Supplement to "Trade, Technology, Size and the Division of Labor"

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A Omitted Proofs

A.1 Omitted Proofs in Proposition 2

A.1.1 Omitted Derivations in Deriving the Multiplier Effect

Expression 1:

$$\frac{d\ln\bar{\alpha}}{d\ln s_1} = \frac{1}{\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i - \bar{\alpha}\right)\frac{d\ln\tilde{Y}_i}{d\ln s_1}.$$

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Proof. Using $\bar{\alpha} = \alpha_e + \delta \sum_{i=e}^{n}(i-e)\lambda_i$, we obtain

$$
\begin{aligned}
&\frac{d \ln \bar{\alpha}}{d \ln s_1} \\
=&\frac{\delta \sum_{i=e}^{n}(i-e)\lambda_i}{\bar{\alpha}} \frac{d \ln}{d \ln s_1}\left[\sum_{i=e}^{n}(i-e)\lambda_i\right] \\
=&\frac{\delta \sum_{i=e}^{n}(i-e)\lambda_i}{\bar{\alpha}} \sum_{i=e}^{n}\left[\frac{(i-e)\lambda_i}{\sum_{r=e}^{n}(r-e)\lambda_r}\right]\frac{d \ln \lambda_i}{d \ln s_1} \\
=&\frac{\delta}{\bar{\alpha}} \sum_{i=e}^{n}(i-e)\lambda_i \left(\frac{d \ln Y_i}{d \ln s_1} - \sum_{r=e}^{n}\lambda_r \frac{d \ln Y_r}{d \ln s_1}\right) \\
=&\frac{1}{\bar{\alpha}}\left[\sum_{i=e}^{n} i\delta\lambda_i \left(\frac{d \ln Y_i}{d \ln s_1} - \sum_{r=e}^{n}\lambda_r \frac{d \ln Y_r}{d \ln s_1}\right) - e\delta \sum_{i=e}^{n}\lambda_i \left(\frac{d \ln Y_i}{d \ln s_1} - \sum_{r=e}^{n}\lambda_r \frac{d \ln Y_r}{d \ln s_1}\right)\right] \\
=&\frac{1}{\bar{\alpha}}\left[\sum_{i=e}^{n} \alpha_i\lambda_i \left(\frac{d \ln Y_i}{d \ln s_1} - \sum_{r=e}^{n}\lambda_r \frac{d \ln Y_r}{d \ln s_1}\right) - e\delta \left(\sum_{i=e}^{n}\lambda_i \frac{d \ln Y_i}{d \ln s_1} - \sum_{i=e}^{n}\lambda_i \sum_{r=e}^{n}\lambda_r \frac{d \ln Y_r}{d \ln s_1}\right)\right] \\
=&\frac{1}{\bar{\alpha}}\left(\sum_{i=e}^{n} \alpha_i\lambda_i \frac{d \ln Y_i}{d \ln s_1} - \sum_{i=e}^{n}\alpha_i\lambda_i \sum_{r=e}^{n}\lambda_r \frac{d \ln Y_r}{d \ln s_1}\right) \\
=&\frac{1}{\bar{\alpha}}\left(\sum_{i=e}^{n} \alpha_i\lambda_i \frac{d \ln Y_i}{d \ln s_1} - \bar{\alpha} \sum_{r=e}^{n}\lambda_r \frac{d \ln Y_r}{d \ln s_1}\right) \\
=&\frac{1}{\bar{\alpha}} \sum_{i=e}^{n}\lambda_i \left(\alpha_i - \bar{\alpha}\right) \frac{d \ln Y_i}{d \ln s_1} \\
=&\frac{1}{\bar{\alpha}} \sum_{i=e}^{n}\lambda_i \left(\alpha_i - \bar{\alpha}\right) \frac{d \ln \tilde{Y}_i}{d \ln s_1},
\end{aligned}
$$

where we use $\sum_{i=e}^{n}\lambda_i = 1$, $\alpha_i \equiv i\delta$, $\bar{\alpha} \equiv \sum_{i=e}^{n}\lambda_i\alpha_i$, and $\sum_{i=e}^{n}\lambda_i\left(\alpha_i - \bar{\alpha}\right) = 0$. ∎

Expression 2:

$$
\frac{1}{\bar{\alpha}} \sum_{i=e}^{n}\lambda_i \left(\alpha_i - \bar{\alpha}\right) \frac{d \ln \tilde{Y}_i}{d \ln s_1} = \frac{\sigma - 1}{\delta\bar{\alpha}} \sum_{i=e}^{n}\lambda_i \left(\alpha_i - \bar{\alpha}\right)^2 + \frac{1}{\bar{\alpha}} \sum_{i=e}^{n}\lambda_i \left(\alpha_i - \bar{\alpha}\right)\left(\sum_{j=e}^{n}\frac{\partial \ln \tilde{Y}_i}{\partial \ln \bar{\varphi}_j}\frac{d \ln \bar{\varphi}_j}{d \ln s_1}\right).
$$

Proof. Using $\frac{d\ln\tilde{Y}_i}{d\ln s_1}=(\sigma-1)(i-e)+\sum_{j=e}^{n}\frac{\partial\ln\tilde{Y}_i}{\partial\ln\bar{\varphi}_j}\frac{d\ln\bar{\varphi}_j}{d\ln s_1}$ we obtain

$$
\begin{aligned}
&\frac{1}{\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)\frac{d\ln\tilde{Y}_i}{d\ln s_1}\\
=&\frac{\sigma-1}{\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)(i-e)+\frac{1}{\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)\left(\sum_{j=e}^{n}\frac{\partial\ln\tilde{Y}_i}{\partial\ln\bar{\varphi}_j}\frac{d\ln\bar{\varphi}_j}{d\ln s_1}\right)\\
=&\frac{\sigma-1}{\delta\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)i\delta+\frac{1}{\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)\left(\sum_{j=e}^{n}\frac{\partial\ln\tilde{Y}_i}{\partial\ln\bar{\varphi}_j}\frac{d\ln\bar{\varphi}_j}{d\ln s_1}\right)\\
=&\frac{\sigma-1}{\delta\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)\alpha_i+\frac{1}{\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)\left(\sum_{j=e}^{n}\frac{\partial\ln\tilde{Y}_i}{\partial\ln\bar{\varphi}_j}\frac{d\ln\bar{\varphi}_j}{d\ln s_1}\right)\\
=&\frac{\sigma-1}{\delta\bar{\alpha}}\left[\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)\alpha_i-\bar{\alpha}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)\right]\\
&+\frac{1}{\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)\left(\sum_{j=e}^{n}\frac{\partial\ln\tilde{Y}_i}{\partial\ln\bar{\varphi}_j}\frac{d\ln\bar{\varphi}_j}{d\ln s_1}\right)\\
=&\frac{\sigma-1}{\delta\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)^2+\frac{1}{\bar{\alpha}}\sum_{i=e}^{n}\lambda_i\left(\alpha_i-\bar{\alpha}\right)\left(\sum_{j=e}^{n}\frac{\partial\ln\tilde{Y}_i}{\partial\ln\bar{\varphi}_j}\frac{d\ln\bar{\varphi}_j}{d\ln s_1}\right).
\end{aligned}
$$

∎

A.1.2 Omitted Proof in Deriving Positive Extensive Margin

Expression 1:

$$
\begin{aligned}
&\sum_{i=e}^{n}\left(\alpha_i-\bar{\alpha}\right)\lambda_i\sum_{j=e}^{n}\frac{\partial\ln\tilde{Y}_i}{\partial\ln\bar{\varphi}_j}\frac{d\ln\bar{\varphi}_j}{d\ln s_1}\\
=&-\frac{\left(\alpha_e-\bar{\alpha}\right)\left(\bar{\varphi}_e\right)^{\sigma}g\left(\bar{\varphi}_e\right)}{\sum_{i=e}^{n}(s_1)^{(i-e)(\sigma-1)}\int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}}\varphi^{\sigma-1}dG(\varphi)}\frac{d\ln\bar{\varphi}_e}{d\ln s_1}\\
&+\sum_{i=e+1}^{n}\frac{\left(\alpha_{i-1}-\bar{\alpha}\right)\left(s_1\right)^{(i-e-1)(\sigma-1)}-\left(\alpha_i-\bar{\alpha}\right)\left(s_1\right)^{(i-e)(\sigma-1)}}{\sum_{i=e}^{n}(s_1)^{(i-e)(\sigma-1)}\int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}}\varphi^{\sigma-1}dG(\varphi)}\left(\bar{\varphi}_i\right)^{\sigma}g\left(\bar{\varphi}_i\right)\frac{d\ln\bar{\varphi}_i}{d\ln s_1}.
\end{aligned}
$$

Proof. Using the sales expression $\tilde{Y}_i = (s_1)^{(i-e)(\sigma-1)} \int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi)$ we obtain

$$
\begin{aligned}
&\lambda_i \sum_{j=e}^{n} \frac{\partial \ln \tilde{Y}_i}{\partial \ln \bar{\varphi}_j} \frac{d \ln \bar{\varphi}_j}{d \ln s_1} \\
=&\lambda_i \sum_{j=e}^{n} \frac{\partial \ln}{\partial \ln \bar{\varphi}_j} \left[\int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi) \right] \frac{d \ln \bar{\varphi}_j}{d \ln s_1} \\
=&\lambda_i \left(\frac{\left(\bar{\varphi}_{i+1}\right)^{\sigma} g\left(\bar{\varphi}_{i+1}\right)}{\int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi)} \frac{d \ln \bar{\varphi}_{i+1}}{d \ln s_1} - \frac{\left(\bar{\varphi}_i\right)^{\sigma} g\left(\bar{\varphi}_i\right)}{\int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi)} \frac{d \ln \bar{\varphi}_i}{d \ln s_1} \right) \\
=&\frac{(s_1)^{(i-e)(\sigma-1)} \left(\bar{\varphi}_{i+1}\right)^{\sigma} g\left(\bar{\varphi}_{i+1}\right)}{\sum_{i=e}^{n} (s_1)^{(i-e)(\sigma-1)} \int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi)} \frac{d \ln \bar{\varphi}_{i+1}}{d \ln s_1} - \frac{(s_1)^{(i-e)(\sigma-1)} \left(\bar{\varphi}_i\right)^{\sigma} g\left(\bar{\varphi}_i\right)}{\sum_{i=e}^{n} (s_1)^{(i-e)(\sigma-1)} \int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi)} \frac{d \ln \bar{\varphi}_i}{d \ln s_1}.
\end{aligned}
$$

Thus, the extensive margin is

$$
\begin{aligned}
&\sum_{i=e}^{n} \left(\alpha_i - \bar{\alpha}\right) \lambda_i \sum_{j=e}^{n} \frac{\partial \ln \tilde{Y}_i}{\partial \ln \bar{\varphi}_j} \frac{d \ln \bar{\varphi}_j}{d \ln s_1} \\
=&\sum_{i=e}^{n} (\alpha_i - \bar{\alpha}) \frac{(s_1)^{(i-e)(\sigma-1)} \left(\bar{\varphi}_{i+1}\right)^{\sigma} g\left(\bar{\varphi}_{i+1}\right)}{\sum_{i=e}^{n} (s_1)^{(i-e)(\sigma-1)} \int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi)} \frac{d \ln \bar{\varphi}_{i+1}}{d \ln s_1} \\
&- \sum_{i=e}^{n} (\alpha_i - \bar{\alpha}) \frac{(s_1)^{(i-e)(\sigma-1)} \left(\bar{\varphi}_i\right)^{\sigma} g\left(\bar{\varphi}_i\right)}{\sum_{i=e}^{n} (s_1)^{(i-e)(\sigma-1)} \int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi)} \frac{d \ln \bar{\varphi}_i}{d \ln s_1} \\
=&- \frac{\left(\alpha_e - \bar{\alpha}\right) \left(\bar{\varphi}_e\right)^{\sigma} g\left(\bar{\varphi}_e\right)}{\sum_{i=e}^{n} (s_1)^{(i-e)(\sigma-1)} \int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi)} \frac{d \ln \bar{\varphi}_e}{d \ln s_1} \\
&+ \sum_{i=e+1}^{n} \frac{\left(\alpha_{i-1} - \bar{\alpha}\right) (s_1)^{(i-e-1)(\sigma-1)} - \left(\alpha_i - \bar{\alpha}\right) (s_1)^{(i-e)(\sigma-1)}}{\sum_{i=e}^{n} (s_1)^{(i-e)(\sigma-1)} \int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}} \varphi^{\sigma-1} dG(\varphi)} \left(\bar{\varphi}_i\right)^{\sigma} g\left(\bar{\varphi}_i\right) \frac{d \ln \bar{\varphi}_i}{d \ln s_1}.
\end{aligned}
$$

∎

Expression 2:

$$
(s_1)^{\sigma-1} > \frac{\frac{\bar{\alpha}}{\delta} - b + 1}{\frac{\bar{\alpha}}{\delta} - b}, \quad \left(\frac{\bar{\alpha}}{\delta} - b - 1 \right) (s_1)^{\sigma-1} < \frac{\bar{\alpha}}{\delta} - b,
$$

where b is a unique positive integer which satifies $\frac{d \ln \bar{\varphi}_b}{d \ln s_1} > 0$ and $\frac{d \ln \bar{\varphi}_{b+1}}{d \ln s_1} < 0$.

Proof. First, from $\frac{d\ln\bar{\varphi}_b}{d\ln s_1} > 0$ we have

$$\begin{aligned}
\frac{d\ln\bar{\varphi}_b}{d\ln s_1} > 0 \;&\Rightarrow\; -\frac{1}{(s_1)^{\sigma-1}-1} - (b-e) + \frac{\bar{\alpha}-\alpha_e}{\delta} > 0 \\
&\Rightarrow\; \frac{1}{(s_1)^{\sigma-1}-1} < \frac{\bar{\alpha}-\alpha_e}{\delta} - (b-e) \\
&\Rightarrow\; \frac{1}{(s_1)^{\sigma-1}-1} < \frac{\bar{\alpha}}{\delta} - b \\
&\Rightarrow\; \left(\frac{\bar{\alpha}}{\delta} - b\right)(s_1)^{\sigma-1} > \frac{\bar{\alpha}}{\delta} - b + 1 \\
&\Rightarrow\; (s_1)^{\sigma-1} > \frac{\frac{\bar{\alpha}}{\delta} - b + 1}{\frac{\bar{\alpha}}{\delta} - b},
\end{aligned}$$

where the last line uses $\frac{\bar{\alpha}}{\delta} - b > 0$, which can be observed from the third line.

Second, from $\frac{d\ln\bar{\varphi}_{b+1}}{d\ln s_1} < 0$ we have

$$\begin{aligned}
\frac{d\ln\bar{\varphi}_{b+1}}{d\ln s_1} < 0 \;&\Rightarrow\; -\frac{1}{(s_1)^{\sigma-1}-1} - (b-e+1) + \frac{\bar{\alpha}-\alpha_e}{\delta} < 0 \\
&\Rightarrow\; \frac{1}{(s_1)^{\sigma-1}-1} > \frac{\bar{\alpha}-\alpha_e}{\delta} - (b-e+1) \\
&\Rightarrow\; \frac{1}{(s_1)^{\sigma-1}-1} > \frac{\bar{\alpha}}{\delta} - (b+1) \\
&\Rightarrow\; \left(\frac{\bar{\alpha}}{\delta} - b - 1\right)(s_1)^{\sigma-1} < \frac{\bar{\alpha}}{\delta} - b.
\end{aligned}$$

∎

A.2 Derivation Details in the Proof of Proposition 3

Derivation details 1

$$\left(\frac{\bar{\varphi}_i}{\bar{\varphi}_e}\right)^{\sigma-1} = \left(s_1^{\sigma-1} - 1\right)^{-1} \left(s_1^{-1}\right)^{(i-e-1)(\sigma-1)} \left(\frac{F_i}{F_e}\right) \tag{A.1}$$

Proof. Using the expressions for productivity cutoffs we obtain

$$\begin{aligned}\left(\frac{\bar{\varphi}_i}{\bar{\varphi}_e}\right)^{\sigma-1} &\equiv \left(\frac{\bar{\varphi}_{e+1}}{\bar{\varphi}_e}\right)\prod_{r=e+2}^{i}\left(\frac{\bar{\varphi}_r}{\bar{\varphi}_{r-1}}\right) \\ &= (S_1)^{-1}\prod_{r=e+2}^{i}\left(\frac{f_r}{f_{r-1}}\right)(s_1^{-(\sigma-1)}) \\ &= (S_1)^{-1}\prod_{r=e+2}^{i}\left(\frac{F_r}{F_{r-1}}\right)(s_1^{-(\sigma-1)}) \\ &= \left(s_1^{\sigma-1}-1\right)^{-1}\hat{f}\left(\frac{F_i}{F_{e+1}}\right)(s_1^{-(\sigma-1)})^{(i-e-1)} \\ &= \left(s_1^{\sigma-1}-1\right)^{-1}\left(\frac{F_i}{F_e}\right)(s_1^{-(\sigma-1)})^{(i-e-1)}.\end{aligned} \tag{A.2}$$

The second equality uses the expressions of cutoffs.

The third equality uses part 2 of the technology assumption, the definition of F_r and re-arranges

$$\begin{aligned}\frac{f_r}{f_{r-1}} &= 1+\hat{f} = \hat{f}\left(1+\frac{1}{\hat{f}}\right) \\ &= \frac{\Delta f_r}{f_{r-1}}\left(1+\frac{f_{r-2}}{\Delta f_{r-1}}\right) = \frac{\Delta f_r}{f_{r-1}}\frac{f_{r-1}}{\Delta f_{r-1}} \\ &= \frac{\Delta f_r}{\Delta f_{r-1}}.\end{aligned}$$

The fourth equality in (A.2) uses the definition of S_1 and expands the product. The fifth agian uses the technology assumption and the definition of $F_{e+1} \equiv \Delta f_{e+1} = \hat{f} f_e$ and $F_e \equiv f_e$. ∎

Derivation details 2

$$f_e\int_{\bar{\varphi}_e}^{\bar{\varphi}_{e+1}}\left(\frac{\varphi}{\bar{\varphi}_e}\right)^{\sigma-1}dG(\varphi) + \frac{{s_1}^{\sigma-1}}{s_1^{\sigma-1}-1}\sum_{i=e+1}^{n}F_i\int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}}\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1}dG(\varphi) = \sum_{i=e}^{n}F_i\int_{\bar{\varphi}_i}^{\infty}\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1}dG(\varphi)$$

Proof. We show it as follows.

$$
\begin{aligned}
& f_e \int_{\bar\varphi_e}^{\bar\varphi_{e+1}} \left(\frac{\varphi}{\bar\varphi_e}\right)^{\sigma-1} dG(\varphi) + \sum_{i=e+1}^{n} \frac{{s_1}^{\sigma-1}}{s_1^{\sigma-1}-1} F_i \int_{\bar\varphi_i}^{\bar\varphi_{i+1}} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi) \\
=& f_e \left[\int_{\bar\varphi_e}^{\infty} \left(\frac{\varphi}{\bar\varphi_e}\right)^{\sigma-1} dG(\varphi) - \int_{\bar\varphi_{e+1}}^{\infty} \left(\frac{\varphi}{\bar\varphi_e}\right)^{\sigma-1} dG(\varphi)\right] \\
& + \sum_{i=e+1}^{n} \frac{{s_1}^{\sigma-1}}{s_1^{\sigma-1}-1} F_i \left[\int_{\bar\varphi_i}^{\infty} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi) - \int_{\bar\varphi_{i+1}}^{\infty} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi)\right] \\
=& f_e \left[\int_{\bar\varphi_e}^{\infty} \left(\frac{\varphi}{\bar\varphi_e}\right)^{\sigma-1} dG(\varphi) - \left(\frac{\bar\varphi_{e+1}}{\bar\varphi_e}\right)^{\sigma-1} \int_{\bar\varphi_{e+1}}^{\infty} \left(\frac{\varphi}{\bar\varphi_{e+1}}\right)^{\sigma-1} dG(\varphi)\right] \\
& + \sum_{i=e+1}^{n} \frac{{s_1}^{\sigma-1}}{s_1^{\sigma-1}-1} F_i \left[\int_{\bar\varphi_i}^{\infty} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi) - \left(\frac{\bar\varphi_{i+1}}{\bar\varphi_i}\right)^{\sigma-1} \int_{\bar\varphi_{i+1}}^{\infty} \left(\frac{\varphi}{\bar\varphi_{i+1}}\right)^{\sigma-1} dG(\varphi)\right] \\
=& \left[F_e \int_{\bar\varphi_e}^{\infty} \left(\frac{\varphi}{\bar\varphi_e}\right)^{\sigma-1} dG(\varphi) - \frac{F_{e+1}}{s_1^{\sigma-1}-1} \int_{\bar\varphi_{e+1}}^{\infty} \left(\frac{\varphi}{\bar\varphi_{e+1}}\right)^{\sigma-1} dG(\varphi)\right] \\
& + \sum_{i=e+1}^{n} \frac{{s_1}^{\sigma-1}}{s_1^{\sigma-1}-1} \left[F_i \int_{\bar\varphi_i}^{\infty} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi) - \frac{F_{i+1}}{s_1^{\sigma-1}} \int_{\bar\varphi_{i+1}}^{\infty} \left(\frac{\varphi}{\bar\varphi_{i+1}}\right)^{\sigma-1} dG(\varphi)\right] \\
=& \left[F_e \int_{\bar\varphi_e}^{\infty} \left(\frac{\varphi}{\bar\varphi_e}\right)^{\sigma-1} dG(\varphi) - \frac{F_{e+1}}{s_1^{\sigma-1}-1} \int_{\bar\varphi_{e+1}}^{\infty} \left(\frac{\varphi}{\bar\varphi_{e+1}}\right)^{\sigma-1} dG(\varphi)\right] \\
& + \sum_{i=e+1}^{n} \left[\frac{{s_1}^{\sigma-1}}{s_1^{\sigma-1}-1} F_i \int_{\bar\varphi_i}^{\infty} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi) - \frac{F_{i+1}}{s_1^{\sigma-1}-1} \int_{\bar\varphi_{i+1}}^{\infty} \left(\frac{\varphi}{\bar\varphi_{i+1}}\right)^{\sigma-1} dG(\varphi)\right] \\
=& \sum_{i=e}^{n} F_i \int_{\bar\varphi_i}^{\infty} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi),
\end{aligned}
$$

where the first equality writes the integral as difference, the second factors out $\frac{\bar\varphi_{i+1}}{\bar\varphi_i}$, the third substitutes in the expression for relative cutoffs derived in (A.1), the last two rearrange terms and combine the $\int_{\bar\varphi_i}^{\infty} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi)$ terms. ∎

Derivation details 3 Here we provide the intermediate steps to show how substituting $d\bar\varphi_i$ into

$$
\sum_{i=e}^{n} \left[\frac{F_i}{\bar\varphi_i} \int_{\bar\varphi_i}^{\infty} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi)\right] d\bar\varphi_i = 0 \tag{A.3}
$$

yields

$$
\frac{d \ln \bar\varphi_e}{d \ln s_1} = \frac{\sum_{i=e+1}^{n} F_e (i-e) (s_1)^{(i-e)(\sigma-1)} \int_{\bar\varphi_i}^{\bar\varphi_{i+1}} \left(\frac{\varphi}{\bar\varphi_e}\right)^{\sigma-1} dG(\varphi)}{\sum_{i=e}^{n} \left[F_i \int_{\bar\varphi_i}^{\infty} \left(\frac{\varphi}{\bar\varphi_i}\right)^{\sigma-1} dG(\varphi)\right]}. \tag{A.4}
$$

Proof. Substituting the expression for $d\bar{\varphi}_i$ into (A.3) and solving for $\frac{\bar{\varphi}_i}{\bar{\varphi}_e}$ in (A.1) yields

$$\frac{d\ln\bar{\varphi}_e}{d\ln s_1}=\frac{\overbrace{\sum_{i=e+1}^{n}\left(\frac{s_1^{\sigma-1}}{s_1^{\sigma-1}-1}+(i-e-1)\right)\left[F_i\int_{\bar{\varphi}_i}^{\infty}\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1}dG(\varphi)\right]}^{\equiv\Upsilon}}{F_e\int_{\bar{\varphi}_e}^{\infty}\left(\frac{\varphi}{\bar{\varphi}_e}\right)^{\sigma-1}dG(\varphi)+\sum_{i=e+1}^{n}\left[F_i\int_{\bar{\varphi}_i}^{\infty}\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1}dG(\varphi)\right]}. \tag{A.5}$$

Next we show that the numerator, defined as Υ, simplifies to what we have in (A.4). For $i>e$, we have

$$\begin{aligned}
&F_i\int_{\bar{\varphi}_i}^{\infty}\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1}dG(\varphi)\\
=&F_i\left[\int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}}\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1}dG(\varphi)+\int_{\bar{\varphi}_{i+1}}^{\infty}\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1}dG(\varphi)\right]\\
=&F_i\left[\int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}}\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1}dG(\varphi)+\left(\frac{\bar{\varphi}_{i+1}}{\bar{\varphi}_i}\right)^{\sigma-1}\int_{\bar{\varphi}_{i+1}}^{\infty}\left(\frac{\varphi}{\bar{\varphi}_{i+1}}\right)^{\sigma-1}dG(\varphi)\right]\\
=&F_i\int_{\bar{\varphi}_i}^{\bar{\varphi}_{i+1}}\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1}dG(\varphi)+\frac{F_{i+1}}{s_1^{\sigma-1}}\int_{\bar{\varphi}_{i+1}}^{\infty}\left(\frac{\varphi}{\bar{\varphi}_{i+1}}\right)^{\sigma-1}dG(\varphi)\\
&\dots\\
=&\sum_{r=i}^{n}\frac{F_r}{(s_1)^{(r-i)(\sigma-1)}}\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}}\left(\frac{\varphi}{\bar{\varphi}_r}\right)^{\sigma-1}dG(\varphi)
\end{aligned}$$

where the penultimate step uses $\left(\frac{\bar{\varphi}_{i+1}}{\bar{\varphi}_i}\right)^{\sigma-1}=F_{i+1}/s_1^{\sigma-1}$ for $i>e$. The last step results from continuing the same procedure all the way to the maximum intensity $\int_{\bar{\varphi}_n}^{\infty}\left(\frac{\varphi}{\bar{\varphi}_n}\right)^{\sigma-1}dG(\varphi)$. Substituting back we have

$$\begin{aligned}
\Upsilon&=\sum_{i=e+1}^{n}\left(\frac{s_1^{\sigma-1}}{s_1^{\sigma-1}-1}+(i-e-1)\right)\left[\sum_{r=i}^{n}\frac{F_r}{(s_1)^{(r-i)(\sigma-1)}}\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}}\left(\frac{\varphi}{\bar{\varphi}_r}\right)^{\sigma-1}dG(\varphi)\right]\\
&=\sum_{r=e+1}^{n}\left[\sum_{i=e+1}^{r}\left(\frac{s_1^{\sigma-1}}{s_1^{\sigma-1}-1}+(i-e-1)\right)(s_1)^{-(r-i)(\sigma-1)}\right]F_r\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}}\left(\frac{\varphi}{\bar{\varphi}_r}\right)^{\sigma-1}dG(\varphi)\\
&=\sum_{r=e+1}^{n}\Xi_rF_r(s_1)^{-(r-e-1)(\sigma-1)}\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}}\left(\frac{\varphi}{\bar{\varphi}_r}\right)^{\sigma-1}dG(\varphi)\\
&=\sum_{r=e+1}^{n}\Xi_rF_r(s_1)^{-(r-e-1)(\sigma-1)}\left(\frac{\bar{\varphi}_e}{\bar{\varphi}_r}\right)^{\sigma-1}\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}}\left(\frac{\varphi}{\bar{\varphi}_e}\right)^{\sigma-1}dG(\varphi)\\
&=\sum_{r=e+1}^{n}\Xi_rF_e\left(s_1^{\sigma-1}-1\right)\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}}\left(\frac{\varphi}{\bar{\varphi}_e}\right)^{\sigma-1}dG(\varphi),
\end{aligned}$$

where $\Xi_r=\sum_{i=e+1}^{r}\left(\frac{1}{s_1^{\sigma-1}-1}+(i-e)\right)(s_1)^{(i-e-1)(\sigma-1)}$. The second equality changes the order of summation (instead of summing first across r, we first sum across i). The last equality uses the expression for $\left(\frac{\bar{\varphi}_e}{\bar{\varphi}_r}\right)^{\sigma-1}$. We can further simplify Ξ_r as

$$\begin{aligned}
\Xi_r &= \sum_{i=e+1}^{r}\left(\frac{1}{s_1^{\sigma-1}-1}+(i-e)\right)(s_1)^{(i-e-1)(\sigma-1)} \\
&= \sum_{i=e+1}^{r}\frac{1}{s_1^{\sigma-1}-1}(s_1)^{(i-e-1)(\sigma-1)}+\sum_{i=e+1}^{r}(i-e)(s_1)^{(i-e-1)(\sigma-1)} \\
&= \frac{(s_1)^{(r-e)(\sigma-1)}-1}{\left(s_1^{\sigma-1}-1\right)^2}+\left[\frac{(r-e)(s_1)^{(r-e)(\sigma-1)}}{s_1^{\sigma-1}-1}-\frac{(s_1)^{(r-e)(\sigma-1)}-1}{\left(s_1^{\sigma-1}-1\right)^2}\right] \\
&= \frac{(r-e)(s_1)^{(r-e)(\sigma-1)}}{s_1^{\sigma-1}-1},
\end{aligned}$$

where the second equality uses

$$\sum_{i=e+1}^{r}(i-e)(s_1)^{(i-e-1)(\sigma-1)}=\frac{(r-e)(s_1)^{(r-e)(\sigma-1)}}{s_1^{\sigma-1}-1}-\frac{(s_1)^{(r-e)(\sigma-1)}-1}{\left(s_1^{\sigma-1}-1\right)^2},$$

which can be shown by noticing that

$$\sum_{i=e+1}^{r}(i-e)(s_1)^{(i-e-1)(\sigma-1)}=\frac{d}{ds_1^{\sigma-1}}\left[\sum_{i=e+1}^{r}(s_1)^{(i-e)(\sigma-1)}\right]=\frac{d}{ds_1^{\sigma-1}}\left[\frac{(s_1)^{(r-e+1)(\sigma-1)}-s_1^{\sigma-1}}{s_1^{\sigma-1}-1}\right].$$

Thus,

$$\begin{aligned}
\Upsilon &= F_e\left(s_1^{\sigma-1}-1\right)\sum_{r=e+1}^{n}\Xi_r\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}}\left(\frac{\varphi}{\bar{\varphi}_e}\right)^{\sigma-1}dG(\varphi) \\
&= F_e\left(s_1^{\sigma-1}-1\right)\sum_{r=e+1}^{n}\left[\frac{(r-e)(s_1)^{(r-e)(\sigma-1)}}{s_1^{\sigma-1}-1}\right]\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}}\left(\frac{\varphi}{\bar{\varphi}_e}\right)^{\sigma-1}dG(\varphi) \\
&= \sum_{r=e+1}^{n}F_e(r-e)(s_1)^{(r-e)(\sigma-1)}\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}}\left(\frac{\varphi}{\bar{\varphi}_e}\right)^{\sigma-1}dG(\varphi).
\end{aligned}$$

Replacing Υ in (A.5) yields (A.4). ∎

A.3 Lemma in the Proof of Proposition 8

Lemma 1. *Under any $g_k(\cdot)$ s.t. $\frac{(\bar{\varphi}_i)^{\sigma}g_k(\bar{\varphi}_i)}{\int_{\bar{\varphi}_i}^{\infty}\varphi^{\sigma-1}dG_k(\varphi)}=\tilde{K}(\sigma)$ for any $\bar{\varphi}_i>0$, $\sigma\geq 1$ and $\tilde{K}(\sigma)>0$ we have $\frac{d\ln\bar{F}}{d\ln s_1}=0$ and thus $\epsilon_M^s>0$ under heterogeneous specialization.*

Proof.

$$
\begin{aligned}
\frac{d\ln \bar{F}}{d\ln s_1} &= \frac{d\ln \Sigma_{i=e}^{n} F_i \int_{\bar{\varphi}_i}^{\infty} dG(\varphi)}{d\ln s_1} \\
&= -\Sigma_{i=e}^{n} \frac{F_i \int_{\bar{\varphi}_i}^{\infty} dG(\varphi)}{\bar{F}} \frac{\bar{\varphi}_i g(\bar{\varphi}_i)}{\int_{\bar{\varphi}_i}^{\infty} dG(\varphi)} \frac{d\ln \bar{\varphi}_i}{d\ln s_1} \\
&= -\Sigma_{i=e}^{n} \frac{F_i \int_{\bar{\varphi}_i}^{\infty} \left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} dG(\varphi)}{\bar{F}} \frac{\bar{\varphi}_i g(\bar{\varphi}_i)}{\int_{\bar{\varphi}_i}^{\infty} \left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} dG(\varphi)} \frac{d\ln \bar{\varphi}_i}{d\ln s_1} \\
&= -\Sigma_{i=e}^{n} \frac{(\bar{\varphi}_i)^{\sigma} g(\bar{\varphi}_i)}{\int_{\bar{\varphi}_i}^{\infty} \varphi^{\sigma-1} dG(\varphi)} \frac{F_i \int_{\bar{\varphi}_i}^{\infty} \left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} dG(\varphi)}{\bar{F}} \frac{d\ln \bar{\varphi}_i}{d\ln s_1} \\
&= -\tilde{K}(\sigma) \frac{\Sigma_{i=e}^{n} F_i \int_{\bar{\varphi}_i}^{\infty} \left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} dG(\varphi) \frac{d\ln \bar{\varphi}_i}{d\ln s_1}}{\bar{F}} \\
&= 0,
\end{aligned}
$$

where the third line divides and multiplies by $\int_{\bar{\varphi}_i}^{\infty} \left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} dG(\varphi)$, the fifth line uses the property of $g_k(\cdot)$, the last line follows from differentiating the free entry condition with respect to s_1 as follows.

$$
\begin{aligned}
\frac{d\ln}{d\ln s_1} \sum_{i=e}^{n} F_i \int_{\bar{\varphi}_i}^{\infty} \left[\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} - 1\right] dG(\varphi) &= 0 \\
\sum_{i=e}^{n} \frac{F_i \int_{\bar{\varphi}_i}^{\infty} \left[\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} - 1\right] dG(\varphi)}{f_E} \frac{(\sigma-1) \int_{\bar{\varphi}_i}^{\infty} \left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} dG(\varphi)}{\int_{\bar{\varphi}_i}^{\infty} \left[\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} - 1\right] dG(\varphi)} \frac{d\ln \bar{\varphi}_i}{d\ln s_1} &= 0 \\
\frac{\sigma-1}{f_E} \sum_{i=e}^{n} \left[F_i \int_{\bar{\varphi}_i}^{\infty} \left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} dG(\varphi)\right] \frac{d\ln \bar{\varphi}_i}{d\ln s_1} &= 0,
\end{aligned}
$$

where the second line follows from

$$
\begin{aligned}
&\frac{d\ln}{d\ln s_1} \left\{F_i \int_{\bar{\varphi}_i}^{\infty} \left[\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} - 1\right] dG(\varphi)\right\} \\
&= - \frac{\left[\left(\frac{\bar{\varphi}_i}{\bar{\varphi}_i}\right)^{\sigma-1} - 1\right] + (\sigma-1) \int_{\bar{\varphi}_i}^{\infty} \left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} dG(\varphi)}{\int_{\bar{\varphi}_i}^{\infty} \left[\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} - 1\right] dG(\varphi)} \frac{d\ln \bar{\varphi}_i}{d\ln s_1} \\
&= - \frac{(\sigma-1) \int_{\bar{\varphi}_i}^{\infty} \left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} dG(\varphi)}{\int_{\bar{\varphi}_i}^{\infty} \left[\left(\frac{\varphi}{\bar{\varphi}_i}\right)^{\sigma-1} - 1\right] dG(\varphi)} \frac{d\ln \bar{\varphi}_i}{d\ln s_1}
\end{aligned}
$$

∎

A.4 Lemma in the Proof of Proposition 9

Lemma 2. *$\tilde{\pi}(\varphi, L)$ is a continuous increasing function of φ and there is some x^* s.t. $\tilde{\pi}(\varphi^*, L) = x^* = \tilde{\pi}(\varphi^*, L')$ with $\tilde{\pi}(\varphi, L) \leq \tilde{\pi}(\varphi, L')$ for $\varphi > \varphi^*$ and $\tilde{\pi}(\varphi, L) \geq \tilde{\pi}(\varphi, L')$ otherwise. Moreover, φ^* is unique iff there is a change in specialization, otherwise $\tilde{\pi}(\varphi, L) = \tilde{\pi}(\varphi, L')$ for all φ.*

Proof. (1) Continuity of $\tilde{\pi}(\cdot)$

Within any technology type $\tilde{\pi}(\varphi) = \hat{\sigma} X P^{\sigma-1} \left[c_i(\varphi)\right]^{1-\sigma}$ so it is differentiable in φ and thus continuous. To show continuity at any adoption threshold $\bar{\varphi}_i$ where firms are indifferent between i and $i-1$ we show $\lim_{\varphi \to \bar{\varphi}_i^-} \tilde{\pi}(\varphi) = \lim_{\varphi \to \bar{\varphi}_i^+} \tilde{\pi}(\varphi)$. Denoting $\hat{\pi}_i(\varphi)$ as the variable profit of firm φ using technology i we have

$$\begin{aligned} \lim_{\varphi \to \bar{\varphi}_i^-} \tilde{\pi}(\varphi) &= \lim_{\varphi \to \bar{\varphi}_i^-} \hat{\pi}_{i-1}(\varphi) - f_{i-1} \\ &= \hat{\pi}_{i-1}(\bar{\varphi}_i) - f_{i-1} = \hat{\pi}_i(\bar{\varphi}_i) - f_i \\ &= \lim_{\varphi \to \bar{\varphi}_i^+} \hat{\pi}_i(\varphi) - f_i = \lim_{\varphi \to \bar{\varphi}_i^+} \tilde{\pi}(\varphi) \end{aligned}$$

where the equalities in the first and third line are respectively because for $\varphi < \bar{\varphi}_i$ firms prefer $i-1$ and i otherwise. The middle equality in the second line reflects the indifference adoption condition that defines $\bar{\varphi}_i$ and the remaining ones the continuity of $\tilde{\pi}_i$ within technology.

(2) Existence of φ^* s.t. $\tilde{\pi}(\varphi^*, L) = x^* = \tilde{\pi}(\varphi^*, L')$. Moreover, for small enough increase in size from L to L', firm φ^* upgrades technology by one step: $\tilde{\pi}_l(\varphi^*, L') = \tilde{\pi}_{l-1}(\varphi^*, L)$, with firm φ^* using technology l under size L'.

For the first part, due to positive selection effect, the entry cutoff is higher under L', $\bar{\varphi}_e^{L'} > \bar{\varphi}_e^L$, so for firms with productivity $\bar{\varphi}_e^L < \varphi < \bar{\varphi}_e^{L'}$, we have $\tilde{\pi}(\varphi, L') = 0 = \tilde{\pi}\left(\bar{\varphi}_e^L, L\right) < \tilde{\pi}(\varphi, L)$. Moreover, as average profits per entering firms do not change by the free entry condition, $\int_{\bar{\varphi}_e^L}^{\infty} \tilde{\pi}(\varphi, L) dG(\varphi) = \int_{\bar{\varphi}_e^{L'}}^{\infty} \tilde{\pi}(\varphi, L') dG(\varphi) = f_E$, there must exist some $\varphi > \bar{\varphi}_e^{L'}$ such that $\tilde{\pi}(\varphi, L') > \tilde{\pi}(\varphi, L)$. Thus by continuity of profit function there exists a φ^* such that $\tilde{\pi}(\varphi^*, L) = \tilde{\pi}(\varphi^*, L') = x^*$.

For the second part, first note that a firm φ^* must have upgraded under L' because its gross profits are unchanged even though markets are more competitive. Thus:

$$\tilde{\pi}_{l-1}(\varphi^*, L) = \tilde{\pi}_l(\varphi^*, L') = x^*. \tag{A.6}$$

Technology downgrading would imply lower profits for all firms φ, $\tilde{\pi}_l(\varphi, L') < \tilde{\pi}_{l+1}(\varphi, L)$ due to positive selection (shown below)

(3) $\tilde{\pi}(\varphi, L) < \tilde{\pi}(\varphi, L')$ for all $\varphi > \varphi^*$ and $\tilde{\pi}(\varphi, L) \geq \tilde{\pi}(\varphi, L')$ otherwise, so φ^* is unique.

Consider any firm with productivity φ that uses technology I under L'. Given (A.6) we consider three cases and show the following:

(3.1) If $\varphi < \bar{\varphi}_b^{L'}$ then $\tilde{\pi}(\varphi, L') < \tilde{\pi}(\varphi, L)$;

(3.2) If $\bar{\varphi}_b^{L'} < \varphi < \varphi^*$ then $\tilde{\pi}(\varphi, L') < \tilde{\pi}(\varphi, L)$;

(3.3) If $\varphi > \varphi^*$ then $\tilde{\pi}(\varphi, L') > \tilde{\pi}(\varphi, L)$,

where b is defined as the technology cutoff for downgrading, $\bar{\varphi}_b^{L'} > \bar{\varphi}_b^L$ and $\bar{\varphi}_{b-1}^{L'} < \bar{\varphi}_{b-1}^L$, and the existence of such b is shown in the proof for positive extensive margin in proposition 2.

Before proving each case we derive expressions useful across them. The first expression is firm profit in terms of productivity cutoffs,

$$\tilde{\pi}(\varphi, L') = \hat{\pi}_I(\varphi, L') - f_I = \hat{\pi}_I(\bar{\varphi}_I^{L'}, L') \left(\frac{\varphi}{\bar{\varphi}_I^{L'}}\right)^{\sigma-1} - f_I, \tag{A.7}$$

where $\hat{\pi}_I(\varphi, L')$ is the operating profit of firm φ under technology I, and the second equality uses $\frac{\hat{\pi}_I(\varphi, L')}{\hat{\pi}_I(\bar{\varphi}_I^{L'}, L')} = \left(\frac{\varphi}{\bar{\varphi}_I^{L'}}\right)^{\sigma-1}$ given a common technology I.

The second expression is the profit of a firm with cutoff $\bar{\varphi}_k^{L'}$ in terms of $\bar{\varphi}_m^{L'}$ with $k > m$:

$$\hat{\pi}_k(\bar{\varphi}_k^{L'}, L') = \hat{\pi}_m(\bar{\varphi}_m^{L'}, L') \left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_m^{L'}}\right)^{\sigma-1} + \sum_{i=m+1}^{k} F_i \left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1}, \tag{A.8}$$

which uses the indifference adoption condition shown in Technote 1. Equivalently,

$$\hat{\pi}_m(\bar{\varphi}_m^{L'}, L') = \hat{\pi}_k(\bar{\varphi}_k^{L'}, L') \left(\frac{\bar{\varphi}_m^{L'}}{\bar{\varphi}_k^{L'}}\right)^{\sigma-1} - \sum_{i=m+1}^{k} F_i \left(\frac{\bar{\varphi}_m^{L'}}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1}. \tag{A.9}$$

The last one uses expression (A.6) so that

$$\hat{\pi}_l(\varphi^*, L') = \hat{\pi}_{l-1}(\varphi^*, L) + f_l - f_{l-1} = \hat{\pi}_{l-1}(\varphi^*, L) + F_l. \tag{A.10}$$

(3.1) Case of downgrading: $\bar{\varphi}_I^{L'} > \bar{\varphi}_I^{L}$

Any firm with $\varphi < \bar{\varphi}_{I+1}^{L'}$, may downgrade after the size increase, i.e. $\frac{d\bar{\varphi}_i}{dL} > 0$ for $i \leq I$. The profit function becomes

$$\begin{aligned}
\tilde{\pi}(\varphi, L') &= \hat{\pi}_I(\bar{\varphi}_I^{L'}, L') \left(\frac{\varphi}{\bar{\varphi}_I^{L'}}\right)^{\sigma-1} - f_I \\
&= \hat{\pi}_e(\bar{\varphi}_e^{L'}, L') \left(\frac{\varphi}{\bar{\varphi}_e^{L'}}\right)^{\sigma-1} + \sum_{i=e+1}^{I} F_i \left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&= \hat{\pi}_e(\bar{\varphi}_e^{L}, L) \left(\frac{\bar{\varphi}_e^{L}}{\bar{\varphi}_e^{L'}}\right)^{\sigma-1} \left(\frac{\varphi}{\bar{\varphi}_e^{L}}\right)^{\sigma-1} + \sum_{i=e+1}^{I} F_i \left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&< \hat{\pi}_e(\bar{\varphi}_e^{L}, L) \left(\frac{\varphi}{\bar{\varphi}_e^{L}}\right)^{\sigma-1} + \sum_{i=e+1}^{I} F_i \left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&= \hat{\pi}_I(\bar{\varphi}_I^{L}, L) \left(\frac{\varphi}{\bar{\varphi}_I^{L}}\right)^{\sigma-1} - \sum_{i=e+1}^{I} F_i \left(\frac{\varphi}{\bar{\varphi}_i^{L}}\right)^{\sigma-1} + \sum_{i=e+1}^{I} F_i \left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&< \hat{\pi}_I(\bar{\varphi}_I^{L}, L) \left(\frac{\varphi}{\bar{\varphi}_I^{L}}\right)^{\sigma-1} - f_I \\
&\leq \tilde{\pi}(\varphi, L),
\end{aligned}$$

where the first line follows from expression (A.7), the second uses (A.8), the third uses the zero profit condition $\hat{\pi}_e(\bar{\varphi}_e^{L}, L) = \hat{\pi}_e(\bar{\varphi}_e^{L'}, L') = f_e$ and divides and multiplies the first term by $\left(\bar{\varphi}_e^{L}\right)^{\sigma-1}$, the fourth line uses positive selection effect after an increase in size, $\bar{\varphi}_e^{L'} > \bar{\varphi}_e^L$, the fifth line follows from expression (A.9), the sixth line uses $\bar{\varphi}_i^{L'} > \bar{\varphi}_i^L$ for $i < I$, as $\frac{d\ln\bar{\varphi}_i}{d\ln L} > \frac{d\ln\bar{\varphi}_I}{d\ln L} > 0$ for $i < I$ by assumption.

The last inequality is obtained as follows. Since $\bar{\varphi}_I^{L'} > \bar{\varphi}_I^L$, firm φ uses technology I or $I+1$ under size L if the size change is small enough. If it uses technology I, then $\hat{\pi}_I(\bar{\varphi}_I^L, L)\left(\frac{\varphi}{\bar{\varphi}_I^L}\right)^{\sigma-1} - f_I = \tilde{\pi}(\varphi, L)$ from (A.7). If on the other hand it uses technology $I+1$, then

$$\hat{\pi}_I(\bar{\varphi}_I^L, L)\left(\frac{\varphi}{\bar{\varphi}_I^L}\right)^{\sigma-1} - f_I < \hat{\pi}_{I+1}(\bar{\varphi}_{I+1}^L, L)\left(\frac{\varphi}{\bar{\varphi}_{I+1}^L}\right)^{\sigma-1} - f_{I+1} = \tilde{\pi}(\varphi, L)$$

the inequality follows because the firm can still use technology I and earn $\hat{\pi}_I(\bar{\varphi}_I^L, L)\left(\frac{\varphi}{\bar{\varphi}_I^L}\right)^{\sigma-1} - f_I$ but choose technology $I+1$, which must therefore yield higher profit.

(3.2) Case of upgrading with $\bar{\varphi}_I^{L'} < \bar{\varphi}_I^L$ and $\bar{\varphi}_I^{L'} < \varphi^*$

For any $\bar{\varphi}_I^{L'} < \varphi < \varphi^*$ we have $\frac{d\bar{\varphi}_i}{dL} < 0$ for $I \le i \le l$. The profit function can be written as

$$\begin{aligned}
\tilde{\pi}(\varphi, L') &= \hat{\pi}_l(\bar{\varphi}_l^{L'}, L')\left(\frac{\varphi}{\bar{\varphi}_l^{L'}}\right)^{\sigma-1} - \sum_{i=I+1}^{l} F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&= \hat{\pi}_l(\varphi^*, L')\left(\frac{\varphi}{\varphi^*}\right)^{\sigma-1} - \sum_{i=I+1}^{l} F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&= \hat{\pi}_{l-1}(\bar{\varphi}_{l-1}^L, L)\left(\frac{\varphi}{\bar{\varphi}_{l-1}^L}\right)^{\sigma-1} + F_l\left(\frac{\varphi}{\varphi^*}\right)^{\sigma-1} - \sum_{i=I+1}^{l} F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&= \hat{\pi}_I(\bar{\varphi}_I^L, L)\left(\frac{\varphi}{\bar{\varphi}_I^L}\right)^{\sigma-1} + \sum_{i=I+1}^{l-1} F_i\left(\frac{\varphi}{\bar{\varphi}_i^L}\right)^{\sigma-1} + F_l\left(\frac{\varphi}{\varphi^*}\right)^{\sigma-1} \\
&\quad - \sum_{i=I+1}^{l} F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&< \hat{\pi}_I(\bar{\varphi}_I^L, L)\left(\frac{\varphi}{\bar{\varphi}_I^L}\right)^{\sigma-1} - f_I \\
&\le \tilde{\pi}(\varphi, L),
\end{aligned}$$

where the first line line uses (A.7) and (A.9), the second uses (A.7) so that $\hat{\pi}_l(\varphi^*, L') = \hat{\pi}_l(\bar{\varphi}_l^{L'}, L')\left(\frac{\varphi^*}{\bar{\varphi}_l^{L'}}\right)^{\sigma-1}$, the third uses (A.10), the fourth uses (A.8), the fifth uses $F_l\left(\frac{\varphi}{\varphi^*}\right)^{\sigma-1} < F_l\left(\frac{\varphi}{\bar{\varphi}_l^{L'}}\right)^{\sigma-1}$ as $\bar{\varphi}_l^{L'} < \varphi^*$ (firm φ^* is using technology l after the size increase), and $F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} > F_i\left(\frac{\varphi}{\bar{\varphi}_i^L}\right)^{\sigma-1}$ for $I+1 \le i \le l-1$ as $\bar{\varphi}_i^L > \bar{\varphi}_i^{L'}$ ($\frac{d\ln\bar{\varphi}_i}{d\ln L} < \frac{d\ln\bar{\varphi}_l}{d\ln L} < 0$ for $i > I$).

The last inequality is obtained as follows. Firm φ either uses $I-1$ or I under size L as $\bar{\varphi}_I^{L'} < \bar{\varphi}_I^L$. If it uses technology I, then

$$\hat{\pi}_I(\bar{\varphi}_I^L, L)\left(\frac{\varphi}{\bar{\varphi}_I^L}\right)^{\sigma-1} - f_I = \hat{\pi}_I(\varphi, L) - f_I = \tilde{\pi}(\varphi, L)$$

from (A.7), so profit is lower under L'. If instead it uses technology $I-1$ then it obtains

$\hat{\pi}_I(\bar{\varphi}_I^L, L)\left(\frac{\varphi}{\bar{\varphi}_I^L}\right)^{\sigma-1} - f_I$ under I, which must be smaller than the profits under the optimally chosen $I-1$ technology. Thus, for $\bar{\varphi}_I^{L'} < \varphi < \varphi^*$ under $\bar{\varphi}_I^{L'} < \bar{\varphi}_I^L$, we have $\tilde{\pi}(\varphi, L') < \tilde{\pi}(\varphi, L)$. Together with the initial result for downgrading, we have shown that

$$\tilde{\pi}(\varphi, L') < \tilde{\pi}(\varphi, L), \quad \text{if } \bar{\varphi}_e^{L'} \le \varphi < \varphi^*.$$

And for $\varphi < \bar{\varphi}_e^{L'}$, $\tilde{\pi}(\varphi, L') = 0$, so $\tilde{\pi}(\varphi, L') \le \tilde{\pi}(\varphi, L)$ if $\varphi < \varphi^*$.

(3.3) Case of upgrading with $\varphi^* < \bar{\varphi}_I^{L'} < \bar{\varphi}_I^L$

For all $\varphi > \varphi^*$ we have $\frac{d\bar{\varphi}_i}{dL} < 0$ for $i > I$. The profit function can be written as

$$\begin{aligned}
\tilde{\pi}(\varphi, L') &= \hat{\pi}_l(\bar{\varphi}_l^{L'}, L')\left(\frac{\varphi}{\bar{\varphi}_l^{L'}}\right)^{\sigma-1} + \sum_{i=l+1}^{I} F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&= \hat{\pi}_l(\varphi^*, L')\left(\frac{\varphi}{\varphi^*}\right)^{\sigma-1} + \sum_{i=l+1}^{I} F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&= \hat{\pi}_{l-1}(\bar{\varphi}_{l-1}^L, L)\left(\frac{\varphi}{\bar{\varphi}_{l-1}^L}\right)^{\sigma-1} + F_l\left(\frac{\varphi}{\varphi^*}\right)^{\sigma-1} + \sum_{i=l+1}^{I} F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&= \hat{\pi}_{I-1}(\bar{\varphi}_{I-1}^L, L)\left(\frac{\varphi}{\bar{\varphi}_{I-1}^L}\right)^{\sigma-1} - \sum_{i=l}^{I-1} F_i\left(\frac{\varphi}{\bar{\varphi}_i^L}\right)^{\sigma-1} + F_l\left(\frac{\varphi}{\varphi^*}\right)^{\sigma-1} \\
&+ \sum_{i=l+1}^{I} F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} - f_I \\
&> \hat{\pi}_{I-1}(\bar{\varphi}_{I-1}^L, L)\left(\frac{\varphi}{\bar{\varphi}_{I-1}^L}\right)^{\sigma-1} + F_I\left(\frac{\varphi}{\bar{\varphi}_I^{L'}}\right)^{\sigma-1} - f_I \\
&\ge \tilde{\pi}(\varphi, L),
\end{aligned}$$

where the first line uses (A.7) and (A.8), the second uses (A.7) so that $\hat{\pi}_l(\varphi^*, L') = \hat{\pi}_l(\bar{\varphi}_l^{L'}, L')\left(\frac{\varphi^*}{\bar{\varphi}_l^{L'}}\right)^{\sigma-1}$, the third uses (A.10), the fourth uses (A.9), the fifth uses $F_l\left(\frac{\varphi}{\varphi^*}\right)^{\sigma-1} > F_l\left(\frac{\varphi}{\bar{\varphi}_l^L}\right)^{\sigma-1}$ as $\bar{\varphi}_l^L > \varphi^*$ (firm φ^* is using $l-1$ under L), and $F_i\left(\frac{\varphi}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} > F_i\left(\frac{\varphi}{\bar{\varphi}_i^L}\right)^{\sigma-1}$ for $l+1 \le i \le I-1$ as $\bar{\varphi}_i^L > \bar{\varphi}_i^{L'}$ ($\frac{d\ln\bar{\varphi}_i}{d\ln L} < \frac{d\ln\bar{\varphi}_l}{d\ln L} < 0$ for $i > l$).

The last step is as follows. Firm φ either uses technology $I-1$ or technology I under size L as $\bar{\varphi}_I^{L'} < \bar{\varphi}_I^L$. If it uses technology $I-1$, then

$$\begin{aligned}
\hat{\pi}_{I-1}(\bar{\varphi}_{I-1}^L, L)\left(\frac{\varphi}{\bar{\varphi}_{I-1}^L}\right)^{\sigma-1} + F_I\left(\frac{\varphi}{\bar{\varphi}_I^{L'}}\right)^{\sigma-1} - f_I &> \hat{\pi}_{I-1}(\bar{\varphi}_{I-1}^L, L)\left(\frac{\varphi}{\bar{\varphi}_{I-1}^L}\right)^{\sigma-1} + F_I - f_I \\
&= \hat{\pi}_{I-1}(\bar{\varphi}_{I-1}^L, L)\left(\frac{\varphi}{\bar{\varphi}_{I-1}^L}\right)^{\sigma-1} - f_{I-1} \\
&= \tilde{\pi}(\varphi, L),
\end{aligned}$$

where the inequality uses $\varphi > \bar{\varphi}_I^{L'}$, and the equalities use $F_I = f_I - f_{I-1}$ and expression (A.7). On the other hand, if firm φ uses technology I, then

$$\begin{aligned}
&\hat{\pi}_{I-1}(\bar{\varphi}_{I-1}^{L}, L)\left(\frac{\varphi}{\bar{\varphi}_{I-1}^{L}}\right)^{\sigma-1} + F_I\left(\frac{\varphi}{\bar{\varphi}_{I}^{L'}}\right)^{\sigma-1} - f_I \\
=&\hat{\pi}_{I}(\bar{\varphi}_{I}^{L}, L)\left(\frac{\varphi}{\bar{\varphi}_{I}^{L}}\right)^{\sigma-1} - F_I\left(\frac{\varphi}{\bar{\varphi}_{I}^{L}}\right)^{\sigma-1} + F_I\left(\frac{\varphi}{\bar{\varphi}_{I}^{L'}}\right)^{\sigma-1} - f_I \\
>&\hat{\pi}_{I}(\bar{\varphi}_{I}^{L}, L)\left(\frac{\varphi}{\bar{\varphi}_{I}^{L}}\right)^{\sigma-1} - f_I \\
=&\tilde{\pi}(\varphi, L),
\end{aligned}$$

where the second line uses (A.9), the third uses $\bar{\varphi}_I^{L'} < \bar{\varphi}_I^{L}$, and the last uses expression (A.7). This shows

$$\tilde{\pi}(\varphi, L') > \tilde{\pi}(\varphi, L) \quad \text{if } \varphi > \varphi^*.$$

∎

Technote 1 In the following we show that when $k > m$ we have

$$\hat{\pi}_k(\bar{\varphi}_k^{L'}, L') = \hat{\pi}_m(\bar{\varphi}_m^{L'}, L')\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_m^{L'}}\right)^{\sigma-1} + \sum_{i=m+1}^{k} F_i\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1}$$

Proof.

$$\begin{aligned}
&\hat{\pi}_k(\bar{\varphi}_k^{L'}, L') \\
=&\hat{\pi}_{k-1}(\bar{\varphi}_{k-1}^{L'}, L')\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_{k-1}^{L'}}\right)^{\sigma-1} + f_k - f_{k-1} \\
=&\hat{\pi}_{k-1}(\bar{\varphi}_{k-1}^{L'}, L')\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_{k-1}^{L'}}\right)^{\sigma-1} + F_k \\
=&\left[\hat{\pi}_{k-2}(\bar{\varphi}_{k-2}^{L'}, L')\left(\frac{\bar{\varphi}_{k-1}^{L'}}{\bar{\varphi}_{k-2}^{L'}}\right)^{\sigma-1} + f_{k-1} - f_{k-2}\right]\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_{k-1}^{L'}}\right)^{\sigma-1} + F_k \\
=&\hat{\pi}_{k-2}(\bar{\varphi}_{k-2}^{L'}, L')\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_{k-2}^{L'}}\right)^{\sigma-1} + \sum_{i=k-1}^{k} F_i\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1} \\
=&\dots \\
=&\hat{\pi}_m(\bar{\varphi}_m^{L'}, L')\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_m^{L'}}\right)^{\sigma-1} + \sum_{i=m+1}^{k} F_i\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_i^{L'}}\right)^{\sigma-1}
\end{aligned}$$

where the second line uses (A.7) and the adoption indifference condition: the profits for firm with productivity $\bar{\varphi}_k^{L'}$ has the same profits from using technology $k-1$ and technology k, i.e.,

$$\hat{\pi}_{k-1}(\bar{\varphi}_{k-1}^{L'}, L')\left(\frac{\bar{\varphi}_k^{L'}}{\bar{\varphi}_{k-1}^{L'}}\right)^{\sigma-1} - f_{k-1} = \hat{\pi}_k(\bar{\varphi}_k^{L'}, L') - f_k$$

the third line uses $F_k = f_k - f_{k-1}$, the fourth uses the adoption indifference condition for cutoff $\bar{\varphi}_{k-1}^{L'}$, and we can continue similarly until we reach the cutoff for technology m. ∎

A.5 Claims in the Proof of Proposition 10

Claim: $\Delta \bar{F} \equiv \bar{F}(\bar{\varphi}, L') - \bar{F}(\bar{\varphi}, L) - \left[\bar{F}(L') - \bar{F}(L)\right] > 0$ for all $\bar{\varphi} > \bar{\varphi}_e^L$ if $\frac{d\gamma(\bar{\varphi})}{d\bar{\varphi}} = 0$.

Proof. In this case, we need to show that

$$\Delta \bar{F} \equiv \bar{F}(\bar{\varphi}, L') - \bar{F}(\bar{\varphi}, L) = \int_{\bar{\varphi}}^{\infty} \left[f(\varphi, L') - f(\varphi, L)\right] dG(\varphi) > 0$$

since $\bar{F}(L') = \bar{F}(L)$ as shown in lemma 1.

The inequality holds if $\bar{\varphi} \geq \bar{\varphi}_{b+1}^L$ where $\bar{\varphi}_{b+1}^L$ is the threshold above which firms will either upgrade or use the same technology after shock (unique as shown in the proof for positive extensive margin in proposition 2), since for those firms $f(\varphi, L') \geq f(\varphi, L)$.

If $\bar{\varphi} < \bar{\varphi}_{b+1}^L$ then we use $\bar{F}(L') = \bar{F}(L)$ to rewrite

$$\Delta \bar{F} \equiv \bar{F}(L) - \bar{F}(\bar{\varphi}, L) - \left(\bar{F}(L') - \bar{F}(\bar{\varphi}, L')\right) = \int_{\bar{\varphi}_e^L}^{\bar{\varphi}} f(\varphi, L) dG(\varphi) - \int_{\bar{\varphi}_e^{L'}}^{\bar{\varphi}} f(\varphi, L') dG(\varphi),$$

where we use $f(\varphi) = 0$ for firms below the production threshold. Thus,

$$\Delta \bar{F} = \int_{\bar{\varphi}_e^L}^{\bar{\varphi}_e^{L'}} f(\varphi, L) dG(\varphi) + \int_{\bar{\varphi}_e^{L'}}^{\bar{\varphi}} \left[f(\varphi, L) - f(\varphi, L')\right] dG(\varphi) > 0 \tag{A.11}$$

The equality follows from selection, $\bar{\varphi}_e^{L'} > \bar{\varphi}_e^L$, which also implies that the first term is positive. The second term is also positive as some firms belonging to $\bar{\varphi} < \bar{\varphi}_{b+1}^L$ have downgraded after the shock so that $f(\varphi, L) > f(\varphi, L')$. ∎

B Size Independence in Calibration

In the following, we provide some intuition as to why our calibration result does not depend on the values of initial size parameters (L, $\varphi_{\min}$, f_0, f_E). We first argue that given the data moments our framework matches, the calibrated values of specialization premium does not depend on initial values of size parameters. Second, we show how the values of fixed productivity boost (ϕ) should adjust to keep the specialization premium unchanged, and derive the expression for specialization potential (A).

B.1 Calibrated Specialization Premium under Different Size Parameters

We follow the steps below to show that given the same data moments we match, varying the size parameters alter only calibrated ϕ but not the specialization premium:

1. Given specialization premium, the intermediates cost share (data moment we match) and top $20V$ firms sales share (data moment we match) do not depend on the initial size parameters;

2. The size parameter affects the free entry and the entry cutoff expressions, which in turn affects the price index;

3. According to specialization schedule, ϕ adjusts with the price index, so the equilibrium with different size parameters have the same values of specialization premium.

First, observe that once specialization premium is known, both the intermediates cost share and top firms sales share do not depend on the values of size parameters, ϕ, and price index. Specifically, the intermediates cost share can be expressed as

$$\bar{\alpha} = \frac{\left(\frac{F_1}{f_0}\right)\left(\frac{\bar{\varphi}_1}{\bar{\varphi}_0}\right)^{-k} + 2\left(\frac{F_2}{f_0}\right)\left(\frac{\bar{\varphi}_2}{\bar{\varphi}_0}\right)^{-k} + \frac{1}{s_1^{\sigma-1}-1}\left[\left(\frac{F_1}{f_0}\right)\left(\frac{\bar{\varphi}_1}{\bar{\varphi}_0}\right)^{-k} + \left(\frac{F_2}{f_0}\right)\left(\frac{\bar{\varphi}_2}{\bar{\varphi}_0}\right)^{-k}\right]}{1 + \left(\frac{F_1}{f_0}\right)\left(\frac{\bar{\varphi}_1}{\bar{\varphi}_0}\right)^{-k} + \left(\frac{F_2}{f_0}\right)\left(\frac{\bar{\varphi}_2}{\bar{\varphi}_0}\right)^{-k}}\left(\frac{\alpha}{2}\right), \tag{B.1}$$

where $F_1 = f_1 - f_0$ and $F_2 = f_2 - f_1$. This only depends on fixed costs relative to f_0 (therefore the absolute values of f_0 is irrelevant) and the specialization premium, since relative productivity thresholds $(\frac{\bar{\varphi}_1}{\bar{\varphi}_0}, \frac{\bar{\varphi}_2}{\bar{\varphi}_0})$ depend only on s_1 and relative fixed cost. Similarly, the top firms sales share can be expressed as

$$\lambda_{20V} = \frac{s_1^{2(\sigma-1)}\left(\frac{\bar{\varphi}_c}{\bar{\varphi}_0}\right)^{\sigma-k-1}}{\sum_{i=0}^{2} s_1^{i(\sigma-1)}\left[\left(\frac{\bar{\varphi}_i}{\bar{\varphi}_0}\right)^{\sigma-k-1} - \left(\frac{\bar{\varphi}_{i+1}}{\bar{\varphi}_0}\right)^{\sigma-k-1}\right]}, \tag{B.2}$$

where $\left(\frac{\bar{\varphi}_c}{\bar{\varphi}_0}\right)^{-k}$ is the fraction of firms consists of the top 20V sales in the data. This again only depends on the specialization premium and relative fixed costs.

Furthermore, the free entry condition in terms of the cutoffs is

$$\frac{\sigma-1}{k-\sigma+1}(\varphi_{\min})^k f_0 \left(\bar{\varphi}_0\right)^{-k}\left[1 + \left(\frac{F_1}{f_0}\right)\left(\frac{\bar{\varphi}_1}{\bar{\varphi}_0}\right)^{-k} + \left(\frac{F_2}{f_0}\right)\left(\frac{\bar{\varphi}_2}{\bar{\varphi}_0}\right)^{-k}\right] = f_E, \tag{B.3}$$

which depends on the size parameters f_E, f_0, $\varphi_{\min}$, and the relative productivity thresholds. This implies that $\bar{\varphi}_0(f_E, f_0, \varphi_{\min})$ is function of f_E, f_0, and $\varphi_{\min}$ given the specialization premium. Moreover, the expression for entry cutoff illustrates that the price index is a function of entry cutoff and total labor force:

$$\bar{\varphi}_0 = \frac{1}{P}\left(\frac{f_0}{\tilde{\sigma}Y}\right)^{\frac{1}{\sigma-1}} = \frac{1}{P}\left(\frac{f_0}{\tilde{\sigma}}\frac{1-\frac{\sigma-1}{\sigma}\bar{\alpha}}{L\left(1+N\tau^{1-\sigma}\right)}\right)^{\frac{1}{\sigma-1}}. \tag{B.4}$$

Therefore, the price index is function of all four size parameters $P(f_E, f_0, \varphi_{\min}, L)$. Finally, from the specialization schedule, in equilibrium the price index is consistent with specialization premium and ϕ:

$$\phi = P s_1^{\frac{2}{\alpha}}. \tag{B.5}$$

The above analysis implies that given the same observed moments on intermediates cost share and initial top $20V$ firms sales share, a different value of size parameter supports the equilibrium with the *same* values of specialization premium, which leaves (B.1) and (B.2) unaffected. Moreover, the price index $P(f_E, f_0, \varphi_{\min}, L)$ adjusts accordingly following (B.3) and (B.4), and from (B.5) ϕ changes following the adjustment of P to leave the specialization premium unchanged.

B.2 Specialization Potential

To derive the expression for specialization potential, we first describe the effects of changes in size parameter on the price schedule and specialization schedule if the framework matches the same data moments.

First, the price schedule can be written as

$$\begin{aligned} P &= \left\{ M \left(1 + N\tau^{1-\sigma}\right) \left[\sum_{r=0}^{2} \int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}} \left[p_r(\varphi)\right]^{1-\sigma} dG(\varphi) \right] \right\}^{\frac{1}{1-\sigma}} \\ &= \left\{ M \left(1 + N\tau^{1-\sigma}\right) \left[p_0(\bar{\varphi}_0)\right]^{1-\sigma} \left[\sum_{r=0}^{2} \int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}} \left[\frac{p_r(\varphi)}{p_0(\bar{\varphi}_0)}\right]^{1-\sigma} dG(\varphi) \right] \right\}^{\frac{1}{1-\sigma}} \\ &= \left\{ \Lambda_2 \left[X(s_1, L)\right]^{\frac{k}{\sigma-1}} \left(1 + N\tau^{1-\sigma}\right)^{\frac{k}{\sigma-1}-1} B(s_1) \right\}^{-\frac{1}{k}}, \end{aligned} \tag{B.6}$$

where $p_r(\varphi)$ denotes the domestic price of firm φ using technology r, $\Lambda_2 = (\tilde{\sigma})^{\frac{k}{\sigma-1}} (f_0)^{\frac{k}{\sigma-1}-1} \frac{(\sigma-1)(\varphi_{\min})^k}{(k-\sigma+1)f_E}$, $X(s_1, L) = \frac{L}{1-\frac{\sigma-1}{\sigma}\tilde{\alpha}(s_1)}$ is total expenditure which captures the effect of economy size, and

$$B(s_1) \equiv \sum_{r=0}^{2} s_1^{r(\sigma-1)} \left[\left(\frac{\bar{\varphi}_r}{\bar{\varphi}_0}\right)^{\sigma-k-1} - \left(\frac{\bar{\varphi}_{r+1}}{\bar{\varphi}_0}\right)^{\sigma-k-1} \right]$$

captures the effect of relative productivity cutoffs, which depends only on the specialization premium. The last equality of (B.6) is shown in Technote 2 at the end of the section.

Second, the specialization schedule can be expressed as

$$s_1 = \left(\frac{\phi}{P}\right)^{\frac{\alpha}{2}}. \tag{B.7}$$

From previous subsection we have shown that under the same data moments the specialization premium is unchanged with different values of size parameters. Therefore, for different size parameters, the changes in calibrated value of ϕ must equal the changes in price index due to different size parameters:

$$\widehat{\phi} = \widehat{P} = \left(\widehat{\Lambda}_2\right)^{-\frac{1}{k}} \left(\widehat{L}\right)^{\frac{1}{\sigma-1}}, \tag{B.8}$$

where $\hat{x} = \frac{x'}{x}$ represents the relative change in endogeneous variable x, and the second equality uses (B.6) in relative change. We obtain the following expression by replacing $\widehat{\Lambda}_2$ in (B.8):

$$A = \frac{\phi L^{\frac{1}{\sigma-1}}}{(f_E)^{1/k}(f_0)^{\frac{1}{\sigma-1}-\frac{1}{k}}} z_{\min}. \tag{B.9}$$

Because $z_{\min}$ is the unit we measure firm productivity, parameter A summarizes the productivity effects from specialization, or the *specialization potential*. It is positively correlated with the productivity effect of specialization (ϕ) and size (L), and negatively correlated with the entry fee (f_E) and operating cost (f_0).[1]

[1] Everything else equal, a larger size L implies a larger number of varieties therefore a lower price index.

Given the data moments we have, we can only pin down the values of specialization potential in the economy, but not each individual force, such as the levels of ϕ. However, given there is no change in other size parameters other than ϕ, the change in ϕ fully reflect the change in specialization potential, which is independent of the initial values of size parameters given the data moments we match.

Technote 2 In the following we derive the last equality in (B.6). Using the expression for mass of entrants, $M = \frac{\sigma-1}{\sigma k}\frac{X}{f_E}$, we simplify the price index as

$$P = \left\{ M \left[p_0(\bar{\varphi}_0)\right]^{1-\sigma} \left[\sum_{r=0}^{2} \int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}} \left[\frac{p_r(\varphi)}{p_0(\bar{\varphi}_0)}\right]^{1-\sigma} dG(\varphi) \right] \right\}^{\frac{1}{1-\sigma}}$$

$$= \left\{ \frac{\sigma-1}{\sigma k}\frac{1}{f_E} X(s_1, L) \left[\sum_{r=0}^{2} \int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}} \left[p_r(\varphi)\right]^{1-\sigma} dG(\varphi) \right] \right\}^{\frac{1}{1-\sigma}}. \tag{B.10}$$

Note that from CES demand and monopolistic competition $p_r(\varphi) = \frac{\sigma}{\sigma-1}c_r(\varphi) = \frac{\sigma}{\sigma-1}\left(s_1^r\varphi\right)^{-1}$, thus,

$$\begin{aligned}
&\sum_{r=0}^{2} \int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}} \left[p_r(\varphi)\right]^{1-\sigma} dG(\varphi) \\
=&\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \left[\int_{\bar{\varphi}_0}^{\bar{\varphi}_1} \varphi^{\sigma-1} dG(\varphi) + (s_1)^{\sigma-1} \int_{\bar{\varphi}_1}^{\bar{\varphi}_2} \varphi^{\sigma-1} dG(\varphi) + (s_1)^{2(\sigma-1)} \int_{\bar{\varphi}_2}^{\infty} \varphi^{\sigma-1} dG(\varphi) \right] \\
=&\frac{k(\varphi_{\min})^k (\frac{\sigma}{\sigma-1})^{1-\sigma}}{k-\sigma+1} (\bar{\varphi}_0)^{\sigma-1-k} \sum_{r=0}^{2} s_1^{r(\sigma-1)} \left[\left(\frac{\bar{\varphi}_r}{\bar{\varphi}_0}\right)^{\sigma-k-1} - \left(\frac{\bar{\varphi}_{r+1}}{\bar{\varphi}_0}\right)^{\sigma-k-1} \right] \\
=&\frac{k(\varphi_{\min})^k \sigma (f_0)^{1-\frac{k}{\sigma-1}} (\hat{\sigma})^{\frac{k}{\sigma-1}}}{k-\sigma+1} P^{k-\sigma+1} \left[X\left(1+N\tau^{1-\sigma}\right)\right]^{\frac{k}{\sigma-1}-1} \sum_{r=0}^{2} s_1^{r(\sigma-1)} \left[\left(\frac{\bar{\varphi}_r}{\bar{\varphi}_0}\right)^{\sigma-k-1} - \left(\frac{\bar{\varphi}_{r+1}}{\bar{\varphi}_0}\right)^{\sigma-k-1} \right] \\
\equiv&\frac{k(\varphi_{\min})^k \sigma (f_0)^{1-\frac{k}{\sigma-1}} (\hat{\sigma})^{\frac{k}{\sigma-1}}}{k-\sigma+1} P^{k-\sigma+1} \left[X\left(1+N\tau^{1-\sigma}\right)\right]^{\frac{k}{\sigma-1}-1} B(s_1),
\end{aligned}$$

where second equality uses the pareto distribution

$$\int_{\bar{\varphi}_r}^{\bar{\varphi}_{r+1}} \varphi^{\sigma-1} dG(\varphi) = \frac{k(\varphi_{\min})^k}{k-\sigma+1} \left[\left(\frac{\bar{\varphi}_r}{\bar{\varphi}_0}\right)^{\sigma-k-1} - \left(\frac{\bar{\varphi}_{r+1}}{\bar{\varphi}_0}\right)^{\sigma-k-1} \right],$$

and multiply and divide the term $(\bar{\varphi}_0)^{\sigma-1-k}$, the third equality uses the entry cutoff expression,

$$\bar{\varphi}_0 = \frac{1}{P} \left[\frac{f_0}{\hat{\sigma} X \left(1+N\tau^{1-\sigma}\right)} \right]^{\frac{1}{\sigma-1}}$$

to substitute out $\bar{\varphi}_0$ and uses $\hat{\sigma} \equiv \frac{1}{\sigma-1}(\frac{\sigma}{\sigma-1})^{-\sigma}$. The last equality defines

$$B(s_1) \equiv \sum_{r=0}^{2} s_1^{r(\sigma-1)} \left[\left(\frac{\bar{\varphi}_r}{\bar{\varphi}_0}\right)^{\sigma-k-1} - \left(\frac{\bar{\varphi}_{r+1}}{\bar{\varphi}_0}\right)^{\sigma-k-1} \right],$$

as it only depends on the specialization premium. Substitute the above expression back to (B.10) we get

$$P = \left\{ \frac{(\sigma-1)(\varphi_{\min})^k (f_0)^{1-\frac{k}{\sigma-1}} (\hat{\sigma})^{\frac{k}{\sigma-1}}}{(k-\sigma+1) f_E} P^{k-\sigma+1} \left[X\left(s_1, L\right)\right]^{\frac{k}{\sigma-1}} \left(1 + N\tau^{1-\sigma}\right)^{\frac{k}{\sigma-1}-1} B(s_1) \right\}^{\frac{1}{1-\sigma}}.$$

Moving $P^{k-\sigma+1}$ to the left hand side and rearranging we obtain (B.6).