Appendix A

Example referred to in Section 3.5; this is also an elaboration of the example referred to in the Introduction

Assume $K = 1.20$, $P = 1.05$, $r = .021$, and (for simplicity) $\delta = 1$.The distributions of $\widetilde{\theta}_i$ are such that $\theta_i = 1.05$ with probability .77, $\theta = .95$ with probability .20, and $\theta_i = \underline{\theta} = .75$ with probability .03. So $m = 1 - \delta K\underline{\theta} = .1$.

If $\theta_1 = .95$, then the collateral is worth $\delta K\theta_1 = 1.14$ at $t = 1$, so the loan will be paid off in full unless $\theta_2 = \underline{\theta}$, and $M_{0,1} = (.97(1.05) + .03(1.14)(.75))/1.021 = 1.0227$, so $M_{0,1} > 1$. Indeed the value of the bank's claim has risen by more than the expected market return ($M_{0,1} > 1 + r$). However, $\delta K\theta_1\underline{\theta} = 1.14(.75) = .855$. Therefore $m_1 = 1 - .855/1.0227 = .1688$, so $m_1 > m$. To make uninsured investors safe, debt must be reduced from .90 to .855, requiring the bank to raise an additional .045 in equity.

So, even though equity has risen by 22.7% (from .10 to .1227 per loan) the bank must raise an additional .045 per loan, requiring either "dilution" of $(.045/(.045 + .1227)) = 26.8\%$, or the sale of that percentage of the bank's assets.

Appendix B

Example of use of ERNs in the model

Suppose, in our model, that a bank that makes L_0 loans at $t = 0$ can take no further actions before the loans are repaid at $t = 2$. Then the amount that it can borrow safely (with either two-period debt or one-period debt that investors can safely roll over) is reduced to $L_0\delta^2K\underline{\theta}^2$, from the $L_0\delta K\underline{\theta}$ it could borrow in our basic model. However, the bank can achieve the equivalent of borrowing the additional $L_0\delta K\underline{\theta} - L_0\delta^2K\underline{\theta}^2$, by issuing ERNs that convert from debt into equity after any sufficiently bad shock at $t = 1$, thus automatically replenishing the equity as required.[91]

For example, Bulow and Klemperer (2015) suggested ERNs that would be convertible into shares at 25% of the share price on the issue date, chosen because the share price of the strongest banks fell by around 75% during the financial crisis. Say that $L_0 = 3000$, $K = 1.2$, and $\delta\underline{\theta} = .75$. Then the bank could borrow $L_0\delta^2K\underline{\theta}^2 = 2025$ on a collateralized basis at $t = 0$. The bank could issue equity of 400 and ERNs, due at $t = 1$, with a face value of

[91] In a more realistic setting, ERNs would be long-term bonds with repayments spread over many periods.

575. If $\theta_1 = \underline{\theta}$, the firm's value would fall from $L_0 = 3000$ to $L_0 K \delta \underline{\theta} = 2700$, so the value of the equity would fall from 400 to 100. Since the share price would then have fallen 75%, the ERNs could then be paid in shares, and their holders would receive equity worth their initial investment of 575. So at $t = 1$, the bank would have 100+575=675 in equity, and 2025 in debt, exactly as in our basic model, after $\theta_1 = \underline{\theta}$. Furthermore, the ERNs would have generated a riskless return, assuming that any shares received could be sold at fair value.

Banks could choose any combination of ERNs and stock they thought would maximize their value; in the context of our model this would involve the lowest amount of equity that assured ERNholders that they would not suffer a loss even if $\theta_1 = \underline{\theta}$, so long as securities could still be sold for fair value (as in the example above).

If the bank's shares fell to less than 25% of the price on the day the ERNs were issued, then ERNholders would own a fixed fraction of the bank's equity (in the example 575/675), regardless of whether this lower price was due to a liquidity shock that caused shares to sell for less than fair value, or because fair value had fallen by more than 75% of the initial share value. Regardless of what happened at $t = 1$ the bank would no go bankrupt, and it would end with a limited number of shares outstanding at a positive price per share (so long as $\theta_1 > 0$ or $\theta_1 = 0$ but $N_1 > 0$).

Appendix C

Lemmas and proofs omitted from main text

Lemma 0 $P > P'(1 + r) > P'/\delta$

Proof of Lemma 0

The second inequality follows straightforwardly from rearranging (A1).

For the first, note that one-period and two-period loans must earn the same expected rate of return, that is, $E\{\min(K\widetilde{\theta}_2, P')\} = E\{\min(K\widetilde{\theta}_1\widetilde{\theta}_2, P)\}$ $/(1 + r) = 1 + r$.

Assume, for contradiction, that $P/(1+r) \leq P'$. Then conditional on any θ_2:

either (i) $\min(K\theta_2, P') = P'$. Our model assumed $K\underline{\theta}\overline{\theta} < P$ so, for all θ_2,we have $K\theta_1\theta_2 < P$ for at least some θ_1, so $E\{\min(K\widetilde{\theta}_1\theta_2, P)\}/(1 + r) < P/(1 + r)$. So in this case, if $P/(1 + r) \leq P'$, the expected return on one-period loans is strictly higher than on two-period loans. Moreover, $E\{K\widetilde{\theta}_2\} = K(1+r) > (1+r) = E\{\min(K\widetilde{\theta}_2, P')\}$, so $K\theta_2 > P'$ for at least some θ_2. So case (i) applies for at least some θ_2.

or (ii) $\min(K\theta_2, P') = K\theta_2$. But $E\{\widetilde{\theta}_1\} = (1+r)$ so $E\{\min(K\widetilde{\theta}_1\theta_2, P)\}/(1+r) \leq K\theta_2$. So in this case, the expected return on one-period loans is at least as high as on two-period loans.

So we have a contradiction.□

Proof of Lemma 4

The payoff from deposit insurance equals the maximum of zero and the difference between the bank's borrowing and the value of its assets at $t = 2$, that is, $\max\{0, L_0^R((1-\widehat{m})M_{0,1} - M_{0,2}) + L_1((1-m) - M_{1,2})\}$, if old loans are financed with a haircut of $\widehat{m}$ in equity. Since $(1-m) - M_{1,2} \leq 0$, an additional new loan weakly reduces this value.

Assume $\widehat{m} < m_1$. If $\theta_2 = \underline{\theta}$, then $L_1((1-m) - M_{1,2}) = L_1((1-m) - \underline{M}_{1,2}) = 0$, so the insurer must pay $L_0^R((1-\widehat{m})M_{0,1} - \underline{M}_{0,2})$, which is strictly positive, and so also increasing in L_0^R. Moreover, if $\theta_2 > \underline{\theta}$ but $(1-\widehat{m})M_{0,1} - M_{0,2} > 0$, then a small increase in L_0^R will either leave the insurer's payment at 0 or increase it by $(1-\widehat{m})M_{0,1} - M_{0,2}$ per old loan. Finally, an additional old loan can never reduce the value of insurance, because if $(1-\widehat{m})M_{0,1} - M_{0,2} \leq 0$, then the value of insurance is 0, independent of the number of old loans. So the expected cost to the insurer, and therefore the expected benefit to shareholders from insurance, is strictly increasing in L_0^R.[92]□

Proof of Lemma 5

Since any draw of θ_1 or θ_2 equal to $\underline{\theta}$ gives the bank economic ownership of the collateral (because $K\underline{\theta}\overline{\theta} < P$), the maximum amount of riskless debt that an uninsured bank can issue is always $\delta\underline{\theta}$ times the current value of the collateral. If $\theta_1 = 1/\delta$, both new and old loans support $\delta K\underline{\theta} = 1 - m$ in debt. New loans have a market value of $E\{\min(K\widetilde{\theta}_2, P')\}/(1+r) = 1$, while old loans have a market value of $M_{0,1} = \delta E\{\min((K/\delta)\widetilde{\theta}_2, P)\}/(1+r)$.

[92] Adding old loans always strictly increases the value of insurance–it suffices that there is positive probability that $\theta_2 < \underline{\theta} + \epsilon$ for all $\epsilon > 0$. But adding new loans may only weakly decrease the value of insurance if there is a discrete distribution of $\widetilde{\theta}_2$–the reason is that insurance does not affect the value of new loans when $\theta_2 = \underline{\theta}$ and, with a discrete distribution of $\widetilde{\theta}_2$, the marginal insurance cost of a new loan may be zero if the second-lowest state of θ_2 and/or N_1 is high enough that the insurance won't pay in the second-lowest state of θ_2.

But from Lemma 0, $\delta P > P'$, so $M_{0,1} > 1$. (The inequality is strict because (i) $K\theta_2 < P'$ for at least some θ_2, by the assumption of our model that $K\underline{\theta} < 1$, and (ii) $E\{\min(K\widetilde{\theta}_2, P')\}/(1+r) = 1$, but also $E\{K\widetilde{\theta}_2\}/(1+r) = K > 1$, so $K\theta_2 > P'$ for at least some θ_2.) Therefore if $\theta_1 = 1/\delta$, then $m_1 = 1 - \delta K\underline{\theta}/M_{0,1} > 1 - \delta K\underline{\theta} = m$. Furthermore, lower values of θ_1 proportionally reduce the debt an old loan can support, proportionally reduce its market value if θ_1 is low enough that the bank is already *de facto* owner of the collateral, and strictly less than proportionally reduce its market value otherwise), and so weakly increase the capital requirement, m_1, on the old loans. And $M_{0,1}$ is strictly increasing in θ_1, since $K\theta_1\underline{\theta} < K\underline{\theta}\overline{\theta} < P$. So if $M_{0,1} \leq 1$, then $m_1 > m$ and also $\theta_1 < 1/\delta$.□

Lemma 6 *If $M_{0,1} \leq 1$ a bank may make new loans financed with m in equity and not old loans financed with m_1 in equity, but not the opposite.*

Proof of Lemma 6

For each uninsured new loan the bank adds at $t = 1$, it will receive a gross return of $\delta \min\{K\theta_2, P'\}$ against which it will have debt of $\delta\underline{\theta}K$. So each uninsured new loan returns $\min\{K\theta_2, P'\}/\underline{\theta}K$ per dollar of debt supported. Each uninsured old loan returns $\delta^2 \min\{K\theta_1\theta_2, P\}$ and supports debt of $\delta^2\theta_1\underline{\theta}K$, so returns $\min\{K\theta_1\theta_2, P\}/\theta_1\underline{\theta}K$ per dollar of debt supported. The proof of Lemma 5 shows that if $M_{0,1} \leq 1$ then $1/\theta_1 > \delta$, so by Lemma 0 $P/\theta_1 > P'$, so the gross return from adding enough old loans to support a dollar of debt weakly exceeds the gross return from adding new loans to support a dollar of debt. But this means that, per dollar of debt, for the old loans (i) a weakly greater amount will be available to the deposit insurer to offset any losses on the insured loans, and (ii) more equity is being used. Since the bank loses $1 - \delta$ in expected present value per dollar of loans financed with equity (it earns a positive expected present value return of $\delta - (1/(1+r))$ per dollar of loans financed with debt), both (i) and (ii) make incremental old loans less profitable (i.e, they have lower net returns), per dollar of debt, than incremental new loans. So the net returns (per dollar of debt, or per loan, or per dollar of equity) may be positive for the new loans and negative for the old loans but not the other way around.□

Proof of Proposition 3a

The regulatory capital that the bank has to have to retain an old loan at $t = 1$ is $mB_{0,1}$, so the amount that it is permitted to borrow against the loan is $(1-m)B_{0,1}$, or $(1-m)(B_{0,1}/M_{0,1})$ per dollar of its market value, which equals or exceeds $(1-m)$ since $B_{0,1} = (1-\alpha_1+\alpha_1 M_{0,1}) \geq M_{0,1}$ when $\alpha_1 \leq 1$.[93]$\square$

Proof of Proposition 3b

Since all assets earn the same expected rate of return but old loans require weakly less equity (Proposition 3a), old loans earn a weakly higher expected rate of return than new loans ignoring insurance effects. Lemma 4 shows that insurance effects strictly increase the difference in expected rate of return..$\square$

Proof of Proposition 3c

Since all assets earn the same expected rate of return but old loans require weakly less equity (Proposition 3a), old loans earn a weakly higher expected rate of return than new loans ignoring insurance effects. Since new loans are profitable on a stand-alone basis (assumption A1), and the insurance effects of adding old loans always benefit the bank (Lemma 4), the fully-insured bank will always retain all its old loans, that is, $L_0^R = N_0$. From Lemma 3, either $L_1 = 0$ or $L_1 = N_1$.

Consider a θ_1 and a distribution of $\widetilde{\theta}_2$ such that there is positive probability that $\theta_2 > \underline{\theta}$ and $M_{0,2}(\theta_2) < (1-m)B_{0,1}$. Then, for these θ_2, the old loans lose more than the market value of the equity that needs to be held against them, so the deposit insurer would benefit from the (strictly positive) returns on any new loans. If $rm = \delta(1+r)-1$ the bank's expected returns from the new loans, ignoring the effects of deposit insurance, would equal its cost of capital. So if r is sufficiently close to (but, to satisfy (A1), greater than) $(1-\delta)/(\delta-m)$, the bank chooses $L_1 = 0$.

On the other hand, the new loans' expected returns strictly exceed the cost of capital, and the maximum loss per old loan is bounded, so if N_0 is sufficiently small relative to N_1, the bank chooses $L_1 = N_1$.$\square$

Proof of Proposition 4

First, note that if $\underline{B}_{0,2} < B_{0,1}(1-m)$ and $\overline{D} \geq L_0 \left[\frac{(1-m)(B_{0,1}\underline{B}_{1,2}-\underline{B}_{0,2})}{\underline{B}_{1,2}-(1-m)}\right]$, then $\overline{D} > L_0 B_{0,1}(1-m)$ (because $\underline{B}_{0,2} < B_{0,1}(1-m) \Longrightarrow B_{0,1}\underline{B}_{1,2} - \underline{B}_{0,2} >$

[93] The fully-insured bank can create *strictly* more riskless debt per dollar of market value of old loans than per dollar of market value of new loans if $M_{0,1} < 1$ and $\alpha_1 < 1$

$B_{0,1}\underline{B}_{1,2} - (1-m)B_{0,1} \Longrightarrow \frac{(1-m)(B_{0,1}\underline{B}_{1,2}-\underline{B}_{0,2})}{\underline{B}_{1,2}-(1-m)} > (1-m)B_{0,1})$. So there is enough insured debt to finance all old loans.

If $L_1(1-m) \leq \left(\overline{D} - L_0B_{0,1}(1-m)\right)$, the bank can also finance all new loans with insured debt, so the bank is only bound by its RCR.

If, instead, $L_1(1-m) \geq \left(\overline{D} - L_0B_{0,1}(1-m)\right) > 0$ then the bank can finance $\left(\overline{D} - L_0B_{0,1}(1-m)\right)/(1-m)$ new loans with insured debt, and each would generate $\underline{B}_{1,2} - (1-m)$ in net regulatory capital, creating a total of $\frac{\left(\underline{B}_{1,2}-(1-m)\right)\left(\overline{D}-L_0B_{0,1}(1-m)\right)}{1-m}$, if $\theta_2 = \underline{\theta}$. So if $\frac{\left(\underline{B}_{1,2}-(1-m)\right)\left(\overline{D}-L_0B_{0,1}(1-m)\right)}{1-m} + L_0(\underline{B}_{0,2} - (1-m)B_{0,1}) \geq 0$, that is, $\overline{D} \geq L_0\left[\frac{(1-m)(B_{0,1}\underline{B}_{1,2}-\underline{B}_{0,2})}{\underline{B}_{1,2}-(1-m)}\right]$, then the bank will be assured of having non-negative regulatory capital at $t = 2$ from its assets backed by insured debt. Furthermore, if apart from assets backed by insured debt the bank only has new loans, and if the regulatory requirement for those uninsured new loans (limiting the bank to $1-m$ of debt per loan) is met, the PLC will also be met, since the uninsured loans can be sold for at least $\underline{M}_{0,2} = 1-m$ at $t = 2$. So if $\overline{D} \geq L_0\left[\frac{(1-m)(B_{0,1}\underline{B}_{1,2}-\underline{B}_{0,2})}{\underline{B}_{1,2}-(1-m)}\right]$, the bank is only bound by its RCR.□

Proof of Proposition 5

For a fully insured bank only the RCR, binds, since $U_1 = 0$. A bank that is not fully insured must also meet the PLC, when it chooses $U_1 = 0$. So its optimization problem will be the same as the fully-insured bank if (PLC) is satisfied whenever (RCR) is satisfied.

If $\overline{D} \leq L_0(1-m)B_{0,1}$, then $L_1 = L_1^U$, so when the RCR and PLC both hold with equality we can write (RCR), (1) and (PLC) as

$$(1-m)(B_{0,1}L_0^R + L_1) = \overline{D} + U_1 \tag{a1}$$

$$L_1\underline{M}_{1,2} + L_0^U\underline{M}_{0,2} = U_1 \tag{a2}$$

$$(L_0^R - L_0^U)\underline{B}_{0,2} = \overline{D} \tag{a3}$$

Rewriting (a1) as

$$\overline{D} + (U_1 - L_1(1-m)) = (L_0^R - L_0^U)(1-m)B_{0,1} + L_0^U(1-m)B_{0,1} \tag{a4}$$

and substituting $(L_0^R - L_0^U)(1-m)B_{0,1} = \overline{D}\frac{(1-m)B_{0,1}}{\underline{B}_{0,2}}$ from (a3), $L_0^U(1-m)B_{0,1} = (U_1 - L_1\underline{M}_{1,2})\frac{(1-m)B_{0,1}}{\underline{M}_{0,2}}$ from (a2), and $\underline{M}_{1,2} = (1-m)$ into (a4), gives

$$\overline{D} + (U_1 - L_1(1-m)) = \overline{D}\frac{(1-m)B_{0,1}}{\underline{B}_{0,2}} + (U_1 - L_1(1-m))\frac{(1-m)B_{0,1}}{\underline{M}_{0,2}}$$

which simplifies to $U_1 = \overline{D}\frac{\underline{M}_{0,2}(\underline{B}_{0,2}-(1-m)B_{0,1})}{\underline{B}_{0,2}((1-m)B_{0,1}-\underline{M}_{0,2})} + L_1(1-m) = \overline{V} + L_1(1-m)$, with $\overline{V} \geq 0$, since we assumed $\underline{B}_{0,2} - (1-m)B_{0,1} \geq 0$.

So if all the old loans can be financed with a debt of exactly $\overline{D} + \overline{V}$, satisfying (RCR) exactly satisfies (PLC).

It is straightforward that if the old loans can be financed with an amount of debt $\in [\overline{D},\ \overline{D} + \overline{V})$, the same amount of "excess regulatory capital" is created by the old loans financed with insured debt (at the regulatory capital rate) but less is eaten up by the old loans financed with uninsured debt (at the regulatory capital rate) so there is then slack in (PLC). Moreover, if all the old loans can be financed with an amount of debt less than $\overline{D}$, there is still positive "excess regulatory capital" created by (all) the loans financed with insured debt (at the regulatory capital rate), and none of it is eaten up by the (new) loans financed with uninsured debt (at the regulatory capital rate) so there is again slack in (PLC).

So if $L_0 B_{0,1}(1-m) \leq \overline{D} + \overline{V}$ and $\underline{B}_{0,2} \geq (1-m)B_{0,1}$ a partly insured bank is constrained only by (RCR) so behaves exactly as a fully insured bank.□

Proof of Proposition 6

The bank will not choose $L_0^R = L_1 = 0$ since, by Proposition 2c, even an uninsured bank would make some loans. So assume, for contradiction, it makes only new loans. Assume it funds these loans with $D_1 \leq \overline{D}$ of insured debt (and possibly some uninsured debt). Then in the worst $t = 2$ state it will have $(\underline{B}_{1,2} - (1-m))D_1/(1-m)$ of regulatory capital, which is strictly positive if $\alpha_2 < 1$. So the bank could instead use some insured debt to finance old loans at the regulatory capital rate without violating its PLC, and without changing the number of new loans financed. Displacing new loans to uninsured debt (if necessary) without violating the PLC has no insurance effects, since all new loans can be sold at $t = 2$ for at least the debt supporting them, however they are financed. The additional old loans would increase the bank's profits, because old loans financed at the regulatory capital rate are always profitable. So the bank will always choose $L_0^R > 0$.□