) > 0,$$

where the last inequality follows from (73) along with the fact that y^\pm satisfy the equation (22). This implies that $w_t^I > 0$.

- iii) In this case, $\omega_t^R < \omega^{(1)}$. Statement 3 of Lemma 2 implies that the quadratic equation (22) has only one solution in $(0, 1)$. This shows that there can only be one equilibrium with shorting. Moreover, this is the unique equilibrium. If w_t^R were zero and the Sharpe

ratio were $\frac{\sigma_D}{1-\omega_t^R} - \eta^I$, then the argument given in (72) would imply that $\frac{\sigma_D}{1-\omega_t^R} - \eta + \frac{\beta}{\sigma_D} < 0$ and investor R would want to deviate from the equilibrium prescription and choose a negative portfolio. ■

Proof of Proposition 3. Case I: Suppose that $w_t^R = 0$ in equilibrium A and $w_t^R < 0$ in equilibrium B . Let $g_t^{R,j}$ denote the logarithmic growth rate of investor R in equilibrium $j \in \{A, B\}$. We have

$$\begin{aligned} g_t^{R,B} - g_t^{R,A} &= -(\kappa^B - \kappa^A) \sigma_D + \max_{w_t \leq 0} \left\{ w_t (\kappa^B \sigma_D + \beta) - \frac{1}{2} w_t^2 \sigma_t^2 \right\} \\ &> (\kappa^A - \kappa^B) \sigma_D \geq 0, \end{aligned}$$

where the first inequality follows from the fact that $w_t^R = 0$ is suboptimal for investor R in equilibrium B (by assumption). Similarly, using (16) gives

$$\begin{aligned} \mu_t^{R,B} - \mu_t^{R,A} &= \omega_t^R \left((\kappa^A - \kappa^B) \sigma_D + w_t^B \sigma_D \left(\kappa^B + \frac{\beta}{\sigma_D} - \sigma_D \right) \right) \\ &= \omega_t^R \left[(\kappa^A - \kappa^B) \sigma_D + (1 - \omega_t^R) w_t^B \sigma_D \left(\frac{\beta}{\sigma_D} (1 - y) - \eta \right) \right] \\ &= \omega_t^R \left[(\kappa^A - \kappa^B) \sigma_D + (1 - \omega_t^R) |w_t^B| \sigma_D \left(\eta - \frac{\beta}{\sigma_D} (1 - y) \right) \right] \\ &> 0, \end{aligned}$$

where the first equality follows from (23), the second equality from $w_t^B < 0$ and the inequality from assumption (19) along with $y < 1$.

Case II: In this case the portfolio choice of investor R is interior in both equilibria. Using the fact that in any interior equilibrium the optimal value of w_t satisfies

$$w_t (\kappa^B \sigma_D + \beta) - \frac{1}{2} w_t^2 \sigma_t^2 = \frac{1}{2} w_t^2 \sigma_t^2,$$

we obtain

$$\begin{aligned}
g_t^{R,B} - g_t^{R,A} &= -(\kappa^B - \kappa^A) \sigma_D + \frac{\sigma_D^2}{2} \left[\left(w_t^{R,B} \right)^2 - \left(w_t^{R,A} \right)^2 \right] \\
&= (\kappa^A - \kappa^B) \sigma_D + \left(w_t^{R,B} + w_t^{R,A} \right) \frac{\sigma_D^2}{2} \left[\left(w_t^{R,B} - w_t^{R,A} \right) \right] \\
&= (\kappa^A - \kappa^B) \sigma_D + \left(w_t^{R,B} + w_t^{R,A} \right) \frac{\sigma_D}{2} (\kappa^B - \kappa^A) \\
&= (\kappa^A - \kappa^B) \sigma_D \left[1 + \left| w_t^{R,B} + w_t^{R,A} \right| \frac{\sigma_D}{2} \right] \\
&> 0.
\end{aligned}$$

Using (16) gives

$$\begin{aligned}
\mu_t^{R,B} - \mu_t^{R,A} &= \omega_t^R \left((\kappa^A - \kappa^B) \sigma_D + w_t^{R,B} \sigma_D \left(\kappa^B + \frac{\beta}{\sigma_D} - \sigma_D \right) - w_t^{R,A} \sigma_D \left(\kappa^A + \frac{\beta}{\sigma_D} - \sigma_D \right) \right) \\
&= \omega_t^R \left((\kappa^A - \kappa^B) \sigma_D + \sigma_D^2 \left[w_t^{R,B} \left(w_t^{R,B} - 1 \right) - w_t^{R,A} \left(w_t^{R,A} - 1 \right) \right] \right) \\
&= \omega_t^R \left((\kappa^A - \kappa^B) \sigma_D + \sigma_D^2 \left[\left(w_t^{R,B} - \frac{1}{2} \right)^2 - \left(w_t^{R,A} - \frac{1}{2} \right)^2 \right] \right) \\
&= \omega_t^R \left((\kappa^A - \kappa^B) \sigma_D + \sigma_D^2 \left[\left(\left| w_t^{R,B} \right| + \frac{1}{2} \right)^2 - \left(\left| w_t^{R,A} \right| + \frac{1}{2} \right)^2 \right] \right) \\
&> 0,
\end{aligned}$$

where the last inequality follows from $w_t^{R,B} < w_t^{R,A} < 0$ (since $0 > \kappa^A > \kappa^B$) and therefore $\left| w_t^{R,B} \right| > \left| w_t^{R,A} \right|$. ■

Proof of Lemma 1. The implicit function theorem states that

$$\frac{d\omega^{(2)}}{d\beta} = -\frac{F_\beta}{F_\omega}.$$

Since $\lim_{\omega \rightarrow \infty} F(\omega) = -\infty$ and the root $F(\omega^{(2)}) = 0$ is unique (by assumption), it follows that $F_\omega(\omega^{(2)}) < 0$. So it suffices to prove that $F_\beta(\omega^{(2)}) < 0$.

Differentiating F with respect to β , multiplying the resulting expression by β and eval-

uating at $\omega^{(2)}$ (which implies that $F(\omega^{(2)}) = 0$) gives

$$\beta F_\beta = -2\omega^{(2)} \frac{\beta}{\sigma_D} (1 + \tau) \left(\sigma_D - \omega^{(2)} \left((1 + \tau) \frac{\beta}{\sigma_D} - \eta \right) \right) - \left(\sigma_D - \omega^{(2)} \left((1 + \tau) \frac{\beta}{\sigma_D} - \eta \right) \right)^2 - 4\tau (\omega^{(2)})^2 \frac{\beta^2}{\sigma_D^2}$$

Completing the square gives

$$\begin{aligned} \beta F_\beta &= - \left(\sigma_D - \omega^{(2)} \left((1 + \tau) \frac{\beta}{\sigma_D} - \eta \right) + \omega^{(2)} \frac{\beta}{\sigma_D} (1 + \tau) \right)^2 + (\omega^{(2)})^2 \frac{\beta^2}{\sigma_D^2} (1 - \tau)^2 \\ &= - (\sigma_D + \omega^{(2)} \eta)^2 + (\omega^{(2)})^2 \frac{\beta^2}{\sigma_D^2} (1 - \tau)^2 \\ &= - \left(\sigma_D + \omega^{(2)} \eta + \omega^{(2)} \frac{\beta}{\sigma_D} (1 - \tau) \right) \left(\sigma_D + \omega^{(2)} \eta - \omega^{(2)} \frac{\beta}{\sigma_D} (1 - \tau) \right) \\ &< 0, \end{aligned}$$

where the last inequality follows from the assumption $\eta \geq \frac{\beta}{\sigma_D}$. ■

Proof of Proposition 4. Differentiating κ_t with respect to β (in an equilibrium where $y > 0$), we obtain

$$\frac{d\kappa_t}{d\beta} = -\frac{\omega_t^R}{\sigma_D} \left\{ 1 + \frac{1 - \omega_t^R}{\omega_t^R} \tau y_t \left[1 + \frac{\beta}{y_t} \frac{dy_t}{d\beta} \right] \right\}. \quad (74)$$

In turn, the implicit function theorem applied to (22) gives

$$\frac{dy_t}{d\beta} = -\frac{\frac{\omega_t^R}{\sigma_D} (1 - \tau y) (1 - y)}{\sigma_D + \omega_t^R \eta - \frac{\omega_t^R}{\sigma_D} \beta (1 + \tau - 2\tau y)} = -\frac{\frac{\omega_t^R}{\sigma_D} (1 - \tau y) (1 - y)}{g'(y)},$$

where $g(y)$ is defined in Proposition 1. Since $g'(y^-) < 0$ and $g'(y^+) > 0$, we have $\frac{dy^-}{d\beta} > 0$ and $\frac{dy^+}{d\beta} < 0$. Combining $\frac{dy^-}{d\beta} > 0$ with (74) implies $\frac{d\kappa_t}{d\beta} < 0$ in the equilibrium associated with y^- . For the equilibrium associated with y^+ we have

$$1 + \frac{\beta}{y^+} \frac{dy^+}{d\beta} = \frac{\left(\sigma_D + \omega_t^R \eta - \frac{\omega_t^R}{\sigma_D} \beta (1 + \tau - 2\tau y^+) \right) y^+ - \frac{\beta \omega_t^R}{\sigma_D} (1 - \tau y^+) (1 - y^+)}{\left(\sigma_D + \omega_t^R \eta - \frac{\omega_t^R}{\sigma_D} \beta (1 + \tau - 2\tau y^+) \right) y^+}. \quad (75)$$

We are interested in the behavior of (75) as ω_t^R approaches zero. Letting $x \equiv \frac{y^+}{\omega_t^R}$, dividing both sides of (22) by ω_t^R and re-arranging terms yields

$$x (\sigma_D + \omega_t^R \eta) + \frac{\beta}{\sigma_D} (1 - \omega_t^R x) (1 - \tau \omega_t^R x) = \frac{1}{1 - \omega_t^R} ((1 - \omega_t^R) \eta^I - \sigma_D).$$

Taking limits as ω_t^R approaches zero, implies

$$\lim_{\omega_t^R \rightarrow 0} x = \frac{\eta - \sigma_D - \frac{\beta}{\sigma_D}}{\sigma_D}. \quad (76)$$

Using (76), as ω_t^R approaches zero we obtain

$$\begin{aligned} \lim_{\omega_t^R \rightarrow 0} \left\{ 1 + \frac{1 - \omega_t^R}{\omega_t^R} \tau y_t \left[1 + \frac{\beta}{y_t} \frac{dy_t}{d\beta} \right] \right\} &= 1 + \tau \lim_{\omega_t^R \rightarrow 0} \{x_t\} \times \lim_{\omega_t^R \rightarrow 0} \left\{ 1 + \frac{\beta}{y_t} \frac{dy_t}{d\beta} \right\} \\ &= \tau \lim_{\omega_t^R \rightarrow 0} \{x_t\} \times \left\{ 1 - \frac{\frac{\beta}{\sigma_D}}{\sigma_D \lim_{\omega_t^R \rightarrow 0} \{x_t\}} \right\} \\ &= \tau \left[\frac{\eta - \sigma_D - 2 \frac{\beta}{\sigma_D}}{\sigma_D} \right] \end{aligned} \quad (77)$$

$$< 0, \quad (78)$$

where we used (19) to derive the last inequality. Combining (78) with (74) implies that, for small ω_t^R , $\frac{d\kappa(y^+)}{d\beta} > 0$. ■

Proof of Proposition 5. The proof essentially repeats the steps from the one-risky asset case, so we provide only a sketch, focusing on the elements that differ. We define

$$\vec{\beta} = \begin{bmatrix} \beta \\ 0 \end{bmatrix}, \vec{\eta} = \begin{bmatrix} \eta \\ 0 \end{bmatrix}.$$

We consider first an equilibrium with $y_t > 0$. Investor R 's and I 's optimal portfolios are

given by

$$\vec{w}_t^R = (\sigma_t \sigma'_t)^{-1} \left(\vec{\mu}_t - r_t \mathbf{1}_{2 \times 1} + \vec{\beta} \right), \quad (79)$$

$$\vec{w}_t^I = (\sigma_t \sigma'_t)^{-1} \left(\vec{\mu}_t - r_t \mathbf{1}_{2 \times 1} + \sigma_{1,t} \vec{\eta} + \tau y_t \vec{\beta} \right). \quad (80)$$

Using (79) inside (36) yields

$$\begin{aligned} (\sigma_t \sigma'_t) \vec{m}_t &= (1 - \widehat{\omega}_t) \left[\widetilde{\omega}_t^R \left(\vec{\mu}_t - r \mathbf{1}_N + \vec{\beta} \right) + (1 - \widetilde{\omega}_t^R) \left(\vec{\mu}_t - r \mathbf{1}_N + \sigma_1 \vec{\eta} + \tau y_t \vec{\beta} \right) \right] \\ &\quad + \widehat{\omega}_t (\sigma_t \sigma'_t) \begin{bmatrix} 0 \\ \frac{\mu_{2,t} - r}{\sigma_{2,t}^2} \end{bmatrix}, \end{aligned} \quad (81)$$

where we introduced the short-hand notation $\widetilde{\omega}_t^R = \frac{\omega_t^R}{1 - \widehat{\omega}_t}$ and $\widetilde{\omega}_t^I = \frac{\omega_t^I}{1 - \widehat{\omega}_t}$. (Since $\omega_t^R + \omega_t^I + \widehat{\omega}_t = 1$, it follows that $\widetilde{\omega}_t^R + \widetilde{\omega}_t^I = 1$.)

Next we use the row selection vector $[0, 1]$ to pre-multiply both sides of (81). Noting that $[0, 1] \vec{\beta} = [0, 1] \vec{\eta} = 0$, and also

$$(\sigma_t \sigma'_t) \begin{bmatrix} 0 \\ \frac{\mu_{2,t} - r}{\sigma_{2,t}^2} \end{bmatrix} = \begin{bmatrix} b_t (\mu_{2,t} - r) \\ \mu_{2,t} - r \end{bmatrix}, \quad (82)$$

leads to (39). We next note that

$$\begin{aligned} [1, -b_t] \sigma \sigma' \begin{bmatrix} m_{1,t} \\ m_{2,t} \end{bmatrix} &= [1, -b_t] \begin{bmatrix} (\sigma_{1,t}^2 + b_t^2 \sigma_{2,t}^2) m_{1,t} + m_{2,t} b_t \sigma_{2,t}^2 \\ m_{1,t} b_t \sigma_{2,t}^2 + m_{2,t} \sigma_{2,t}^2 \end{bmatrix} \\ &= \sigma_{1,t}^2 m_{1,t}. \end{aligned} \quad (83)$$

Pre-multiplying both sides of (81) with the row vector $[1, -b_t]$, using (82) and (83) and

the definition of $\kappa_{1,t}$ re-arranging yields

$$\kappa_{1,t} = \tilde{m}_{1,t}\sigma_{1,t} - (1 - \tilde{\omega}_t^R)\eta - \tilde{\omega}_t^R \frac{\beta}{\sigma_{1,t}} \left(1 + \frac{1 - \tilde{\omega}_t^R}{\tilde{\omega}_t^R} y_t \right). \quad (84)$$

Using the definition of $\kappa_{1,t}$ inside (79) gives

$$w_{1,t}^R = \frac{\kappa_{1,t}}{\sigma_{1,t}} + \frac{\beta}{\sigma_{1,t}^2}, \text{ and } w_{1,t}^I = \frac{\kappa_{1,t} + \eta}{\sigma_1} + \frac{\tau y_t \beta}{\sigma_{1,t}^2}, \quad (85)$$

where we used the notation $w_t^{1,R}, w_t^{1,I}$ to denote the first element of w_t^R and w_t^I respectively.

Using the market clearing condition $y_t = \frac{\omega_t^R w_{1,t}^R}{\omega_t^I w_{1,t}^I} = \frac{\tilde{\omega}_t^R w_{1,t}^R}{\tilde{\omega}_t^I w_{1,t}^I} = \frac{\tilde{\omega}_t^R w_{1,t}^R}{(1 - \tilde{\omega}_t^R) w_{1,t}^I}$ leads to (85) leads to (37).

If agent R chooses not to short then the market clearing condition becomes

$$\omega_t^I w_t^I + (\hat{\omega}_t + \omega_t^R) \begin{bmatrix} 0 \\ \hat{w}_t \end{bmatrix} = m_t. \quad (86)$$

Substituting (80), pre-multiplying by $(\sigma_t \sigma_t')$ gives

$$(\sigma_t \sigma_t') \vec{m}_t = \omega_t^I (\vec{\mu}_t - r \mathbf{1}_N + \sigma_1 \vec{\eta}) + (\hat{\omega}_t + \omega_t^R) (\sigma_t \sigma_t') \begin{bmatrix} 0 \\ \frac{\mu_{2,t} - r}{\sigma_{2,t}^2} \end{bmatrix}. \quad (87)$$

Pre-multiplying (87) by the row $[1, -b_t]$ and using (82) and (83) gives

$$\sigma_{1,t}^2 m_{1,t} = \omega_t^I \sigma_{1,t} (\kappa_{1,t} + \eta).$$

Divding both sides by $\omega_t^I \sigma_{1,t}$ yields $\kappa_{1,t} = \sigma_{1,t} \frac{m_{1,t}}{\omega_t^I} - \eta = \sigma_{1,t} \frac{\tilde{m}_{1,t}}{\tilde{\omega}_t^I} - \eta = \sigma_{1,t} \frac{\tilde{m}_{1,t}}{1 - \tilde{\omega}_t^R} - \eta$. Finally, when both agents hold positive portfolios, the optimal portfolios are $\vec{w}_t^R = (\sigma_t \sigma_t')^{-1} (\vec{\mu}_t - r_t \mathbf{1}_{2 \times 1})$, $\vec{w}_t^I = (\sigma_t \sigma_t')^{-1} (\vec{\mu}_t - r_t \mathbf{1}_{2 \times 1} + \sigma_{1,t} \vec{\eta})$. Repeating the arguments in equations (79)-(84), $\kappa_{1,t}$

becomes $\kappa_{1,t} = \tilde{m}_{1,t}\sigma_{1,t} - (1 - \tilde{\omega}_t^R)\eta$. ■

Proof of Proposition 6. The proof of this Proposition is contained in Appendix B. ■

Proof of Proposition 7. Since this proof is essentially identical to the proof of Proposition 2, we only provide a sketch. Combining (12) with $\sum_i \omega_t^i w_t^i = 1$ implies that in any equilibrium with $w_t^R < 0$ and $w_t^I > 0$ the Sharpe ratio is

$$\kappa_t + (1 - \omega^R)\eta + \omega_t^R \frac{1}{\sigma_D} f_t + (1 - \omega_t^R) \tau y_t \frac{1}{\sigma_D} f_t = \sigma_D. \quad (88)$$

Re-arranging (88) and using $f_t = l(y_t)$ gives (47). Substituting (47) back into the investors' optimal portfolios (12) and the fact that $y_t = -\frac{\omega_t^R w_t^R}{(1-\omega_t^R)w_t^I}$ leads to (46).

We next study the roots of (46). Let $Z(y) \equiv \frac{\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(y)}{\eta + \frac{\sigma_D}{\omega_t^R} - \frac{1}{\sigma_D} z(y)}$, so that equation (46) can be expressed as $y = Z(y)$. The assumption of the proposition is that there exists at least one y such that $y = Z(y)$. Let $y^{(*)}$ be the largest root of (46). We consider two cases i) $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(y) > 0$ for all $y \in [y^{(*)}, 1]$ and ii) $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(\bar{y}) = 0$ for some $\bar{y} \in [y^{(*)}, 1]$. In case i) it must be that $Z'(y^{(*)}) \leq 1$, since $y^{(*)} = Z(y^{(*)})$ and $1 > Z(1)$. In case ii) it must also be that $Z'(y^{(*)}) \leq 1$ since $\bar{y} > Z(\bar{y}) = 0$. The assumption that $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(0) < 0$ implies that there exists some interval $[y, y^{(*)}]$ such that $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(y) > 0$ for all $y \in [y, y^{(*)}]$. Since $Z'(y^{(*)}) \leq 1$ it must be the case that there exists at least one more root in the interval $y \in [y, y^{(*)}]$.

Finally, to confirm that a no-shorting equilibrium is also an equilibrium, $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(0) < 0$ implies that $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} l(0) < 0$. If the Sharpe ratio is given by $\kappa_t = \frac{\sigma_D}{1-\omega_t^R} - \eta < 0$, the assumption that $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} l(0) < 0$ implies that $\kappa_t + \frac{1}{\sigma_D} l(0) > 0$. Accordingly, investor R does not wish to short when the fee is $f_t = l(0)$ and the lending market clears with $y = 0$ at the lending fee $l(0)$. Moreover, $w_t^I = \frac{\kappa_t + \eta}{\sigma_D} = \frac{1}{1-\omega_t^R}$. Therefore $\omega_t^R \times 0 + (1 - \omega_t^R) \times w_t^I = 1$ and the stock market clears. ■

D Additional Data Discussion

D.1 Methodology

D.1.1 Measuring ticker discussion on WallstreetBets

Our measure of ticker mentions on WallstreetBets is constructed as follows. We use the PushshiftAPI to collect all submissions posted on WallstreetBets subreddit from January 1, 2020 through February 7, 2021 (Baumgartner et al., 2020). For each submission, we observe the title text, the body of the submission, the author of the submission, and the time of the submission.

In order to identify which tickers are discussed in the submission, we take advantage of the fact that users often tag tickers with a leading \$ (i.e. \$TSLA or \$AAPL). This practice is entirely voluntary and is therefore insufficient for identifying all mentions of a ticker. We use regular expressions to identify all words tagged in this way and match those words to CRSP tickers that were traded on the NYSE, AMEX, and NASDAQ exchanges in 2020. This gives us a set S of roughly 4,000 tickers that are mentioned on WSB between January 2020 and February 2021.

We then identify all cases in which these tickers are mentioned in submissions, irrespective of whether they are prefixed with a dollar sign. To address the possibility of falsely identifying tickers, we require that, if the ticker is a common word in the written English language, it must be prefaced by a dollar sign. For example, AT&T’s ticker T is also a common word in written English, and thus we require that the text “\$T” appear in a submission for it to be considered as mentioned AT&T. We consider a ticker as being mentioned in a submission if it appears in either the title or the body of the submission. We identify common word-stems based on the Google Trillion Word Corpus (Michel et al., 2011). In a robustness check, we account for the downward bias this restriction introduces by scaling common-word tickers

by an in-sample estimated adjustment factor. This adjustment leaves the relative ranking of ticker mentions largely unchanged. We estimate the adjustment factor by comparing the frequency of tagged ticker mentions versus untagged ticker mentions for the set of tickers which do not commonly appear in written English.

Revised submissions and comments. Authors of Reddit comments have the ability to edit their comments even after the comment has been posted. The PushshiftAPI records the comment text as of a certain day, and does not update to reflect potential revised comments. The same constraint applies to the content body of submissions. Titles of submissions cannot be revised and thus do not have this measurement problem.

Missed tickers Tickers that, for whatever reason, are never tagged with a leading dollar sign will be omitted from our dataset. Similarly, we under-count the occurrences of tickers that are common words, owing to requiring they appear with a leading “\$” We attempt to correct for this by scaling the observed counts for common word tickers. For AAPL and GME, which are not common word tickers, the ticker appears with the leading “\$” roughly 20% of the time. We can thus simply multiply our observed frequencies by a factor of five to adjust for the more stringent matching procedure. As can be seen in Figures 16a and 16b, the adjustment does not have a significant impact on the relative popularity of the top tickers.

In some cases, users may choose to refer to the company by its name, rather than by its ticker. We do not attempt to identify mentions of companies by name.

D.2 Measuring retail trading

We adopt the methodology of Boehmer et al. (2020) to identify retail trades in the TAQ data. We briefly summarize the methodology here and refer readers to the paper for details.

The intuition behind the methodology is the knowledge that retail trades are often executed by wholesalers or via broker internalization, rather than on the major trading exchanges. These trades appear in the TAQ consolidated tape data under the exchange code “D”. These trades are given a small price improvement on the order of tenths of a penny as a means to induce brokers to route orders to the wholesaler. Similarly, brokers which internalize retail trades offer a subpenny price improvement in order to comply with Regulation 606T. Importantly, institutional trades are rarely, if ever, internalized or directed to wholesalers and their trades are usually in round penny prices, with the notable exception of midpoint trades.

The methodology of Boehmer et al. (2020) uses these institutional details to identify retail trades in the TAQ consolidated tape data. Trades flagged with exchange code “D” and with a subpenny amount in the set $(0, 0.40) \cup (0.60, 1.00)$ are identified as retail trades. Splitting these trades further, retail trades with subpenny amounts between zero- and forty-hundredths of a penny are labeled as “sell orders”, whereas subpenny amounts between sixty- and one hundred-hundredths are considered “buy orders”. The midpoint trades are excluded to avoid mis-classifying institutional trades executed at midpoints as retail trades.

D.2.1 Challenges

Derivatives The TAQ data only contains trades of equities. Options offer another way to benefit for investors to benefit from increases in the price of stock. As an added advantage for retail investors, options offer embedded leverage greater than what might otherwise be available through their broker. The Boehmer et al. (2020) methodology relies on institutional details to identify off-exchange retail trades, and thus cannot reliably identify replication trades by market makers.

D.3 Betting against the shorts portfolio

As is standard in the literature, we restrict attention to common shares of COMPUSTAT firms which trade on the NYSE, AMEX, and NASDAQ exchanges. We further exclude companies for whom no share class has a price exceeding \$1. The strategy equally weights each firm in the top decile, shorts the market index, and reconstitutes 8 trading days following the disclosure date, which is the first opportunity following the public dissemination of the short interest data.

E Additional Figures and Tables

E.1 Analyzing Reddit Discussion

Consistent with the views expressed in the press, there is evidence that stocks with comparatively higher short interest were more likely to be discussed on the WSB subreddit. To substantiate this claim, we analyze the text of all WSB posts and identify all mentions of individual stocks. In Table 2, we investigate which stock characteristics make a stock a likely topic for discussion on the WSB subreddit over the two week period from January 15 to January 25, 2021. Consistent with the popular press, short interest as of January 15, 2021 is strongly correlated with these mentions. We show this using two specifications. In columns (1)–(3), we use ticker-mention percentile as the y-variable. The most discussed ticker, GME, takes a value of 1, while a stock that is never discussed takes a value of 0. By using the ranked ticker-mentions, rather than the raw counts, we avoid over-weighting right-tail observations in the data.

Since several stocks are not discussed at all, we estimate a Tobit model to account for the left-censoring of the y-variable at 0. In column (1), we first include only a concise set of explanatory variables: short interest and index inclusion for both the Russell 3000 and the S&P 500. The reported coefficient on short interest is for a standardized version of the variable, and has the interpretation that a one standard-deviation increase in short interest is associated with a 34 percent increase in the distribution of ticker-mentions. The coefficient on short interest is economically large and significant even when we include financial ratios (column (2)) and factor betas (column (3)). Both financial ratios and factor betas are calculated as of December 2020. A one standard deviation increase in short interest leads to a 24 to 34 percent increase in the distribution of ticker mentions. Short interest is measured as shares shorted divided by total shares outstanding, value-weighted across share classes of a given firm. In columns (4)–(6), we repeat the analysis with a different y-variable: an

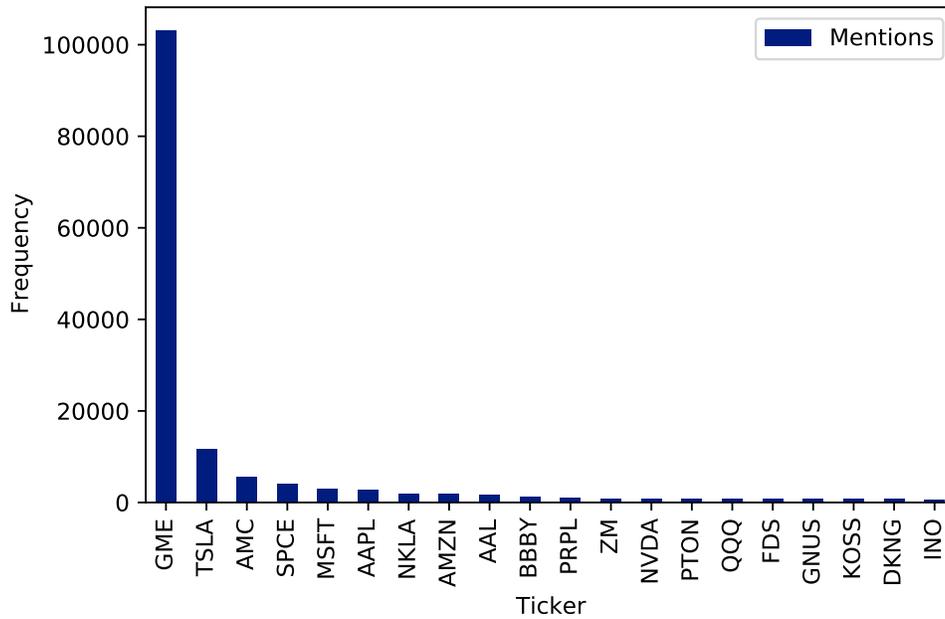
indicator for whether the ticker was discussed at all on WSB. This linear probability model, which we estimate using OLS, uses only the extensive margin of ticker discussion on WSB, whereas the Tobit model uses both the extensive and intensive margins of ticker-mentions. We find that a one standard deviation increase in short interest is associated with a 8 to 11 percent higher probability of being discussed on WSB. As can be seen from the count of uncensored observations in our Tobit model, roughly 13 percent of tickers are ever mentioned on WSB, so this coefficient is large in magnitude.

Table 2: Characteristics of highly-discussed tickers

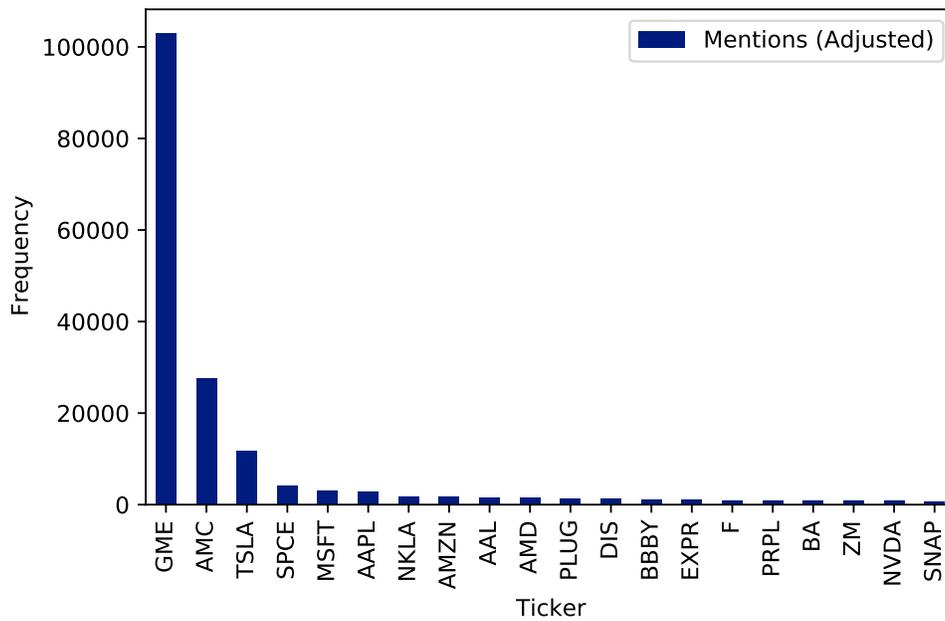
	Tobit			OLS		
	Rank	Rank	Rank	Indicator	Indicator	Indicator
Short Interest	0.335*** (0.023)	0.309*** (0.024)	0.241*** (0.024)	0.107*** (0.006)	0.099*** (0.006)	0.081*** (0.007)
Russell Constituent	-0.482*** (0.065)	-0.502*** (0.071)	-0.197** (0.086)	-0.113*** (0.014)	-0.115*** (0.015)	-0.044** (0.020)
SPX Constituent	0.905*** (0.077)	0.977*** (0.086)	1.024*** (0.091)	0.244*** (0.018)	0.242*** (0.019)	0.249*** (0.020)
Book-Market		-0.171*** (0.042)	-0.067* (0.037)		-0.022*** (0.006)	-0.012* (0.007)
Debt-Assets		0.028 (0.040)	0.107** (0.042)		0.011 (0.009)	0.029*** (0.009)
Dividend Yield		-0.002 (0.047)	0.026 (0.045)		0.002 (0.009)	0.008 (0.009)
Dividend Yield Missing		0.282** (0.112)	0.144 (0.111)		0.042* (0.022)	0.008 (0.023)
β_{MKT}			0.091*** (0.029)			0.015** (0.007)
β_{SMB}			-0.019 (0.032)			-0.003 (0.007)
β_{HML}			-0.213*** (0.038)			-0.046*** (0.009)
β_{UMD}			0.167*** (0.035)			0.037*** (0.008)
Idio. Vol.			0.229*** (0.035)			0.058*** (0.008)
N_{unc}	428	390	390			
N	3100	2947	2947	3100	2947	2947
R^2	0.118	0.132	0.167	0.125	0.128	0.154

Columns (1)-(3) report results estimated using a left-censored tobit model on ticker discussion percentile as the left hand side variable. Columns (4)-(6) report results estimated using OLS on a dummy variable indicating whether a ticker was discussed on the WallstreetBets sub-reddit. Both Rank and Indicator variables are based on tickers mention between January 15 and January 25. β and volatility co-variates are calculated as of January 15, 2021. Financial ratios are calculated as of most recent reporting date available on WRDS.

Note: Coefficients reported are for continuous co-variates standardized to have mean zero and unit standard deviation. N_{unc} denotes number of uncensored observations in the tobit specification. R^2 for columns (1)-(3) denotes McFadden's pseudo R-squared. * p < 0.1, ** p < 0.05, *** p < 0.01. Standard errors in parentheses.

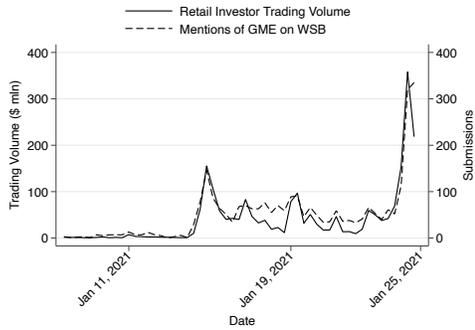


(a) Submissions mentioning each Ticker

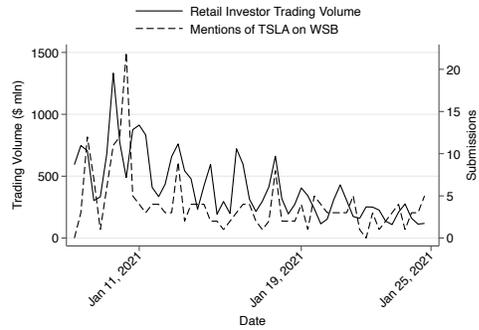


(b) Submissions mentioning each Ticker, adjusted for word-ticker overlap

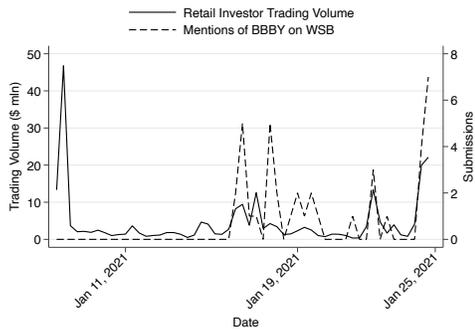
Figure 16: Popular Tickers on WallstreetBets (January 1, 2020 - February 7, 2021).



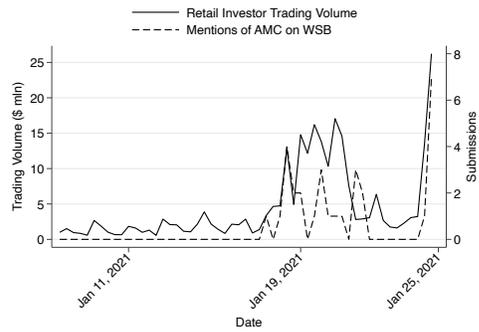
(a) Gamestop (GME)



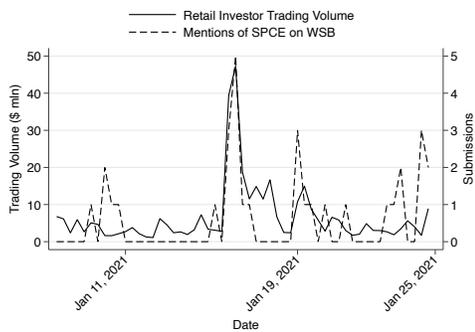
(b) Telsa (TSLA)



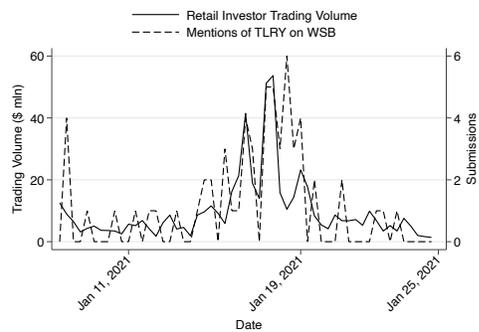
(c) Bed, Bath and Beyond (BBBY)



(d) AMC (AMC)

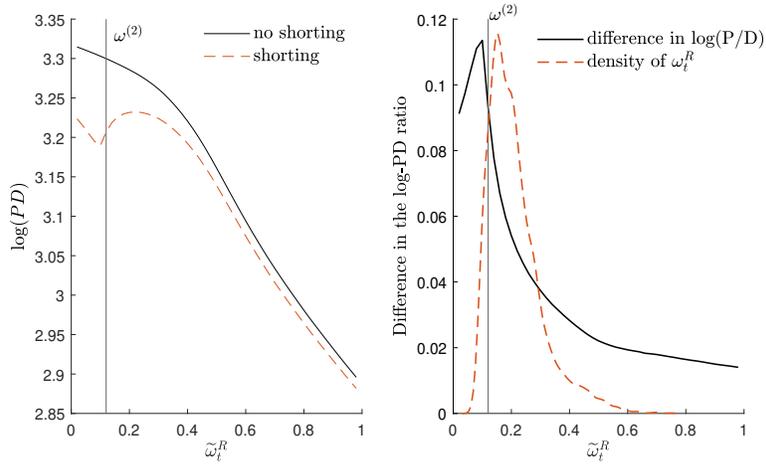


(e) Virgin Galactic (SPCE)

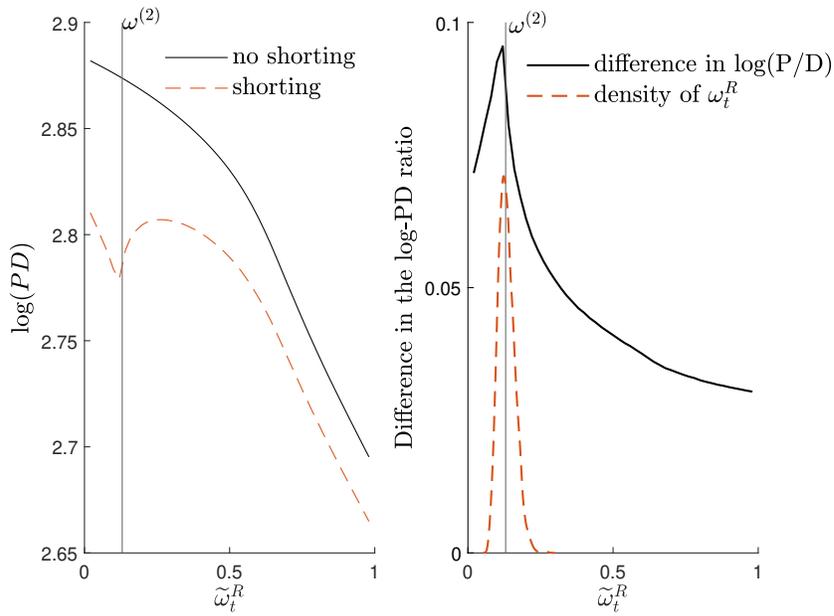


(f) Tilray (TLRY)

Figure 17: Retail Trading Volume and Reddit Discussion (January 7 - January 25, 2021).



(a) $\delta = 0.1$



(b) $r + \delta_1 = 0.10$

Figure 18: This figure repeats the same quantitative exercise as Figure 13 but with modified $r + \delta_1$ and δ .