All Appendices are Intended for Online Publication

Appendix A Proofs

Proof of Lemma 1. Using (2) we obtain

$$\begin{aligned}\frac{P_{t,t}}{\sum_{s\leq t} P_{t,s}} &= \frac{\pi_{t,t}}{\sum_{s\leq t}\left(\frac{h(t-s)}{h(0)}\right)\pi_{t,s}} = \frac{p_{t,t}y_{t,t} - w_t l_{t,t}}{\sum_{s\leq t}\left(\frac{h(t-s)}{h(0)}\right)(p_{t,s}y_{t,s} - w_t l_{t,s})} \\ &= \left(\frac{l_{t,t}}{\sum_{s\leq t} l_{t,s}}\right) \times \frac{\left(\frac{p_{t,t}y_{t,t}}{l_{t,t}} - w_t\right)}{\sum_{s\leq t}\left(\frac{h(t-s)}{h(0)}\right)\left(\frac{l_{t,s}}{\sum_{s\leq t} l_{t,s}}\right)\left(\frac{p_{t,s}y_{t,s}}{l_{t,s}} - w_t\right)}\end{aligned}$$

Taking logarithms and then first differences on both sides

$$\begin{aligned}\Delta \log\left(\frac{P_{t,t}}{\sum_{s\leq t} P_{t,s}}\right) - \Delta\log\left(\frac{l_{t,t}}{\sum_{s\leq t} l_{t,s}}\right) &= \Delta \log \frac{\frac{p_{t,t}y_{t,t}}{l_{t,t}} - w_t}{\left(\sum_{s\leq t}\omega\left(t-s\right)\left(\frac{p_{t,s}y_{t,s}}{l_{t,s}}\right)\right) - w_t} \\ &= \Delta\log\left(\frac{\delta_{t,t} - 1}{\sum_{s\leq t}\omega_{t,s}\delta_{t,s} - 1}\right)\end{aligned}$$

■

Proof of Proposition 1. Multiplying (15) by k_{it}, (16) by l_{it} and adding the resulting equations implies that

$$\pi\left(Z_{i,t}\right) \equiv p_{it}y_{it} - w_t\left(l_{i,t} + \bar{l}\right) - r_t^K k_{it} = (1-\xi)\, p_{i,t}y_{i,t} - w_t\bar{l}.$$

In turn, (16) implies

$$\pi\left(Z_{i,t}\right) = (1-\xi)\, p_{u,t}y_{u,t} - w_t\bar{l} = \frac{1-\xi}{\xi}\frac{w_t l_{it}}{\alpha} - w_t\bar{l}.$$

In steady state $\varphi_t = \varphi$, $w_t = w$ and $r_t^K = r^K = \rho + \delta$. Using these constants, equation (20) and the definition of Ξ gives

$$\pi\left(z_{i,t}\right) = \frac{1-\xi}{\xi}\frac{w l_{it}}{\alpha} - w\bar{l} = w\left(\Xi e^{\frac{\xi}{1-\xi}z} - \bar{l}.\right) \quad \text{(A.1)}$$

Before its termination, the value function of the firm solves the following differential equation:

$$V_{zz}\frac{\sigma^2}{2}+V_z\left(\mu-\frac{\sigma^2}{2}\right)-(r+\lambda)V+\pi\left(z_{i,t}\right)=0. \tag{A.2}$$

A particular solution V^P of this differential equation is

$$V^P(z)=\frac{2}{\sigma^2}\frac{1}{\omega_2-\omega_1}\left(\int_{z^*}^{z}e^{\omega_1(z-s)}\pi(s)\,ds+\int_{z}^{\infty}e^{\omega_2(z-s)}\pi(s)\,ds\right), \tag{A.3}$$

which can be verified by substituting (A.3) into (A.2). As a result, the general solution of (A.2) is

$$V(z)=D_1e^{\omega_1 z}+D_2e^{\omega_2 z}+V^P(z). \tag{A.4}$$

By standard arguments (value matching, smooth pasting, no bubble condition) we have that

$$V(z^*) = 0 \tag{A.5}$$

$$V_z(z^*) = 0 \tag{A.6}$$

$$\lim_{z\to\infty}V(z) = V^P(z) \tag{A.7}$$

Condition (A.7) implies that $D_2=0$ and condition (A.5) implies that

$$D_1=-e^{-\omega_1 z^*}\frac{2}{\sigma^2}\frac{1}{\omega_2-\omega_1}\left(\int_{z^*}^{\infty}e^{\omega_2(z^*-s)}\pi(s)\,ds\right). \tag{A.8}$$

Differentiating (A.4) with respect to z, evaluating the resulting expression at z^* and using (A.6) gives

$$\omega_1D_1e^{\omega_1 z^*}+\frac{2}{\sigma^2}\frac{\omega_2}{\omega_2-\omega_1}\int_{z^*}^{\infty}e^{\omega_2(z-s)}\pi(s)\,ds=0. \tag{A.9}$$

Using (A.8) inside (A.9) and re-arranging implies that

$$\int_{z^*}^{\infty}e^{-\omega_2 s}\pi(s)\,ds=0. \tag{A.10}$$

Substituting (A.1) into (A.10) and integrating leads after some simplifications to (22). ■

Proof of Proposition 2. Letting $g(z)$ denote the mass of firms with log-productivity z in the steady state, the forward Kolmogorov equation implies that the density $g(z)$ obeys the differential equation

$$\frac{\sigma^2}{2} g_{zz} - \left(\mu - \frac{\sigma^2}{2}\right) g_z - \lambda g + \phi m(z) = 0 \tag{A.11}$$

subject to the boundary condition $g(z^*) = 0$ and $\lim_{z\to\infty} g(z) = 0$. Similar to the proof of Proposition 1, a particular solution of (A.11) is

$$g^P(z) = \frac{2}{\sigma^2} \frac{\phi}{\eta_2 - \eta_1} \left(\int_{z^*}^{z} e^{\eta_1 (z-s)} m(s)\, ds + \int_{z}^{\infty} e^{\eta_2 (z-s)} m(s)\, ds \right). \tag{A.12}$$

The general solution is therefore

$$g(z) = K_1 e^{\eta_1 z} + K_2 e^{\eta_2 z} + g^P(z) \tag{A.13}$$

The two boundary conditions $g(z^*) = 0$ and $\lim_{z\to\infty} g(z) = 0$ imply that $K_2 = 0$ and

$$K_1 = \frac{-g^P(z^*)}{e^{\eta_1 z^*}}. \tag{A.14}$$

Substituting (A.14) and $K_2 = 0$ into (A.13) leads to

$$g(z; z^*) = g^P(z) - g^P(z^*)\, e^{\eta_1 (z - z^*)}, \tag{A.15}$$

which leads to (23). ■

Proof of Lemma 2. The left hand side of (24) is an increasing function of $\frac{\varphi}{w}$. The right hand side is decreasing in $\frac{\varphi}{w}$, which can be shown by differentiating the right hand side of (24) with respect to z^* and then using the fact that z^* is equal to $\log\left(\frac{\varphi}{w}\right)$ plus an additive constant, by (22). Specifically, using the fact that $g(z^*) = 0$, we have that

$$\frac{d}{dz^*} \int_{z^*}^{\infty} g(z; z^*)\, dz = \int_{z^*}^{\infty} \frac{\partial g(z; z^*)}{\partial z^*} dz =$$

$$= -\left(\int_{z^*}^{\infty} e^{\eta_2 (z^* - s)} m(s)\, ds \right) \left(\int_{z^*}^{\infty} e^{\eta_1 (z - z^*)} dx \right) \frac{2\phi}{\sigma^2} < 0 \tag{A.16}$$

and

$$\frac{d}{dz^*} \int_{z^*}^{\infty} \exp\left(\frac{\xi}{1-\xi} z\right) g(z; z^*)\, dz = \int_{z^*}^{\infty} \exp\left(\frac{\xi}{1-\xi} z\right) \frac{\partial g(z; z^*)}{\partial z^*} dz =$$

$$= -\frac{2\phi}{\sigma^2}\left(\int_{z^*}^{\infty} e^{\eta_2(z^*-s)} m(s)\, ds\right)\int_{z^*}^{\infty} e^{\eta_1(z-z^*)+\frac{\xi}{1-\xi}z} dz < 0 \tag{A.17}$$

Using (A.16) and (A.17) implies that the right hand side of (24) is increasing in z^*. Since z^* is decreasing in $\log\left(\frac{\varphi}{w}\right)$, the right-hand side of (24) is decreasing in $\frac{\varphi}{w}$. By inspection, the right hand side becomes strictly positive as $\frac{\varphi}{w} \to 0$. Combining all the above facts implies that the difference between the right and the left hand side of (24) is decreasing in $\frac{\varphi}{w}$, becomes positive as $\frac{\varphi}{w} \to 0$, and tends to negative infinity as $\frac{\varphi}{w} \to \infty$. By continuity, we conclude that there is a unique positive $\frac{\varphi}{w}$, for which equation (24) holds. ■

Proof of Proposition 3. To prove proposition 3, we start by proving the following Lemma, which shows the correspondence of the decentralized equilibrium with an (appropriately distorted) planning problem.

Lemma 3 *Assume that $\rho = 0$ and consider the optimization problem of maximizing H, where*

$$H \equiv \max_{l(z),k(z),x^*} Y - \frac{\delta}{\xi} K \tag{A.18}$$

subject to the constraint

$$\int_{x^*}^{\infty} g(z, x^*)\left(\bar{l} + l(z)\right) dz = L, \tag{A.19}$$

where

$$Y \equiv \left(\int_{x^*}^{\infty} g(z, x^*)\, y_i(z)^{\xi} dz\right)^{\frac{1}{\xi}}, \tag{A.20}$$

$y_i(z) = \exp(z)\, k(z)^{1-\alpha}\, l(z)^{\alpha}$ *and*

$$K \equiv \int_{x^*}^{\infty} g(z, x^*)\, k(z)\, dz$$

The optimization problem (A.18) has the same solution as the market equilibrium, namely $x^* = z^*, k^*(z) = k^{market}(z)$ and $l^*(z) = l^{market}(z)$, where $k^{market}(z)$ and $l^{market}(z)$ are the capital and labor values in the decentralized equilibrium, and stars indicate the optimal solution to the planning problem. Accordingly, $Y^* = Y^{market}$ and $C^* = C^{market}$. The maximized objective H is related to Y^* and C^* as follows

$$H = \alpha Y^* \text{ and } C^* = \frac{1-\xi(1-\alpha)}{\alpha} H. \tag{A.21}$$

Proof of Lemma 3. Maximizing (A.18) over $k(z)$ gives

$$(1-\alpha) Y^{1-\xi} y(z)^{\xi-1} \exp(z) \left(\frac{k(z)}{l(z)}\right)^{-\alpha} = \frac{\delta}{\xi}, \tag{A.22}$$

while maximizing over $l(z)$ gives

$$\alpha Y^{1-\xi} y(z)^{\xi-1} \exp(z) \left(\frac{k(z)}{l(z)}\right)^{1-\alpha} = \zeta, \tag{A.23}$$

where ζ is a Lagrange multiplier associated with the constraint (A.19). Maximizing over x^* leads to[27]

$$\int_{x^*}^{\infty} \frac{\partial}{\partial x^*} g(z, x^*) \left[\frac{1}{\xi} Y^{1-\xi} (y(z))^{\xi} dz - \frac{\delta}{\xi} k(z) - \zeta \left(l(z) + \bar{l}\right)\right] dz = 0 \tag{A.24}$$

Using (23) and differentiating implies

$$\frac{\partial}{\partial x^*} g(z; x^*) = -\left(\int_{x^*}^{\infty} e^{\eta_2(x^*-s)} m(s)\, ds\right) \frac{2\phi}{\sigma^2} \left[e^{\eta_1(z-x^*)}\right]. \tag{A.25}$$

Using (A.25) inside (A.24) and noting that $g(x^*; x^*) = 0$, leads after some simplifications to

$$\int_{x^*}^{\infty} e^{\eta_1 z} \left[\frac{1}{\xi} Y^{1-\xi} y(z)^{\xi} dz - \frac{\delta}{\xi} k(z) - \zeta \left(l(z) + \bar{l}\right)\right] dz = 0. \tag{A.26}$$

We next argue that $Y = Y^{\text{market}}, l(z) = l^{\text{market}}(z), k(z) = k^{\text{market}}(z)$, $x^* = z^*$ and $\zeta = \frac{w}{\xi}$ satisfies (A.22), (A.23) and (A.26). Indeed, using equation (11), equation (A.22) coincides with (15) when $\rho = 0$ (since $r_K = \delta$). Similarly, (A.23) coincides with (16) since $\zeta = \frac{w}{\xi}$. The expression inside square brackets in (A.26) is equal to the profits, $\frac{1}{\xi}\pi(z)$, in the decentralized market. When $\rho = 0$, we have that $\eta_1 = -\omega_2$ and hence (A.26) coincides with $\int_{x^*}^{\infty} e^{\eta_1 z} \pi(z)\, dz = 0$, i.e., the optimality condition for z^* (equation (A.10)). Moreover, since the market allocation is feasible (it clears the labor market), $\zeta = \frac{w}{\xi}$ is the Lagrange multiplier associated with the constraint (A.19).

Multiplying both sides of (A.22) with $k(z)$, and aggregating across firms gives

$$\frac{\delta}{\xi} K = (1-\alpha) Y^*, \tag{A.27}$$

[27]Note that $g(x^*, x^*) = 0$.

which implies $H = \alpha Y^*$ and $C = Y^* - \delta K = Y^* - \xi \frac{\delta}{\xi} K = [1 - \xi(1-\alpha)] Y^* = \frac{1-\xi(1-\alpha)}{\alpha} H$. ■

Having established the equivalence between the market equilibrium and the planning problem (A.18), we use the simpler notation $Y = Y^*, C = C^*$, etc. to refer to the output, consumption, etc. that arise in market equilibrium (equivalently, at an optimum of (A.18)).

We start by determining the partial derivatives of the objective H, which captures the steady-state value of $Y - \frac{\delta}{\xi} K$ (equation (A.18)) with respect to various parameters.. Letting $z^c = \log(Z^c)$, $y(z) = \exp(z^c + z) k^{1-\alpha}(z) l(z)$, and applying the envelope theorem (around $z^c = 0$) to (A.18) gives

$$\frac{\partial H}{\partial z^c} = Y.$$

Similarly, using the definition

$$\widetilde{g}(z) \equiv \frac{g(z, x^*) e^{\xi \log y(z)}}{\int_{x^*}^{\infty} g(z, x^*) e^{\xi \log y(z)} dz},$$

and equation (A.27) yields

$$\frac{\partial H}{\partial \xi} = \frac{1}{\xi} Y \left[\int_{x^*}^{\infty} \widetilde{g}(z) \log y(z) \, dz - \log(Y) + (1-\alpha) \right]. \tag{A.28}$$

Similarly,

$$\frac{\partial H}{\partial \alpha} = Y \log\left(\frac{l(z)}{k(z)} \right). \tag{A.29}$$

With these partial derivatives and using (A.21) and $H = \alpha Y$ we next proceed to compute dC. To that end we compute $\frac{\partial\left(\frac{1-\xi(1-\alpha)}{\alpha}\right)}{\partial \xi} = -\frac{1-\alpha}{\alpha}$, and $\frac{\partial\left(\frac{1-\xi(1-\alpha)}{\alpha}\right)}{\partial \alpha} = -\frac{1-\xi}{\alpha^2}$ and hence

$$\begin{aligned} \frac{dC}{C} &= \left[-\frac{1-\xi}{\alpha} \frac{1}{1-\xi(1-\alpha)} + \frac{1}{\alpha} \log\left(\frac{l(z)}{k(z)} \right) \right] d\alpha + \frac{1}{\alpha} dz^c \\ &\quad + \left\{ -\frac{1-\alpha}{1-\xi(1-\alpha)} + \frac{1}{\alpha\xi} \left[\int_{x^*}^{\infty} \widetilde{g}(z) \log y(z) \, dz - \log(Y) + (1-\alpha) \right] \right\} d\xi. \end{aligned} \tag{A.30}$$

We next show how to relate the steady-state change in $\frac{dC}{C}$ to the change in the revenue produced by the incoming cohort of firms at the beginning of the transition to the new steady state. We start by assuming that arriving firms have a coefficient $\xi = \xi^{\text{old}} + d\xi$, where ξ^{old} pertains to the old firms and $d\xi$ is a marginal change. Similarly, we assume that the new firms' productivity is multiplied by e^{dz^c} for a marginal change in dz^c, and their labor share is $\alpha = \alpha^{\text{old}} + d\alpha$.

The production function from the onset of the transition onward is given by

$$1 = \int_{i \in I_{\text{old}}} \left(\frac{y_i^{\text{old}}}{Y} \right)^{\xi^{\text{old}}} di + \int_{i \in I_{\text{new}}} \left(\frac{y_i^{\text{new}}}{Y} \right)^{\xi} di.$$

Using the implicit function theorem and (10) leads to

$$p_{it}^{\text{old}} = \frac{\xi^{\text{old}} \left(\frac{y_{it}^{\text{old}}}{Y_t} \right)^{\xi^{\text{old}}-1}}{\int_{i \in I_{\text{old}}} \xi^{\text{old}} \left(\frac{y_{it}}{Y_t} \right)^{\xi_{\text{old}}} di + \int_{i \in I_{\text{new}}} \xi \left(\frac{y_{it}}{Y_t} \right)^{\xi} di}, \tag{A.31}$$

$$p_{it}^{\text{new}} = \frac{\xi \left(\frac{y_{it}^{\text{new}}}{Y_t} \right)^{\xi-1}}{\int_{i \in I_{\text{old}}} \xi^{\text{old}} \left(\frac{y_{it}}{Y_t} \right)^{\xi_{\text{old}}} di + \int_{i \in I_{\text{new}}} \xi \left(\frac{y_{it}}{Y_t} \right)^{\xi} di}. \tag{A.32}$$

Note that at the onset of the transition the second integral in the denominator of (A.31) and (A.32) is zero, since the total measure of arriving firms with coefficient $\xi = \xi^{\text{old}} + d\xi$ are of zero measure. In the long run, the first integral becomes measure zero, and (A.31) becomes identical to (11).

While of measure zero at the beginning of the transition, the percentage change in the revenue of incoming firms normalized by the measure of these firms is well defined and given by

$$Y^{\text{new}} \equiv \int_{z^*}^{\infty} m(z) \, p^{\text{new}}(z) \, y^{\text{new}}(z) \, dz,$$

where p_{it}^{new} is shorthand notation for $p_{it}^{\text{new}}\left(z; z + z^c, \xi^{\text{old}} + d\xi, \alpha^{\text{old}} + d\alpha\right)$ and similarly for $y^{\text{new}}(z)$. Multiplying both sides of (A.32) by $y^{\text{new}}(z)$, aggregating across entering firms (and noting that at the onset of the transition the measure of firms employing the new technologies is of measure zero) gives

$$\begin{aligned} d \log Y^{\text{new}} = & \left(\frac{1}{\xi} - \log Y + \left(\int_{z^*}^{\infty} \widetilde{m}(z) \log y^{\text{new}}(z) \, dz \right) \right) d\xi \\ & + \xi \left(dz^c + \log \left(\frac{l^{\text{new}}}{k^{\text{new}}} \right) d\alpha \right) \\ & + \xi \int_{z^*}^{\infty} \widetilde{m}(z) \left(\alpha d \log l^{\text{new}}(z) + (1-\alpha) \, d \log k^{\text{new}}(z) \right) dz \end{aligned} \tag{A.33}$$

where $\widetilde{m}(z)$ is defined as

$$\widetilde{m}(z) = \frac{m(z)\, e^{\xi \log y^{\text{new}}}}{\int_{z^*}^{\infty} m(z)\, e^{\xi \log y^{\text{new}}} dz} = \frac{m(z)\, p^{\text{new}}(z)\, y^{\text{new}}(z)}{\int_{z^*}^{\infty} m(z)\, p^{\text{new}}(z)\, y^{\text{new}}(z)\, dz}, \tag{A.34}$$

where the second equality follows from equation (A.32).

The first order conditions for labor and capital (16) and (15) give $w_t l^{\text{new}}(z) = \alpha \xi p^{\text{new}}(z)\, y^{\text{new}}(z)$ and $r_t^K k^{\text{new}}(z) = (1-\alpha)\, \xi p^{\text{new}}(z)\, y^{\text{new}}(z)$ and therefore

$$d \log l^{\text{new}}(z) = \frac{d\alpha}{\alpha} + \frac{d\xi}{\xi} + \frac{d\left[p^{\text{new}}(z)\, y^{\text{new}}(z)\right]}{p^{\text{new}}(z)\, y^{\text{new}}(z)}, \tag{A.35}$$

and

$$d \log k^{\text{new}}(z) = -\frac{d\alpha}{(1-\alpha)} + \frac{d\xi}{\xi} + \frac{d\left[p^{\text{new}}(z)\, y^{\text{new}}(z)\right]}{p^{\text{new}}(z)\, y^{\text{new}}(z)}. \tag{A.36}$$

Combining (A.35) and (A.36) gives

$$\alpha d \log l^{\text{new}} + (1-\alpha)\, d \log k^{\text{new}} = \frac{d\xi}{\xi} + \frac{d\left[p^{\text{new}}(z)\, y^{\text{new}}(z)\right]}{p^{\text{new}}(z)\, y^{\text{new}}(z)}. \tag{A.37}$$

Combining (A.33) with (A.37) gives

$$\begin{aligned} \frac{dY^{\text{new}}}{Y^{\text{new}}} &= \left(1 + \frac{1}{\xi} - \log Y + \left(\int_{z^*}^{\infty} \widetilde{m}(z) \log y^{\text{new}}(z)\, dz\right)\right) d\xi \\ &\quad + \xi \left(dz^c + \log\left(\frac{l^{\text{new}}}{k^{\text{new}}}\right) d\alpha\right) + \xi \frac{dY^{\text{new}}}{Y^{\text{new}}}, \end{aligned} \tag{A.38}$$

or after re-arranging

$$\left(\frac{1-\xi}{\xi}\right) \frac{dY^{\text{new}}}{Y^{\text{new}}} = \left(1 + \frac{1}{\xi} - \log Y + \int_{z^*}^{\infty} \widetilde{m}(z) \log y^{\text{new}} dz\right) \frac{d\xi}{\xi} + dz^c + \log\left(\frac{l^{\text{new}}}{k^{\text{new}}}\right) d\alpha. \tag{A.39}$$

Solving for dz^c in (A.39), substituting the result inside (A.30) and evaluating around $z^c = 0, \xi = \xi^*, \alpha = \alpha^*$, and noting that the labor-capital ratio is equalized across all firms when $z^c = 0, \xi = \xi^*, \alpha = \alpha^*$ leads to

$$\frac{dC}{C} = \frac{1}{\alpha} \frac{1-\xi}{\xi} \frac{dY^{\text{new}}}{Y^{\text{new}}} - \left(\frac{1-\xi}{1-\xi(1-\alpha)}\right) \frac{d\alpha}{\alpha} \tag{A.40}$$

$$+ \frac{1}{\alpha}\left(\int_{x^*}^{\infty} \widetilde{g}(z) \log y(z)\, dz - \int_{x^*}^{\infty} \widetilde{m}(z) \log y(z)\, dz - D\right) \frac{d\xi}{\xi} \tag{A.41}$$

where $D \equiv \left(1+\frac{1}{\xi}\right)+\alpha\frac{\xi(1-\alpha)}{1-\xi(1-\alpha)}-(1-\alpha)$. ■

Proof of Proposition 4. To start, we prove the next Lemma, which extends Lemma 3 to a multi-sector setup.

Lemma 4 *Assume that $\rho = 0$ and consider the optimization problem of maximizing H, where*

$$\widehat{H} \equiv \max_{l^S(z),k^S(z),x^{S,*}} Y - \delta \sum_S \frac{1}{\xi^S} \int_{x^{S,*}}^{\infty} g^S\left(z, x^{S,*}\right) k^S(z)\, dz, \tag{A.42}$$

subject to the constraints

$$\int_{x^*}^{\infty} g\left(z^S, x^{S,*}\right)\left(\bar{l} + l^S(z)\right) dz = L^{S,market}, \tag{A.43}$$

where $L^{S,market}$ is the amount of labor employed in sector S in the market equilibrium. The production function for Y is given by

$$1 = \sum_S \int_{x^*}^{\infty} g\left(z^S, x^{S,*}\right)\left(\frac{y_i^S(z)}{Y}\right)^{\xi^S} dz, \tag{A.44}$$

with $y_i^S(z) = \exp(z)\, k^S(z)^{1-\alpha^S} l^S(z)^{\alpha^S}$. The optimization problem (A.42) has the same solution as the market equilibrium, namely $x^{S,*} = z^{S,*}, k^{S,*}(z) = k^S(z)$ and $l^{S,*}(z) = l^S(z)$, where $k^S(z)$ and $l^S(z)$ are the capital and labor values in the decentralized equilibrium, and stars indicate the optimal solution to the planning problem. Accordingly, $Y^* = Y$ and $C^* = C$.

Proof of Lemma 4. The proof is essentially the same as the proof of Lemma 3, so we only provide a sketch. The first-order conditions for capital by firm i in sector S is

$$\left(1-\alpha^S\right)\frac{\partial Y}{\partial y_i^S}\exp(z)\left(\frac{k^S(z)}{l^S(z)}\right)^{-\alpha} = \frac{\delta}{\xi^S}. \tag{A.45}$$

Letting ζ^S denote the Lagrange multiplier on the constraint (A.43), the respective first-order condition for labor is

$$\alpha^S \frac{\partial Y}{\partial y_i^S}\exp(z)\left(\frac{k^S(z)}{l^S(z)}\right)^{-\alpha} = \zeta^S. \tag{A.46}$$

Using the implicit function theorem in equation (A.44), the first-order condition for the termination cut-off, $x^{*,S}$, in sector S is

$$\frac{\partial Y}{\partial x^{*,S}} = \int_{x^{S,*}}^{\infty} \frac{\partial g^S\left(z, x^{S,*}\right)}{\partial x^{*,S}} \left[\frac{Y^{1-\xi^S}\left(y_i^S(z)\right)^{\xi^S}}{\sum_S \xi^S Y^{-\xi^S} \int_{i\in S}^{\infty} \left(y_i^S(z)\right)^{\xi^S} g^S\left(z, x^{S,*}\right) dz} - \frac{\delta}{\xi^S} k^S(z) + \zeta^S \left(\bar{l} + l^S(z)\right) \right] dz = 0, \tag{A.47}$$

In addition, implicit differentiation of equation (A.44) gives

$$\frac{\partial Y}{\partial y_i^S} = \frac{\xi^S Y^{1-\xi^S}\left(y_i^S(z)\right)^{\xi^S - 1}}{\sum_S \xi^S Y^{-\xi^S} \int_{i\in S}^{\infty} \left(y_i^S(z)\right)^{\xi^S} di},$$

and since $p_i^S = \frac{\partial Y}{\partial y_i^S}$, we can re-write (A.47)

$$\int_{x^{S,*}}^{\infty} \frac{\partial g^S\left(z, x^{S,*}\right)}{\partial x^{*,S}} \left[\frac{p_i(z)\, y_i(z)}{\xi^S} - \frac{\delta}{\xi^S} k^S(z) + \zeta_S \left(\bar{l} + l^S(z)\right) \right] dz = 0. \tag{A.48}$$

From this point onward, the argument is exactly the same as in the proof of Lemma 3. Multiplying both sides of (A.48) by ξ^S and setting $\zeta_S = \frac{w}{\xi^S}$, we recognize that the term inside square brackets in expression (A.48) captures profits, and the equations (A.45) and (A.46) are just the first order conditions for capital and labor in a market equilibrium. ■

Let w^S denote the revenue weight of sector S, defined as

$$\omega^S \equiv \frac{\int_{i\in S} p_i y_i di}{Y} = \frac{\xi^S Y^{-\xi^S} \int_{i\in S} \left(y_i^S(z)\right)^{\xi^S} di}{\sum_S \xi^S Y^{-\xi^S} \int_{i\in S}^{\infty} \left(y_i^S(z)\right)^{\xi^S} di}.$$

Using implicit differentiation on (A.44) and the envelope theorem (fixing a given labor allocation across sectors) gives

$$\begin{aligned} \frac{\partial \widehat{H}}{\partial z^{c,S}} &= \omega^S Y,\ \frac{\partial \widehat{H}}{\partial \alpha^S} = \ln\left(\frac{l^S}{k^S}\right)\omega^S Y, \text{ and} \\ \frac{\partial \widehat{H}}{\partial \xi^S} &= \frac{1}{\xi^S}\omega^S Y \left[\int_{x^{S,*}}^{\infty} \widetilde{g}^S\left(z, x^{S,*}\right) \ln y_i^S(z) dz - \ln Y - \left(1 - \alpha^S\right) \right] \end{aligned} \tag{A.49}$$

where $\widetilde{g}^S\left(z, x^{S,*}\right)$ are within-sector revenue weights defined as

$$\widetilde{g}^S\left(z, x^{S,*}\right) \equiv \frac{p_i y_i}{\int_{i\in S} p_i y_i di} = \frac{g^S\left(z, x^{S,*}\right)\left(y_i^S(z)\right)^{\xi^S}}{\int_{x^{S,*}}^{\infty} g^S\left(z, x^{S,*}\right)\left(y_i^S(z)\right)^{\xi^S} dz}.$$

Note that the partial derivatives of $\widehat{H}$ with respect to the various parameters are analogous to their counterparts in the single-sector economy, except for the presence of the sector weights ω^S. Using the relation $C = \widehat{H} - \delta \sum_S \left(1 - \frac{1}{\xi^S}\right) K^S$, totally differentiating C and using the Envelope theorem gives[28]

$$
\begin{aligned}
dC &= Y \sum_S \omega^S \left\{ dz^{c,S} + \ln\left(\frac{l^S}{k^S}\right) d\alpha^S + \frac{1}{\xi^S}\left(\int_{x^{S,*}}^{\infty} \widetilde{g}^S\left(z, x^{S,*}\right) \ln\left(\frac{y_i^S(z)}{Y}\right) dz + \left(1-\alpha^S\right)\right) d\xi^S \right\} \\
&\quad + \zeta^S \sum_S dL^S - \delta \sum_S \left(1 - \frac{1}{\xi^S}\right) dK^S - \sum_S \frac{\delta K^S}{\xi^S} \frac{d\xi^S}{\xi^S}.
\end{aligned} \tag{A.50}
$$

The proof of Lemma 4 shows that the Lagrange multipliers, ζ^S, associated with the constraints (A.43) obey the relation $\zeta_S = \frac{w}{\xi^S}$. Moreover, aggregating the first-order conditions for capital within a sector implies that $\xi^S\left(1-\alpha^S\right)\int_{i\in S} p_i y_i di = \delta K^S$. Accordingly, (A.50) simplifies to

$$
\begin{aligned}
dC &= Y \sum_S \omega^S \left\{ dz^{c,S} + \ln\left(\frac{l^S}{k^S}\right) d\alpha^S + \left(\int_{x^{S,*}}^{\infty} \widetilde{g}^S\left(z, x^{S,*}\right) \ln\left(\frac{y_i^S(z)}{Y}\right) dz\right) \frac{d\xi^S}{\xi^S} \right\} \\
&\quad + \sum_S \frac{1}{\xi^S}\left(w dL^S + \delta dK^S\right) - \delta K\left(\frac{dK}{K}\right).
\end{aligned}
$$

Let $\widehat{\sigma} \equiv \sum_S \omega^S \left(1-\alpha^S\right) \xi^S$. Aggregating the first-order conditions for capital across sectors gives $\delta K = \widehat{\sigma} Y$ and by implication $C = Y - \delta K = (1-\widehat{\sigma}) Y$. Therefore,

$$
\begin{aligned}
dC &= Y \sum_S \omega^S \left\{ dz^{c,S} + \ln\left(\frac{l^S}{k^S}\right) d\alpha^S + \left(\int_{x^{S,*}}^{\infty} \widetilde{g}^S\left(z, x^{S,*}\right) \ln\left(\frac{y_i^S(z)}{Y}\right) dz\right) \frac{d\xi^S}{\xi^S} \right\} \\
&\quad + Y \sum_S \left(\frac{wL^S}{\xi^S Y}\frac{dL^S}{L^S} + \omega^S\left(1-\alpha^S\right)\frac{dK^S}{K^S}\right) - \widehat{\sigma} Y\left(\frac{dK}{K}\right),
\end{aligned}
$$

[28] In applying the envelope theorem, we used the fact that the Lagrange multipliers are the solution to the min-max problem

$$
\min_{\zeta^S} \left\{ \begin{array}{c} \max_{l^S(z), k^S(z), x^{S,*}} Y - \delta \sum_S \frac{1}{\xi^S} \int_{x^{S,*}}^{\infty} g^S\left(z, x^{S,*}\right) k^S(z)\, dz \\ -\sum_S \left(\int_{x^*}^{\infty} g\left(z^S, x^{S,*}\right)\left(\bar{l} + l^S(z)\right) dz - L^S\right). \end{array} \right\}
$$

which implies

$$dC = Y\sum_S \omega^S \left\{dz^{c,S} + \ln\left(\frac{l^S}{k^S}\right)d\alpha^S + \left(\int_{x^{S,*}}^{\infty} \widetilde{g}^S\left(z, x^{S,*}\right)\ln\left(\frac{y_i^S(z)}{Y}\right)dz\right)\frac{d\xi^S}{\xi^S}\right\}$$
$$+Y\sum_S\left\{\frac{wL^S}{\xi^S Y}\frac{dL^S}{L^S} + \omega^S\left(1-\alpha^S\right)\left(\frac{dK^S}{K^S} - \frac{dK}{K}\right)\right\} + Y\left(1-\widehat{\alpha}-\widehat{\sigma}\right)\left(\frac{dK}{K}\right),$$

where $K = \sum_S K^S$, $\widehat{\alpha} \equiv \sum_S \omega^S\alpha^S$. Using $C = (1-\widehat{\sigma})Y$ and $\delta K = \widehat{\sigma}Y$ implies $\frac{C}{\delta K} = \frac{1-\widehat{\sigma}}{\widehat{\sigma}}$ and hence $\frac{dC}{C} - \frac{dK}{K} = -\frac{1}{1-\widehat{\sigma}}\frac{d\widehat{\sigma}}{\widehat{\sigma}}$ and therefore we obtain after some re-arranging

$$(1-\widehat{\sigma})\frac{dC}{C} = \sum_S \omega^S \left\{dz^{c,S} + \ln\left(\frac{l^S}{k^S}\right)d\alpha^S + \left(\int_{x^{S,*}}^{\infty} \widetilde{g}^S\left(z, x^{S,*}\right)\ln\left(\frac{y_i^S(z)}{Y}\right)dz\right)\frac{d\xi^S}{\xi^S}\right\}$$
$$+\sum_S\left\{\frac{wL^S}{\xi^S Y}\frac{dL^S}{L^S} + \omega^S\left(1-\alpha^S\right)\left(\frac{dK^S}{K^S} - \frac{dK}{K}\right)\right\} + \left(1-\widehat{\alpha}-\widehat{\sigma}\right)\left(\frac{dC}{C} + \frac{1}{(1-\widehat{\sigma})}\frac{d\widehat{\sigma}}{\widehat{\sigma}}\right)$$

Solving for $\frac{dC}{C}$ gives

$$\frac{dC}{C} = \sum_S \omega^S \left\{\frac{dz^{c,S}}{\widehat{\alpha}} + \log\left(\frac{l^S}{k^S}\right)\frac{d\alpha^S}{\widehat{\alpha}} + \left(\int_{x^{S,*}}^{\infty} \widetilde{g}^S\left(z, x^{S,*}\right)\ln\left(\frac{y_i^S(z)}{Y}\right)dz\right)\frac{d\xi^S}{\widehat{\alpha}\xi^S}\right\}$$
$$+\frac{1}{\widehat{\alpha}}\sum_S\left\{\frac{wL^S}{\xi^S Y}\frac{dL^S}{L^S} + \omega^S\left(1-\alpha^S\right)\left(\frac{dK^S}{K^S} - \frac{dK}{K}\right)\right\} + \frac{1}{\widehat{\alpha}}\frac{1-\widehat{\alpha}-\widehat{\sigma}}{1-\widehat{\sigma}}\frac{d\widehat{\sigma}}{\widehat{\sigma}}. \tag{A.51}$$

The remainder of the proof follows exactly the same steps as the proof of Proposition 29. Specifically, the following identity continues to be true

$$\left(\frac{1-\xi^S}{\xi^S}\right)\frac{dY^{S,\text{new}}}{Y^{S,\text{new}}} = \left(1 + \frac{1}{\xi^S} + \int_{z^{*,s}}^{\infty}\widetilde{m}^S(z)\log\left(\frac{y^{S,\text{new}}(z)}{Y}\right)dz\right)\frac{d\xi^S}{\xi^S} + dz^{c,S} + \log\left(\frac{l^{S,\text{new}}}{k^{S,\text{new}}}\right)d\alpha^S. \tag{A.52}$$

Combining (A.52) with (A.51) leads to (38). ■

Proof of Corollary 1. If $S = 1$ then $\omega^S = 1, \alpha^S = \alpha$ and $\xi^S = \xi$. This implies that that the term on the second line of (38) is zero since $\sum_S dL_S = dL = 0$ and $\frac{dK^S}{K^S} = \frac{dK}{K}$. Therefore to prove that (29) and (38) are identical, it suffices to show that

$$\frac{1}{\alpha}\frac{1-\alpha-\sigma}{1-\sigma}\frac{d\sigma}{\sigma} = -\left(\frac{1-\xi}{1-\xi(1-\alpha)}\right)\frac{d\alpha}{\alpha} - \left(\alpha\frac{\xi(1-\alpha)}{1-\xi(1-\alpha)} - (1-\alpha)\right)\frac{d\xi}{\alpha\xi} \tag{A.53}$$

To prove (A.53), note that when $S = 1$, $\sigma = (1-\alpha)\,\xi$. Therefore

$$\frac{d\sigma}{\sigma} = -\frac{\alpha\xi}{(1-\alpha)\,\xi}\frac{d\alpha}{\alpha} + \frac{d\xi}{\xi}. \tag{A.54}$$

Substituting $\sigma = (1-\alpha)\,\xi$ and (A.54) into the left-hand side (A.53) and re-arranging gives the right-hand side of (A.53). ■

Appendix B Numerical algorithm for computing the transition dynamics

In this section we provide a brief description of our numerical algorithm to solve for the transition path. The key difficulty preventing a closed from solution along the transition path is that both the wage, the interest rate and the threshold level of productivity that leads to endogenous bankruptcy (for the two different kinds of firms) are now functions of time rather than constants. To solve for the transition dynamics, we start with an initial guess for the productivity thresholds that trigger bankruptcy. With that guess in hand, we simulate an economy whereby new firms arrive each year, with idiosyncratic shocks that follow the dynamics (13). The number of new firms is in principle irrelevant for the model, with a higher number reducing simulation error at the expense of computational power. We verified the validity of Monte Carlo by checking (analytically) that for our chosen parameters the numerically integrated quantities posses finite moments.[29,30]

Using the cross-section of simulated productivities we determine the market clearing prices, wages and output. Fixing these time-series of prices, wages and output, we then use a binomial tree with 200 years and time increment $dt = 0.1$ to solve for the optimal termination thresholds for the pre- and post-transition firms separately. Using these optimal termination policies, we repeat the wage and output calculation and iterate to convergence (which typically takes two-three iterations of the algorithm).

[29]For our simulations we set the number of incoming firms to 10,000 per year and per computer processor We use parallel computing, repeat our calculations on a 12-processor parallel cluster (resulting in 120,000 draws in total) and report the average value.

[30]An alternative to Monte Carlo would be to numerically solve the forward Kolmogorov equations.

Appendix C Elastic Labor supply

Throughout the paper we maintained the assumption of inelastic labor supply for simplicity. Here we show how to extend the key results of the paper to the case where labor supply is elastic. Specifically, suppose that workers have a utility of the form

$$E_t \int_t^{\infty} e^{-\rho(s-t)} \left[u\left(c_s\right) + h\left(L^e - L_s\right)\right] ds,$$

where L^e is the household's endowment of hours, L denotes the hours supplied by the representative household, and $h\left(\cdot\right)$ is an increasing and concave function. The first-order condition for labor supply is

$$\frac{h'\left(L^e - L_t\right)}{u'\left(C_t\right)} = w_t. \tag{C.1}$$

To determine the steady-state labor, L – which equations (24) - (28) took as fixed – we start by noting that Lemma 2 implies that $\frac{\varphi}{w}$ is a function of L; by implication, equations (26) and (27) imply that $w = w(L)$ and $Y = Y\left(L\right)$ are functions of L. In a steady state we have that $C = Y - \delta K = Y\left(1 - \delta \frac{K}{Y}\right)$, where the capital-to-output ratio, $\frac{K}{Y} = \frac{\xi(1-\alpha)}{\rho+\delta}$, is a constant independent of L. Therefore, consumption, C, is proportional to output, and therefore $C = C(L)$. With these observations, the equilibrium L amounts to solving the labor-supply equation

$$\frac{h'\left(L^e - L\right)}{u'(Y(L)\left(1 - \delta \frac{K}{Y}\right))} = w\left(L\right)$$

for L.

We next prove the following generalization of Lemma 3.

Lemma 5 *Assume that $\rho = 0$ and consider the optimization problem of maximizing H, where*

$$H \equiv \max_{L,l(z),k(z),x^*} u\left(Y - \frac{\delta}{\xi} K\right) + bh(L^e - L) \tag{C.2}$$

subject to the constraints (A.19), and the definition (A.20), where

$$b \equiv \frac{u'\left(\frac{\alpha}{1-\xi(1-\alpha)}\right)}{\xi}.$$

The optimization problem (C.2) has the same solution as the market equilibrium, namely $L = L^*$ $x^* = z^*, k^*\left(z\right) = k^{market}\left(z\right)$ and $l^*\left(z\right) = l^{market}\left(z\right)$, where $k^{market}\left(z\right)$ and $l^{market}\left(z\right)$

are the capital and labor values in the decentralized equilibrium, and stars indicate the optimal solution to the planning problem. Accordingly, $Y^{*}=Y^{market}$ and $C^{*}=C^{market}$.

Proof of Lemma 5. We provide a sketch since the proof is essentially identical to the proof of Lemma 3. Equation (A.22) remains unchanged, while maximizing over $l(z)$ gives

$$\alpha Y^{1-\xi}y(z)^{\xi-1}\exp(z)\left(\frac{k(z)}{l(z)}\right)^{-\alpha}=\frac{\zeta}{u'\left(Y-\frac{\delta}{\xi}K\right)}, \tag{C.3}$$

where ζ is a Lagrange multiplier associated with the constraint (A.19). Maximizing over L gives

$$bh'(L-L^{e})=\zeta. \tag{C.4}$$

Maximizing over x^{*}leads to[31]

$$\int_{x^{*}}^{\infty}\frac{\partial}{\partial x^{*}}g(z,x^{*})\left[\frac{1}{\xi}Y^{1-\xi}(y(z))^{\xi}dz-\frac{\delta}{\xi}k(z)-\frac{\zeta}{u'\left(Y-\frac{\delta}{\xi}K\right)}\left(l(z)+\bar{l}\right)\right]dz=0 \tag{C.5}$$

From this point on one can repeat the same steps as in Lemma 3 in order to confirm that the market equilibrium corresponds to a planning optimum. The arguments are identical to Lemma 3, except that the relation between the Lagrange multiplier, ζ, and the wage is now $\frac{\zeta}{U'\left(Y-\frac{\delta}{\xi}K\right)}=\frac{w}{\xi}$. The only new step is to confirm that the market equilibrium L and the planner-chosen L coincide. To confirm this, use the relation $\frac{\zeta}{U'\left(Y-\frac{\delta}{\xi}K\right)}=\frac{w}{\xi}$ together with $\frac{\delta}{\xi}K=(1-\alpha)Y$, $C=Y-\delta K=Y-\xi\frac{\delta}{\xi}K=\ [1-\xi(1-\alpha)]Y$ and (C.4) to obtain

$$\frac{w}{\xi}=\frac{\zeta}{u'\left(Y-\frac{\delta}{\xi}K\right)}=\frac{bh'(L^{e}-L)}{u'(C)\,u'\left(\frac{\alpha}{1-\xi(1-\alpha)}\right)}=\frac{h'(L^{e}-L)}{\xi u'(C)}, \tag{C.6}$$

which is equation (C.1), the equation that determines labor supply in the market equilibrium. Therefore the optimal labor L in the planning problem and the free market equilibrium coincide. ■

To generalize proposition 3, let $z^{c}=\log(Z^{c})$, $y(z)=\exp(z^{c}+z)\,k^{1-\alpha}(z)\,l(z)$, and

[31]Note that $g(x^{*},x^{*})=0$.

apply the envelope theorem (around $z^c = 0$) to (C.2) to obtain

$$\frac{\partial H}{\partial z^c} = u'\left(Y - \frac{\delta}{\xi}K\right)Y.$$

Similarly,

$$\frac{\partial H}{\partial \xi} = u'\left(Y - \frac{\delta}{\xi}K\right)\times\frac{1}{\xi}Y\left[\int_{x^*}^{\infty}\widetilde{g}\left(z\right)\log y\left(z\right)dz - \log\left(Y\right) + \left(1-\alpha\right)\right] + \frac{\partial b}{\partial \xi}h(L^e - L), \quad \text{(C.7)}$$

Similarly,

$$\frac{\partial H}{\partial \alpha} = u'\left(Y - \frac{\delta}{\xi}K\right)\times Y\log\left(\frac{l\left(z\right)}{k\left(z\right)}\right) + \frac{\partial b}{\partial \alpha}h(L^e - L). \quad \text{(C.8)}$$

Since $U(\cdot)$ is monotone, its inverse exists and therefore

$$u^{-1}\left(H - bh\left(L\right)\right) = Y - \frac{\delta}{\xi}K = \alpha Y, \quad \text{(C.9)}$$

where the last equality follows from $\frac{\delta}{\xi}K = \left(1-\alpha\right)Y$. Total differentiation of (C.9) gives

$$\begin{aligned} d\alpha Y + \alpha dY &= u^{-1\prime}\left(H - bh\left(L^e - L\right)\right)\left(dH + h'\left(L^e - L\right)dL\right) \\ &= \frac{1}{u'\left(Y - \frac{\delta}{\xi}K\right)}\left(dH + bh'\left(L^e - L\right)dL\right) \\ &= Y\left\{dz^c + \log\left(\frac{l\left(z\right)}{k\left(z\right)}\right)d\alpha + \frac{1}{\xi}\left(\int_{x^*}^{\infty}\widetilde{g}\left(z\right)\log\left(\frac{y\left(z\right)}{Y}\right)dz + \left(1-\alpha\right)\right)d\xi\right\} \\ &+ \frac{bh'\left(L^e - L\right)}{u'\left(Y - \frac{\delta}{\xi}K\right)}dL, \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{dY}{Y} &= \frac{1}{\alpha}\left\{dz^c + \left(\log\left(\frac{l\left(z\right)}{k\left(z\right)}\right) - 1\right)d\alpha + \frac{1}{\xi}\left(\int_{x^*}^{\infty}\widetilde{g}\left(z\right)\log\left(\frac{y\left(z\right)}{Y}\right)dz + \left(1-\alpha\right)\right)d\xi\right\} \\ &+ \frac{bh'\left(L^e - L\right)}{\alpha Y u'\left(Y - \frac{\delta}{\xi}K\right)}dL. \end{aligned}$$

Note that $\frac{dY}{Y}$ is the same as in equation (36), except for the presence of the term

$$\frac{bh'\left(L^e - L\right)}{\alpha Y u'\left(Y - \frac{\delta}{\xi}K\right)}dL = \frac{\zeta}{\alpha Y u'\left(Y - \frac{\delta}{\xi}K\right)}dL = \frac{wL}{\alpha\xi Y}\frac{dL}{L},$$

where we used (C.6). The term $\frac{wL}{\alpha\xi Y}\frac{dL}{L}$, which encapsulates the effects of elastic labor, is comprised of the percentage change in labor $\frac{dL}{L}$and the term $\frac{wL}{\alpha\xi Y}$ ≈1, where the approximation is accurate as long as the fraction of labor that is due to overhead, $\bar{l}$, is close to zero. To determine $\frac{dL}{L}$, let $\Phi \equiv \int_{z^*}^{\infty} g\left(z, z^*\right) dz$ denote the measure of firms in the stationary distribution and note that equations (28) and (C.1) imply

$$log\xi\alpha + logY = logw + log\left(L - \bar{l}\Phi\right), \tag{C.10}$$

$$logh'\left(L^e - L\right) - logU'\left(C\right) = logw. \tag{C.11}$$

Because of our additively separable specification for labor utility, we assume for the remainder of this section that $u(c) = log(c)$, in order to ensure balanced growth. This implies that

$$b = \frac{\alpha\xi}{1 - \xi\left(1 - \alpha\right)}.$$

Combining (C.10) with (C.11) gives

$$log\xi\alpha = logh'\left(L^e - L\right) + log\left(1 - \xi\left(1 - \alpha\right)\right) + logL + log\left(\frac{L - \bar{l}\Phi}{L}\right). \tag{C.12}$$

Letting $\frac{1}{f} \equiv -\frac{h''(L^e-L)L}{h'(L^e-L)}$ denote the inverse of the Frisch elasticity of labor supply (f), totally differentiating both sides of (C.12), using the definition of b and simplifying gives

$$\begin{aligned}\frac{dL}{L} &= -\frac{1}{1+\frac{1}{f}}\left\{dlogb + dlog\left(\frac{L - \bar{l}\Phi}{L}\right)\right\} \\ &= -\frac{1}{1+\frac{1}{f}}\left\{dlogb + \frac{\bar{l}\Phi}{L}\left(1 - \frac{\bar{l}\Phi}{L}\right)^{-1}\left(\frac{dL}{L} - \frac{d\Phi}{\Phi}\right)\right\}\end{aligned}$$

Assuming that the fraction of overhead in production is small ($\frac{\bar{l}\Phi}{L} \approx 0$) we obtain the simple formula

$$\frac{dL}{L} \approx -\frac{1}{1+\frac{1}{f}} dlogb,$$

where

$$dlogb = \frac{(1-\xi)}{1-\xi(1-\alpha)}\frac{d\alpha}{\alpha} - \frac{1}{1-\xi(1-\alpha)}\frac{d\xi}{\xi}.$$

A decline in $log\xi$ and a decline in $log\alpha$ have opposite effects. In particular a decline in ξ (higher rent share) results in lower labor supply. Moreover, $\frac{1}{1-\xi(1-\alpha)} > \frac{(1-\xi)}{1-\xi(1-\alpha)}$, so that the magnitude of the elasticity of labor supply with respect to a change in ξ is higher than with respect to a change in α.

The quantitative magnitude of $\frac{dL}{L}$ depends on the assumed elasticity of labor supply, f. As $f \to 0$, $\frac{dL}{L} \to 0$. If one were to use an elasticity around 0.5, then a decline in $\frac{d\xi}{\xi}$ of approximately 0.1 would result in a drop in $\frac{dL}{L}$ of approximately 0.05 for $\xi \approx 1$ and $\alpha \approx \frac{2}{3}$.

Appendix D The decomposition of the labor share

To start, we observe that the aggregate labor share can be expressed as

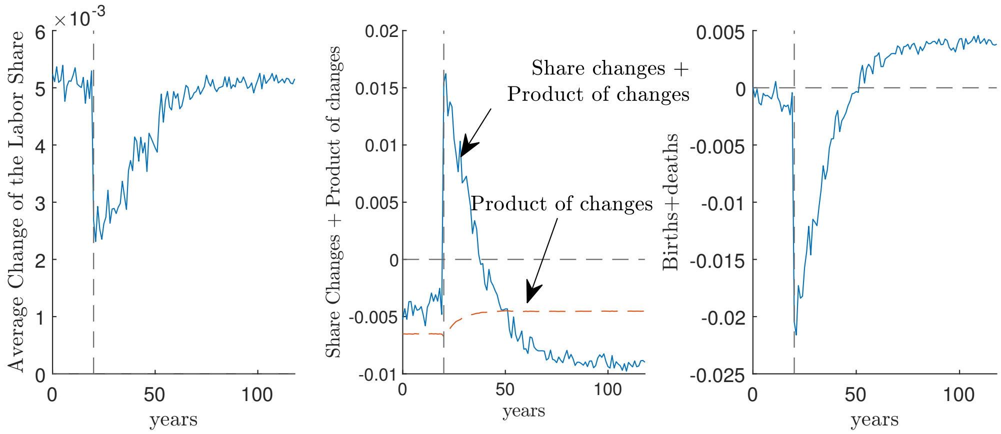
$$\frac{w_t L}{Y_t} = \frac{w_t \int_i l_{it} di}{Y_t} = \int_i \left(\frac{p_{it} y_{it}}{Y_t}\right)\left(\frac{w_t l_{it}}{p_{it} y_{it}}\right) di = \int_i \omega_{it} a_{it} di, \tag{D.1}$$

where $\omega_{it} \equiv \frac{y_{it}}{Y_t}$ is the output weight of firm i and $a_{it} \equiv \frac{w_t l_{it}}{p_{it} y_{it}}$ is firm $i's$ labor share. In light of (D.1), we obtain the "time-share" decomposition

$$\begin{aligned} \frac{w_{t+1}L}{Y_{t+1}} - \frac{w_t L}{Y_t} &= \int_i \omega_{it+1} a_{it+1} di - \int_i \omega_{it} a_{it} di \\ &= \underbrace{\int_i \omega_{it}\left(a_{it+1} - a_{it}\right) di}_{\text{Average change of the labor share}} + \\ &\quad \underbrace{\int_i \left(\omega_{it+1} - \omega_{it}\right) a_{it} di}_{\text{Share changes}} + \underbrace{\int_i \left(\omega_{it+1} - \omega_{it}\right)\left(a_{it+1} - a_{it}\right) di.}_{\text{Product of changes}} \end{aligned} \tag{D.2}$$

The above equation shows that the change in the labor share can be decomposed into three distinct terms. The first term is the output-share-weighted change in individual labor shares. The second term captures the effect of changing shares and the third term is a term that resembles a covariance term. Figure D.1 shows that the first term is always positive both in the old steady state and in the transition phase. This is driven by the fact that in

Figure D.1: **Model-Implied Decomposition of the Labor share decline.** The left plot depicts the evolution of the term labeled "Average change of the labor share" in equation (D.2). The middle plot depicts the sum of "Share changes" and "Product of changes" (solid line). The dashed line depicts the term "Product of changes". The last plot depicts the effect of deaths and births, i.e., the difference between the change in the labor share and the sum of the three components in (D.2). The vertical dashed line in all three plots depicts the onset of the transition. The "noise" in the figures is due to numerical approximation.



our model $\mu - \frac{\sigma^2}{2} < 0$ and hence for the "median" firm the log productivity declines slightly. Due to the presence of a fixed labor cost, the labor share for the average firm increases. The decline in the labor share is driven mostly by the sum of the second and the third components of equation (D.2) (middle plot), which capture the effects of changing firm weights as the output weight of the more productive firms (which have the smaller labor shares) increases. In equation (D.2) we aggregate only over firms that are alive both at time t and $t+1$. The impact of births and deaths on labor share changes is depicted in the third plot of the figure.

Appendix E Additional Compustat Results

E.1 Compustat by Sector

Our measures of employment, sales, and market value contribution measures in Section 2.1 show that the ratio of the ratio of market value (or sales) contribution to employment contribution has increased over time. In this section we show that the same patterns hold when we perform the analysis at the level of individual sectors.

We construct two additional measures of employment, sales, and market value contribution that account for firms' sector. Our first alternative measure, presented in equation (E.1), separately measures the contribution of young firms from each sector relative to the universe of public firms. Our second alternative measure, presented in equation (E.2), separately measures the contribution of young firms from each sector relative to the mature public firms in the same sector.

In equation form, letting X denote either employment, sales, or market value, we define the contribution of the year t IPO cohort from sector s as:

$$\text{X Contribution in Total}_{s,t} = \frac{\text{X of IPO Firms (Excluding Mature Firms)}_{s,t}}{\text{Total X}_{t-1}} \tag{E.1}$$

$$\text{X Contribution in Sector}_{s,t} = \frac{\text{X of IPO Firms (Excluding Mature Firms)}_{s,t}}{\text{Sector X}_{s,t-1}} \tag{E.2}$$

Continuing with the format of our main results, for each of the two measures of sector specific contributions, we construct the cumulative employment, sales, and market value contributions of 5-year IPO cohort bin as follows:

$$\text{X Contribution in Total}_{s,bin} = \sum_{i \in Bin} \text{X Contribution in Total}_{s,i} \tag{E.3}$$

$$\text{X Contribution in Sector}_{s,bin} = \sum_{i \in Bin} \text{X Contribution in Sector}_{s,i} \tag{E.4}$$

Figure E.2 presents the ratio of the sales and market value contributions to the employment contributions for each sector. In Panels A and B, we measure the contribution of young firms from each sector of the economy relative to the universe of public firms. Panel A presents the logarithm of the ratio of the sales and market value contributions to the employment contributions. Panel B presents the normalized (1985–1989 cohort = 0) logarithm of the ratio of the sales and market value contributions to the employment contributions. In Panels C and D, we measure the contribution of young firms from each sector of the economy relative to mature public firms in the same sector. Panel C presents the logarithm of the ratio of the sales and market value contributions to the employment contributions. Panel D presents the normalized (1985–1989 cohort = 0) logarithm of the ratio of the sales and market value contributions to the employment contributions. Panel E presents the number of firms going public in each sector and IPO cohort bin, after excluding firms that were founded more than 10 years prior to their IPO.

E.2 Operating Income

Figure E.3 presents a slightly modified version of the analysis of Section 2.1, in which we present results for employment, operating income, and market value. Operating income is Compustat variable OIBDP.

We measure the employment, operating income, and market value contribution of an IPO cohort as a share of the total market value and employment of public firms in the prior year. We then measure the contribution of an IPO cohort bin as the sum of the contributions of the different IPO cohorts in the bin.

Figure E.2: **Contribution of IPO Cohorts, By Sector**
Data on employment, sales, and market values of US public firms are taken from Compustat. Data on firm founding years are described in the text. We exclude from IPO cohorts all firms that were founded more than 10 years prior to their IPO. In Panels A and B, we measure the contribution of young firms from each sector of the economy relative to the universe of public firms. Panel A presents the logarithm of the ratio of the sales and market value contributions to the employment contributions. Panel B presents the normalized (1985–1989 cohort = 0) logarithm of the ratio of the sales and market value contributions to the employment contributions. In Panels C and D, we measure the contribution of young firms from each sector of the economy relative to mature public firms in the same sector. Panel C presents the logarithm of the ratio of the sales and market value contributions to the employment contributions. Panel D presents the normalized (1985–1989 cohort = 0) logarithm of the ratio of the sales and market value contributions to the employment contributions. Panel E presents the number of firms going public in each sector and IPO cohort bin, after excluding firms that were founded more than 10 years prior to their IPO. See Section E.1 for further details. [Images are on the next five pages.]

Figure E.2: **Contribution of IPO Cohorts, By Sector (Continued from Previous Page)**

(a) Ratio of Contributions, Relative to All of Public Firms

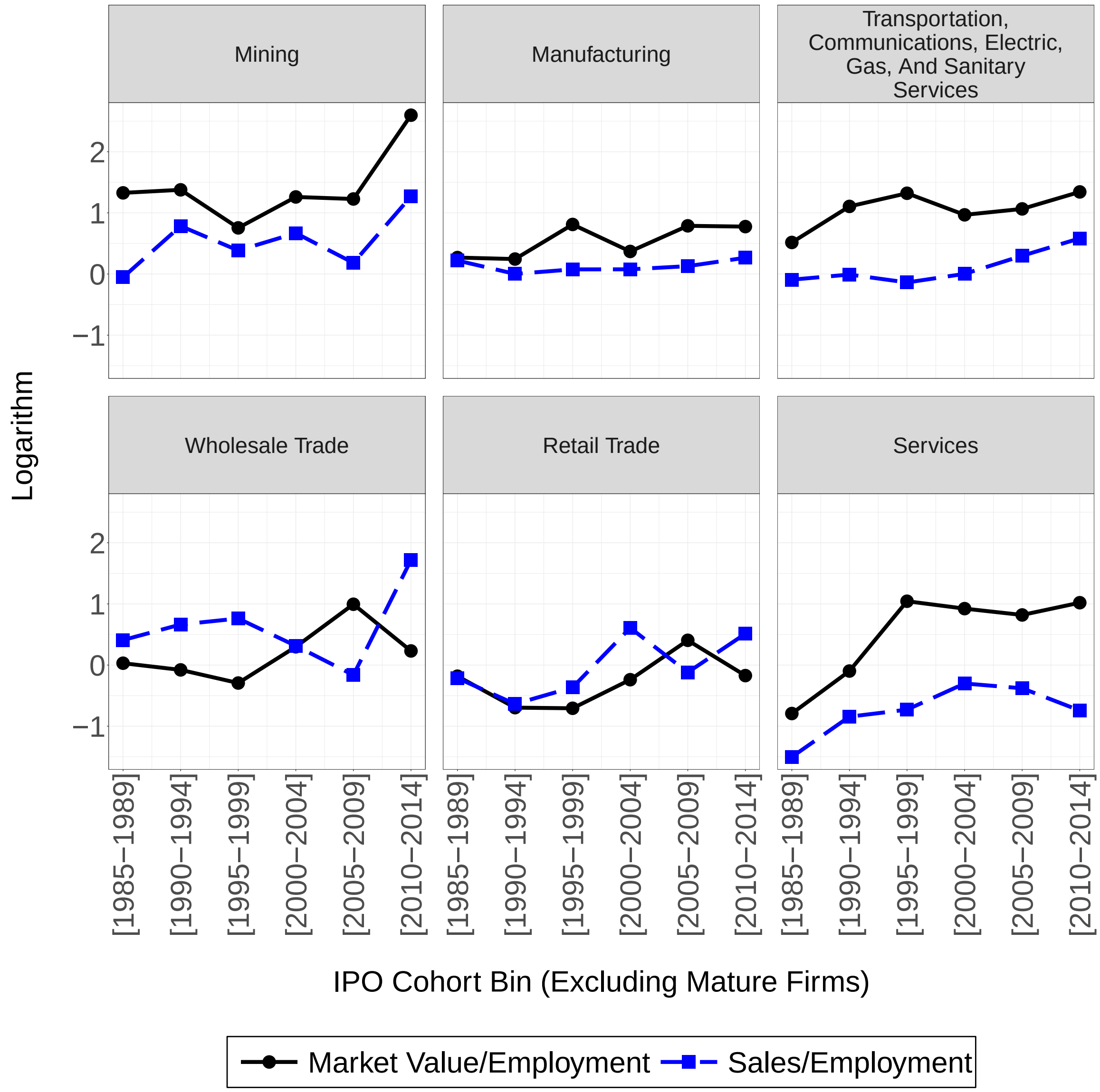


Figure E.2: **Contribution of IPO Cohorts, By Sector (Continued from Previous Page)**

(b) Ratio of Contributions Normalized, Relative to All of Public Firms

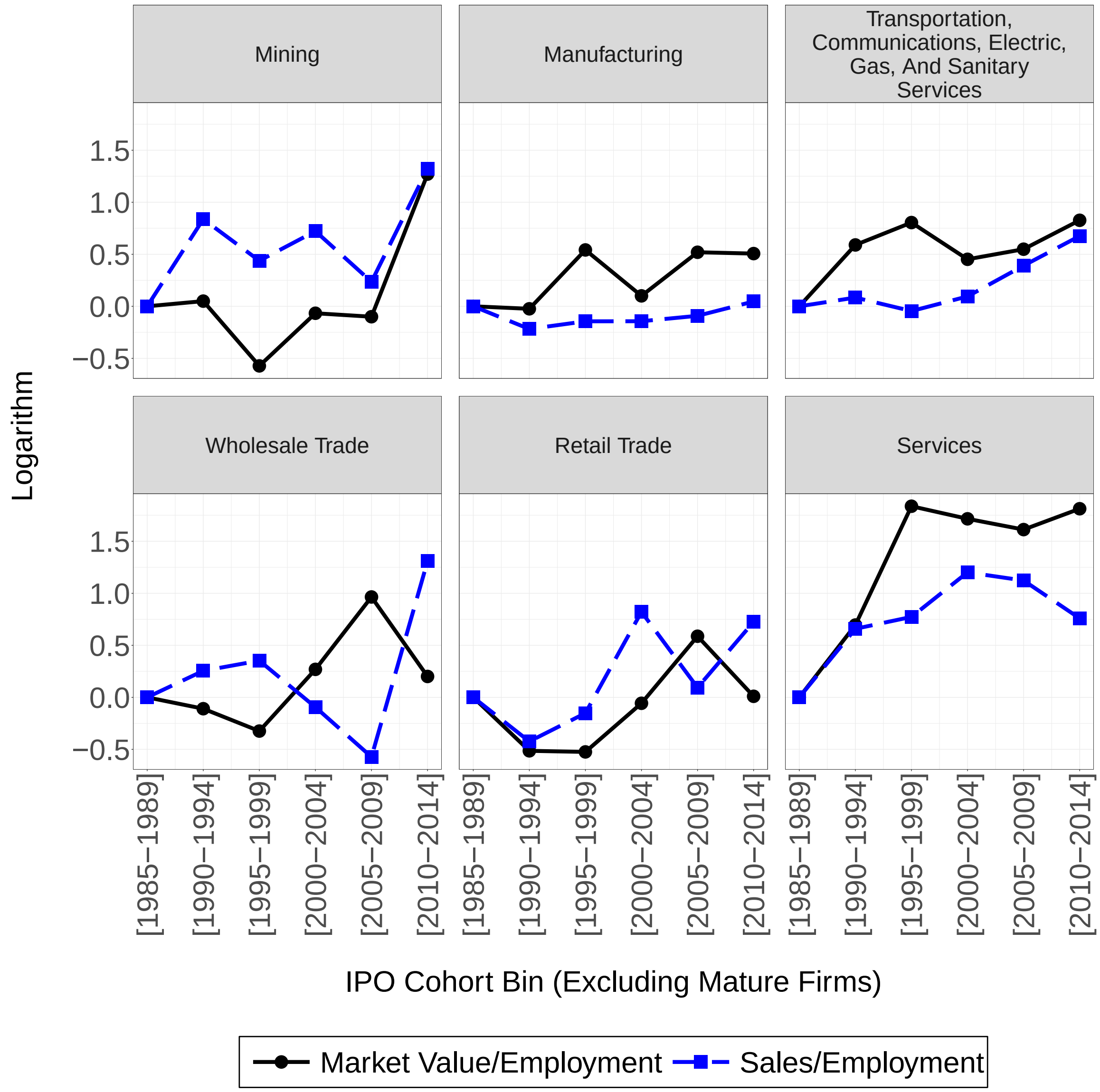


Figure E.2: **Contribution of IPO Cohorts, By Sector (Continued from Previous Page)**

(c) Ratio of Contributions, Relative to Public Firms in Sector

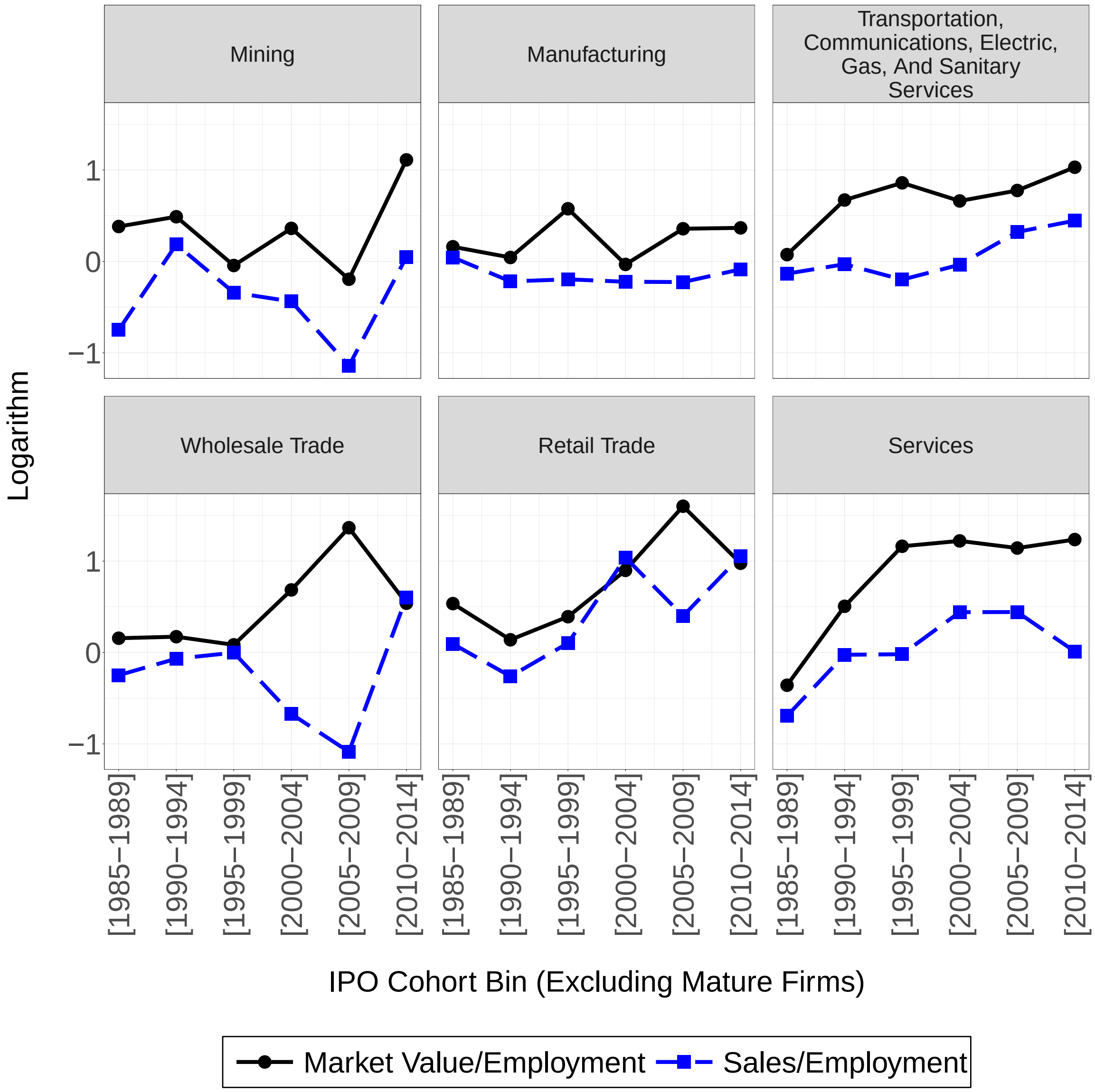


Figure E.2: **Contribution of IPO Cohorts, By Sector (Continued from Previous Page)**

(d) Ratio of Contributions Normalized, Relative to Public Firms in Sector

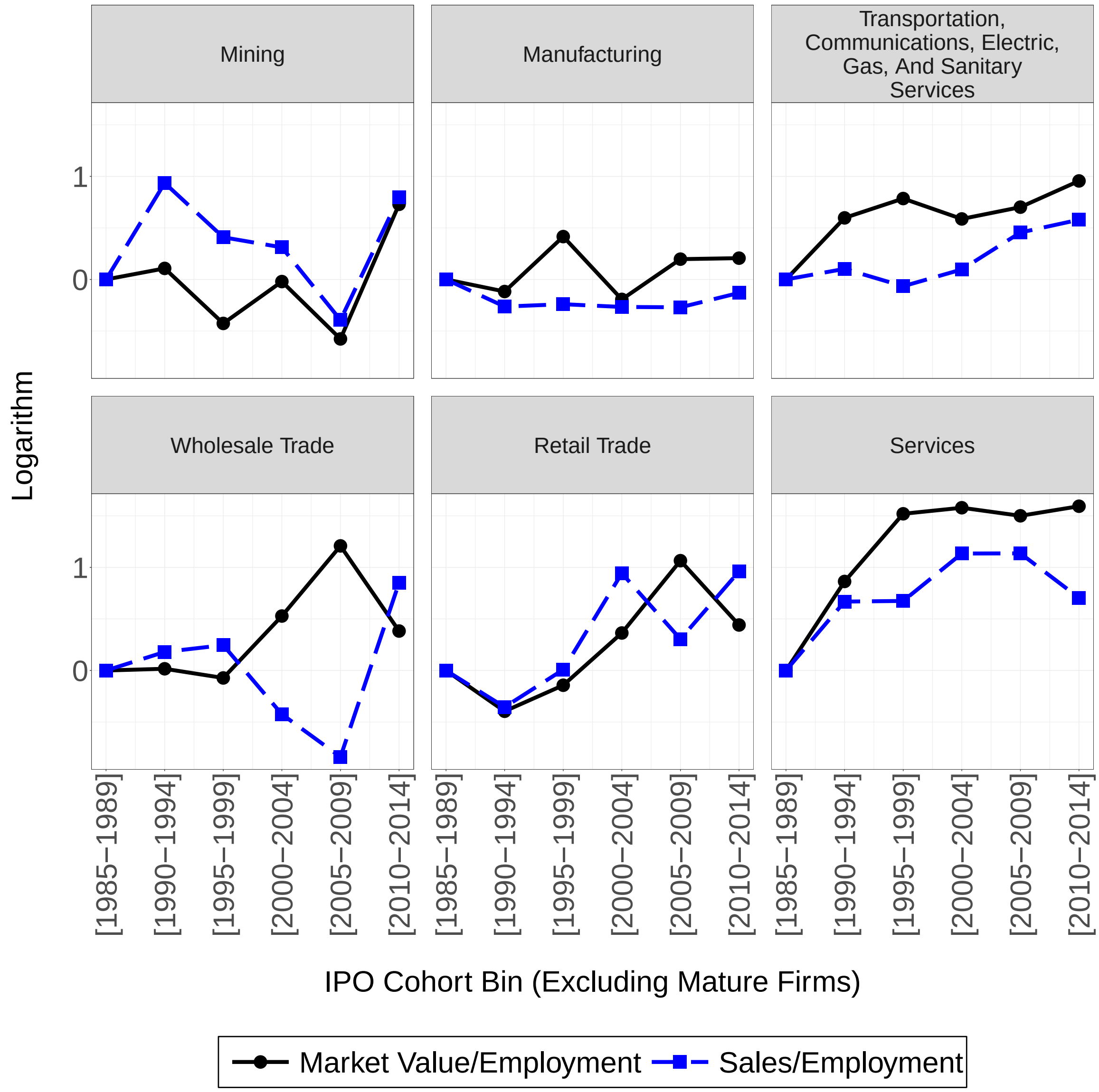


Figure E.2: **Contribution of IPO Cohorts, By Sector (Continued from Previous Page)**

(e) Number of Firms in each IPO Cohort, Excluding Mature Firms

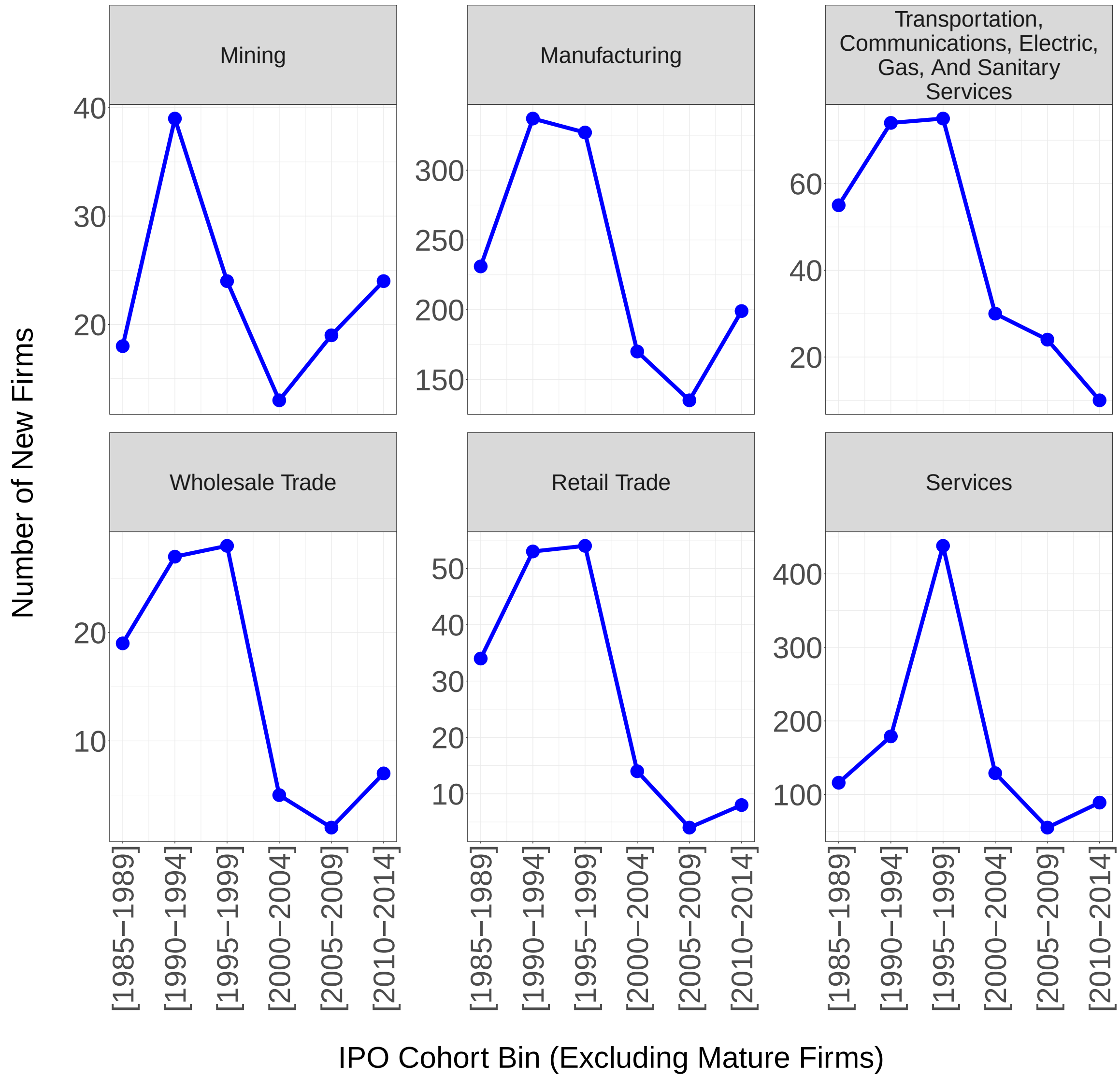
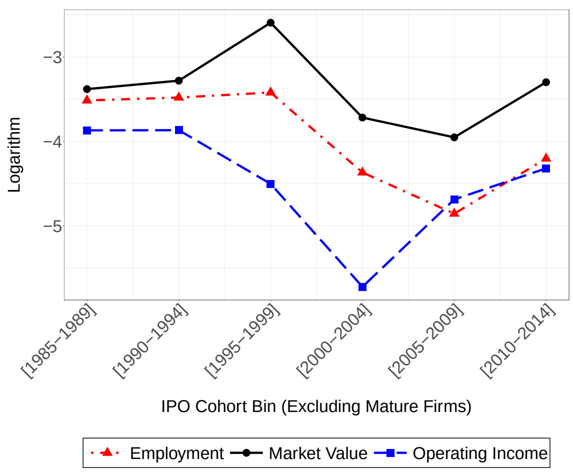


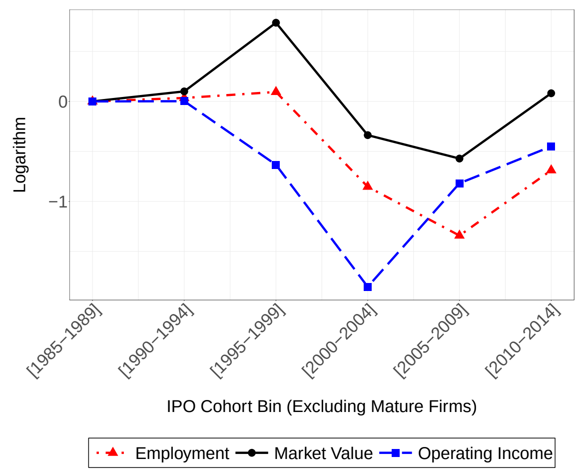
Figure E.3: **Employment, Operating Income, and Market Value Contributions of IPO Cohorts**

Data on employment, operating income, and market values of US public firms are taken from Compustat. Operating income is Compustat variable OIBDP. Data on firm founding years are described in the text. We exclude from IPO cohorts all firms that were founded more than 10 years prior to their IPO. We measure the employment, operating income, and market value contribution of an IPO cohort as a share of the total employment, operating income, and market value of public firms in the prior year. We then measure the contribution of an IPO cohort bin as the sum of the contributions of the different IPO cohorts in the bin. Panel A presents the logarithm of the employment, operating income, and market value contributions of each IPO cohort bin since 1985. Panel B presents the normalized (1985–1989 cohort = 0) logarithm of the employment, operating income, and market value contributions. Panel C presents the logarithm of the ratio of the operating income and market value contributions to the employment contributions. Panel D present the normalized (1985–1989 cohort = 0) logarithm of the ratio of the operating income and market value contributions to the employment contributions. See Section 2.1 for further details.

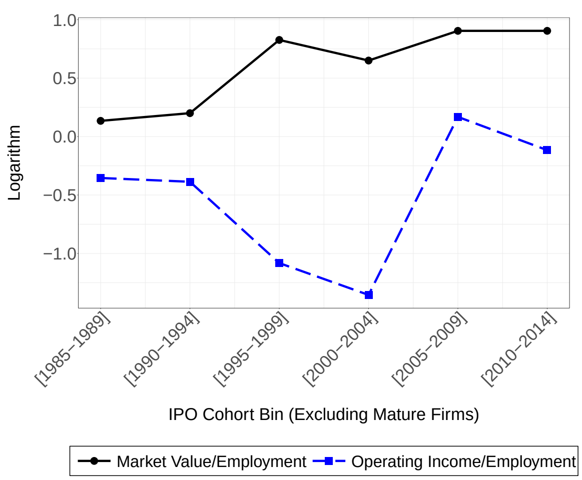
(a) Contribution



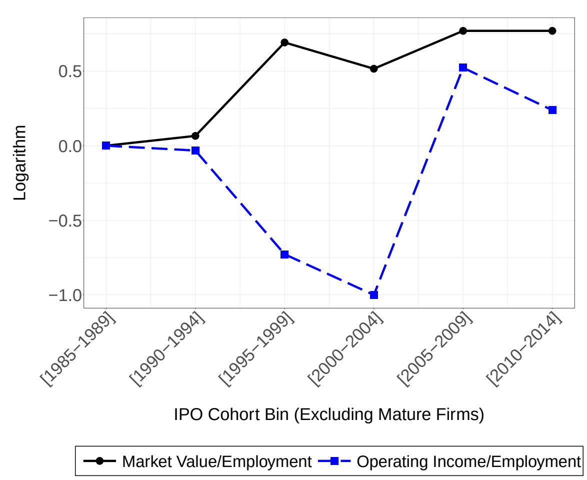
(b) Contribution Normalized



(c) Ratio of Contributions



(d) Ratio of Contributions Normalized



Appendix F Constructing Founding Year in NETS

This appendix describes our classification of changes in ownership and our adjustments to firm founding year that account for firm reorganizations, spin-offs, and mergers.

F.1 Classifying Changes in Ownership

We construct the set of all firms in year t that are destination of a switcher (destination) and all firms in year t-1 that are home of a switcher (home). Each home-destination pair is classified as one of the following mutually exclusive transactions.

1. **Reorganization** A home-destination pair is defined as a reorganization if all of the following are true:
   (a) The destination is a firm that had no establishments in year t-1.
   (b) The establishments of the destination firm are precisely the continuing establishments of the home firm.
2. **Spin-Off** A home-destination pair is defined as a spin-off if all of the following are true:
   (a) The destination is a firm that had no establishments in year t-1.
   (b) The establishments of the destination firm are a strict subset of the continuing establishments of the home firm.
3. **Merger** A home-destination pair is defined as a merger if all of the following are true:
   (a) The destination is a firm that had no establishments in year t-1.
   (b) The destination acquired establishments from more than one firm.
4. **Acquisition** A home-destination pair is defined as part of an acquisition if it is not a reorganization, spin-off, or merger. These are cases in which the destination is not a new firm.

F.2 Adjusting Firm Founding Year

We repeat the following process sequentially from the start to the end of the sample.

1. **Reorganization** In the case of a reorganization the destination firm is assigned the founding year of home firm.

2. **Spin-Off** In the case of a spin-off we distinguish between two possibilities. (1) If the spun-off destination is a new firm we assign the founding year of home firm. (2) If the spun-off destination had existed in the past we assign the minimum of the founding year of the home firm and the founding year of the previously existed firm. This second possibility arises in cases where a firm is purchased and then spun-off several years later.

3. **Merger** In the case of a merger the destination firm is assigned the founding year of largest of the home firms (measured by employment in year t-1).

F.3 Sample of Changes in Ownership

We exclude reorganizations from our sample of changes in ownership. The results are robust to including these in the sample.