Appendix to "Demographics, Wealth and Global Imbalances in the Twenty-First Century"

A Appendix to Section 1

The total return on wealth r_t for the US from 1950–2016 in panel C of figure 1 is constructed as follows. We take:

- Capital K_t as total private fixed assets at current cost from line 1 of Table 2.1 in the BEA's Fixed Assets Accounts (FA).
- Output Y_t as gross domestic product from line 1 of Table 1.1.5 in the BEA's National Income and Product Accounts (NIPA).
- Wealth W_t as "net private wealth" from the World Inequality Database (WID).
- Net foreign assets NFA_t as the net worth of the "rest of the world" sector from line 147 of Table S.9.a in the Integrated Macroeconomic Accounts (IMA).[60]
- Government bonds B_t as gross federal debt held by the public, from the Economic Report of the President (accessed via FRED at FYGFDPUB).
- The safe real interest rate r_t^{safe} as the 10-year constant maturity interest rate—from Federal Reserve release H.15 (accessed via FRED at GS10), extended backward from 1953 to 1950 by splicing with the NBER macrohistory database's yield on long-term US bonds (accessed via FRED at M1333BUSM156NNBR)—minus a slow-moving inflation trend, calculated as the trend component of annual HP-filtered inflation in the PCE deflator, with smoothing parameter $\lambda = 100$.
- Net capital income $(s_K Y - \delta K)_t$ as corporate profits plus net interest and miscellaneous payments of the corporate sector (sum of lines 7 and 8 in NIPA Table 1.13), plus imputed net capital income from the noncorporate business sector, under the assumption that the ratio of net capital income to net factor income (line 11 minus line 17) in the noncorporate business sector is the same as the ratio of net capital income to net factor income (line 3 minus line 9) in the corporate sector.[61]

We then calculate our baseline total return on wealth series as

$$r_t \equiv \frac{(s_K Y - \delta K)_t + r_t^{safe} B_t}{W_t - NFA_t} \tag{A.1}$$

i.e. as the ratio of net capital income plus real interest income on government debt to domestic assets. This calculation gives the total return on private wealth, excluding changes in asset valua-

[60] This is very similar to the standard net international investment position computed by the BEA, but is chosen because it offers a longer time series.

[61] This imputation is a common way of splitting mixed income within the noncorporate sector between labor and capital, used e.g. by Piketty and Zucman (2014).

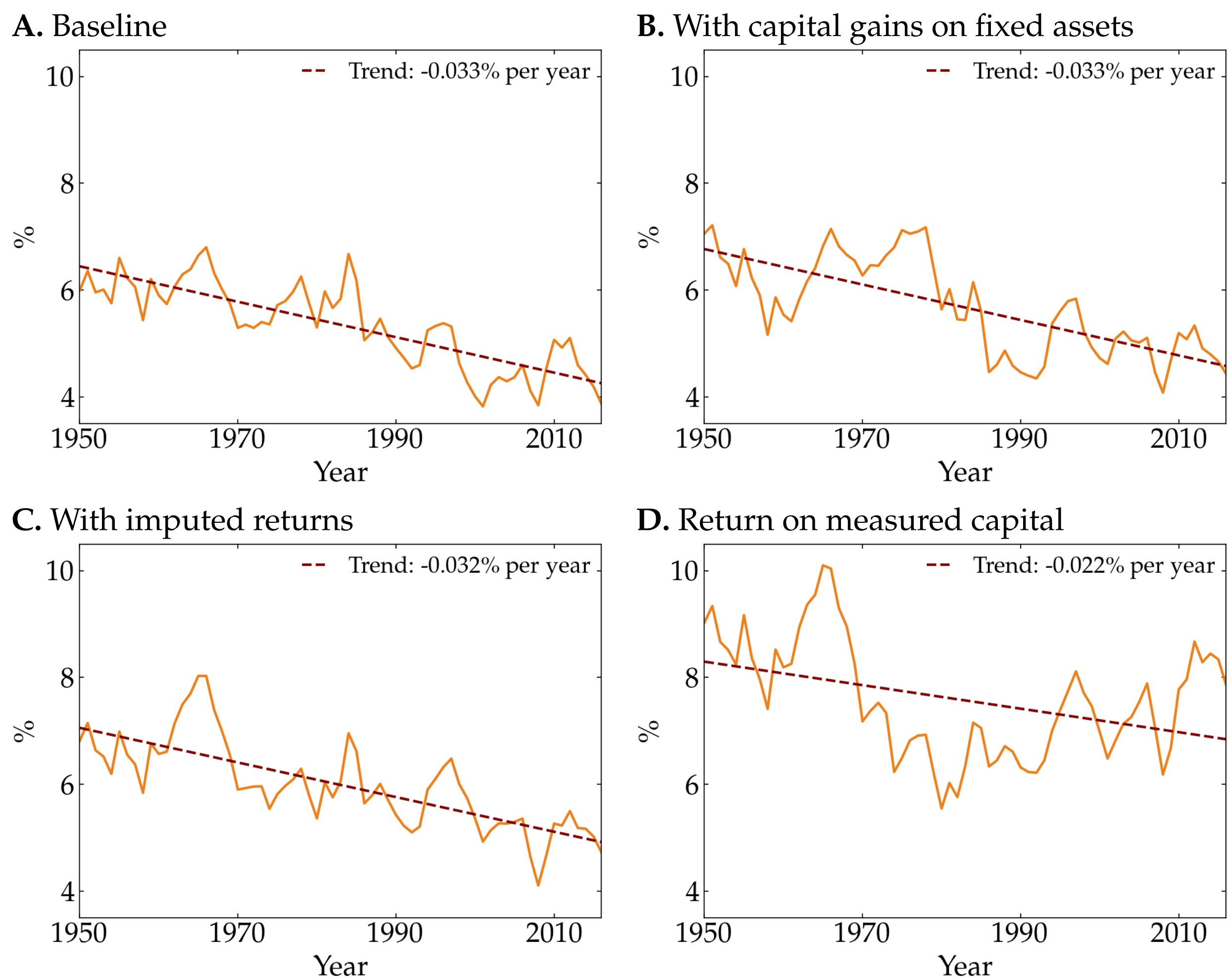

Figure A.1: Alternative ways of constructing the total return on wealth in the US

Notes: Panel A gives our baseline series for the total return on wealth in the US, as described in the text. Panel B adds capital gains on fixed assets, as measured in the fixed assets accounts. Panel C imputes an additional return on unmeasured wealth $W_t - K_t - B_t - NFA_t$ equal to trend growth. Panel D takes our baseline capital income series and divides it by capital measured in the fixed assets accounts.

tion, under the assumption that the average return on net foreign assets is the same as the average return on private wealth.[62]

This baseline r_t and its trend are displayed in panel A of figure A.1. The other three panels provide alternative ways to calculate r_t.

Panel B adds a slow-moving trend of capital good inflation minus PCE inflation, which we denote by π_{Kt}:

$$r_t \equiv \frac{(s_K Y - \delta K)_t + r_t^{safe} B_t + \pi_{Kt} K_t}{W_t - NFA_t}$$

Average inflation of goods in the capital stock is inferred by taking the ratio of changes in the nominal stock (FA Table 2.1, line 1) and changes in the quantity index (FA Table 2.2, line 1), and as with PCE inflation above, we take the slow-moving trend component using the HP filter with $\lambda = 100$. This accounts for expected capital gains on fixed capital (assuming that the expectation follows the trend).

Panel C assumes that there is some unmeasured return on the portion of wealth $W_t - K_t - B_t - NFA_t$ that cannot be accounted for by capital, bonds, or net foreign assets, which it sets equal to the trend real GDP growth rate g_t:

$$r_t \equiv \frac{(s_K Y - \delta K)_t + r_t^{safe} B_t + g_t (W_t - K_t - B_t - NFA_t)}{W_t - NFA_t}$$

where g_t is again calculated using the HP filter with $\lambda = 100$. If $W_t - K_t - B_t - NFA_t$ is the capitalized value of pure rents in the economy, for instance, its value might be expected to grow in line with output.

Finally, panel D simply divides net capital income by the measured capital stock:

$$r_t \equiv \frac{(s_K Y - \delta K)_t}{K_t}$$

Note that despite these alternative constructions, the 1950–2016 trends in panels A, B, and C of figure A.1 are almost identical: -.033, -.033, and -.032 percentage points, respectively. All show a steady decline.

The return on capital in panel D, on the other hand, is quite different: it has a smaller long-term trend decline, of -.022 percentage points per year, and since roughly 1980 it actually displays a mild increase. This post-1980 pattern of a constant or increasing return on capital has been widely remarked upon in the literature—for instance, Gomme, Ravikumar and Rupert (2011), Farhi and Gourio (2018), and Eggertsson et al. (2018). The main source of the disparity between panels A–C and panel D is that the former divides by wealth, while the latter divides only by measured capital. Since our primary object of interest is wealth, we prefer the former convention. Another advantage of using wealth in the denominator is that capital may be imperfectly measured in the fixed assets accounts.

[62]This can be seen by rearranging (A.1) as $r_t = \frac{s_K Y - \delta K + r^{safe} B + rNFA}{W}$, which gives the total return r_t on private wealth if r_t equals the return on NFA_t. We take this route because data on capital income from foreign assets is not comparable to domestic data; for instance, the national accounts only measure dividend payments, not the total net capital income, on foreign equities (other than FDI) held in the US, and also only measure nominal rather than real interest payments on bonds. The trend in r_t, however, is not very sensitive to alternative assumptions on the average rate for NFA_t.

B Appendix to Section 2

B.1 Contribution of changing fertility to aging, 1950-2100

Figure A.2 uses our model of the age distribution of the population in each country to decompose population aging into contributions from fertility, mortality, migration and the so-called momentum effect. Our measure of population aging is the changes in the share of the population aged 50 or above. Denote by $\Delta\pi$ the change in this share between two periods t_0 and t_1. To isolate the role of primitive forces for $\Delta\pi$, we start with an initial age distribution in year t_0. We obtain the contribution of fertility plus "momentum" by simulating the population distribution holding mortality and migration constant until t_1, and then computing the counterfactual change $\Delta^f \pi$ in the share of the 50+ year-old in this scenario. The ratio $\Delta^f \pi / \Delta\pi$ gives us the contribution of fertility and momentum to population aging, which our baseline model of section 2 includes, with the remainder accounted for by mortality and migration, which the baseline model abstracts from. We conduct this exercise over two separate time periods t_0-t_1: 1950-2016 and 2016-2100.

Figure A.2 presents the results, showing $\Delta^f \pi / \Delta\pi$ over these two time periods for the 25 countries in our sample. The top panel shows that, between 1950-2016, fertility and momentum contributed an average of 63.5% of population aging. The bottom panel shows that, between 2016 and 2100, their contributions are projected to shrink a little to an average of 55.9%, but still constitute the majority of the contribution. Hence, our baseline assumption of fixed mortality and migration is a useful first pass at the data, although decreasing mortality becomes increasingly important to population aging as we look towards the 21st century. Our model of section 4 allows for time variation in mortality and models the savings response to it.

B.2 Proofs of lemma 1 and proposition 1

The ratio $K_t / Z_t L_t$ of capital to effective labor is constant over time, pinned down by constant r and the condition $r_t + \delta = F_K(K_t/(Z_t L_t), 1)$. From the condition $w_t = Z_t F_L(K_t/(Z_t L_t), 1)$, w_t is then proportional to Z_t and grows at the constant rate γ. It follows immediately that average pre-tax labor income $h_{jt} \equiv \mathbb{E} w_t \ell_j = (1+\gamma)^t w_0 \mathbb{E} \ell_j$ grows at the constant rate γ.

Letting hats denote normalization of time-subscripted variables by $(1+\gamma)^t$, and defining $\hat{\beta}_j \equiv (1+\gamma)^{j(1-\frac{1}{\sigma})}\beta_j$, the household utility maximization problem (1) becomes

$$
\begin{aligned}
&\max_{\hat{c}_{jt}, \hat{\mathfrak{a}}_{j+1,t+1}} \mathbb{E}_k \left[\sum_{j=0}^{J} \hat{\beta}_j \Phi_j \frac{\hat{c}_{jt}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \right] \\
\text{s.t.} \quad & \hat{c}_{jt} + (1+\gamma)\phi_j \hat{\mathfrak{a}}_{j+1,t+1} \leq w_0 \left((1-\tau)\ell(z_j) + tr(z^j) \right) + (1+r)\hat{\mathfrak{a}}_{jt} \\
& \hat{\mathfrak{a}}_{j+1,t+1} \geq -Z_0 \bar{a}
\end{aligned}
\tag{A.2}
$$

This problem is no longer time-dependent: given the same asset holdings $\hat{a}_j$, state z^j and age j, households optimally choose the same $(\hat{c}_j, \hat{\mathfrak{a}}_{j+1})$ regardless of t. Regardless of their date of birth, every cohort born in this environment will have the same distribution of normalized assets $\hat{a}_j$ at each age j. Hence, once t is high enough that all living agents were born in this environment, there exists a balanced-growth distribution of assets at each age that grows at rate γ. Average assets normalized by productivity satisfy $a_{jt}/Z_t = (\mathbb{E}\mathfrak{a}_{jt})/Z_t = (\mathbb{E}\hat{\mathfrak{a}}_j)/Z_0 \equiv a_j(r)$ for some function

A. 1952-2016 change in the share of 50+ : percentage due to fertility and momentum

B. 2016-2100 change in the share of 50+ : percentage due to fertility and momentum

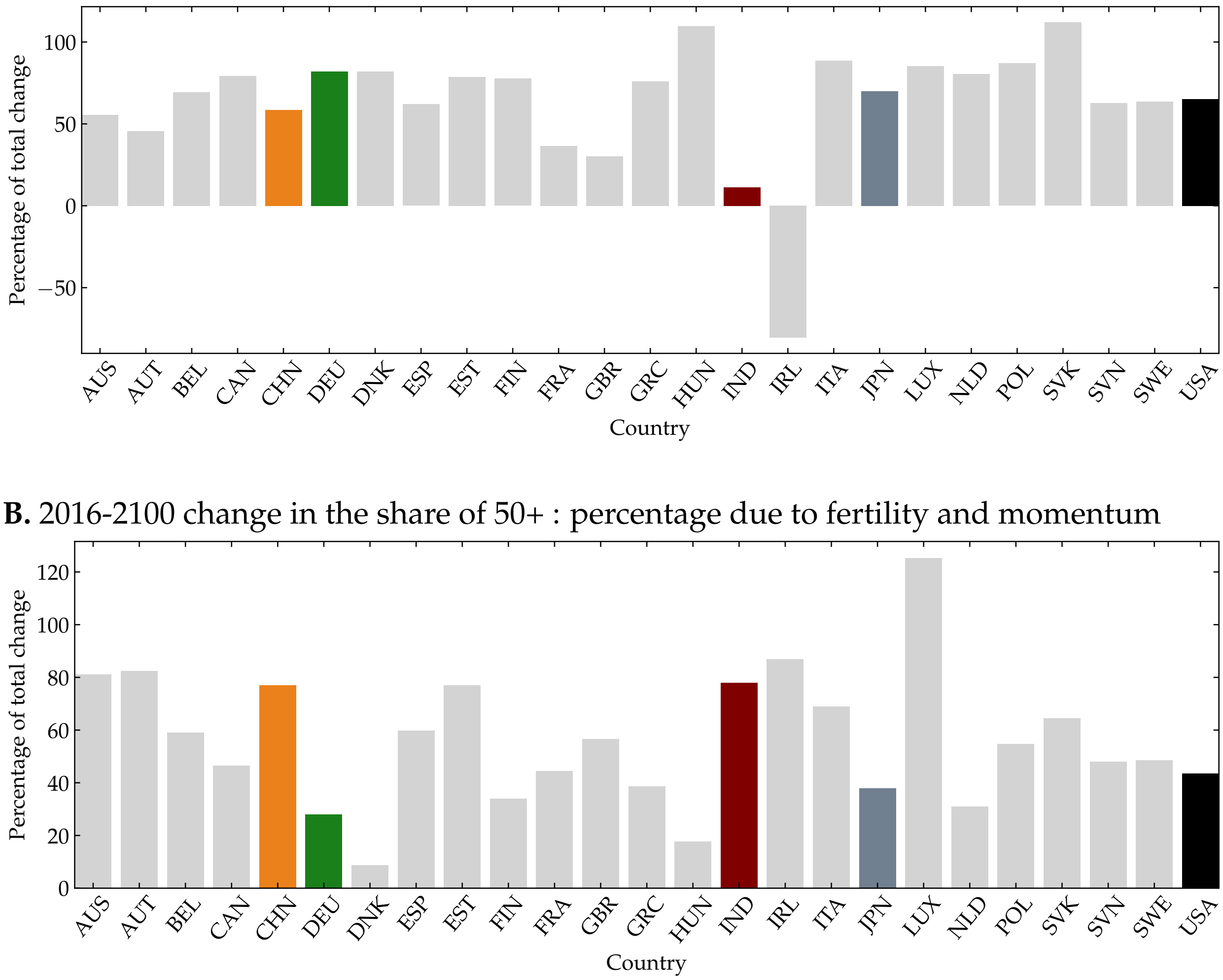

Figure A.2: Contribution of fertility and momentum to population aging

Notes: This figure presents the percentage of the change in the share of 50+ that is due to fertility changes and momentum. It is computed as the ratio between the change in this share under the assumptions of constant mortality rates and migration flows, and under the baseline assumptions for 1952-2016 (panel A) and 2016-2100 (panel B).

$a_j(r)$. If, at date 0, already-living agents start with the joint balanced-growth distribution of assets and states, then this holds immediately.

The ratio of aggregate wealth to aggregate labor at time t is

$$\frac{W_t}{L_t} = \frac{\sum_j N_{jt} a_{jt}}{\sum_j N_{jt} \mathbb{E}\ell_j} = \frac{\sum_j N_{jt}(1+\gamma)^t a_{j0}}{\sum_j N_{jt} h_{j0}/w_0} = (1+\gamma)^t w_0 \frac{\sum_j \pi_{jt} a_{j0}}{\sum_j \pi_{jt} h_{j0}} \tag{A.3}$$

The ratio of output to aggregate labor is

$$\frac{Y_t}{L_t} = \frac{F(K_t, Z_t L_t)}{L_t} = Z_t F\left(\frac{K_t}{Z_t L_t}, 1\right) = Z_t F\left(\frac{K_0}{Z_0 L_0}, 1\right) \tag{A.4}$$

where we use the fact that the capital-to-effective-labor ratio is constant. Dividing (A.3) and (A.4), the wealth-to-output ratio is

$$\frac{W_t}{Y_t} = \frac{w_0}{Z_0 F(K_0/Z_0 L_0, 1)} \frac{\sum_j \pi_{jt} a_{j0}}{\sum_j \pi_{jt} h_{j0}} \tag{A.5}$$

where the first factor is constant with time. We conclude that $\frac{W_t}{Y_t}$ grows in proportion to $\frac{\sum_j \pi_{jt} a_{j0}}{\sum_j \pi_{jt} h_{j0}}$.

B.3 Proofs of propositions 2 and 3

Proof of proposition 2. Within each country c, for a constant rate of return r, lemma 1 shows that there exists a balanced-growth distribution of assets normalized by productivity. Assuming we start with this balanced-growth distribution, then at each t, (A.5) implies

$$\begin{aligned}
\frac{W_t^c}{Y_t^c} &= \frac{w_0^c}{Z_0^c F^c(K_0^c/Z_0^c L_0^c, 1)} \frac{\sum_j \pi_{jt}^c a_{j0}^c}{\sum_j \pi_{jt}^c h_{j0}^c} \\
&= \frac{F_L^c(K_0^c/Z_0^c L_0^c, 1)}{F^c(K_0^c/Z_0^c L_0^c, 1)} \frac{\sum_j \pi_{jt}^c a_{j0}^c}{\sum_j \pi_{jt}^c h_{j0}^c} \\
&= \frac{F_L^c(k^c(r), 1)}{F^c(k^c(r), 1)} \frac{\sum_j \pi_{jt}^c a_{j0}^c}{\sum_j \pi_{jt}^c h_{j0}^c} \equiv \frac{W^c}{Y^c}(r, \pi_t^c)
\end{aligned}$$

where $\pi_t^c \equiv \{\pi_{jt}^c\}_j$, and $k(r)$ is the capital-to-effective-labor ratio associated with r, defined implicitly by $F_K^c(k(r), 1) = r + \delta$.

Each country's share of world GDP is then given by

$$\frac{Y_t^c}{Y_t} = \frac{Z_t^c L_t^c y^c(r)}{\sum Z_t^c L_t^c y^c(r)} = \frac{Z_0^c \nu_t^c y^c(r) \sum \pi_{jt}^c \ell_j^c}{\sum Z_0^c \nu_t^c y^c(r) \sum \pi_{jt}^c \ell_j^c} \equiv \frac{Y^c}{Y}(r, \pi_t, \nu_t),$$

where $\nu_t^c \equiv N_t^c / N_t$ and π_t and ν_t denote vectors across all countries, and $y^c(r) \equiv F^c(k^c(r), 1)$.

The capital-to-output ratio in every country can also be written as a function of r, $\frac{K^c}{Y^c}(r) \equiv k^c(r)/F^c(k^c(r), 1)$, and we assume that government policy maintains a constant $\frac{B^c}{Y^c}$ in each country.

We assume that the economy is in balanced growth corresponding to long-run r_0 at date 0, which means that the initial wealth-to-output ratio is $\frac{W^c}{Y^c}(r_0, \pi_0^c)$ and that the initial capital-output ratio is $\frac{K^c}{Y^c}(r_0)$. We also assume that net foreign asset positions in each country are 0 at time 0, i.e.

that

$$\frac{W^c}{Y^c}(r_0, \pi_0^c) - \frac{K^c}{Y^c}(r_0) - \frac{B^c}{Y^c} = 0.$$

In the long run, π_t^c and ν_t^c converge to constants π_{LR}^c and ν_{LR}^c in each country. Suppose that the real return r_t converges to a long-run value r_{LR}. Then the world asset market clearing condition is

$$0 = \sum_c \frac{Y^c}{Y}(r, \pi, \nu) \left[\frac{W^c}{Y^c}(r, \pi^c) - \frac{K^c}{Y^c}(r) - \frac{B^c}{Y^c} \right] \tag{A.6}$$

which holds for both $(r, \pi, \nu) = (r_0, \pi_0, \nu_0)$ and $(r, \pi, \nu) = (r_{LR}, \pi_{LR}, \nu_{LR})$. Subtracting the former from the latter, we have

$$\begin{aligned}
0 &= \sum_c \frac{Y^c}{Y}(r_{LR}, \pi_{LR}, \nu_{LR}) \left[\frac{W^c}{Y^c}(r_{LR}, \pi_{LR}^c) \right. \\
&\quad \left. - \frac{K^c}{Y^c}(r_{LR}) - \frac{B^c}{Y^c} \right] - \sum_c \frac{Y^c}{Y}(r_0, \pi_0, \nu_0) \left[\frac{W^c}{Y^c}(r_0, \pi_0^c) - \frac{K^c}{Y^c}(r_0) - \frac{B^c}{Y^c} \right] \\
&= \sum_c \left[\frac{Y^c}{Y}(r_{LR}, \pi_{LR}, \nu_{LR}) - \frac{Y^c}{Y}(r_0, \pi_0, \nu_0) \right] \left[\frac{W^c}{Y^c}(r_{LR}, \pi_{LR}^c) - \frac{K^c}{Y^c}(r_{LR}) - \frac{B^c}{Y^c} \right] \\
&\quad + \sum_c \frac{Y^c}{Y}(r_0, \pi_0, \nu_0) \left[\frac{W^c}{Y^c}(r_{LR}, \pi_{LR}^c) - \frac{K^c}{Y^c}(r_{LR}) - \frac{B^c}{Y^c} - \left(\frac{W^c}{Y^c}(r_0, \pi_0^c) - \frac{K^c}{Y^c}(r_0) - \frac{B^c}{Y^c} \right) \right]
\end{aligned}$$

Note that $\frac{W^c}{Y^c}(r_0, \pi_0^c) - \frac{K^c}{Y^c}(r_0) - \frac{B^c}{Y^c}$ is 0 by the assumption of zero initial NFA. To first-order, therefore, the product of $\left[\frac{Y^c}{Y}(r_{LR}, \pi_{LR}, \nu_{LR}) - \frac{Y^c}{Y}(r_0, \pi_0, \nu_0) \right]$ and $\left[\frac{W^c}{Y^c}(r_{LR}, \pi_{LR}^c) - \frac{K^c}{Y^c}(r_{LR}) - \frac{B^c}{Y^c} \right]$ is zero as well. To first-order, the above then simplifies to the equivalent

$$\begin{aligned}
0 &= \sum_c \frac{Y_0^c}{Y_0} \left[\frac{W^c}{Y^c}(r_{LR}, \pi_{LR}^c) - \frac{K^c}{Y^c}(r_{LR}) - \frac{B^c}{Y^c} - \left(\frac{W^c}{Y^c}(r_0, \pi_0^c) - \frac{K^c}{Y^c}(r_0) - \frac{B^c}{Y^c} \right) \right] && \text{(A.7)} \\
&= \sum_c \frac{Y_0^c}{Y_0} \left[\frac{W^c}{Y^c}(r_{LR}, \pi_{LR}^c) - \frac{W^c}{Y^c}(r_0, \pi_{LR}^c) + \frac{W^c}{Y^c}(r_0, \pi_{LR}^c) - \frac{W^c}{Y^c}(r_0, \pi_0^c) - \left(\frac{K^c}{Y^c}(r_{LR}) - \frac{K^c}{Y^c}(r_0) \right) \right] \\
&\simeq \sum_c \frac{Y_0^c}{Y_0} \left[\frac{\partial \frac{W^c}{Y^c}(r_0, \pi_{LR}^c)}{\partial r}(r_{LR} - r_0) + \frac{W^c}{Y^c}(r_0, \pi_0^c) \left(\exp(\Delta_{LR}^{comp,c}) - 1 \right) - \frac{\partial \frac{K^c}{Y^c}(r_0)}{\partial r}(r_{LR} - r_0) \right] \\
&\simeq \sum_c \frac{W_0^c}{Y_0} \left[\frac{\partial \log \frac{W^c}{Y^c}(r_0, \pi_{LR}^c)}{\partial r}(r_{LR} - r_0) + \Delta_{LR}^{comp,c} - \frac{1}{\frac{W^c}{Y^c}(r_0, \pi_0)} \frac{\partial \frac{K^c}{Y^c}(r_0)}{\partial r}(r_{LR} - r_0) \right], && \text{(A.8)}
\end{aligned}$$

where we write $\frac{Y_0^c}{Y_0}$ and $\frac{W_0^c}{Y_0}$ to denote $\frac{Y^c}{Y}(r_0, \pi_0, \nu_0)$ and $\frac{W^c}{Y}(r_0, \pi_0^c, \nu_0^c)$.

Let us also define

$$\begin{aligned}
\epsilon^{d,c} &\equiv \frac{\partial \log \frac{W^c}{Y^c}(r_0, \pi_{LR}^c)}{\partial r} \\
\epsilon^{s,c} &\equiv -\frac{1}{\frac{W^c}{Y^c}(r_0, \pi_0^c)} \frac{\partial \frac{K^c}{Y^c}(r_0)}{\partial r} \\
\omega^c &\equiv \frac{W^c}{W}(r_0, \pi_0, \nu_0)
\end{aligned}$$

and divide both sides of (A.8) by $\frac{W}{Y}(r_0, \pi_0, \nu_0)$ to obtain the first-order result

$$\begin{aligned} 0 &\simeq \sum_c \omega^c \left[\Delta_{LR}^{comp,c} + (\epsilon^{d,c} + \epsilon^{s,c})(r_{LR} - r_0)\right] \\ &= \bar{\Delta}_{LR}^{comp} + (\bar{\epsilon}^d + \bar{\epsilon}^s)(r_{LR} - r_0) \end{aligned} \tag{A.9}$$

where we let bars denote averages across countries with initial wealth weights ω^c. The equations (12) and (13) are rearrangements of (A.9).

Now, the change in W^c/Y^c in each country can be written to first-order as

$$\Delta_{LR} \log\left(\frac{W^c}{Y^c}\right) = \Delta_{LR}^{comp,c} + \epsilon^{d,c}(r_{LR} - r_0)$$

Summing up both sides with weights ω^c, this becomes

$$\overline{\Delta_{LR} \log\left(\frac{W^c}{Y^c}\right)} = \bar{\Delta}_{LR}^{comp} + \bar{\epsilon}^d(r_{LR} - r_0)$$

and using (A.9) to substitute out for $r_{LR} - r_0$, we obtain (14),

$$\overline{\Delta_{LR} \log\left(\frac{W^c}{Y^c}\right)} = \frac{\bar{\epsilon}^s}{\bar{\epsilon}^s + \bar{\epsilon}^d} \bar{\Delta}_{LR}^{comp} \tag{A.10}$$

Proof of proposition 3. The change in $NFA^c/Y^c = W^c/Y^c - K^c/Y^c - B^c/Y^c$ is given by

$$\begin{aligned} \Delta_{LR} \frac{NFA^c}{Y^c} &= \frac{W_0^c}{Y_0^c}\left[\exp\left(\Delta_{LR}^{comp,c} + (\epsilon^{d,c} + \epsilon^{s,c})(r_{LR} - r_0)\right) - 1\right] \\ &= \frac{W_0^c}{Y_0^c}\left[\exp\left(\Delta_{LR}^{comp,c} + (\epsilon^{d,c} + \epsilon^{s,c})\frac{\bar{\Delta}_{LR}^{comp}}{\bar{\epsilon}^d + \bar{\epsilon}^s}\right) - 1\right] \\ &= \frac{W_0^c}{Y_0^c}\left[\exp\left(\Delta_{LR}^{comp,c} - \bar{\Delta}_{LR}^{comp} + (\epsilon^{d,c} + \epsilon^{s,c} - (\bar{\epsilon}^d + \bar{\epsilon}^s))\frac{\bar{\Delta}_{LR}^{comp}}{\bar{\epsilon}^d + \bar{\epsilon}^s}\right) - 1\right] \\ &= \frac{W_0^c}{Y_0^c}\left[\exp\left(\Delta_{LR}^{comp,c} - \bar{\Delta}_{LR}^{comp} + (\epsilon^{d,c} + \epsilon^{s,c} - (\bar{\epsilon}^d + \bar{\epsilon}^s))(r_{LR} - r_0)\right) - 1\right] \end{aligned}$$

Rearranged, this gives the desired result, which is

$$\log\left(1 + \left(\Delta_{LR}\frac{NFA^c}{Y^c}\right) \Big/ \frac{W_0^c}{Y_0^c}\right) = \Delta_{LR}^{comp,c} - \bar{\Delta}_{LR}^{comp} + (\epsilon^{d,c} + \epsilon^{s,c} - (\bar{\epsilon}^d + \bar{\epsilon}^s))(r_{LR} - r_0)$$

B.4 Relaxing assumptions in propositions 2 and 3

In the more general case, we allow initial NFA's to be non-zero and debt-to-output ratios to vary over time. Below, we show how the formulas are modified in this case, and some discussions of how particular sequences of debt-to-output ratios can mitigate or even undo the general equilibrium effects on interest rates.

Allowing for nonzero initial NFAs. With non-zero initial NFAs, there is a compositional effect of aging on net asset demand insofar as the change in relative GDP across countries is correlated with initial NFAs.

If NFA_0^c is not zero in every country c, we would retain an additional term in (A.7), equal to first-order to

$$\sum_c \left[\frac{Y^c}{Y}(r_{LR}, \pi_{LR}, \nu_{LR}) - \frac{Y_0^c}{Y_0}\right] \frac{NFA_0^c}{Y_0^c} = \sum_c \frac{Y_0^c}{Y_0} \Delta_{LR} \log \frac{Y^c}{Y}(r_{LR}, \pi_{LR}, \nu_{LR}) \frac{NFA_0^c}{Y_0^c}$$

When we divide by $\frac{W_0}{Y_0}$ as in our derivation of (A.9), this becomes

$$\sum_c \omega^c \frac{NFA_0^c}{W_0^c} \Delta_{LR} \log \frac{Y^c}{Y}(r_{LR}, \pi_{LR}, \nu_{LR}) \tag{A.11}$$

which will show up as an additional term in (A.9). Since the wealth-weighted average of $\frac{NFA_0^c}{W_0^c}$ is zero by global market clearing, this can be written as a wealth-weighted covariance

$$\text{Cov}_{\omega^c}\left(\frac{NFA_0^c}{W_0^c}, \Delta_{LR} \log \frac{Y^c}{Y}(r_{LR}, \pi_{LR}, \nu_{LR})\right) \tag{A.12}$$

If we define

$$\Delta_L^{demog} R \frac{Y^c}{Y} \equiv \frac{\partial(\log \frac{Y^c}{Y})}{\partial \pi} \Delta_{LR} \pi + \frac{\partial(\log \frac{Y^c}{Y})}{\partial \nu} \Delta_{LR} \nu$$

to be the change in GDP shares caused by demographic change alone, holding r constant, and

$$\bar{\epsilon}^{weight} \equiv \text{Cov}_{\omega^c}\left(\frac{NFA_0^c}{W_0^c}, \frac{\partial(\log \frac{Y^c}{Y})}{\partial r}\right) \tag{A.13}$$

then the modified (A.9) becomes

$$\bar{\Delta}_{LR}^{comp} + \text{Cov}_{\omega^c}\left(\frac{NFA_0^c}{W_0^c}, \Delta_{LR}^{demog} \frac{Y^c}{Y}\right) + (\bar{\epsilon}^d + \bar{\epsilon}^s + \bar{\epsilon}^{weight})(r_{LR} - r_0) = 0 \tag{A.14}$$

and we can solve to obtain

$$r_{LR} - r_0 = \frac{\bar{\Delta}_{LR}^{comp} + \text{Cov}_{\omega^c}\left(\frac{NFA_0^c}{W_0^c}, \Delta_{LR}^{demog} \frac{Y^c}{Y}\right)}{\bar{\epsilon}^d + \bar{\epsilon}^s + \bar{\epsilon}^{weight}}$$

Note that the two departures from our previous result, the covariance in (A.14) and the covariance in the definition (A.13) of $\bar{\epsilon}^{weight}$, both involve wealth-weighted covariances between initial net foreign asset positions as shares of wealth, $\frac{NFA_0^c}{W_0^c}$, and some change in each country's GDP weight (either in response to demographics or endogenously in response to r). A priori, there is no particular reason to have a covariance in either direction here, and indeed we have found that these terms seem quite small in practice, to the point that they can be disregarded in our main analysis without risk for non-trivial errors.

Our previous simplification for the average change in wealth-to-GDP no longer holds, but we can still write

$$\overline{\Delta_{LR} \log \frac{W^c}{Y^c}} \simeq \bar{\Delta}^{comp} + \bar{\epsilon}^d (r_{LR} - r_0).$$

The change in NFA in each country is

$$\Delta \log\left(1+\frac{\Delta_{LR}NFA^c/Y^c}{W_0^c/Y_0^c}\right) = \Delta_{LR}^{comp,c} + (\epsilon^{d,c}+\epsilon^{s,c})(r_{LR}-r_0)$$

Change in debt-to-output ratios. Suppose that each country operates a fiscal rule that targets an exogenous sequence $\frac{B_t^c}{Y_t^c}$ which converges to some long-run value $\frac{B_{LR}^c}{Y_{LR}^c}$ in every country. The average change in bonds is a shifter of asset supply, and the new version of (12) is

$$\bar{\Delta}_{LR}^{comp} - \overline{\frac{\Delta_{LR}B^c/Y^c}{W_0^c/Y_0^c}} + \bar{\epsilon}^d(r_{LR}-r_0) \simeq -\bar{\epsilon}^s(r_{LR}-r_0), \tag{A.15}$$

where $\overline{\frac{\Delta_{LR}B^c/Y^c}{W_0^c/Y_0^c}} \equiv \sum_c \omega^c\left(\frac{B_{LR}^c}{Y_{LR}^c}-\frac{B_0^c}{Y_0^c}\right)$ is the average log change in debt-to-output ratios.

We can solve (A.15) to obtain $r_{LR}-r_0$, which is simply the original formula with this shifter in supply subtracted from the compositional effect:

$$r_{LR}-r_0 = \frac{\bar{\Delta}_{LR}^{comp} - \overline{\frac{\Delta_{LR}B^c/Y^c}{W_0^c/Y_0^c}}}{\bar{\epsilon}^d+\bar{\epsilon}^s} \tag{A.16}$$

The average change in wealth-to-GDP now becomes

$$\overline{\Delta_{LR}\log\frac{W^c}{Y^c}} \simeq \frac{\bar{\epsilon}^s}{\bar{\epsilon}^d+\bar{\epsilon}^s}\bar{\Delta}_{LR}^{comp} + \frac{\bar{\epsilon}^d}{\bar{\epsilon}^d+\bar{\epsilon}^s}\overline{\frac{\Delta_{LR}B^c/Y^c}{W_0^c/Y_0^c}} \tag{A.17}$$

which adds the direct impact of increasing debt to (14), and the change in NFA in each country is

$$\log\left(1+\frac{\Delta_{LR}NFA_{LR}^c}{W_0^c/Y_0^c}\right) \simeq \left(\Delta_{LR}^{comp,c} - \frac{\Delta_{LR}B^c/Y^c}{W_0^c/Y_0^c}\right) - \left(\overline{\Delta}_{LR}^{comp} - \overline{\frac{\Delta_{LR}B^c/Y^c}{W_0^c/Y_0^c}}\right) + \left(\epsilon^{c,d}+\epsilon^{c,s}-\left(\bar{\epsilon}^d+\bar{\epsilon}^s\right)\right)(r_{LR}-r_0) \tag{A.18}$$

which now subtracts the change in asset supply from bonds in each country from the compositional effect on asset demand, but is otherwise the same formula as (15).

Neutralizing debt-to-output policy. The equations (A.16) and (A.18) show that effects of demographics on interest rates and NFAs can be neutralized if governments conduct a debt policy that absorbs the shift in aggregate asset demand. More precisely, if all governments expand debt in line with their compositional effect

$$\Delta_{LR}^{comp,c} = \frac{\Delta_{LR}B^c/Y^c}{W_0^c/Y_0^c}$$

we obtain $r_{LR}-r_0 \simeq 0$ and $\log\left(1+\frac{\Delta_{LR}NFA_{LR}^c}{W_0^c}\right) \simeq 0$ for every country c. Intuitively, if governments in every country expand debt to perfectly meet the new demand for assets, there is no change in net asset demand, so interest rates stay constant and NFAs do not change. In this case, the change in wealth equals the compositional effect in every country, since there is no general equilibrium feedback reducing the impact of increased asset demand on wealth.

An alternative specification is if each government increases the *level* of its debt-to-output ratio in line with the *average* compositional effect, so that for all c

$$\frac{W}{Y}\bar{\Delta}^{comp}_{LR} = \Delta_{LR}\frac{B^c}{Y^c}$$

In this case, we still have $r_{LR} - r_0 = 0$, but now (A.18) implies that NFAs change in line with the demeaned compositional effect across countries $\Delta^{comp,c} - \bar{\Delta}^{comp,c}$.

Strikingly, these findings are also true in the transition, not just in the long run. That is, if the sequence of debt holdings satisfies $\frac{\Delta_t B^c/Y^c}{W_0^c/Y_0^c} = \Delta_t^{comp,c}$ for every t, then interest rates and NFAs are constant over time, and the path of wealth-to-output ratios equals the path of the compositional effect. Moreover, if $\frac{\Delta_t B^c/Y^c}{W_0^c/Y_0^c} = \bar{\Delta}^{comp}$, then the interest rate change is zero at every point in time, and NFAs at every time period for each country is the demeaned compositional effect.

B.5 Proof of proposition 4

B.5.1 Framework

Dropping idiosyncratic risk and the borrowing constraint, and writing assets (which are now common to all individuals of the same age at a given time) as $a_{j,t}$ for convenience, the individual problem is

$$\max_{\{c_{jt},a_{j+1,t+1}\}} \sum_{j=0}^{J} \beta_j \Phi_j \frac{c_{jt}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$$
$$\text{s.t. } c_{j,t} + \phi_j a_{j+1,t+1} \le w_t\left((1-\tau)\ell_j + tr_j\right) + (1+r_t)a_{jt} \tag{A.19}$$

where $t \equiv k + j$ is time. Note that we assume agents start and end the lifecycle with zero assets: $a_{0,t} = 0$ and $\Phi_{J+1}a_{J+1,t} = 0$.

The only way in which time-varying macroeconomic aggregates enter this problem is through the real wage w_t and real interest rate r_t. Suppose that we have a balanced growth path by age with technology growth γ, so that $r_t = r$, $w_t = w(1+\gamma)^t$ for some w, and we can also write $a_{jt} = a_j(1+\gamma)^t$ and $c_{jt} = c_j(1+\gamma)^t$. Then (A.19) becomes

$$\max_{\{c_j,a_{j+1}\}} \sum_{j=0}^{J} \tilde{\beta}_j \Phi_j \frac{c_j^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$$
$$\text{s.t. } c_j + \phi_j(1+\gamma)a_{j+1} \le wy_j + (1+r)a_j \tag{A.20}$$

where we define $\tilde{\beta}_j \equiv \beta_j(1+\gamma)^{j(1-\frac{1}{\sigma})}$ and $y_j \equiv (1-\tau)\ell_j + tr_j$. Again, we have the initial and terminal conditions $a_0 = 0$ and $a_{J+1} = 0$.

B.5.2 Effects on wealth-to-GDP

We are interested in characterizing the semielasticity of steady-state W/Y with respect to steady-state r. Using balanced growth by age and a demographic steady state, we have both $W_t = W(1+g)^t$ and $Y_t = Y(1+g)^t$, where $W = \sum_{j=0}^{J} \pi_j a_j$ and $Y = F(k(r),1)L_0$.

Thanks to linearity of the budget constraint and homotheticity of intertemporal preferences, the entire problem (A.20) scales in w. Hence, if we use W to denote aggregate wealth given the normalization $w = 1$, then for a different w, steady-state wealth will be wW.

We can now write the semielasticity of wealth-to-GDP with respect to r as

$$\frac{\partial \log(w(r)W(r)/Y(r))}{\partial r} = \frac{\partial \log W(r)}{\partial r} + \frac{\partial \log(w(r)/F(k(r),1))}{\partial r} \tag{A.21}$$

where the first term $\frac{\partial \log W(r)}{\partial r}$ is the semielasticity of wealth with respect to r, holding fixed wages at $w = 1$. Note that the second term, the semielasticity of the wage-output ratio with respect to r, will be zero in the Cobb-Douglas case. We will return to this term for the non-Cobb-Douglas case later, and focus on evaluating the first term $\frac{\partial \log W(r)}{\partial r}$ for now.

B.5.3 Budget constraint, Euler equation, and wealth

At the optimum, the budget constraint in (A.20) will hold with equality, and (recalling that we are now using the normalization $w = 1$) can be rewritten as

$$a_{j+1} = \frac{1}{\phi_j}\frac{1}{1+\gamma}\left(y_j - c_j + (1+r)a_j\right)$$

Multiply both sides by the survival probability $\Phi_{j+1} = \phi_j \cdot \Phi_j$ to obtain

$$\Phi_{j+1}a_{j+1} = \frac{1}{1+\gamma}\Phi_j\left(y_j - c_j + (1+r)a_j\right) \tag{A.22}$$

Now, a demographic steady-state implies that $\frac{\pi_{j+1}}{\Phi_{j+1}} = \frac{1}{1+n}\frac{\pi_j}{\Phi_j}$. Multiplying (A.22) by this gives

$$\begin{aligned}\pi_{j+1}a_{j+1} &= \frac{1}{1+g}\pi_j\left(y_j - c_j + (1+r)a_j\right) \\ &= \frac{1}{1+g}\pi_j(y_j - c_j) + \pi_j(1+\hat{r})a_j\end{aligned} \tag{A.23}$$

where we use the steady-state relationship $1+g = (1+n)(1+\gamma)$ and the definition $1+\hat{r} = \frac{1+r}{1+g}$.

Also using $\Phi_{j+1} = \phi_j \cdot \Phi_j$, the optimization problem (A.20) has the Euler equation $\tilde{\beta}_j c_j^{-1/\sigma} = \tilde{\beta}_{j+1}\frac{1+r}{1+\gamma}c_{j+1}^{-1/\sigma}$. (Note that survival probablities drop out, since they appear symmetrically in the price of an annuity and in preferences.) This can be iterated forward to obtain

$$c_j = \left(\frac{\tilde{\beta}_j}{\tilde{\beta}_0}\left(\frac{1+r}{1+\gamma}\right)^j\right)^\sigma c_0 \tag{A.24}$$

We can solve for the $2J+1$ unknowns $c_0, \ldots, c_J$ and $a_1, \ldots, a_J$ (recalling $a_0 = a_{J+1} = 0$) using $2J+1$ equations, specifically (A.24) for $j = 1, \ldots, J$ and (A.23) for $j = 0, \ldots, J$.

Note that r enters these equations in two places: on the right in (A.23) (inside $1+\hat{r} = \frac{1+r}{1+g}$), and on the right in (A.24). To find the derivative of $\log W$ with respect to r, we will separately perturb r in each of these two places, find the effect on $\log W$, and then sum to find the overall derivative. The part of the derivative from perturbing r inside the Euler equation (A.24) can be thought of as the substitution effect, since it takes into account the effect of intertemporal substitution but ignores the effect of r in the budget constraint, and part from perturbing r inside (A.23) can be thought of as the income effect.

We will consider two cases of increasing complexity: first the special case where steady-state $\hat{r} = 0$ and preferences are Cobb-Douglas, and then the general case, for which we will also need to evaluate the second term in (A.21).

B.5.4 Special case with $\hat{r} = 0$ and Cobb-Douglas

Substitution effect. Given steady-state $\hat{r} = 0$, we can sum (A.23) from 0 to j to obtain

$$\pi_j a_j = \sum_{k=0}^{j-1} \pi_k \frac{1}{1+g}(y_k - c_k) \tag{A.25}$$

which for $j = J+1$ becomes the lifetime budget constraint

$$0 = \sum_{j=0}^{J} \pi_j \frac{1}{1+g}(y_j - c_j) \tag{A.26}$$

Summing up (A.25), we obtain

$$\begin{aligned} W = \sum_{j=0}^{J} \pi_j a_j &= \sum_{j=0}^{J} \sum_{k=0}^{j-1} \pi_k \frac{1}{1+g}(y_k - c_k) \\ &= \sum_{j=0}^{J} (J-j)\pi_j \frac{1}{1+g}(y_j - c_j) = \sum_{j=0}^{J} \pi_j j \frac{1}{1+g}(c_j - y_j) \end{aligned} \tag{A.27}$$

This simple result states that total wealth is the gap between the ages at which consumption occurs and the ages at which (after-tax-and-transfer) income is earned.[63] The intuition is simple: every year that income is deferred for later consumption requires holding an asset.[64]

Now suppose that we perturb r in (A.24). Log-differentiating gives

$$\frac{dc_j}{c_j} = \sigma j \frac{dr}{1+r} + \frac{dc_0}{c_0} \tag{A.28}$$

and substituting into (A.26) we get

$$0 = \frac{dc_0}{c_0} \sum_{j=0}^{J} \pi_j c_j + \sigma \sum_{j=0}^{J} j \pi_j c_j \frac{dr}{1+r}$$

which we can solve out to obtain

$$\frac{dc_0}{c_0} = -\sigma \frac{\sum_{j=0}^{J} \pi_j j c_j}{\sum_{j=0}^{J} \pi_j c_j} \frac{dr}{1+r}$$

[63] This is multiplied by $1/(1+g)$, since W is incoming wealth, which when normalized by GDP growth is $1/(1+g)$ times smaller than the outgoing wealth from income exceeding consumption in prior periods.

[64] See, for instance, Willis (1988) and Lee (1994).

and hence, plugging back into (A.28),

$$\frac{dc_j}{c_j} = \sigma \left(j - \frac{\sum_{k=0}^{J} \pi_k k c_k}{\sum_{k=0}^{J} \pi_k c_k} \right) \frac{dr}{1+r} \tag{A.29}$$

i.e. the proportional change in consumption at a given age j due to the substitution response to an interest rate shock $\frac{dr}{1+r}$ equals the elasticity of intertemporal substitution σ times the difference between age j and the average age of consumption. Here, the schedule of consumption by age rotates counterclockwise around the average age of consumption: in response to a rising r, individuals substitute so that their consumption increases at high ages and decreases at low ages, increasing by more as we get further from the average age.

Plugging (A.29) into (A.27) gives

$$\begin{aligned} dW &= \sigma \frac{1}{1+g} \sum_{j=0}^{J} \pi_j j c_j \left(j - \frac{\sum_k \pi_k k c_k}{\sum_k \pi_k c_k} \right) \frac{dr}{1+r} \\ &= \sigma \frac{1}{1+g} \sum_{j=0}^{J} \pi_j c_j \left(j - \frac{\sum_k \pi_k k c_k}{\sum_k \pi_k c_k} \right)^2 \frac{dr}{1+r} \end{aligned} \tag{A.30}$$

where in the second step we use the fact that $\sum_{j=0}^{J} \pi_j c_j \left(j - \frac{\sum_k \pi_k k c_k}{\sum_k \pi_k c_k} \right) = 0$. Finally, dividing both sides of (A.30) by W and multiplying and dividing the right by $C = \sum_{j=0}^{J} \pi_j c_j$, we get

$$d \log W = \sigma \frac{C}{(1+g)W} \sum_{j=0}^{J} \pi_j c_j \left(j - \frac{\sum_k \pi_k k c_k}{\sum_k \pi_k c_k} \right)^2 \frac{dr}{1+r}$$

Now, if we let Age_c be a random variable distributed across ages j with mass proportional to $\pi_j c_j$, then this becomes simply

$$d \log W = \sigma \frac{C}{(1+g)W} \frac{\text{Var} Age_c}{1+r} dr \tag{A.31}$$

which gives us the substitution effect of dr.

Note that $\text{Var} Age_c$, which grows *quadratically* with the dispersion of consumption across ages, appears in (A.31). This reflects two forces. First, from (A.29) we see that when consumption is further from the average age, it changes by proportionally more in response to a change in r. Second, financing higher consumption later in life (and correspondingly lower consumption earlier in life) requires holding assets for longer, leading to a larger effect on aggregate assets. Together, these produce the quadratic effect in (A.31).[65]

[65] Although the $1+g$ and $1+r$ factors in the denominator of (A.31) are equal in this $r = g$ special case, we retain them to highlight their separate origin. The $1+r$ originates with (A.28), since $d\log(1+r) = dr/1+r$. Meanwhile, the $1+g$ originates with (A.23), since wealth is measured at the beginning of the period, and yesterday's net saving by a $1/(1+n)$ smaller generation when productivity was $1/(1+\gamma)$ as high translates into normalized beginning-of-period wealth today that is $1/(1+g)$ smaller relative to the

Income effect. Write $1 + r = (1 + r_{ss})(1 + \tilde{r})$, so that $1 + \hat{r} = (1 + \hat{r}_{ss})(1 + \tilde{r})$. Substituting this into (A.23) and assuming that $\hat{r}_{ss} = 0$, we get

$$\pi_{j+1}a_{j+1} = \frac{1}{1+g}\pi_j(y_j - c_j) + \pi_j a_j \tilde{r} + \pi_j a_j \tag{A.32}$$

Noting that $a_j\tilde{r}$ enters (A.32) in the same way as $\frac{1}{1+g}(y_j - c_j)$ (i.e. this extra asset income acts as another form of net income), we can redo the same steps to obtain modified versions of (A.26) and (A.27):

$$0 = \sum_{j=0}^{J} \pi_j \left(a_j\tilde{r} + \frac{1}{1+g}(y_j - c_j) \right) \tag{A.33}$$

$$W = \sum_{j=0}^{J} \pi_j j \left(\frac{1}{1+g}(c_j - y_j) - a_j\tilde{r} \right) \tag{A.34}$$

Since the interest rate in the Euler equation (A.24) is unchanged, we must have $dc_j/c_j \equiv \hat{c}$ for some common $\hat{c}$ across all j. Totally differentiating, (A.33) thus becomes

$$\frac{1}{1+g}\hat{c}\sum_{j=0}^{J} \pi_j c_j = d\tilde{r}\sum_{j=0}^{J} \pi_j a_j \tag{A.35}$$

and (A.34) becomes

$$dW = \frac{1}{1+g}\hat{c}\sum_{j=0}^{J} \pi_j j c_j - d\tilde{r}\sum_{j=0}^{J} \pi_j j a_j \tag{A.36}$$

Dividing both sides of (A.36) by (A.35) (and recalling that $W = \sum_{j=0}^{J} \pi_j a_j$), we get

$$\begin{aligned} d\log W &= \left(\frac{\sum_{j=0}^{J} \pi_j j c_j}{\sum_{j=0}^{J} \pi_j c_j} - \frac{\sum_{j=0}^{J} \pi_j j a_j}{\sum_{j=0}^{J} \pi_j a_j} \right) d\tilde{r} \\ &= (\mathbb{E}Age_c - \mathbb{E}Age_a)\frac{dr}{1+r} \end{aligned} \tag{A.37}$$

where we define the random variable Age_c as before, and analogously Age_a as a variable with mass at each age j proportional to $\pi_j a_j$.

The basic intuition behind (A.37) is the same as in (A.27): total wealth is the gap between the ages at which consumption occurs and the ages at which income is earned. For the income effect, we can think of a rise in r as an increase in income proportional to assets in each period. Consumption will increase proportionally in every period in response to this extra income; this increased consumption will occur, on average, at the same age as existing consumption. The marginal change in wealth is proportional to the gap between the average age of the marginal consumption ($\mathbb{E}Age_c$) and the average age of the marginal income ($\mathbb{E}Age_a$).

Overall special-case result. Evaluating the semielasticity of wealth-to-GDP with respect to r in (A.21), noting that the second term is zero because of the Cobb-Douglas assumption, we

normalized savings yesterday. (Of course, both factors will tend to be fairly small.)

combine (A.31) and (A.37) to obtain

$$\underbrace{\sigma \frac{C}{(1+g)W} \frac{\text{Var}Age_c}{1+r}}_{\equiv \epsilon^d_{substitution}} + \underbrace{\frac{\mathbb{E}Age_c - \mathbb{E}Age_a}{1+r}}_{\equiv \epsilon^d_{income}} \tag{A.38}$$

which is $\partial \log W(r) / \partial r$.

B.5.5 General case

Substitution effect. For the case $\hat{r} \neq 0$, sum both sides of (A.23) from $j = 0$ to $j = J$, making use of the boundary conditions $a_{J+1} = 0$ and $a_0 = 0$ and the definition $W = \sum_{j=0}^{J} \pi_j a_j$, to obtain

$$W = \frac{1}{1+g} \pi_j (y_j - c_j) + (1 + \hat{r}) W$$

which can be rearranged as

$$W = \frac{1}{\hat{r}} \sum_{j=0}^{J} \pi_j \frac{1}{1+g} (c_j - y_j) \tag{A.39}$$

Applying (A.23), we can obtain the general version of the lifetime budget constraint (A.26)

$$0 = \sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} \frac{1}{1+g} (y_j - c_j) \tag{A.40}$$

The consumption response to a r shock in the Euler equation is still given by (A.28). Substituting into (A.40), we obtain

$$0 = \frac{dc_0}{c_0} \sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} c_j + \frac{dr}{1+r} \sigma \sum_{j=0}^{J} j \pi_j (1+\hat{r})^{-j} c_j$$

which we can solve out to obtain

$$\frac{dc_0}{c_0} = -\sigma \frac{\sum_{j=0}^{J} j \pi_j (1+\hat{r})^{-j} c_j}{\sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} c_j} \frac{dr}{1+r}$$

and hence, plugging back into (A.28),

$$\frac{dc_j}{c_j} = \sigma \left(j - \frac{\sum_{k=0}^{J} k \pi_k (1+\hat{r})^{-j} c_k}{\sum_{k=0}^{J} \pi_k (1+\hat{r})^{-j} c_k} \right) \frac{dr}{1+r} \tag{A.41}$$

which is a slight generalization of (A.29), replacing the average age of consumption $\frac{\sum_{k=0}^{J} k \pi_k c_k}{\sum_{k=0}^{J} \pi_k c_k}$ with the average age in present value terms discounted by $\hat{r}$, $\frac{\sum_{k=0}^{J} k \pi_k (1+\hat{r})^{-j} c_k}{\sum_{k=0}^{J} \pi_k (1+\hat{r})^{-j} c_k}$.

Plugging (A.41) into (A.39), we have

$$
\begin{aligned}
dW &= \frac{1}{\hat{r}} \sum_{j=0}^{J} \pi_j \frac{1}{1+g} \sigma \left(j - \frac{\sum_{k=0}^{J} k\pi_k (1+\hat{r})^{-j} c_k}{\sum_{k=0}^{J} \pi_k (1+\hat{r})^{-j} c_k} \right) c_j \frac{dr}{1+r} \\
&= \sigma \frac{dr}{1+r} \frac{1}{1+g} \frac{1}{\hat{r}} \left(\sum_{j=0}^{J} j\pi_j c_j - \sum_{j=0}^{J} \pi_j c_j \cdot \frac{\sum_{j=0}^{J} j\pi_j (1+\hat{r})^{-j} c_j}{\sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} c_j} \right)
\end{aligned}
$$

Dividing both sides by W and multiplying and dividing the right by $C = \sum_{j=0}^{J} \pi_j c_j$, we obtain

$$
\begin{aligned}
d \log W &= \frac{dr}{1+r} \frac{C}{(1+g)W} \frac{1}{\hat{r}} \left(\frac{\sum_{j=0}^{J} j\pi_j c_j}{\sum_{j=0}^{J} \pi_j c_j} - \frac{\sum_{j=0}^{J} j\pi_j (1+\hat{r})^{-j} c_j}{\sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} c_j} \right) \\
&= \frac{dr}{1+r} \frac{C}{(1+g)W} \frac{\mathbb{E}Age_c - \mathbb{E}Age_c^{PV}}{\hat{r}} \qquad \text{(A.42)}
\end{aligned}
$$

where we define Age_c^{PV} as the random variable with probability mass on each age j proportional to $\pi_j (1+\hat{r})^{-j} c_j$.

Income effect. We define $\tilde{r}$ as before, so that $1+r = (1+r_{ss})(1+\tilde{r})$ and $1+\hat{r} = (1+\hat{r}_{ss})(1+\tilde{r})$, and the budget constraint (A.32) becomes

$$
\pi_{j+1} a_{j+1} = \frac{1}{1+g} \pi_j (y_j - c_j) + \pi_j (1+\hat{r}) a_j \tilde{r} + (1+\hat{r}) \pi_j a_j \qquad \text{(A.43)}
$$

Since $(1+\hat{r}) a_j \tilde{r}$ enters into the budget constraint the same way as income net of consumption, $\frac{1}{1+g}(y_j - c_j)$, we can write modified versions of (A.40) and (A.39) that incorporate this term:

$$
0 = \sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} \left((1+\hat{r}) a_j \tilde{r} + \frac{1}{1+g} (y_j - c_j) \right) \qquad \text{(A.44)}
$$

$$
W = \frac{1}{\hat{r}} \sum_{j=0}^{J} \pi_j \left(\frac{1}{1+g} (c_j - y_j) - (1+\hat{r}) a_j \tilde{r} \right) \qquad \text{(A.45)}
$$

Now totally differentiate with respect to $\tilde{r}$. Since we are not perturbing the r in the Euler equation, (A.28) implies that $dc_j / c_j \equiv \hat{c}$ for some common $\hat{c}$ across all j. (A.44) can be solved out to obtain

$$
\hat{c} = d\tilde{r} \frac{(1+\hat{r}) \sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} a_j}{\frac{1}{1+g} \sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} c_j}
$$

Plugging this into the totally differentiated (A.45), we obtain

$$dW = \frac{1}{\hat{r}}\left(\sum_{j=0}^{J}\frac{1}{1+g}\pi_j c_j \hat{c} - d\tilde{r}(1+\hat{r})\sum_{j=0}^{J}\pi_j a_j\right)$$
$$= d\tilde{r}\left(\left(\sum_{j=0}^{J}\pi_j c_j\right)\frac{(1+\hat{r})\sum_{j=0}^{J}\pi_j(1+\hat{r})^{-j}a_j}{\sum_{j=0}^{J}\pi_j(1+\hat{r})^{-j}c_j} - (1+\hat{r})\sum_{j=0}^{J}\pi_j a_j\right)$$

Dividing both sides by W, this becomes

$$d\log W = \frac{dr}{1+r}(1+\hat{r})\frac{\frac{C/C^{PV}}{W/W^{PV}}-1}{\hat{r}} = \frac{dr}{1+g}\frac{\frac{C/C^{PV}}{W/W^{PV}}-1}{\hat{r}} \tag{A.46}$$

where we identify $C^{PV} \equiv \sum_{j=0}^{J}\pi_j(1+\hat{r})^{-j}c_j$ and $A^{PV} \equiv \sum_{j=0}^{J}\pi_j(1+\hat{r})^{-j}a_j$, and also write $d\tilde{r} = \frac{dr}{1+r}$.

Labor share effect. In the general, non-Cobb-Douglas case, the $\frac{\partial \log(w(r)/F(k(r),1))}{\partial r}$ term in (A.45), which is the semielasticity of the labor share with respect to r, is nonzero.

Normalizing $L = 1$ and letting $s_L \equiv w/F(k,1)$ be the labor share and $1 - s_L \equiv (r+\delta)k/F(k,1)$ be the capital share, we log-differentiate and use the definition of the local elasticity of substitution η to write

$$d\log s_L - d\log(1-s_L) = (1-\eta)\left(d\log w - d\log(r+\delta)\right) \tag{A.47}$$

Since F has constant returns to scale, the log change in output price (zero here, since output is the numeraire) must be the share-weighted log change in input prices, so that

$$s_L d\log w + (1-s_L)d\log(r+\delta) = 0 \tag{A.48}$$

implying that $d\log w = -\frac{1-s_L}{s_L}d\log(r+\delta)$. Using this and other simplifications, we can rewrite (A.47) as

$$\frac{1}{1-s_L}d\log s_L = -(1-\eta)\frac{1}{s_L}\frac{dr}{r+\delta}$$
$$d\log s_L = (\eta-1)\frac{1-s_L}{s_L}\frac{dr}{r+\delta} \tag{A.49}$$

giving us the semielasticity of the labor share.

Overall result. Combining (A.42), (A.46), and (A.49), the semielasticity (A.21) of wealth-to-GDP with respect to r is

$$\underbrace{\sigma\frac{1}{1+r}\frac{C}{(1+g)W}\frac{\mathbb{E}Age_c - \mathbb{E}Age_c^{PV}}{\hat{r}}}_{\equiv\epsilon^d_{substitution}} + \underbrace{\frac{1}{1+g}\frac{\frac{C/C^{PV}}{W/W^{PV}}-1}{\hat{r}}}_{\equiv\epsilon^d_{income}} + \underbrace{(\eta-1)\frac{(1-s_L)/s_L}{r+\delta}}_{\equiv\epsilon^d_{laborshare}} \tag{A.50}$$

which is our main result.

Continuity in the $\hat{r} \to 0$ limit. Taking the limit of $\frac{\mathbb{E}Age_c - \mathbb{E}Age_c^{PV}}{\hat{r}}$ as $\hat{r} \to 0$ using L'Hospital's rule, we get:

$$\lim_{\hat{r}\to 0} \frac{1}{\hat{r}} \left(\frac{\sum_{j=0}^{J} j\pi_j c_j}{\sum_{j=0}^{J} \pi_j c_j} - \frac{\sum_{j=0}^{J} j\pi_j (1+\hat{r})^{-j} c_j}{\sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} c_j} \right) = \mathbb{E}Age_c \left(\frac{\sum_{j=0}^{J} j^2 \pi_j c_j}{\sum_{j=0}^{J} j\pi_j c_j} - \frac{\sum_{j=0}^{J} j\pi_j c_j}{\sum_{j=0}^{J} \pi_j c_j} \right)$$
$$= \mathbb{E}Age_c \left(\frac{\mathbb{E}Age_c^2}{\mathbb{E}Age_c} - \mathbb{E}Age_c \right) = EAge_c^2 - (EAge_c)^2 = \text{Var}Age_c$$

which makes the $\epsilon^d_{substitution}$ term in (A.50) identical to (A.38).

Similarly, taking the limit of $\frac{\frac{C/C^{PV}}{W/W^{PV}} - 1}{\hat{r}}$ as $\hat{r} \to 0$ using L'Hospital's rule, we get:

$$\lim_{\hat{r}\to 0} \frac{1}{\hat{r}} \left(\frac{\sum_{j=0}^{J} \pi_j c_j \Big/ \sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} c_j}{\sum_{j=0}^{J} \pi_j a_j \Big/ \sum_{j=0}^{J} \pi_j (1+\hat{r})^{-j} a_j} - 1 \right) = \frac{\sum_{j=0}^{J} j\pi_j c_j}{\sum_{j=0}^{J} \pi_j c_j} - \frac{\sum_{j=0}^{J} j\pi_j a_j}{\sum_{j=0}^{J} \pi_j a_j} = \mathbb{E}Age_c - \mathbb{E}Age_a$$

which, when also using the fact that $\hat{r} = 0$ implies $1 + g = 1 + r$, makes the ϵ^d_{income} term in (A.50) identical to (A.38).

C Appendix to Section 3

C.1 Data sources

Demographics. Our population data and projections comes from the 2019 UN World Population Prospects.[66] We gather data between 1950 and 2100 on total number of births, number of births by age-group of the mother, population by 5-year age groups, and mortality rates by 5-year age groups. We interpolate to construct population distributions N_{jt} and mortality rates ϕ_{jt} in every country, every year, and for every age. We compute total population as $N_t = \sum_j N_{jt}$, population distributions as $\pi_{jt} = N_{jt}/N_t$, and population growth rates as $1 + n_t = N_t/N_{t-1}$. Finally, we compute the number of migrants by age M_{jt} as the residual of the population law of motion

$$N_{jt} = (N_{j-1,t-1} + M_{j-1,t-1})\phi_{j-1,t-1}.$$

Age-income profiles. We use the LIS to construct the base-year age-income profiles for all the countries we consider. For Australia, the LIS is based on the Survey of Income and Housing (SIH) and the Household Expenditure Survey (HES), for Austria on the Survey on Income and Living Conditions (SILC), for Canada on the Canadian Income Survey (CIS), for China on the Chinese Household Income Survey (CHIP), for Denmark on the Law Model (based on administrative records), for Estonia on the Estonian Social Survey (ESS) and the Survey on Income and Living Conditions (SILC), for Finland on the Income Distribution Survey (IDS) and the Survey on Income and Living Conditions (SILC), for France on the Household Budget Survey (BdF), for Germany on the German Socio-Economic Panel (GSOEP), for Greece on the Survey of Income and Living Conditions (SILC), for Hungary on the Tárki Household Monitor Survey, for India on the India Human Development Survey (IHDS), for Ireland on the Survey on Income and Living Conditions (SILC), for Italy on the Survey of Household Income and Wealth (SHIW), for Japan on the Japan

[66] https://population.un.org/wpp/

Household Panel Survey (JHPS), for Luxembourg on the Socio-economic Panel "Living in Luxembourg" (PSELL III) and the Survey on Income and Living Conditions (SILC), for Netherlands on the Survey on Income and Living Conditions (SILC), for Norway on the Household Income Statistics, for Poland on the Household Budget Survey, for Slovakia on the Survey of Income and Living Conditions (SILC), for Slovenia on the Household Budget Survey (HBS), for Estonia on the Survey on Income and Living Conditions (SILC), for Sweden on the Household Income Survey (HINK/HEK), and for the United Kingdom on the Family Resources Survey (FRS).

Age-wealth profiles. Our wealth data for the United States comes from the 2016 Survey of Consumer Finance. We gather data from other countries as follows. First, we take data from the Luxembourg Wealth Study (LWS)[67] for Australia in 2010, Canada in 2012, Germany in 2012, United Kingdom in 2011, Italy in 2010, and Sweden in 2005. For Australia the LWS is based on the Survey of Income and Housing (SIH) and the Household Expenditure Survey (HES), for Canada on the Survey of Financial Securities (SFS), for Germany on the German Socio-Economic Panel (GSOEP), for Italy on the Survey of Household Income and Wealth (SHIW), for Sweden on the Household Income Survey (HINK/HEK), and for United Kingdom on the Wealth and Assets Survey (WAS). We rescale the survey weights such that they sum up to the correct number of households according to, respectively, the Australian Bureau of Statistics, Statistics Canada, Statistisches Bundesamt, the Office for National Statistics, the Instituto Nazionale di Statistica, and the United Nations Economic Commission for Europe (UNECE). Next, we use the Household Finance and Consumption Survey (HFCS)[68] for Austria in 2010, Belgium in 2010, Estonia in 2014, Spain in 2010, Finland in 2010, France in 2010, Greece in 2010, Hungary in 2014, Ireland in 2014, Luxembourg in 2014, Netherlands in 2010, Poland in 2014, Slovenia in 2014, and Slovakia in 2014. For China, we rely on the 2013 China Household Finance Survey (CHFS).[69] For India, we use the National Sample Survey (NSS).[70] For Japan, we construct a measure of total wealth by age of household head from Table 4 of the 2009 National Survey of Family Income and Expenditure (NFSIE) available on the online portal of Japanese Government Statistics[71]. This table provides average net worth and total number of households by age groups for single person households and households with two or more members, which we aggregate to obtain total household net worth by age group. For Denmark, we use the 2014 table "Assets and liabilities per person by type of components, sex, age and time" produced by Statistics Denmark that provides a measure of average net wealth per person by age group produced from tax data.

Aggregation. We cross-check the wealth data aggregated from the household survey with the aggregate wealth-to-GDP ratio provided by the WID or the OECD. Table A.1 provides details on the source of both survey and aggregate data, as well as the wealth-to-GDP ratio computed from the survey, compared to the official statistic.

C.2 Robustness

In this section, we show that our results are robust to some of our main assumptions behind the calculation of compositional effects. In the interest of space, we focus here on the United States;

[67] https://www.lisdatacenter.org/our-data/lws-database/
[68] https://www.ecb.europa.eu/stats/ecb_surveys/hfcs/
[69] http://www.chfsdata.org/
[70] http://microdata.gov.in
[71] https://www.e-stat.go.jp

Table A.1: Wealth-to-GDP ratios from survey data and aggregate data

	Wealth survey data			Aggregate data		
Country	Year	Source	$\frac{W^c}{Y^c}$	Year	Source	$\frac{W^c}{Y^c}$
AUS	2014	LWS	3.59	2016	WID	5.09
AUT	2014	HFCS	2.79	2016	OECD	3.90
BEL	2014	HFCS	3.84	2016	OECD	5.74
CAN	2016	LWS	6.98	2016	WID	4.63
CHN	2013	CHFS	3.27	2016	WID	4.20
DEU	2017	LWS	3.8	2016	WID	3.64
DNK	2014	SD	2.54	2016	WID	3.42
ESP	2014	HFCS	4.96	2016	WID	5.33
EST	2014	HFCS	2.78	2016	OECD	2.64
FIN	2014	HFCS	2.33	2016	WID	2.78
FRA	2014	HFCS	3.30	2016	WID	4.85
GBR	2011	LWS	4.01	2016	WID	5.35
GRC	2014	HFCS	2.65	2016	WID	4.25
HUN	2014	HFCS	1.84	2016	OECD	2.19
IND	2013	NSS	4.01	2016	-	-
IRL	2014	HFCS	3.39	2016	CBI	2.32
ITA	2016	LWS	3.35	2016	WID	5.83
JPN	2009	NSFIE	6.11	2016	WID	4.85
LUX	2014	HFCS	3.80	2016	OECD	3.92
NLD	2014	HFCS	1.80	2016	WID	3.92
POL	2014	HFCS	3.31	2016	OECD	1.50
SVK	2014	HFCS	1.80	2016	OECD	2.17
SVN	2014	HFCS	3.11	2016	OECD	2.82
SWE	2005	LWS	2.00	2016	WID	3.81
USA	2016	SCF	4.38	2016	WID	4.28

Notes: This table summarizes our sources of wealth survey data and aggregate data. Abbreviations are described in the text. The survey-based wealth to GDP ratio W^c/Y^c is computed by aggregating household wealth using survey weights and dividing by GDP per household from the national accounts.

conclusions are similar when repeating this exercise in other countries.

Alternative allocation of household wealth across individual members. All our surveys measure wealth at the household level. In the main text, we obtain individual wealth by splitting up all assets equally between all members of the household that are at least as old as the head or spouse. The orange line in figure A.3, labeled "baseline", reproduces the projection from the United States under the main fertility scenario (cf figures 2 and 5). The red line shows that allocating all household wealth to the head increases the compositional effect a little, since heads tend to be older on average; the grey line shows that allocating all wealth equally to head as spouse, as in Poterba (2001), or equally to all household members aged 20 or older. This delivers results extremely close to our baseline.

Constructing compositional effects at the household level. All our exercises in the main text of section 3.1, as well as the alternative considered in the previous paragraph, are conducted at the individual level. To gauge the importance of the household vs individual distinction, here we calculate compositional effects at the household level instead.

We first obtain the age-wealth and labor income profiles at the household level, summing the pre-tax labor income of each household member. To convert the age distribution of the population over individuals to an age distribution over households, we use the PSID to estimate a mapping that gives, for each age j, the age of the household head than an average individual of age j lives with.

With this data in hand, we recompute the compositional effect Δ^{comp}. Figure A.3 reports the projected change in W/Y from this exercise under the baseline fertility scenarios. The dashed line reproduces the central individual-level compositional effect from the main text. Overall, the timing of the projected changes in W/Y change slightly, but the overall magnitude remains close.

Alternative choice of base year profiles. Tabled A.2 and A.3 explores how the magnitude of the compositional effects Δ^{comp} changes when we change the base year 0 we use to construct the age profiles a_{j0} and h_{j0} in equation (10).

In the last row and column, labeled "DH", we use the age effects extracted from a time-age-cohort decomposition in the style of Hall (1968) and Deaton (1997), imposing that all growth loads on time effects. It is important to load growth on time effects to recover the age profiles that are the correct input into Proposition 1.

Using earlier data for age-wealth profiles tends to imply smaller effects, since the age-wealth profile has steepened over time. (The 1977 data stands out as an outlier implying especially small effects; the age-wealth profile in that year declined much more rapidly at higher ages.) Using earlier data for age-labor income profiles tends to imply slightly larger effects, since the hump-shape in the age-labor income profile has moved to the right over time as generations retire later. Overall, using earlier data for both profiles implies mildly smaller effects. In contrast, using the age effects from our time-age-cohort decomposition ("DH-t") implies a slightly larger compositional effect.

C.3 Additional results for section 3.1

Historical predicted change in W/Y from composition effects vs actual change in W/Y. Table A.4 contrasts, for a range of countries for which the World Inequality Database contains a sufficiently long time series of measured wealth-to-GDP ratios, the measured change in the log of

Table A.2: Sensitivity of predicted change in US log W/Y to choice of base year

Panel A. Predicted change in log W/Y from composition between 2016 and 2100

	h_j year												
a_j year	1974	1979	1986	1991	1994	1997	2000	2004	2007	2010	2013	2016	DH
1989	26.5	26.1	25.8	25.5	25.5	25.2	25.2	24.7	24.3	24.0	23.6	23.5	27.6
1992	22.5	22.1	21.8	21.5	21.5	21.2	21.2	20.7	20.3	20.0	19.6	19.4	23.6
1995	25.5	25.1	24.8	24.6	24.5	24.2	24.2	23.7	23.3	23.0	22.6	22.5	26.7
1998	22.7	22.4	22.0	21.8	21.7	21.4	21.4	20.9	20.5	20.2	19.8	19.7	23.9
2001	22.5	22.1	21.8	21.5	21.4	21.2	21.1	20.7	20.2	20.0	19.5	19.4	23.6
2004	25.7	25.3	25.0	24.7	24.7	24.4	24.4	23.9	23.5	23.2	22.7	22.6	26.8
2007	24.5	24.1	23.8	23.5	23.5	23.2	23.1	22.7	22.3	22.0	21.5	21.4	25.6
2010	28.7	28.4	28.1	27.8	27.7	27.5	27.4	26.9	26.5	26.3	25.8	25.7	29.9
2013	28.2	27.9	27.5	27.3	27.2	26.9	26.9	26.4	26.0	25.7	25.3	25.2	29.4
2016	30.9	30.5	30.2	29.9	29.9	29.6	29.5	29.1	28.7	28.4	27.9	27.8	32.0
DH-t	28.0	27.7	27.3	27.1	27.0	26.7	26.7	26.2	25.8	25.5	25.1	25.0	29.2

Panel B. Predicted change in log W/Y from composition between 1950 and 2016

	h_j year												
a_j year	1974	1979	1986	1991	1994	1997	2000	2004	2007	2010	2013	2016	DH
1989	22.8	22.5	22.3	21.7	21.2	20.7	21.1	20.1	19.8	19.1	19.0	19.3	29.3
1992	22.7	22.4	22.2	21.6	21.1	20.6	21.0	20.1	19.8	19.0	18.9	19.2	29.2
1995	24.3	24.0	23.8	23.2	22.7	22.2	22.6	21.7	21.4	20.6	20.5	20.8	30.8
1998	23.6	23.3	23.1	22.6	22.0	21.5	21.9	21.0	20.7	20.0	19.9	20.1	30.2
2001	23.8	23.5	23.3	22.8	22.2	21.8	22.2	21.2	20.9	20.2	20.1	20.3	30.4
2004	25.4	25.1	24.9	24.3	23.8	23.3	23.7	22.8	22.5	21.7	21.6	21.9	31.9
2007	24.8	24.5	24.3	23.8	23.2	22.7	23.1	22.2	21.9	21.2	21.1	21.3	31.4
2010	27.9	27.6	27.4	26.9	26.3	25.8	26.2	25.3	25.0	24.3	24.2	24.4	34.5
2013	26.7	26.4	26.2	25.6	25.1	24.6	25.0	24.0	23.7	23.0	22.9	23.2	33.2
2016	28.1	27.9	27.6	27.1	26.5	26.1	26.5	25.5	25.2	24.5	24.4	24.6	34.7
DH-t	29.2	29.0	28.8	28.2	27.6	27.2	27.6	26.6	26.3	25.6	25.5	25.8	35.8

Notes: This table reports the US Δ^{comp}, the predicted change in $\log W/Y$ from compositional effects as defined in equation (10), for alternative base years of the age-wealth and the age-labor income profiles, reported in percent. Panel A considers our main period of interest 2016 to 2100, and panel B considers 1950 to 2016. Every column corresponds to an alternative base year for the age-labor income profile, and every row to an alternative base year for the age-wealth profile. The last row and column correspond to the cases where we use the average age effect from a time-age-cohort decomposition on the 1989–2016 SCF data, with all growth loading on time effects (DH, for "Deaton-Hall").

Table A.3: Sensitivity of predicted change in US log W/Y to choice of earlier base year

Panel A. Predicted change in log W/Y from composition between 2016 and 2100

	h_j year												
a_j year	1974	1979	1986	1991	1994	1997	2000	2004	2007	2010	2013	2016	DH
1958	20.8	20.4	20.1	19.8	19.8	19.5	19.5	19.0	18.6	18.3	17.8	17.7	21.9
1959	17.0	16.6	16.3	16.0	16.0	15.7	15.7	15.2	14.8	14.5	14.1	14.0	18.2
1960	17.5	17.2	16.9	16.6	16.5	16.3	16.2	15.7	15.3	15.0	14.6	14.5	18.7
1962	17.4	17.0	16.7	16.4	16.3	16.1	16.0	15.6	15.1	14.9	14.4	14.3	18.5
1965	19.0	18.6	18.3	18.0	18.0	17.7	17.6	17.2	16.8	16.5	16.0	15.9	20.1
1967	21.5	21.1	20.8	20.5	20.5	20.2	20.1	19.7	19.3	19.0	18.5	18.4	22.6
1968	19.5	19.1	18.8	18.6	18.5	18.2	18.2	17.7	17.3	17.0	16.6	16.5	20.7
1969	19.8	19.4	19.1	18.9	18.8	18.5	18.5	18.0	17.6	17.3	16.9	16.8	21.0
1970	24.0	23.6	23.3	23.0	23.0	22.7	22.6	22.2	21.8	21.5	21.0	20.9	25.1
1977	11.5	11.1	10.8	10.5	10.5	10.2	10.2	9.7	9.3	9.0	8.6	8.5	12.7
1983	23.8	23.5	23.2	22.9	22.8	22.6	22.5	22.0	21.6	21.3	20.9	20.8	25.0
2016	30.9	30.5	30.2	29.9	29.9	29.6	29.5	29.1	28.7	28.4	27.9	27.8	32.0

Panel B. Predicted change in log W/Y from composition between 1950 and 2016

	h_j year												
a_j year	1974	1979	1986	1991	1994	1997	2000	2004	2007	2010	2013	2016	DH
1958	12.7	12.4	12.2	11.6	11.1	10.6	11.0	10.1	9.8	9.0	8.9	9.2	19.3
1959	15.9	15.6	15.4	14.8	14.3	13.8	14.2	13.3	13.0	12.2	12.1	12.4	22.5
1960	17.5	17.2	17.0	16.5	15.9	15.4	15.8	14.9	14.6	13.9	13.8	14.0	24.1
1962	18.0	17.7	17.5	16.9	16.4	15.9	16.3	15.4	15.1	14.3	14.2	14.5	24.5
1965	14.7	14.4	14.2	13.6	13.1	12.6	13.0	12.1	11.8	11.0	10.9	11.2	21.2
1967	18.0	17.7	17.5	16.9	16.4	15.9	16.3	15.4	15.1	14.3	14.2	14.5	24.5
1968	17.2	16.9	16.7	16.2	15.6	15.1	15.5	14.6	14.3	13.6	13.4	13.7	23.8
1969	18.0	17.7	17.5	17.0	16.4	15.9	16.3	15.4	15.1	14.4	14.3	14.5	24.6
1970	21.9	21.6	21.4	20.8	20.2	19.8	20.2	19.2	18.9	18.2	18.1	18.4	28.4
1977	10.6	10.3	10.1	9.5	9.0	8.5	8.9	8.0	7.7	6.9	6.8	7.1	17.1
1983	21.4	21.1	20.9	20.4	19.8	19.3	19.7	18.8	18.5	17.8	17.6	17.9	28.0
2016	28.1	27.9	27.6	27.1	26.5	26.1	26.5	25.5	25.2	24.5	24.4	24.6	34.7

Notes: This table reports the US Δ^{comp}, the predicted change in log W/Y from compositional effects as defined in equation (10), for alternative base years of the age-wealth and the age-labor income profiles, reported in percent. Compared to table A.2, this table considers earlier SCF waves for the age-wealth profile, as constructed by Kuhn et al. (2020). Panel A considers our main period of interest 2016 to 2100, and panel B considers 1950 to 2016. Every column corresponds to an alternative base year for the age-labor income profile, and every row to an alternative base year for the age-wealth profile. The last column corresponds to the case where we use the average age effect from a time-age-cohort decomposition on the 1989–2016 SCF data, with all growth loading on time effects (DH, for "Deaton-Hall").

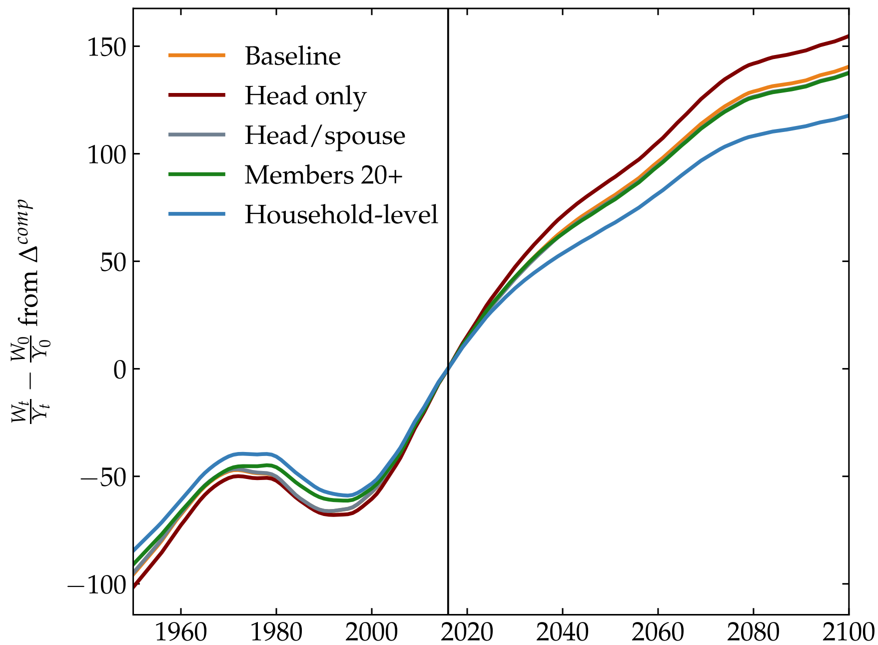

Figure A.3: Predicted change in US W/Y from composition: alternative assumptions

Notes: This figure depicts the evolution of the predicted change in the wealth-to-GDP ratio from the compositional effect, calculated using equation (21) from t =1950 to 2100. The orange line corresponds to our baseline case, where the wealth of households is allocated equally to all members at least as old as the head or the spouse. The red line shows the outcome when wealth is allocated to the head of household only, the gray line to the head and the spouse equally, and the green line to all members aged 20 or more. The blue line presents the outcome when the analysis is conducted at the household-level rather than at the individual level.

W/Y (labelled "Data") relative to the compositional effect Δ_t^{comp} (labelled "Comp"). The latter is constructed from equation 10 using age profiles from the baseline year interacted with the actual change in population distributions over the period reported. Both columns are multiplied by 100 to be interpretable as percentage points. The compositional effect predicts an increase in W/Y in every country, consistent with what occurred. For countries like the United States and the Netherlands, the magnitudes also line up closely; for Spain, the compositional effect overpredicts the historical magnitude, while for most other countries the historical increase in W/Y is greater than the compositional effect alone would predict. If demographics was the only force driving wealth-to-GDP ratios then our theory suggests that the rise in W/Y should be less than what is predicted by the compositional effect due to the endogenous response of asset returns; the fact that many countries experienced larger increases suggests that other forces, such as declining productivity growth, have also been at play.

Role of heterogeneity in demographic change vs age profiles. Figure A.4 presents the predicted change in W/Y between 2016 and 2100 from the compositional effect and isolates the contributions from demographic forces and from the age-profiles. Panel A repeats the results from section 3.1, ranking countries from lowest to highest compositional effect. It also presents the results under the two UN fertility scenarios. To isolate the contribution from demographic forces, panel B computes the compositional effect where age-profiles in all countries are identical to the

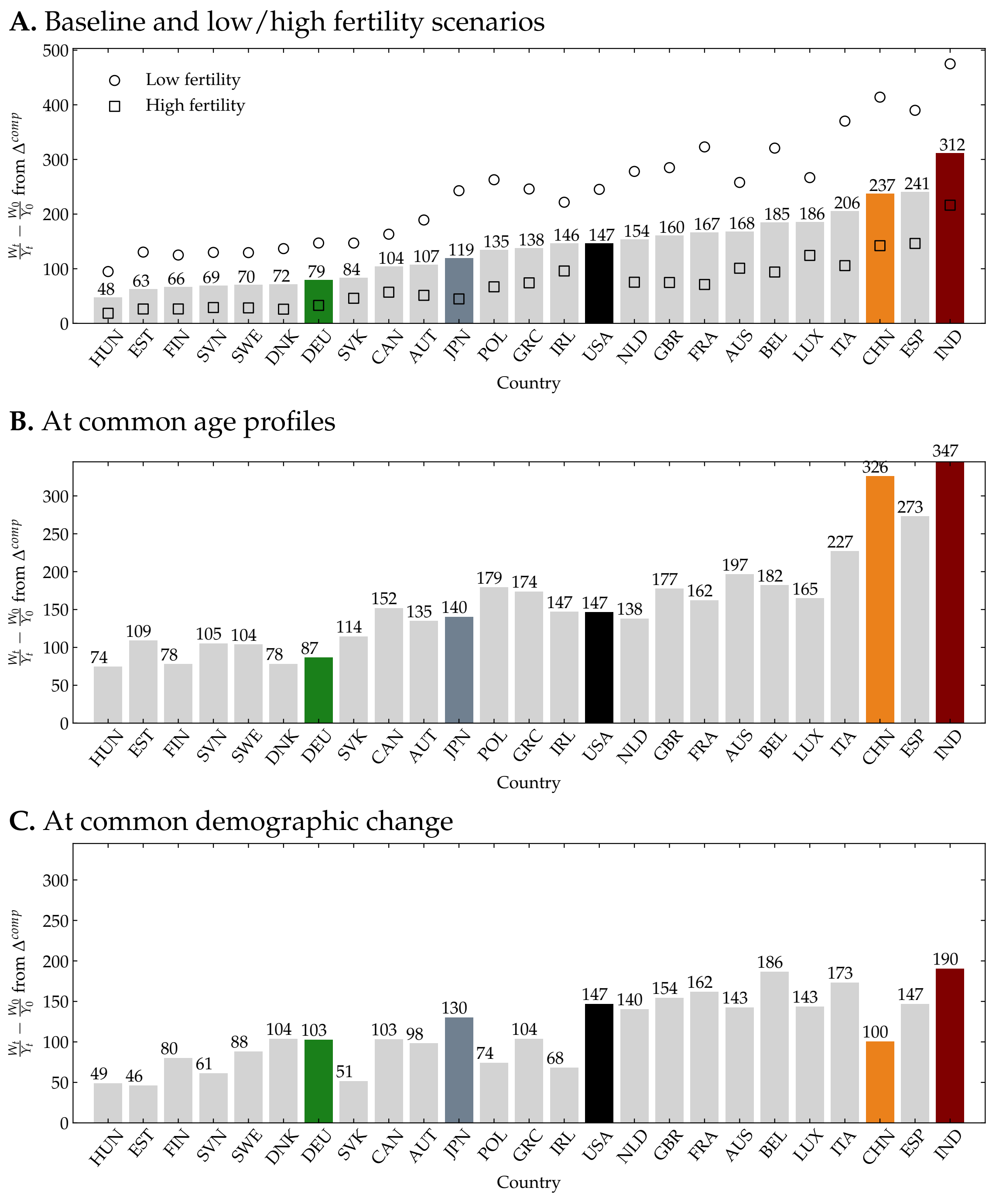

Figure A.4: Predicted change in W/Y from composition between 2016 and 2100

Notes: Panel A presents the change in W/Y between 2016 and 2100 from equation (21) as well as its value using the low fertility (circles) and high fertility (squares) scenarios. Panel B does this calculation again, assuming that all countries have US age profiles of assets and income. Panel C does this calculation again, assuming all countries have the US age distribution in every year.

Table A.4: Historical change in $\log(W/Y)$ vs predicted change from Δ^{comp} (in log %)

Country	Period	Data	Comp.
AUS	1960-2016	59.8	15.6
CAN	1970-2016	82.6	19.2
CHN	1978-2016	140.9	16.8
DEU	1950-2016	67.4	23.7
DNK	1973-2016	80.2	13.8
ESP	1950-2016	19.1	27.6
FIN	2011-2016	9.2	5.5
FRA	1950-2016	109.3	21.4
GBR	1950-2016	37.5	18.9
GRC	1997-2016	17.3	8.7
IND	1950-2016	23.2	10.9
ITA	1966-2016	108.8	23.5
JPN	1970-2016	66.0	42.5
NLD	1997-2016	23.4	21.1
SWE	1950-2016	48.8	19.6
USA	1950-2016	31.6	27.5

US profile. To isolate the contribution from the profiles, panel C computes the compositional effect where population distributions of the US are used in every country. Panels B and C show that both the shapes of the profiles and the changes in population distributions matter to the compositional effect, but that the demographic forces play a much more important role in generating shift-shares that are high and heterogeneous across countries.

C.4 Additional results for sections 3.2 and 3.3

Age profiles of consumption and assets. Figure A.5 presents the age distributions of consumption (orange lines) and asset holdings (red lines), constructed using the procedure described in section 3.2. The consumption profile is backed out of the asset profile and the profile of net income. Net income includes all taxes and transfers; since this measure is not available in most surveys, we back it out of aggregate information on taxes and transfers. In practice, we use net income from our quantitative model of section 4, which is constructed using that information for each country.

Applying equation (15) at each point in time to predict NFAs. Figure A.6 reproduces Figure 7, but we applying equation (15) at each point in time to predict NFAs. Specifically, we apply equation

$$\log\left(1+\frac{NFA_t^c/Y_t - NFA_0^c/Y_0}{W_0^c/Y_0^c}\right) \simeq \Delta_t^{comp,c} - \overline{\Delta}_t^{comp} + \left(\epsilon^{c,d} + \epsilon^{c,s} - \left(\bar{\epsilon}^d + \bar{\epsilon}^s\right)\right)(r_t - r_0) \quad \text{(A.51)}$$

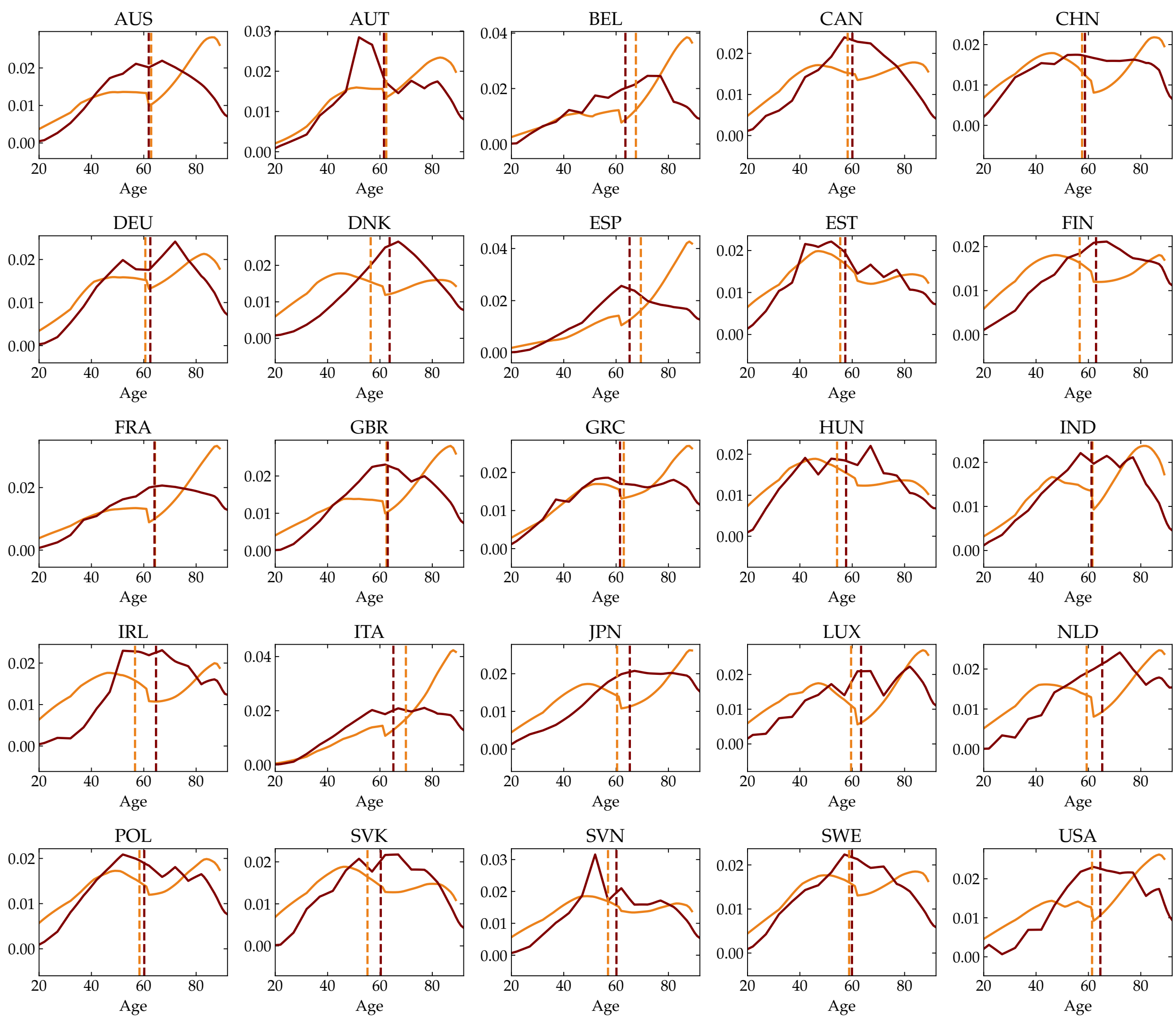

Figure A.5: Distribution of the ages of consumption and wealth in each country.

Notes: This figure presents the age distributions of consumption (orange lines) and asset holdings (red lines). The dashed vertical lines depict the average ages of consumption and asset holdings.

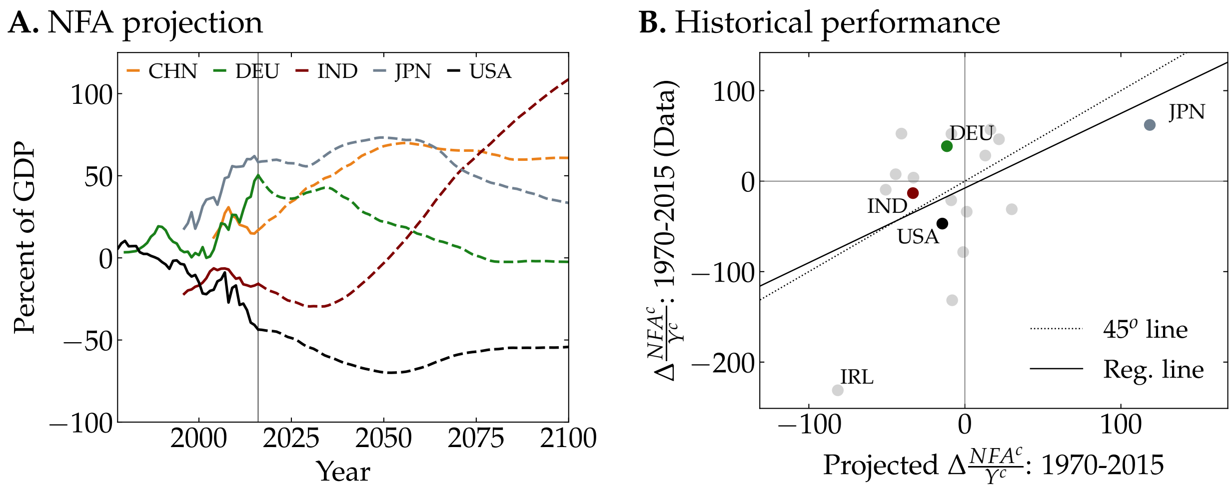

Figure A.6: Using a dynamic version of equation (15) to project NFAs

Notes: This reproduces figure 7, but uses (A.51)–(A.52), rather than $\Delta_t^{comp,c} - \overline{\Delta}_t^{comp}$, to project $\Delta_t \frac{NFA^c}{Y^c}$.

where $r_t - r_0$ is, in turn, calculated by applying equation (13) at each point in time,

$$r_t - r_0 \simeq -\frac{1}{\bar{\epsilon}^d + \bar{\epsilon}^s} \bar{\Delta}_t^{comp} \tag{A.52}$$

and, in equations (A.51)–(A.52), we take $\bar{\epsilon}^d$ and $\bar{\epsilon}^s$ to be the steady state elasticities calculated using our sufficient statistics.[72]

The main findings from figure A.6 are unchanged relative to those from figure 7, indicating that the interest rate adjustment term does not play a major role when it comes to forecasting NFAs. This is because this interest adjustment only matters to the extent that elasticities of supply and demand differ across countries, and the heterogeneity we calculate from our sufficient statistics is relatively limited.

D Appendix to Section 4

D.1 Full model setup

Here, we describe the model in section 4. We first describe the full model for one country, omitting the country superscript c, and define a small open economy equilibrium for a fixed sequence $\{r_t\}$. The world equilibrium is defined as a sequence $\{r_t\}$ that clears the global asset market.

Demographics. The demographics are given by a sequence of births $\{N_{0t}\}_{t\geq -1}$, a sequence of age- and time-specific survival rates $\{\phi_{jt}\}_{t\geq -1}$ for individuals between age j and $j+1$, a sequence of net migration levels $\{M_{jt}\}_{-1\leq t, 0\leq j\leq T-1}$, as well as an initial number of agents by age $N_{j,-1}$. The

[72]The exact first-order approximation involves a sequence-space Jacobian matrix (Auclert, Bardóczy, Rognlie and Straub 2021). In practice, however, we are unaware of a sufficient statistic expression for the Jacobian that underlies $\bar{\epsilon}^d$. Figure 8 shows that the approximation in (A.51)–(A.52) works fairly well in the context of our structural model.

assumption that demographic variables start at $t = -1$ is made for technical reasons; it allows us to correctly account for migration and bequests received at time $t = 0$. Given these parameters, the population variables evolve according to the exogenous N_{0t} and

$$N_{jt} = (N_{j-1,t-1} + M_{j-1,t-1})\phi_{j-1,t-1}, \quad \forall t \geq 0, j > 0. \tag{A.53}$$

As in section 2, we write $N_t \equiv \sum_j N_{jt}$ for the total population at time t, and $\pi_{jt} \equiv \frac{N_{jt}}{N_t}$ for the age distribution of the population.

Agents' problem. The basic setup is the same as in section 2, with heterogeneous individuals facing idiosyncratic income risk. We restrict the income process so that effective labor supply ℓ_{jt} is the product of a deterministic term ℓ_j that varies across ages, a fixed effect θ, and a transitory component ϵ, where both the fixed effect and the transitory component have a mean of 1. The log transitory component follows a finite-state Markov process with a transition matrix across years $\Pi^\epsilon(\epsilon|\epsilon_-)$ from ϵ_- to ϵ, calibrated to have a persistence χ_ϵ and a standard deviation v_ϵ, while the log permanent component follows a discrete Markov process across generations with a transition matrix $\Pi(\theta|\theta_-)$ from θ_- to θ calibrated to have a persistence χ_θ and a standard deviation v_θ. The processes are independent, and we write $\pi^\epsilon(\epsilon)$ and $\pi^\theta(\theta)$ for the corresponding stationary probability mass functions.[73]

We assume that individuals become economically active at age J^w, so that labor income at age j at time t is $w_t(1-\rho_{jt})\theta\epsilon\ell_{jt}$, where w_t is the wage per efficiency unit as in section 2, and $\rho_{jt} \in [0,1]$ is a parameter of the retirement system indicating the fraction of labor that households of age j are allowed to supply at time t. After retirement, agents receive social security payments $w_t\rho_{jt}\theta d_t$ in proportion to their permanent type, where d_t encodes a time-varying social security replacement rate.

The state for an individual at age j and time t is given by the fixed effect θ, the transitory effect ϵ, and asset holdings $\mathfrak{a}$, and their value function is given by

$$\begin{aligned} V_{jt}(\theta,\epsilon,\mathfrak{a}) = \max_{c,\mathfrak{a}'} & \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \Upsilon Z_t^{\nu-\frac{1}{\sigma}}\left(1-\phi_{jt}\right)\frac{\left(\mathfrak{a}'\right)^{1-\nu}}{1-\nu} + \phi_{jt}\frac{\beta_{j+1}}{\beta_j}\mathbb{E}\left[V_{j+1,t+1}(\theta,\epsilon',\mathfrak{a}')|\epsilon\right] \\ & c + \mathfrak{a}' \leq w_t\theta\left[(1-\rho_{jt})(1-\tau_t)\ell_{jt}\epsilon + \rho_{jt}d_t\right] + (1+r_t)[\mathfrak{a} + b_{jt}^r(\theta)] \\ & -\bar{\mathfrak{a}}Z_t \leq \mathfrak{a}', \end{aligned} \tag{A.54}$$

which determines the decision function $c = c_{jt}(\theta,\epsilon,\mathfrak{a})$ and $a' = a_{j+1,t+1}(\theta,\epsilon,\mathfrak{a})$ for consumption and next-period assets.

The term $\frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$ represents the flow utility of consumption, and $\Upsilon Z_t^{\nu-\frac{1}{\sigma}}\left(1-\phi_{jt}\right)\left(\mathfrak{a}'\right)^{1-\nu}/(1-\nu)$ represents the utility from giving bequests $\mathfrak{a}'$. The bequest utility is scaled by mortality risk $1-\phi_{jt}$, since agents only give bequests if they die, and $\nu \geq \frac{1}{\sigma}$ captures potential non-homotheticities in bequests, which has been shown to generate more realistic levels of wealth inequality (De Nardi, 2004). The scaling factor $Z_t^{\nu-\frac{1}{\sigma}}$ ensures balanced growth in spite of this non-homotheticity. The term $b_{jt}^r(\theta)$ represents bequests received, and is allowed to vary according to the agent's permanent type.

[73] Discrete processes are used to facilitate notation. The calibration to the persistence and standard deviation is done using Tauchen's method applied to a Gaussian AR(1) process with a given persistence, standard deviation, and mean.

State distribution. To determine the evolution of states, we assume that the distribution of individuals across θ and ϵ is in the stationary distribution for all ages and times, as well as for arriving and leaving migrants. This implies that the joint distribution across $(\theta, \epsilon, \mathfrak{a})$ is fully characterized by

$$H_{jt}(\mathfrak{a}|\theta,\epsilon) = \mathbb{P}(\mathfrak{a}_{jt} \leq \mathfrak{a}|\theta,\epsilon),$$

where H_{jt} is the conditional probability distribution of assets given θ and ϵ.[74]

Over time, the distribution evolves according to

$$H_{j+1,t+1}(\mathfrak{a}|\theta,\epsilon) = \sum_{\epsilon_-} \frac{\Pi^\epsilon(\epsilon|\epsilon_-)\pi^\epsilon(\epsilon_-)}{\pi^\epsilon(\epsilon)} \int_{\mathfrak{a}'} \mathbb{I}(\mathfrak{a}_{j+1,t+1}(\mathfrak{a}',\theta,\epsilon) \leq \mathfrak{a}) dH_{j,t}(\mathfrak{a}'|\theta,\epsilon) \quad \forall j > J^w, \tag{A.55}$$

where $\mathfrak{a}_{j+1,t+1}$ is the decision function for assets implied by the agents' problem (A.54). Note that (A.55) implicitly assumes that death is independent of asset holdings and that migrants have the same distribution of assets as residents. At time zero, there is an exogenous distribution of assets $H_{j0}(\cdot|\theta,\epsilon)$ for each age group. As a boundary condition, we assume that individuals do not have any assets before working life starts:

$$H_{j,t}(\mathfrak{a}|\theta,\epsilon) = \mathbb{I}(\mathfrak{a} \geq 0) \quad \forall \theta, \epsilon, j \leq J^w, 0 \leq t, \tag{A.56}$$

where $\mathbb{I}$ is the indicator function.

Bequest distribution. We model partial intergenerational wealth persistence by assuming that all bequests from individuals of type θ are pooled and distributed across the types θ' of survivors in accordance with the intergenerational transmission of types. Formally, the total amount of bequests received by agents of type θ of age j at time t satisfies

$$\begin{aligned} N_{jt} b^r_{jt}(\theta) =& F_j \sum_{\theta_-} \left(\frac{\Pi^\theta(\theta|\theta_-)\pi^\theta(\theta_-)}{\pi^\theta(\theta)} \right) \times \\ & \sum_{k=0}^{T} \left[N_{k,t-1} + M_{k,t-1} \right] (1 - \phi_{k,t-1}) \times \sum_{\epsilon} \pi^\epsilon(\epsilon) \int_{\mathfrak{a}} \mathfrak{a} dH_{kt}(\mathfrak{a}|\theta_-,\epsilon) \end{aligned} \tag{A.57}$$

Here, $\sum_k [N_{k,t-1} + M_{k,t-1}](1 - \phi_{k,t-1}) \sum_\epsilon \int_{\mathfrak{a}} \pi^\epsilon(\epsilon) \mathfrak{a} dH_{kt}(\mathfrak{a}; \theta_-, \epsilon)$ captures the total amount of bequests given by individuals of type θ_-. The timing is that migrants arrive before the death event and that interest rate accrues after the death event. A share $\frac{\Pi^\theta(\theta|\theta_-)\pi^\theta(\theta_-)}{\pi^\theta(\theta)}$ of these bequests is given to agents of type θ, capturing partial intergenerational transmission by using the probability that an agent of type θ has a parent of type θ_-.

Note that an aging population alters the relative number of agents that give relative to the number of agents that receive bequests, which ceteris paribus increases bequest sizes. The migrants are included, assuming that migrants have the same mortality as the overall population, and that migrants who plan to arrive at t but die between $t-1$ and t augment the bequest pool in the receiving country.

[74]Formally, given H_{jt}, the joint distribution function $\tilde{H}_{jt}$ of $(\theta, \epsilon, \mathfrak{a})$ can be written $\tilde{H}_{jt}(\theta, \epsilon, \mathfrak{a}) = \sum_{\theta' \leq \theta, \epsilon' \leq \epsilon} \pi^\theta(\theta') \pi^\epsilon(\epsilon') H_{jt}(\mathfrak{a}|\theta', \epsilon')$.

Aggregation. Given the decision functions c_{jt} and $\mathfrak{a}_{j+1,t+1}$ and the distribution across states, aggregate consumption and assets satisfy

$$W_t = \sum_{j=0}^{J} N_{jt} \times \sum_{\epsilon,\theta} \pi^\epsilon(\epsilon)\pi^\theta(\theta) \int_{\mathfrak{a}} [\mathfrak{a} + b^r_{jt}(\theta)] dH_{jt}(\mathfrak{a};\theta,\epsilon)$$

$$C_t = \sum_{j=0}^{J} N_{jt} \times \sum_{\epsilon} \pi^\epsilon(\epsilon) \int_{\mathfrak{a}} c_{jt}(\theta_-,\epsilon,\mathfrak{a}) dH_{jt}(\mathfrak{a}|\theta_-,\epsilon). \tag{A.58}$$

Note that bequests received are included in the definition of today's ingoing assets.

Production. As in section 2, markets are competitive, there are no adjustment costs in capital, and there is labor-augmenting growth at a constant rate γ. Production is given by a CES aggregate production function. We obtain the following equations:

$$Y_t = F(K_t, Z_t L_t) \equiv \left(\alpha K_t^{\frac{\eta-1}{\eta}} + (1-\alpha)[Z_t L_t]^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} \tag{A.59}$$

$$Z_t = (1+\gamma)^t Z_0 \tag{A.60}$$

$$r_t = F_K(K_t, Z_t L_t) - \delta \tag{A.61}$$

$$w_t = Z_t F_L(K_t, Z_t L_t) \tag{A.62}$$

$$K_{t+1} = (1-\delta)K_t + I_t \tag{A.63}$$

$$L_t = \sum_{j=0}^{J} N_{jt}(1-\rho_{jt})\ell_j, \tag{A.64}$$

where the last line uses that $\mathbb{E}\theta\epsilon = 1$ to obtain that average effective labor supply is ℓ_j of individuals of age j.

Government. The government purchases G_t goods and sets the retirement policy ρ_{jt}, the tax rate τ_t, and the benefit generosity d_t. It faces the flow budget constraint

$$G_t + \sum_{j=0}^{J} N_{jt} w_t \rho_{jt} d_t + (1+r_t)B_t = \tau_t w_t \sum_{j=0}^{J} N_{jt}(1-\rho_{jt})\ell_{jt} + B_{t+1}, \tag{A.65}$$

where a positive B_t denotes government borrowing. In the aggregation, we again use that $\mathbb{E}\theta\epsilon = \mathbb{E}\theta = 1$ for each j to obtain that average benefits and labor income per age-j person are $w_t\rho_{jt}d_t$ and $w_t(1-\rho_{jt})\bar{\ell}_{jt}$ respectively.

The government targets an eventually converging sequence $\left\{\frac{B_{t+1}}{Y_{t+1}}\right\}_{t\geq 0}$. To reach this target, we assume that the government uses a fixed sequence of retirement policies $\{\rho_{jt}\}_{t\geq 0}$, and adjusts the other instruments using a fiscal rule defined in term of the "fiscal shortfall" SF_t, defined as

$$\frac{SF_t}{Y_t} \equiv \frac{\bar{G}}{Y} + \frac{\sum_{j=0}^{T}[\rho_{j,t}\bar{d} - \bar{\tau}(1-\rho_{j,t})\bar{\ell}_{jt}]N_{jt}w_t}{Y_t} + (r_t - g_t)\frac{B_t}{Y_t} - (1+g_t)\left[\frac{B_{t+1}}{Y_{t+1}} - \frac{B_t}{Y_t}\right], \tag{A.66}$$

where $g_t = \frac{Y_{t+1}}{Y_t} - 1$. The fiscal shortfall is positive at time t if expenditures minus revenues is too high to reach the debt target when the instruments G, d, and τ are set at some reference levels $\bar{G}$, $\bar{d}$

and $\bar{\tau}$. Given a non-zero fiscal shortfall, the fiscal rule consists of three weights $\varphi^G, \varphi^\tau, \varphi^d$ and an updating rule for instruments

$$\varphi^G SF_t = -(G_t - \bar{G}) \quad \forall t \geq 0 \tag{A.67}$$

$$\varphi^\tau SF_t = (\tau_t - \bar{\tau}) \times w_t \sum_{j=0}^{J} N_{jt}\ell_{jt}(1-\rho_{jt}) \quad \forall t \geq 0 \tag{A.68}$$

$$\varphi^d SF_t = -(d_t - \bar{d}) \times \left(w_t \sum_{j=0}^{J} N_{jt}\rho_{jt} \right) \quad \forall t \geq 0 \tag{A.69}$$

$$1 = \varphi^G + \varphi^\tau + \varphi^d, \tag{A.70}$$

where the weights capture the share of the shortfall covered by each instrument.

Market clearing. The assets in the economy consist of capital K_t, government bonds B_t, and foreign assets NFA_t. The asset market clearing condition is

$$K_t + B_t + NFA_t = W_t. \tag{A.71}$$

Given the other equilibrium conditions, (A.71) can be used to derive the goods market clearing condition[75]

$$NFA_{t+1} - NFA_t = NX_t + r_t NFA_t + W_{t+1}^{mig}, \tag{A.72}$$

where $NX_t \doteq Y_t - I_t - C_t - G_t$ is net exports at time t and

$$W_{t+1}^{mig} \doteq \sum_{j=1}^{J} M_{j-1,t} \sum_{\theta,\epsilon} \pi^\theta(\theta)\pi^\epsilon(\epsilon) \left(\int_{\mathfrak{a}} \mathfrak{a} dH_{j,t+1}(\mathfrak{a},\theta,\epsilon) + b_{j,t+1}^\theta(\theta) \right)$$

is the assets at time t that comes from migrants.

Small open economy equilibrium. A small open economy equilibrium is defined for:

- A sequence of interest rates $\{r_t\}_{t=0}^\infty$
- A government fiscal rule $\{B_{t+1}/Y_{t+1}, \rho_{jt}, \varphi^G, \varphi^\tau, \varphi^d, \bar{G}, \bar{\tau}, \bar{d}\}_{t=0}^\infty$
- A sequence of average effective labor supplies $\{\bar{\ell}_{jt}\}_{0 \leq t, J^w \leq j \leq J}$
- An initial distribution of assets $\{H_{j0}(a|\theta,\epsilon)\}_{j=0}^J$
- Technology parameters $\{Z_0, \gamma, \delta, \nu, \alpha\}$
- Demographics: initial $\{N_{j,-1}\}_{j=0}^J$ and forcing parameters $\{M_{jt}, \phi_{jt}, N_{0,t+1}\}_{-1 \leq t, 0 \leq j \leq J}$
- Initial aggregate variables K_0, B_0, A_0

The equilibrium consists of:

- Individual decision functions: $c_{jt}(\theta,\epsilon,\mathfrak{a}), \mathfrak{a}'_{jt}(\theta,\epsilon,\mathfrak{a})$

[75] Combine the aggregated household budget constraint with the government budget constraint (A.65), the capital evolution equation (A.63), and the asset market clearing condition (A.71).

- A sequence of asset distribution functions $\{H_{jt}(a;\theta,\epsilon)\}_{1\leq t,J^w\leq j\leq J}$
- Government policy variables $\{G_t,\tau_t,d_t\}_{t\geq 0}$
- A sequence of wages $\{w_t\}_{t\geq 0}$
- A sequence of bequests received $\{b_{jt}(\theta)\}_{t\geq 0}$
- A sequence of aggregate quantities $\{Y_t,L_t,I_t,K_{t+1},W_t,C_t,NFA_t\}_{t\geq 0}$

It is characterized by that

- r_0 is consistent with $K_0 \Longrightarrow$ (A.61) holds given K_0 and $L_0=\sum_j N_{j0}(1-\rho_{j0})\ell_{j0}$
- W_0 is consistent with H_{j0}, that is, (A.58) holds
- Individual decision functions solve (A.54).
- The set of H_{jt}'s satisfies the evolution equation (A.55) and the boundary condition (A.56)
- The government policy variables satisfy (A.66)-(A.69).
- Equations (A.59)-(A.64) hold.
- A_t satisfies (A.58) for $t\geq 0$
- $NFA_t=W_t-K_t-B_t$, with B_0 given by the initial condition, and B_{t+1}/Y_{t+1} by the government fiscal rule.
- Bequests received $b_{jt}(\theta)$ satisfy (A.57)

World-economy equilibrium. Given a set of countries $c\in\mathcal{C}$, a *world-economy equilibrium* is a sequence of returns $\{r_0,\{r_t\}_{t\geq 1}\}$ and a set of corresponding sequences of prices and allocations $\mathcal{S}^c$ for each economy c such that each $\mathcal{S}^c$ is a small open economy equilibrium, and that their NFAs satisfy

$$\sum_{c\in\mathcal{C}} NFA_t^c=0 \quad \forall t\geq 0 \tag{A.73}$$

D.2 Proof of proposition 5

Let Φ_t^c capture all demographic variables in a country: population shares, fertility, mortality, migration. Given fixed r and B^c/Y^c, long-run government policy only depends on Φ^c. Wages per unit of effective labor only depend on r. Assuming that the steady state of the household problem is unique conditional on demographics, wages, and government policy, we can express it as a function of (r,Φ^c).[76] Let $\frac{W^c}{Y^c}(r,\Phi^c)$ denote the resulting steady-state wealth-to-output ratio.

Output, normalized by technology, only depends on aggregate effective labor supply, which is a function of Φ^c (both directly through the number of people at each age and indirectly through

[76] Aside from bequests, we have a standard incomplete markets household problem and this would be a standard result. Bequests introduce some complication, since bequests depend on the endogenous distribution of assets, but household asset policy also depends on realized and expected bequests. The solution to the household problem is a fixed point of this process. We assume that the fixed point is unique and a global attractor; in practice, we have found that this assumption is always satisfied.

government retirement policy), and the capital-to-effective-labor ratio, which is a function of r. Hence we can write each country's share of global GDP as $\frac{Y^c}{Y}(r, \nu, \Phi)$.

From here on, the proofs of propositions 2 and 3 in appendix B.3 apply, provided that, in equations (A.6) and later, we replace π with Φ, $\frac{W^c}{Y^c}(r_0, \Phi^c_{LR}) - \frac{W^c}{Y^c}(r_0, \Phi^c_0)$ with $\Delta^{soe,c}_{LR}$, as well as $\sum_c \omega^c \left(\frac{W^c}{Y^c}(r_0, \Phi^c_{LR}) - \frac{W^c}{Y^c}(r_0, \Phi^c_0)\right)$ with $\bar{\Delta}^{soe}_{LR}$ everywhere.

D.3 Steady-state equations and calibration details

Steady-state equations. Our calibration targets a stationary equilibrium associated with a constant rate of return r. Most elements are standard: we assume constant technology parameters $\{\gamma, \delta, \nu, \alpha, \ell_j\}$, a constant bond-to-output ratio $\frac{B}{Y}$, retirement policy ρ_j, tax rate τ, social security generosity d, and government consumption-to-output ratio G/Y. We also assume that there is a fixed distribution of assets $H_j(\tilde{a}|\theta, \epsilon)$, where $\tilde{a}$ is assets normalized by technology (again, we drop the country superscripts in the description of each country, and reintroduce them when we define the world equilibrium).

The non-standard element is that we introduce a counterfactual flow of migrants to ensure a time-invariant population distribution and growth rate at their 2016 levels. In particular, demography consists of constant mortality rates, a fixed age distribution, a constant population growth rate, and a constant rate of migration by age m_j:

$$\phi_{jt} \equiv \phi_j, \quad \pi_{jt} \equiv \phi_j, \quad N_t = (1+n)^t N_0, \quad m_j \equiv \frac{M_j}{N},$$

and the net migration by age is given by

$$m_{j-1} \equiv \frac{M_{j-1}}{N} = \pi_j \frac{1+n}{\phi_{j-1}} - \pi_{j-1}, \tag{A.74}$$

which ensures that (A.53) holds given a fixed age distribution of population. The notation $\frac{M_{j-1}}{N}$ without a time index is used to indicate the constant ratio $\frac{M_{j-1,t}}{N_t}$. Throughout, we use an analogous notation whenever the ratio of two variables is constant over time.

In normalized form, the consumer problem is

$$\begin{aligned} \tilde{V}_j(\theta, \epsilon, \tilde{a}) = \max_{\tilde{c}, \tilde{a}'} \frac{\tilde{c}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \Upsilon(1+\gamma)^{1-\nu}\left(1-\phi_j\right)\frac{(\tilde{a}')^{1-\nu}}{1-\nu} + \frac{\beta_{j+1}}{\beta_j}(1+\gamma)^{1-\frac{1}{\sigma}}\phi_j \mathbb{E}\left[\tilde{V}_{j+1}(\theta, \epsilon', \tilde{a}')|\epsilon\right] \\ c + (1+\gamma)\tilde{a}' \leq \tilde{w}_t\theta\left[(1-\rho_j)(1-\tau)\bar{\ell}_j\epsilon + \rho_j d\right] + (1+r)\tilde{a} + \tilde{b}^r_j(\theta) \qquad \text{(A.75)} \\ -\tilde{a} \leq a'(1+\gamma), \end{aligned}$$

where a variable with a $\sim$ denotes normalization by Z_t, except for $\tilde{V}_j \equiv \frac{V_{jt}}{Z_t^{1-\frac{1}{\sigma}}}$. As elsewhere in the paper, we write g for the overall growth rate of the economy

$$1+g \equiv (1+n)(1+\gamma).$$

The consumer problem implies decision functions $\tilde{c}_j(\cdot)$ and $\tilde{a}'_j(\cdot)$, where the latter denotes the choice of next period's normalized assets as a function of the state at age j. From the evolution and boundary conditions of assets (A.55) and (A.56), the stationary distribution of assets satisfies

$$H_j(\tilde{a}|\theta,\epsilon) = \begin{cases} \sum_{\epsilon_-} \frac{\Pi^\epsilon(\epsilon|\epsilon_-)\times\pi^\epsilon(\epsilon_-)}{\pi^\epsilon(\epsilon)} \int_{\tilde{a}'} \mathbb{I}\left[\tilde{a}'_{j-1}(\tilde{a}',\theta,\epsilon) \leq \tilde{a}\right] dH_{j-1}(\tilde{a}'|\theta,\epsilon) & \text{if } j > J^w \\ \mathbb{I}(\tilde{a} \geq 0) & \text{if } j = J^w \end{cases},$$

Normalized bequests satisfy

$$\begin{aligned} \pi_j \tilde{b}^r_j(\theta) =& F_j \sum_{\theta_-} \left(\frac{\Pi^\theta(\theta|\theta_-)\pi^\theta(\theta_-)}{\pi^\theta(\theta)} \right) \times \\ & \sum_{k=0}^{T} \frac{[\pi_k + m_k](1-\phi_k)}{1+n} \times \\ & \int_{\tilde{a}} \sum_{\epsilon} \pi^\epsilon(\epsilon)\tilde{a} dH_k(\tilde{a};\theta_-,\epsilon) \end{aligned} \tag{A.76}$$

Aggregate consumption and assets are

$$\begin{aligned} \frac{C}{NZ} &= \sum_{j=0}^{T} \pi_j \sum_{\theta,\epsilon} \pi^\theta(\theta)\pi^\epsilon(\epsilon) \int_{\tilde{a}} c_j(\tilde{a},\theta,\epsilon) dH_j(\tilde{a},\theta,\epsilon) \\ \frac{W}{NZ} &= \sum_{j=0}^{T} \pi_j \sum_{\theta,\epsilon} \pi^\theta(\theta)\pi^\epsilon(\epsilon) \left(\int_{\tilde{a}} \tilde{a} dH_j(\tilde{a},\theta,\epsilon) + b^r_j(\theta) \right) \end{aligned}$$

Finally, since we assume that steady state migrants have the same distribution of assets as regular households, we have

$$\frac{A^{mig}}{NZ} = \sum_{j=1}^{T} m_{j-1} \sum_{\theta,\epsilon} \pi^\theta(\theta)\pi^\epsilon(\epsilon) \left(\int_{\tilde{a}} \tilde{a} dH_j(\tilde{a},\theta,\epsilon) + b^r_j(\theta) \right) \tag{A.77}$$

where we recall that m_j is the number of migrants as a share of age group j at time t, and W_j is the total amount of assets of age-j individuals.

The stationary analogues of the production sector equations (A.59)-(A.64) are

$$\frac{Y}{ZN} = F\left[\frac{K}{ZN}, \frac{L}{N}\right] \tag{A.78}$$

$$r + \delta = F_K\left[\frac{K}{ZN}, \frac{L}{N}\right] = \alpha \left(\frac{K}{Y}\right)^{-1/\eta} \tag{A.79}$$

$$\frac{w}{Z} = F_L\left[\frac{K}{ZN}, \frac{L}{N}\right] \tag{A.80}$$

$$(g+\delta)\frac{K}{Y} = \frac{I}{Y} \tag{A.81}$$

$$\frac{L}{N} = \sum_{j=0}^{T} \pi_j (1-\rho_j)\ell_j, \tag{A.82}$$

The steady-state government budget constraint is derived from (A.65) given a fixed debt-to-output ratio

$$\frac{G}{Y} + \frac{w \times d \times \sum_j N_j \rho_j}{Y} + (r-g)\frac{B}{Y} = \tau \times \frac{wL}{Y}, \tag{A.83}$$

and the asset market and goods market clearing conditions are derived from (A.71) and (A.72):

$$\frac{W}{Y} = \frac{K}{Y} + \frac{B}{Y} + \frac{NFA}{Y} \tag{A.84}$$

$$0 = \frac{NX}{Y} + (r-g)\frac{NFA}{Y} + \frac{A^{mig}}{Y(1+g)}. \tag{A.85}$$

The world asset market clearing condition is

$$\sum_c \omega^c \frac{NFA^c}{Y^c} = 0, \qquad \omega^c \equiv \frac{Y^c}{\sum_c Y^c} \tag{A.86}$$

D.4 Calibration details

All demographic data is from the UN World Population Prospects, interpolated across years and ages to obtain data for each combination of year and age. For each country, we use the 2016 values for age-specific survival rates ϕ_j^c and population shares π_j^c. The population growth rate is defined as

$$1 + n^c = \frac{N_{2016}^c}{N_{2015}^c}$$

where N_{2016}^c and N_{2015}^c are the populations of country c in 2016 and 2015.

Debt-to-output is from the October 2019 IMF World Economic Outlook, and the net foreign asset position from the IMF Balance of Payments and International Investment Positions Statistics, deflated by nominal GDP from the Penn World Table 9.1.

For each country, the labor-augmenting productivity growth γ^c is defined as the average growth rate between 2000 and 2016 in real GDP divided by effective labor supply. For each country, we measure real GDP as expenditure-side real GDP from the Penn World Table 9.1, effective labor supply as $L_t^c = \sum_j N_{jt}^c h_j^c$, with N_{jt}^c taken from the UN World Population Prospects, and h_j^c given by the labor income profiles defined in section 3. We define the world γ as the average of γ^c, weighted by real GDP.

Given γ^c and the elasticity of substitution between capital and labor η, the growth rate of each economy is

$$g^c = (1+n^c)(1+\gamma^c) - 1,$$

and we calibrate the investment-to-output ratios, the share parameter in the production function, and the labor share

$$\frac{I^c}{Y^c} = \frac{K^c}{Y^c}(\delta + g_c)$$

$$\alpha^c = (r+\delta)\left(\frac{K^c}{Y^c}\right)^{\frac{1}{\eta}}$$

$$s^{L,c} = 1 - (r+\delta)\frac{K^c}{Y^c},$$

where the expression for investment and α use (A.81) and (A.79). Note that this calibration ensures that the world asset market clearing condition (A.86) holds for r.

For government policy, we use the average labor wedge from the OECD Social Expenditure Database 2019 to target τ. This measure includes both employer and employee social security contribution, which is consistent with treating w_t as the labor cost for employers. For d, we use data on the share of GDP spent on old age benefits, using data on benefits net of taxes from the OECD Social Expenditure Database.[77] Our main source for the retirement age is OECD's data on "Effective Age of Labor Market Exit" from the OECD Pensions at a Glance guide.[78] For some countries, the age provided by the OECD implies that labor market exit happens after the age at which aggregate labor income falls below implied benefit income. In those cases, we define the latter age as the date of labor market exit. Formally, this is done by calibrating the implied benefit levels for each possible retirement age and choosing the highest age at which retirement benefits are weakly lower than net-of-tax income. Last, G/Y is calibrated residually to target (A.83) given B/Y, τ, d, and the retirement age.

For individuals, we use Auclert and Rognlie (2018) and De Nardi (2004) to target the standard deviations v_ϵ, v_θ and the persistence parameters $\chi_\epsilon, \chi_\theta$. The processes are discretized using Tauchen's method, using three states for θ and 11 states for ϵ. Both processes are rescaled to ensure that they have a mean of 1.

Model outcomes and fit. Figure A.7 and A.8 show the model fit of age profiles of wage and labor income across all countries. For the labor income profile, the orange depicts labor income $(1-\rho_{j0})\ell_j$ in the initial steady state, and the white hollow dots depict ℓ_j which become relevant as the retirement age increases.

Table A.5 provides the main parameters for all countries, table A.6 provides additional parameters for all countries. Last, figure A.9 shows the outcomes for bequests and wealth inequality in the US. Panel A compares the distribution of bequests in the model to the empirical distribution in the data. We measure it as the value of bequests at certain percentiles divided by average bequests. We take the empirical distribution from Table 1 in Hurd and Smith (2002). The legend also reports the resulting model aggregate bequests-to-GDP ratio $\frac{Beq}{Y} = 8.8\%$. Panel B compares the model Lorenz curve to the one obtained in the SCF. We see that our model produces substantial wealth inequality, with the richest 20% holding roughly 70% of wealth. However, it does not go all the way to fit the wealth inequality in the US data.

D.5 Simulating demographic change

Solution method. We solve for the perfect foresight transition path between 2016 ($t = 0$) and 2300 ($t = 284 \equiv T$) as follows.

In every country, we simulate demographics forward using the initial population distribution $\{N_{j,-1}\}_{j=0}^{J}$ and the forcing variables $\{M_{jt}, \phi_{jt}, N_{0,t+1}\}_{-1\le t\le T, 0\le j\le J}$ to obtain $\{\pi_{j,t}, N_{j,t}\}_{0\le t\le T, 0\le j\le J}$ and population growth rates $\{n_t\}_{j=0}^{T}$. The forcing variables are obtained from the UN World Population Prospects until 2100. From 2100 on, we assume that the survival rates ϕ_{jt} and migration rates $\frac{M_{jt}}{N_t}$ are kept constant at their 2100 levels. We further assume that the growth rate of the number of births, $N_{0,t+1}/N_{0t}$, adjusts linearly in every country from its 2100 level to a common long-run rate of -0.5% by 2200. Given the effective labor supply profile and the retirement policy, the demographic projections imply a path for aggregate labor $\{L_t\}_{t=0}^{T}$ from (A.64).

Next, given a path for the interest rate $\{r_t\}_{t=0}^{T}$, technological parameters, and aggregate labor, we can obtain the optimal capital-labor ratio from (A.61) and other production aggregates as well as the wage rate $\{\frac{K_t}{L_t}, K_t, Y_t, I_t, w_t\}_{t=0}^{T}$ follow from (A.59)-(A.63).

[77] OECD SOCX Manual, 2019 edition.

[78] Pensions at a Glance 2019: OECD and G20 Indicators.

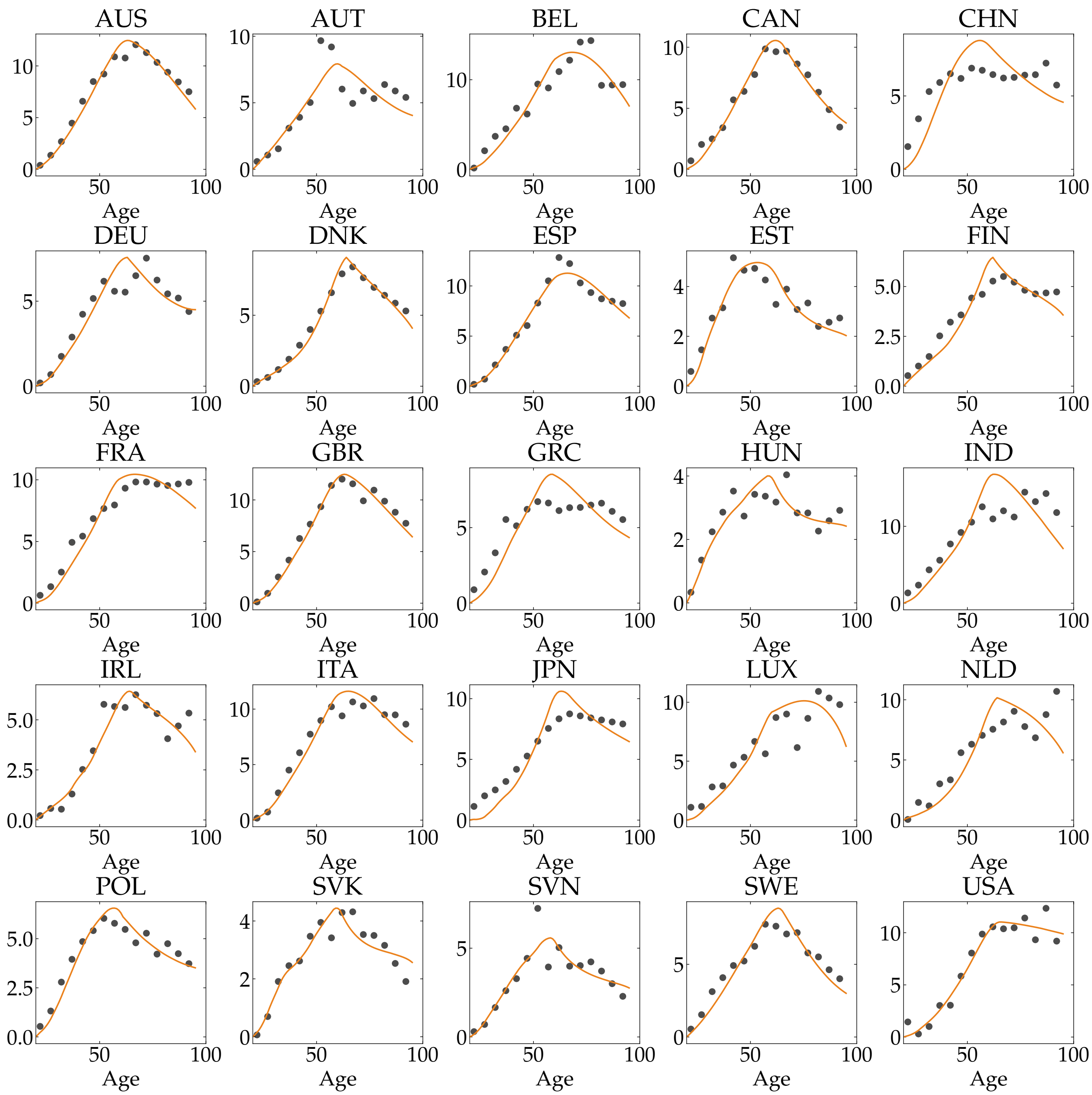

Figure A.7: Calibration outcomes: wealth

Notes: This figure presents the empirical age-wealth profiles (gray dots) and the calibrated model age-wealth profiles in the baseline calibration (orange line) for the 25 countries we consider.

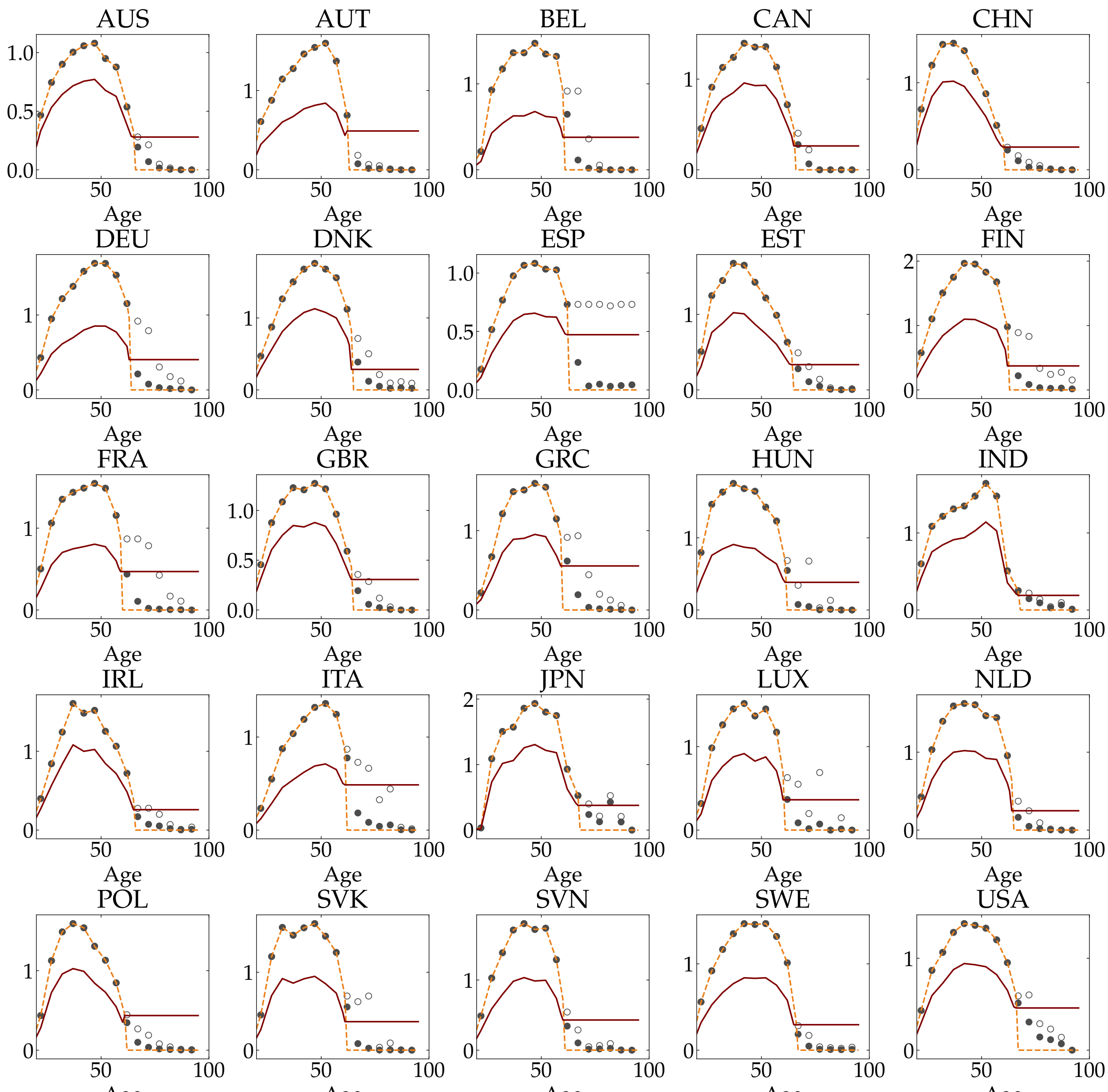

Figure A.8: Calibration outcomes: labor income

Notes: This figure presents the empirical age-labor supply profile from LIS used in section 2 (black dots), as well as the model gross age-labor supply profile (dashed orange line) and the net-of-taxes profile (red line).

Table A.5: World economy calibration

	$\Delta^{comp,c}$		Components of wealth			Government policy	
Country	Model	Data	$\frac{W^c}{Y^c}$	$\frac{B^c}{Y^c}$	$\frac{NFA^c}{Y^c}$	τ^c	$\frac{Ben^c}{Y^c}$
AUS	1.72	1.68	5.09	0.40	-0.46	0.29	0.04
AUT	1.14	1.07	3.90	0.83	0.12	0.47	0.11
BEL	1.91	1.85	5.74	1.06	0.65	0.54	0.09
CAN	1.07	1.04	4.63	0.92	0.20	0.31	0.04
CHN	2.50	2.37	4.20	0.44	0.25	0.30	0.04
DEU	0.82	0.79	3.64	0.69	0.58	0.50	0.10
DNK	0.75	0.72	3.42	0.37	0.46	0.36	0.06
ESP	2.77	2.41	5.33	0.99	-0.74	0.39	0.10
EST	0.64	0.63	2.64	0.09	-0.33	0.39	0.07
FIN	0.65	0.66	2.78	0.63	0.16	0.44	0.09
FRA	1.72	1.67	4.85	0.98	-0.05	0.48	0.13
GBR	1.64	1.60	5.35	0.88	0.08	0.31	0.06
GRC	1.56	1.38	4.25	1.81	-1.25	0.40	0.16
HUN	0.49	0.48	2.19	0.76	-0.54	0.48	0.09
IND	3.75	3.12	4.16	0.68	-0.08	0.30	0.01
IRL	1.49	1.46	2.32	0.74	-1.65	0.33	0.03
ITA	2.33	2.06	5.83	1.31	-0.02	0.48	0.13
JPN	1.32	1.19	4.85	2.36	0.66	0.32	0.09
LUX	2.05	1.86	3.92	0.21	0.64	0.40	0.07
NLD	1.57	1.54	3.92	0.62	0.70	0.37	0.05
POL	1.38	1.35	3.31	0.54	-0.52	0.36	0.10
SVK	0.85	0.84	2.17	0.52	-0.59	0.42	0.07
SVN	0.70	0.69	2.82	0.79	-0.21	0.43	0.11
SWE	0.67	0.70	3.81	0.42	0.08	0.43	0.06
USA	1.62	1.47	4.38	1.07	-0.36	0.32	0.06

Table A.6: World economy calibration

Country	$\bar{\beta}^c$	ξ^c	Y^c	ν^c	α^c	G^c/Y^c
AUS	0.984	0.00022	118.269	1.681	0.500	9.9%
AUT	0.996	-0.00012	118.269	1.681	0.287	22.0%
BEL	0.983	0.00065	118.269	1.681	0.391	22.2%
CAN	1.001	-0.00017	118.269	1.681	0.341	15.5%
CHN	1.024	-0.00003	118.269	1.681	0.341	15.1%
DEU	1.006	-0.00037	118.269	1.681	0.230	27.6%
DNK	1.161	0.00239	118.269	1.681	0.252	20.4%
ESP	0.939	-0.00044	118.269	1.681	0.494	7.6%
EST	1.177	0.00024	118.269	1.681	0.280	21.0%
FIN	1.195	0.00255	118.269	1.681	0.193	25.4%
FRA	1.001	0.00040	118.269	1.681	0.380	15.6%
GBR	1.000	0.00029	118.269	1.681	0.426	10.7%
GRC	1.015	0.00024	118.269	1.681	0.359	6.0%
HUN	1.178	0.00116	118.269	1.681	0.191	28.1%
IND	0.997	0.00041	118.269	1.681	0.347	18.5%
IRL	1.199	0.00284	118.269	1.681	0.314	18.6%
ITA	0.930	-0.00071	118.269	1.681	0.441	11.1%
JPN	1.089	0.00098	118.269	1.681	0.177	12.7%
LUX	1.195	0.00341	118.269	1.681	0.299	20.8%
NLD	1.144	0.00248	118.269	1.681	0.253	21.9%
POL	1.055	0.00057	118.269	1.681	0.319	13.6%
SVK	1.233	0.00199	118.269	1.681	0.218	24.4%
SVN	1.171	0.00076	118.269	1.681	0.219	20.9%
SWE	1.010	-0.00003	118.269	1.681	0.322	23.0%
USA	1.044	0.00063	118.269	1.681	0.356	12.5%

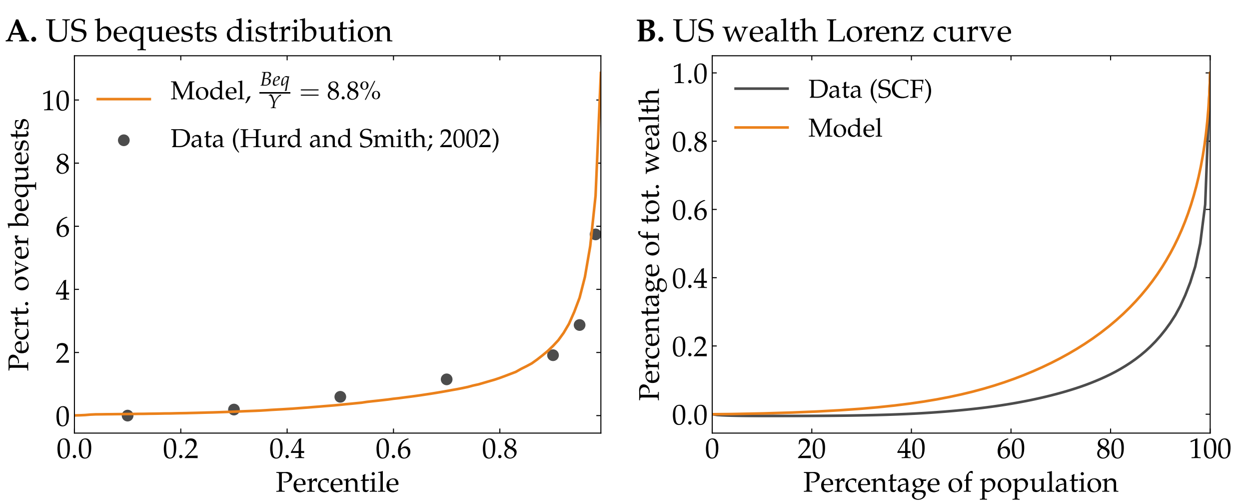

Figure A.9: Distribution of bequests and wealth Lorenz curve in the US

Given a government fiscal rule $\{B_{t+1}/Y_{t+1}, \rho_{jt}, \varphi^G, \varphi^\tau, \varphi^d, \bar{G}, \bar{\tau}, \bar{d}\}_{t=0}^T$, we obtain the path for the policies $\{G_t, \tau_t, d_t\}_{t=0}^T$ from (A.67)-(A.69) such that the government budget constraint (A.65) is satisfied for every t.

Then, we solve the household problem as follows. Given a guess for total bequests received by type θ across all ages $\{Beq_t^r(\theta)\}_{t\geq 0,\theta}$.[79], a path of prices $\{r_t, w_t\}_{t=0}^T$, government policy $\{\rho_{jt}, \tau_t, d_t\}_{t=0}^T$, demographic variables $\{n_t, \pi_{j,t}, \phi_{j,t}\}_{0\leq t\leq T, 0\leq j\leq J}$, we solve the household problem (A.54) in two steps. First, we use Carroll (2006)'s Endogenous Grid Point Method (EGM) to determine the decision functions $\{c_{jt}(\theta,\epsilon,\mathfrak{a})\}_{t\geq 0, 0\leq j\leq T}$ and $\{a_{j+1,t+1}(\theta,\epsilon,\mathfrak{a})\}_{t\geq 0, 0\leq j\leq T}$, assuming constant prices after 2300. Second, we obtain the distributions following Young (2010). We start from an initial distribution, which we take from the 2016 steady-state, and iterate forward using the asset decision function and the law of motion of the state (θ,ϵ). We then compute aggregates following (A.58).

To solve for the world economy equilibrium, we use a Newton-based method to ensure that bequests received equals bequests given by type θ and that the asset market clearing condition (A.73) is satisfied. We iterate on a 285×1 path for the interest rate by year $\{r_t\}_t$, and a $285\times 25\times 3$ path for bequest by year, country and type $\{B^{r,c}(\theta)\}_{t,c,\theta}$ until convergence.

To solve for the small open economy, we hold fixed the path of the interest rate, i.e. $r_t = r_0, \forall t > 0$.

Details on table 4. Below, we provide details on the results in table 4, starting with the construction of each column, and then the details on the various experiments. The description of the columns applies to the full model analyses; for the pure compositional analysis, some columns have a slightly different interpretation, which is clarified when we discuss this experiment. For all columns, the changes refer to differences between 2016 and 2100. In the left panel, Δr is the change in the rate of return, $\overline{\Delta\log\frac{W}{Y}} \equiv \sum_c \omega^c \Delta_{2100}\log\left(\frac{W^c}{Y^c}\right)$ is the average change in the wealth-to-output ratio, weighted by initial shares of wealth.

In the right panel, $\bar{\Delta}^{comp} \equiv \sum_c \omega^c \Delta_{2100}^{c,comp}$ is the average compositional effect between 2016 and 2100, weighted by initial wealth levels. The term $\bar{\Delta}^{soe} \equiv \sum_c \omega^c \Delta_{2100}^{c,soe}$ is the equivalent average for the small open economy effect. For each country c, $\Delta^{c,soe}$ is defined as the change in $\frac{W^c}{Y^c}$ between 2016 and 2100 in a small open economy equilibrium with a fixed interest rate r_{2016}.

The asset supply and demand semielasticities $\bar{\epsilon}^d = \sum_c \omega^c \epsilon^{c,d}$ and $\bar{\epsilon}^s = \sum^c \omega_c \epsilon^{c,s}$ are the averages of the country semielasticities weighted by initial wealth levels. For each country c, the asset demand sensitivity $\epsilon^{c,d}$ is defined as the semielasticity of the steady-state $\frac{W^c}{Y^c}$ with respect to the steady state interest rate r.[80] The asset supply sensitivites are given by $\epsilon^{c,s} = \frac{1}{W^c/Y^c}\frac{\eta}{r+\delta}\frac{K^c}{Y^c}$.

The list below describes the pure compositional analysis and the various model experiments. All model experiments feature a retirement age increased by 1 month per year for the first 60 year of the simulation. All start from the steady-state equilibrium calibration.

- **Pure compositional effect.** This row reproduces the exercise in section 3. That is, all changes in r, wealth, and NFAs are defined using proposition 2 and 3 given the initial wealth weights ω^c, the compositional effects $\Delta^{comp,c}$, and the set of sensitivities $\epsilon^{c,d}$ and $\epsilon^{c,s}$. The supply sensitivities are given by $\epsilon^{c,s} = \frac{1}{W^c/Y^c}\frac{\eta}{r+\delta}\frac{K^c}{Y^c}$, where $\frac{K^c}{Y^c}$ is the calibrated capital stock from the steady-state calibration. The demand sensitivities are defined using the expression in

[79] $Beq_t^r(\theta) \equiv \sum_{\theta_-}\left(\frac{\Pi^\theta(\theta|\theta_-)\pi^\theta(\theta_-)}{\pi^\theta(\theta)}\right) \times \sum_{k=0}^T [N_{k,t-1} + M_{k,t-1}](1-\phi_{k,t-1}) \times \sum_\epsilon \pi^\epsilon(\epsilon)\int_{\mathfrak{a}} \mathfrak{a}\, dH_{kt}(\mathfrak{a}|\theta_-,\epsilon)$, so that bequests per age-j person of type θ is $b_{jt}^r(\theta) = \frac{F_j}{N_{jt}} Beq_t^r(\theta)$.

[80] In practice, we calibrate a steady-state to 2100 demographics, and perturb r_{2016} and resolve for a new stationary equilibrium, using the resulting perturbation to $\frac{W^c}{Y^c}$ to calculate the derivative.

proposition 4, using the same method as in section 3, but instead using the calibrated profiles of assets and income to back out the consumption profile and calculate the relevant moments of the asset and consumption profiles.

- **Preferred model specification.** The fiscal rule places equal weight on consumption, taxes, and retirement benefits.
- **Constant bequests.** The process $\frac{b^{jt}(\theta)}{w_t}$ of bequests received normalized by wages is kept constant over time. This removes a source of non-compositional increases in asset holdings which comes from an older population implying that people receive more bequests over time. To make a constant sequence of bequests consistent with equilibrium, we assume that it is implemented with an age-type specific lump sum tax/transfer that keeps bequests over wages constant at their 2016 level once these additional taxes/transfers are netted out. To prevent this tax from having second order effects on individual behavior through the government budget constraint, we assume that it is neutralized by variation in government consumption.
- **Constant mortality.** The subjective mortality risk of individuals is kept fixed at their 2016 values, while the population evolution still follows the objective mortality risks.
- **Constant taxes and transfers.** The fiscal rule places all weight on adjustments in government consumption, so that taxes and benefits are constant over time.
- **Constant retirement age.** The retirement age is kept fixed at its 2016 level.
- **No income risk.** The idiosyncratic income risk is switched off and the model is recalibrated.
- **Annuities.** Households get access to annuities, the bequest preference is set to zero: $\Upsilon = 0$, and the model is recalibrated.
- **Fiscal rules.** The full adjustment weight is placed on either G, d, or τ.

Changes to net foreign asset positions. Appendix Figure A.10 summarizes the model's predictions for the change in net foreign asset position in each country from 2016–2100. Panel A compares the full model findings to the method used in section 3 by plotting the full model results on the vertical axis, and the prediction based on demeaned compositional effects $\Delta^{comp,c} - \bar{\Delta}^{comp}$ on the horizontal axis. The compositional predictions are generally quite accurate, and the line of best fit excluding India is close to 45 degrees. In India, however, the model predicts even larger net foreign asset position growth than expected from the compositional effect.

Panel B shows that this discrepancy disappears, and the fit is even closer, when we use the demeaned small open economy effect $\Delta^{soe,c}$ for predictions on the horizontal axis instead. This shows that discrepancies in panel A, including for India, are mostly due to the non-compositional effects $\Delta^{soe,c} - \Delta^{comp,c}$ of aging in our model, rather than non-linearities or heterogeneity in elasticities.

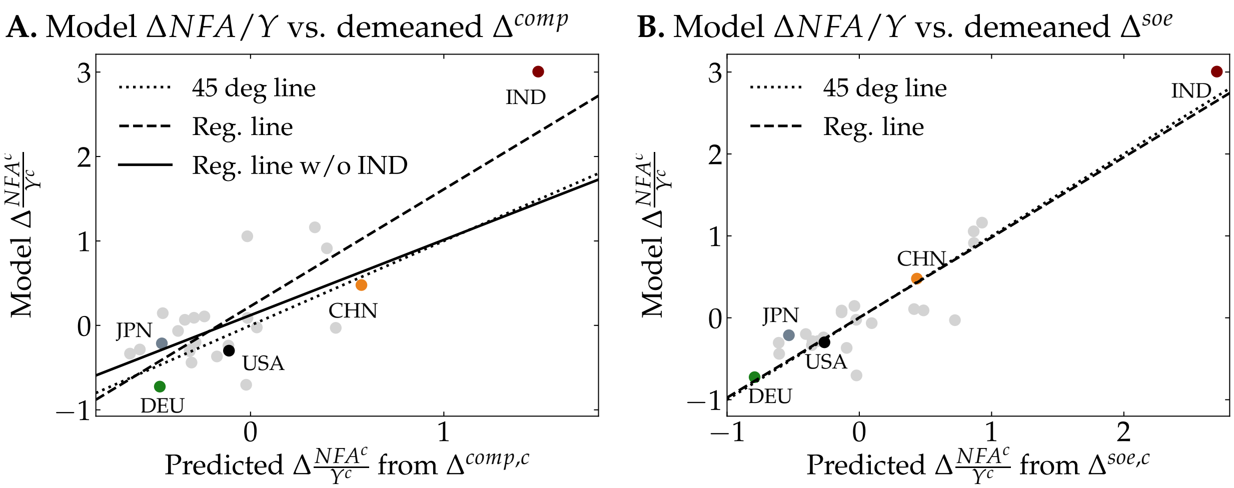

Figure A.10: Predicting change in net foreign asset position

Notes: Panel A presents the model-implied change in NFA/Y between 2016 and 2100 on the y-axis, and on the x-axis the change in NFA/Y predicted from the demeaned model compositional effect, $NFA/Y \approx \exp(\Delta^{comp,c} - \bar{\Delta}^{comp}) - 1$, over the same period. The dotted line is a 45 deg line. The dashed line is a regression line, and the solid line is this same regression line when India is excluded. Panel B also shows the model $\Delta NFA/Y$ on the y-axis, but the x-axis presents the change in NFA/Y predicted from the demeaned model small open economy effect, $NFA/Y \approx \exp(\Delta^{soe,c} - \bar{\Delta}^{soe}) - 1$.

E Appendix to Section 5

We first prove the results in the main text. Defining savings for an individual of age j in state (z^j, a_{jt}) at time t as

$$s_{jt} \equiv r a_{jt} + w_t \left((1-\tau)\ell(z_j) + tr(z^j) \right) - c_{jt}$$

and using the budget constraint (1), we see that aggregate savings for agents of age j is given by

$$s_{jt} = \mathbb{E} s_{jt} = \phi_j a_{j+1,t+1} - a_{jt} \tag{A.87}$$

Next, since lemma 1 implies $a_{jt} = a_j(r) Z_t$, we have

$$s_{jt} = \left(\phi_j (1+\gamma) a_{j+1} - a_j(r) \right) Z_t = s_j(r) Z_t$$

Hence, defining aggregate savings as

$$S_t \equiv \sum N_{jt} s_{jt} \tag{A.88}$$

we have that

$$\frac{S_t}{N_t} = \sum \pi_{jt} s_{jt} = \sum \pi_{jt} \underbrace{s_j(r) Z_0}_{s_{j0}} (1+\gamma)^t = \sum \pi_{jt} s_{j0} (1+\gamma)^t$$

Taking the ratio of this expression to equation (8), we obtain the equivalent of Proposition 1,

$$\frac{S_t}{Y_t} = \frac{F_L(k(r),1)}{F(k(r),1)} \cdot \frac{\sum \pi_{jt} s_{j0}}{\sum \pi_{jt} h_{j0}} \tag{A.89}$$

which delivers equation (28).

Next, combining (A.87), (A.88), and the population dynamics equation $N_{j+1t+1} = \phi_j N_{jt}$, we have

$$S_t \equiv \sum N_{jt} s_{jt} = \sum N_{jt} \phi_j a_{j+1,t+1} - \sum N_{jt} a_{jt} = \sum N_{j+1,t+1} a_{j+1,t+1} - \sum N_{jt} a_{jt} = W_{t+1} - W_t$$

where the last line uses the initial and terminal condition on wealth by age. Hence, the aggregate savings rate is:

$$\frac{S_t}{Y_t} = \frac{W_{t+1} - W_t}{Y_t} = \frac{Y_{t+1}}{Y_t}\frac{W_{t+1}}{Y_{t+1}} - \frac{W_t}{Y_t} = (1 + g_{t+1})\frac{W_{t+1}}{Y_{t+1}} - \frac{W_t}{Y_t}$$

where g_t is the growth rate of aggregate GDP, the sum of productivity growth, population growth and changing composition,

$$1 + g_{t+1} \equiv \frac{Y_{t+1}}{Y_t} = (1+\gamma)\frac{N_{t+1}}{N_t}\frac{\sum_j \pi_{jt+1} h_{j0}}{\sum_j \pi_{jt} h_{j0}} = (1+\gamma)(1+n_{t+1})\frac{\sum_j \pi_{jt+1} h_{j0}}{\sum_j \pi_{jt} h_{j0}}$$

In steady state, therefore, we have

$$\frac{S}{Y} = g\frac{W}{Y}$$

where $1 + g = (1+\gamma)(1+n)$. This is the famous Solow (1956)–Piketty and Zucman (2014) formula for the relationship between the net savings rate W/Y, the growth rate of GDP g, and the wealth-to-GDP ratio W/Y.

Finally, towards our implementation, we show that S_t/Y_t can be calculated from the cross-sectional profiles of assets a_{jt} and demographic projections alone. We first show that S_t/Y_t in equation (28) can be calculated from cross-sectional age profiles of assets $a_{j,0}$. Indeed, we have, starting from $S_t = W_{t+1} - W_t$, we have

$$\begin{aligned}
\frac{S_t}{N_t(1+\gamma)^t} &= \frac{W_{t+1}}{N_t(1+\gamma)^t} - \sum \pi_{jt} a_{j0} \\
&= (1+n_{t+1})(1+\gamma)\sum \pi_{jt+1} a_{j0} - \sum \pi_{jt} a_{j0} \\
&= ((1+n_{t+1})(1+\gamma) - 1)\sum \pi_{jt} a_{j0} + (1+n_{t+1})(1+\gamma)\sum(\pi_{jt+1} - \pi_{jt}) a_{j0} \\
&= g^{ZN}_{t+1}\sum \pi_{jt} a_{j0} + \left(1 + g^{ZN}_{t+1}\right)\sum(\Delta\pi_{jt+1}) a_{j0}
\end{aligned}$$

where we have defined $1 + g^{ZN}_{t+1} \equiv (1+n_{t+1})(1+\gamma)$. Taking the ratio of this expression to equation (8), we have the following expression for the aggregate savings rate:

$$\frac{S_t}{Y_t} = \frac{F_L(k(r),1)}{F(k(r),1)}\left(\frac{g^{ZN}_{t+1}\sum \pi_{jt} a_{j0} + (1 + g^{ZN}_{t+1})\sum(\Delta\pi_{jt+1}) a_{j0}}{\sum \pi_{jt} h_{j0}}\right) \tag{A.90}$$

which is an alternative to equation (A.89).

In principle, to project savings rates from demographic composition, we could equally well implement equation (A.89) or equation (A.90). Summers and Carroll (1987), Auerbach and Kotlikoff (1990), and Bosworth et al. (1991) follow the first route. We prefer to follow the second because it only requires only information that we have already used so far in the paper, and because the computation of age-specific savings rates is subject to a large amount of measurement error.

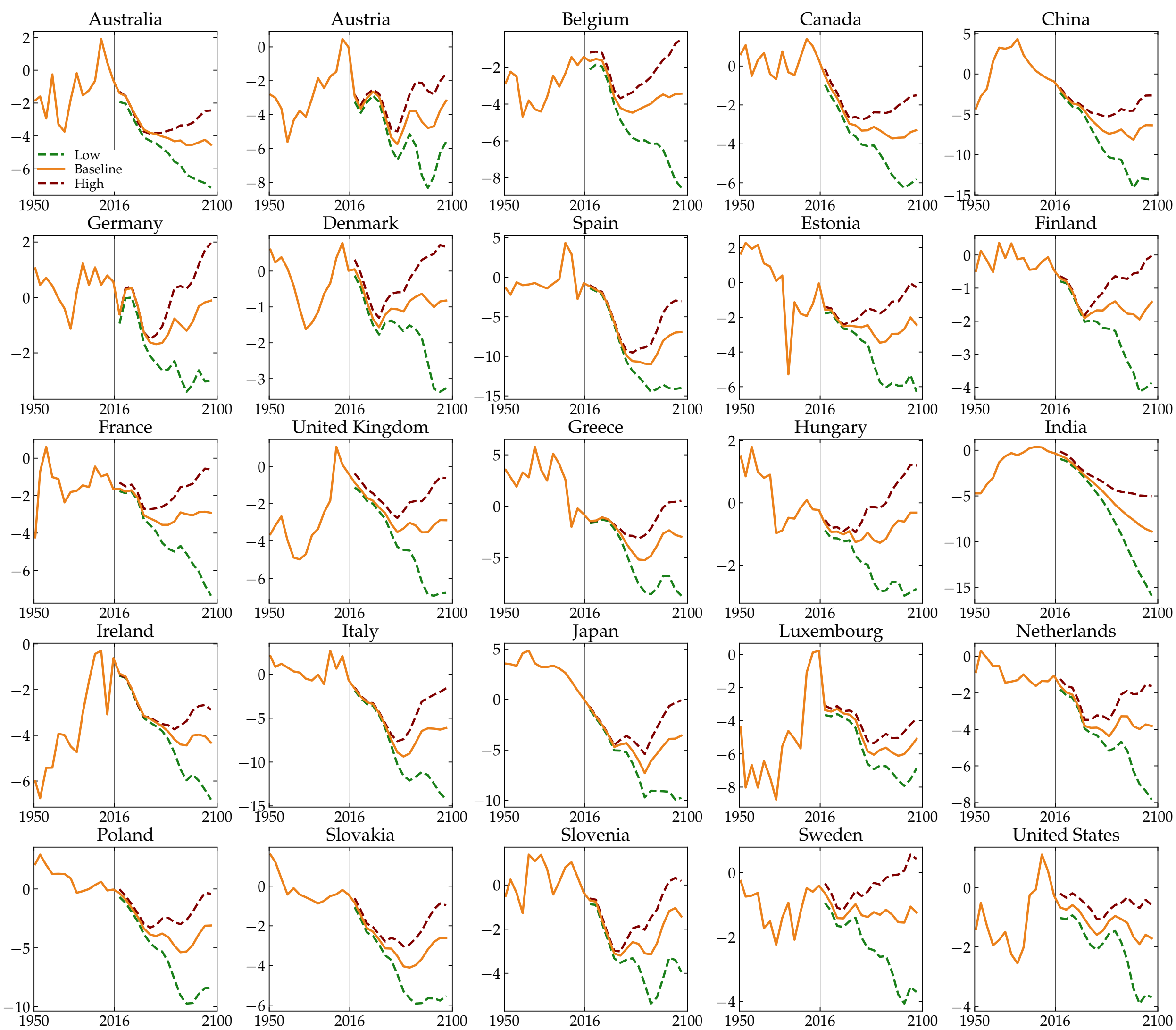

Figure A.11: Predicted change in savings-to-GDP from compositional effects

Notes: This figure depicts the evolution of the predicted change in the savings-to-GDP ratio from the compositional effect for $t = 1950$ to 2100, reported in percentage points. The base year is 2016 (vertical line). The solid orange line corresponds to the medium fertility scenario from the UN, the dashed green line to the low fertility scenario, and the dashed red line to the high fertility scenario.

F Interpreting literature findings

In this appendix, we show that our results are useful to understand existing findings in the literature. First, across papers that conduct a similar exercise, we trace results back to their inputs, and show why different assumptions about the compositional effect are a critical driver of the differences in general equilibrium outcomes. Second, within papers that consider the role of parameter changes, we show that our results are useful in explaining the functional form relationship between these parameters and general equilibrium outcomes. In the interest of space, we focus on the effect of demographic change on the total return r (sometimes referred to as the natural interest rate, or r^*, in the literature).

F.1 Explaining different magnitudes across papers

Eggertsson et al. (2019) (EMR) and Gagnon et al. (2021) (GJLS) are two recent papers that find very different effects of demographics on real interest rates. Both study the US economy using closed-economy general equilibrium models, but EMR find that demography reduced real interest rates by 3.44 percentage points between 1970 and 2015, while GJLS only find an effect of 0.92 percentage points, a difference of 2.52 percentage points. We use publicly available replication files[81] to create table A.7, which applies the framework of proposition 5 to explain these results in terms of the underlying differences in compositional effects Δ^{comp}, non-compositional effects $\Delta^{soe} - \Delta^{comp}$, and semielasticities ϵ^d and ϵ^s.

The single most important difference is that the compositional effect in EMR is more than three times as large as that in GJLS. If EMR had the same compositional effect as GJLS, more than half of the gap between the two estimates would be closed. EMR also have a far lower asset supply semielasticity ϵ^s, one-fourth as large as GJLS. If EMR also had the same ϵ^s as GJLS, 86% of the gap would be closed.[82]

The results on compositional effects can be interpreted using figure A.12, which shows the asset profiles by age and the population distribution shifts in the two papers and in the data. Two forces explain the large compositional effect in EMR. First, the age-wealth profile is much steeper than in the data, staying below zero until age 46 and then rising sharply. This inflates the effect of shifting the age distribution toward older ages. Second, the shift in age composition itself is very large, because the exercise compares a steady state based on 2015 fertility and mortality with a steady state based on 1970 demographics (for which EMR take 1970 mortality and, since agents in the model come of age after 25 years, 1945 fertility). Due to the slow convergence rate of the empirical age distribution, these two steady states have larger differences in age distribution than the actual change that occurred between 1970 and 2015.[83]

[81] Replication repositories: `https://www.openicpsr.org/openicpsr/project/114159/version/V1/view` (EMR) and `https://sites.google.com/site/etigag/gjls-replication-materials` (GJLS).

[82] The difference in ϵ^d in table A.7 also appears substantial, at 12.7 in EMR vs. 28.5 in GJLS. However, the asset demand curve exhibits some non-linearity in response to EMR's very large change in r, so that if ϵ^d is taken around the 1970 steady state instead, it is 19.7, considerably closer to GJLS. If we move toward a second-order approximation by taking the average $\epsilon^d = (12.7 + 19.7)/2 = 16.2$, then Δr in table A.7 becomes an extremely accurate approximation, at 3.5%. With this ϵ^d, the compositional effect and ϵ^s together explain 87% of the difference between the two papers.

[83] In addition to this comparison of steady states, EMR also perform an exercise with explicit transitional dynamics. This exercise features a smaller Δ^{comp} for 1970 to 2015—albeit one that is still somewhat overstated, due to the steep age-wealth profile and since the exercise starts with the 1970 steady state. Overall, however, the decline in r in this exercise from 1970 to 2015 is quite similar to the decline in r in the steady

Table A.7: Decomposing the change in equilibrium r in existing papers

	Eggertsson et al. (2019)	Gagnon et al. (2021)	Sufficient statistic
Time-period	1970–2015	1970–2015	1970–2015
GE transition			
Δr^{GE}	-3.44%	-0.92%	
First-order approximation $\Delta r = \frac{-\Delta^{soe}}{\epsilon^d + \epsilon^s}$			
Δr	-4.30%	-0.97%	-0.49%
Δ^{comp}	45.4%	13.4%	12.4%
$\Delta^{soe} - \Delta^{comp}$	21.1%	25.3%	0%
ϵ^s	2.8	11.1	8.0
ϵ^d	12.7	28.5	17.5
σ	0.75	0.5	0.5
η	0.6	1.0	1.0

Notes: This table analyzes two key results from Eggertsson et al. (2019) (EMR) and Gagnon et al. (2021) (GJLS) using the framework of proposition 5. In GJLS, we analyze the 1970 to 2015 segment of the paper's main experiment, which is a simulation of the effects of demographic change between 1900 and 2030. In EMR, we analyze jointly the two demographic experiments from table 6 ("mortality rate" and "total fertility rate"). These are steady state experiments that consider the effect of changing fertility and mortality from their 2015 to their 1970 level. For both experiments, Δr^{GE} is the general equilibrium change in r from 1970 to 2015, Δ^{comp} is our compositional effect measure, implemented using the two papers' 2015 age profiles and the age distributions for 1970 and 2015, and ϵ^s is the semielasticity of asset supply $(B+K)/W$ in 2015 with respect to r. For EMR, Δ^{soe} is given by the change in W/Y between the 1970 and 2015 steady state when both have $r = r_{2015}$ and ϵ^d is the derivative of $\log W/Y$ to r in the 2015 steady state. For GJLS, Δ^{soe} is the counterfactual change in W/Y in a simulation where r is fixed after 1970, and ϵ^d is the derivative of $\log W/Y$ to r around a steady state defined to have the same population age distribution as the one observed in 2015. The sufficient statistic column applies the method in section 3 to 1970-2015, constructing Δ^{comp} from observed changes in the age distribution from 1970 to 2015 together with age profiles of assets and labor income from 2016, and asset semielasticities from (23) and proposition 4, for ϵ^s using the 2016 value of K/W, and for ϵ^d using the 2016 profiles of assets and labor income, together with $\sigma = 0.5$ and $\eta = 1$.

For the asset supply semielasticity ϵ^s, the lower value in EMR partly reflects their assumption of a lower elasticity of substitution between capital and labor relative to GJLS ($\eta = 0.6$ versus $\eta = 1$). However, even with $\eta = 1$, EMR would only have $\epsilon^s = 4.6$, less than half that of GJLS. The remaining difference reflects a second, more subtle, reason for EMR having a low ϵ^s, namely that ϵ^s scales with the share of capital in total wealth K/W, which is 1 in GJLS and only 0.51 in EMR. Capital is a small part of wealth in EMR because high (uncapitalized) markups mean that capital owners only receives $\sim 10\%$ of total output, with a resulting low capital-output ratio of $K/Y = 124\%$. Combined with a high level of bonds $B/Y = 117\%$, capital becomes a small part of total wealth, lowering the responsiveness of asset supply to changes in r.

state exercise. This is for a reason we saw in figure 8. In equilibrium, r tends to overshoot what current demographics would imply, incorporating future demographic change as well; if r is only allowed to vary from its initial steady state starting in 1970, as in this exercise (but not GJLS), much of the effect of long-run demographic change is compressed into the 1970–2015 period. Because of this difficulty in interpretation, and because the steady-state exercise is the only one for which EMR explicitly do a breakdown into demographic causes of the decline in r, we focus on the steady-state exercise in table A.7.

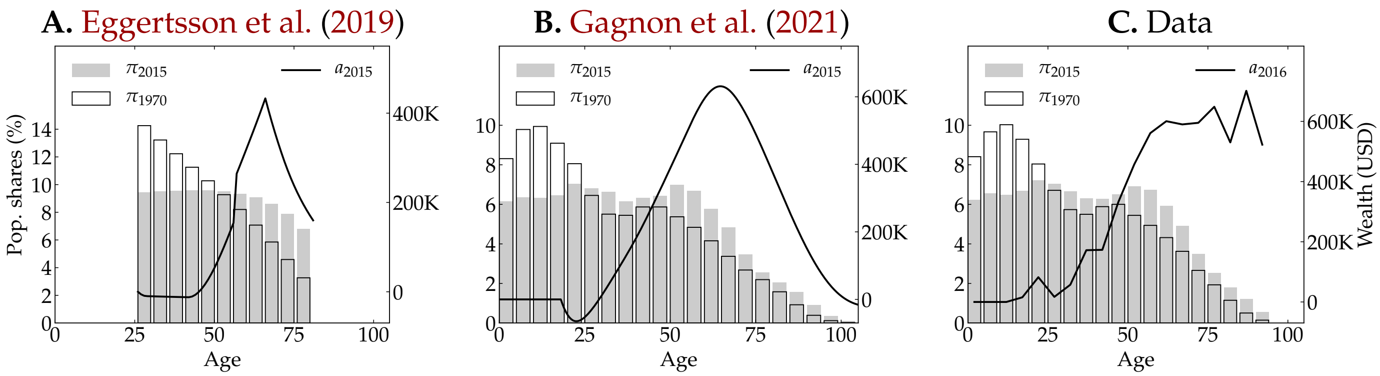

Figure A.12: Age-wealth profiles in papers vs the data

For comparison, we also include the results of the sufficient statistic analysis from section 3 applied to the same time period. For Δ^{comp}, the sufficient statistic result comes directly from the data and is closer to GJLS than to EMR. This reflects the fact that GJLS closely target the change in age distribution over time, and also do a good job fitting the age profile of wealth for all but the highest ages, which are of limited quantitative importance before 2015. For ϵ^s, the results in the sufficient statistic analysis lie above EMR and below GJLS. Apart from having a higher η than EMR, this mainly reflects the fact that our assumed share of capital in wealth $K/W = 0.76$ is between the values in GJLS and EMR.

While the non-compositional effects $\Delta^{soe} - \Delta^{comp}$ are zero in the sufficient statistic analysis, they are positive in EMR (21.1%) and GJLS (25.3%), and relatively large compared to what we find in the quantitative analysis in section 4. The non-compositional effect is especially pronounced in GJLS, where it is twice as large as the compositional effect. This reflects a very strong response of asset accumulation to falling mortality. This is largely due to the lack of bequest motive in GJLS, which implies that all saving is for personal consumption needs, which scale proportionally with survival probabilities. In our model in section 4, the bequest motive scales with mortality and counterbalances this effect; the role of saving for personal consumption in retirement is further diluted by the presence of a social security system.

F.2 Understanding the role of parameter changes

Our results in section 2 uncover a structural relationship between primitive parameters, calibration moments, and general equilibrium counterfactuals. For instance, combining the results in equations (13) and (17), the inverse effect on the interest rate of a change in demographics that creates a compositional effect of $\bar{\Delta}^{comp}$ is given by a simple affine function,

$$\frac{1}{dr} = -\frac{\bar{\epsilon}^{income} - \bar{\epsilon}^{laborshare}}{\bar{\Delta}^{comp}} - \sigma\frac{\bar{\epsilon}^{substitution}}{\bar{\Delta}^{comp}} - \eta\frac{\bar{\epsilon}^{laborshare} + \frac{1}{r+\delta}\frac{\bar{K}}{\bar{W}}}{\bar{\Delta}^{comp}} \tag{A.91}$$

Plugging in the elasticity values from section 3.2, we obtain

$$\frac{1}{dr} = \frac{7.5}{\bar{\Delta}^{comp}} - \sigma\frac{39.5}{\bar{\Delta}^{comp}} - \eta\frac{13.5}{\bar{\Delta}^{comp}}$$

Table A.8: Understanding the functional form relationship between σ, η and dr

$1/\sigma$	σ	dr for $\eta = 1$	$1/dr$	η	dr for $1/\sigma = 2.5$	$1/dr$
1	1.00	-0.41	-2.42	0.4	-1.70	-0.59
1.5	0.67	-0.65	-1.54	0.6	-1.44	-0.69
2	0.50	-0.84	-1.19	0.8	-1.20	-0.83
2.5	0.40	-1.00	-1.00	1	-1.00	-1.00
3	0.33	-1.14	-0.88	1.2	-0.84	-1.18
3.5	0.29	-1.25	-0.80	1.4	-0.73	-1.37
4	0.25	-1.35	-0.74	1.6	-0.66	-1.52

Notes: This table presents Papetti (2019)'s findings for the equilibrium change in the real interest rate between 1990 and 2030 (dr) as a function of risk aversion $1/\sigma$ and capital-labor substitution η. The numbers are taken from his Figures 10 and 12, and then transformed to make the additively linear relationship between $1/dr$ and σ and η, which is implied by our framework, appear.

For the 2016-2100 period, we can take $\bar{\Delta}^{comp} = 32\%$ from section 3, and obtain (for r in %)

$$\frac{1}{dr} = 0.23 - 1.23 \cdot \sigma - 0.42 \cdot \eta$$

Equation (A.91) shows that, conditional on having recalibrated the model to hit the same data moments and therefore the same $\bar{\Delta}^{comp}$ and $\bar{\epsilon}$'s,[84] the effects of σ and η are additively separable for the inverse general equilibrium effect on interest rates, $1/dr$.

To illustrate the potential of this equation for interpreting findings in other papers, we study the results in Papetti (2019), who provides a comprehensive structural OLG quantitative model of the Euro Area. In Figures 10 and 12, the author reports his model's predicted effect of demographics on the change in the real interest rate change over the period 1990 – 2030, which we call dr, first as a function of risk aversion $1/\sigma$, and then as a function of capital-labor substitution η. We reproduce his results in table A.8. Observe that all his estimates of the effect of demographics on interest rates over this period are all negative.

Note further that the inverse effect on the interest rate, $1/dr$, appears to be linear in both σ and η, just like equation (A.91) predicts. To confirm this, we run a linear regression of $1/dr$ on σ and η and obtain:

$$\frac{1}{dr} = 0.67 - 2.22 \cdot \sigma - 0.81 \cdot \eta$$

with an R^2 of 0.993. The quality of the fit of the functional form is remarkable. The coefficients are around two times larger than our coefficient for 2016-2100, so the interest rate effects are about half in our model what they are in his. One obvious distinction is that our results are for an 80 year period, while his are for a 40 year period. In addition, the fundamental inputs into $\epsilon^{substitution}$, ϵ^{income} are different, and the compositional effects $\bar{\Delta}^{comp}$ in his model appear to be lower than in ours, perhaps because Papetti (2019) does not does not directly target wealth profiles in his calibration.

[84] In practice, changing η does does not change the steady state so does not require a recalibration, while changing σ requires adjusting parameters to keep r and the age profiles of assets unchanged.