Online Appendix

Sufficient Statistics for Nonlinear Tax Systems With Preference Heterogeneity

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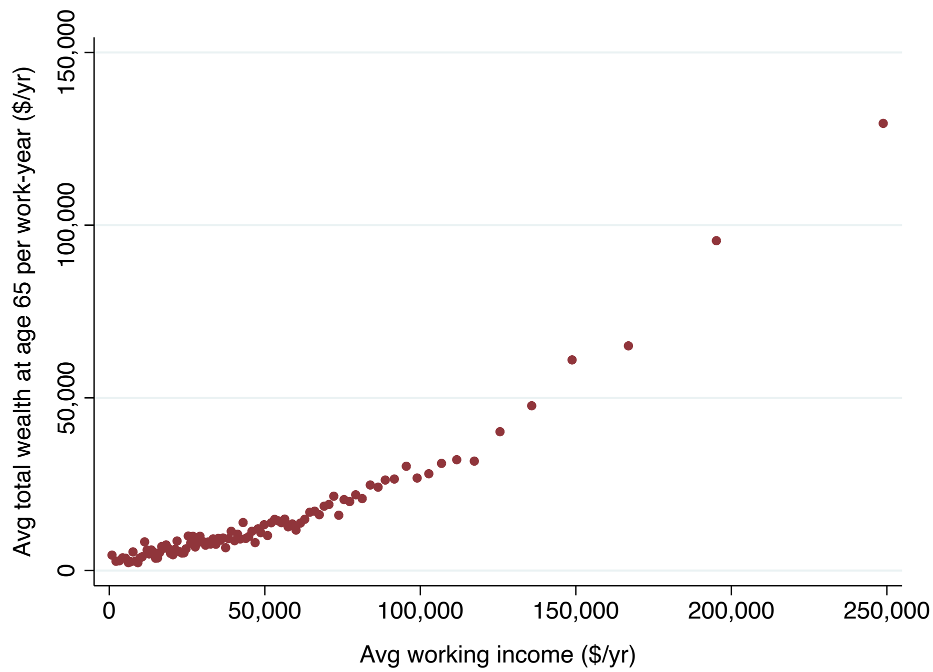
A Supplementary Tables & Figures

Figure A1: Savings Across Incomes in the United States



Notes: The earnings and savings distribution in the U.S. is calibrated based on the Distributional National Accounts micro-files of Piketty et al. (2018). We use 2019 measures of pretax income (*plinc*) and net personal wealth (*hweal*) at the individual level, as well as the age category (20 to 44 years old, 45 to 64, and above 65), to impute gross savings at the time of retirement, which we normalize by the number of work years. See Appendix D.1 for further details.

B Supplementary Theoretical Results

B.1 Monotonicity with Preference Heterogeneity

Lemma B.1. *Under Assumption 1 and 2, earnings z are strictly increasing with type θ in the optimal incentive-compatible allocation $\mathcal{A}$.*

B.2 Implementation Results for Simple Tax Systems

We proceed in three steps to provide sufficient conditions under which some SN and LED tax systems decentralize the optimal incentive-compatible allocation $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$.

First, we define candidate SN and LED tax systems that satisfy type-specific feasibility and individuals' first-order conditions at the optimal allocation. Second, in Proposition B.1, we present sufficient conditions under which these SN and LED tax systems also satisfy individuals' second-order conditions at the optimal allocation, implying local optimality. Third, in Proposition B.2, we present sufficient conditions under which local optima are ensured to be global optima, implying that the candidate SN and LED systems are indeed implementing the optimal allocation.

There are interesting differences between SN and LED tax systems in their ability to implement the optimal allocation. Under our baseline assumptions, we have shown that $z^*(\theta)$ is strictly increasing with type (Lemma B.1). However, $s^*(\theta)$ may not be monotonic. When the optimal incentive-compatible allocation $\mathcal{A}$ features a monotonic $s^*(\theta)$, we show that implementation by a SN tax system requires relatively weaker conditions than implementation by a LED tax system. However, when the optimal incentive-compatible allocation $\mathcal{A}$ features non-monotonicity in $s^*(\theta)$, we show that a LED tax system may be able to implement the optimal allocation, whereas a SN tax system cannot – the candidate SN tax system is not even well defined. Hence, all implementation results for SN tax systems are made under the following assumption:

Assumption 4. *When the SN system is studied, $s^*(\theta)$ is assumed strictly monotonic (increasing or decreasing) in type θ.*

Step 1: Defining candidate tax systems. We first define a candidate SN tax system $\mathcal{T}(s,z) = T_s(s)+T_z(z)$, with the nonlinear functions T_s and T_z defined across all savings and earnings bundles of the optimal allocation $\mathcal{A} = (c^*(\theta), s^*(\theta), z^*(\theta))_\theta$ as follows:

$$T_s(s^*(\theta)) := \int_{\vartheta=\theta_{min}}^{\theta} \left(U_s'(\vartheta)/U_c'(\vartheta) - 1\right) s^{*\prime}(\vartheta)d\vartheta, \tag{45}$$

$$T_z(z^*(\theta)) := z^*(\theta_{min}) - s^*(\theta_{min}) - c^*(\theta_{min}) + \int_{\vartheta=\theta_{min}}^{\theta} \left(U_z'(\vartheta)/U_c'(\vartheta) + 1\right) s^{*\prime}(\vartheta)d\vartheta. \tag{46}$$

where θ_{min} denotes the lowest earning type of the compact type space Θ, and the derivatives are evaluated at the bundle assigned in the optimal allocation (e.g., $U_s'(\vartheta) = U_s'(c^*(\vartheta), s^*(\vartheta), z^*(\vartheta); \vartheta)$). Under this tax system, the optimal allocation satisfies by definition each type's first-order conditions for individual optimization given in Equations (7) and (8). By Lemma C.1, this tax system thus satisfies type-specific feasibility.

We similarly define a candidate LED tax system $\mathcal{T}(s,z) = \tau_s(z)\cdot s + T_z(z)$ as follows:

$$\tau_s(z^*(\theta)) := U_s'(\theta)/U_c'(\theta) - 1, \tag{47}$$

$$\begin{aligned} T_z(z^*(\theta)) :=& z^*(\theta_{min}) - s^*(\theta_{min}) - c^*(\theta_{min}) \\ &+ \int_{\vartheta=\theta_{min}}^{\theta} \left(U_z'(\vartheta)/U_c'(\vartheta) + 1\right) s^{*\prime}(\vartheta)d\vartheta - s^*(z)\cdot\left(\tau_s(z) - \tau_s(z^*(\theta_{min}))\right). \end{aligned} \tag{48}$$

This tax system also satisfies local first-order conditions for individual optimization and type-specific feasibility.

Step 2: Local maxima. We can now derive sufficient conditions under which the above candidate SN and LED tax systems satisfy the second-order conditions for individual optimization, implying that under these conditions assigned bundles are local optima. These conditions can be simply stated in terms of the marginal rates of substitution between consumption and, respectively, savings $\mathcal{S}(c,s,z;\theta)$ and earnings $\mathcal{Z}(c,s,z;\theta)$. These marginal rates of substitutions are smooth functions of c, s, z, and θ by the smoothness of the allocation and the utility function, and sufficient conditions for local second-order conditions are given by the following proposition.

Proposition B.1. *Suppose that an allocation satisfies the conditions in Theorem 1. Under the SN tax system defined by Equations* (45) *and* (46)*, each individual's assigned choice of savings and earnings is a local optimum if the following conditions hold at each point in the allocation:*

$$\mathcal{S}'_c \geq 0,\ \mathcal{S}'_z \geq 0,\ \mathcal{S}'_\theta \geq 0 \tag{49}$$

and

$$\mathcal{Z}'_c \leq 0,\ \mathcal{Z}'_s \geq 0,\ \mathcal{Z}'_\theta \geq 0. \tag{50}$$

Under the LED tax system defined by Equations (47) *and* (48)*, each individual's assigned choice of savings and earnings is a local optimum if the utility function is additively separable in consumption, savings, and earnings ($U''_{cs} = 0$, $U''_{cz} = 0$, and $U''_{sz} = 0$), and additionally the following conditions hold at each point in the allocation:*

$$\mathcal{S}'_\theta \geq 0,\ \mathcal{S}'_\theta \leq \frac{z^{*\prime}(\theta)}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta,\ \mathcal{S}'_\theta \leq s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right). \tag{51}$$

The sufficiency conditions (49) and (50) are quite weak; they are satisfied under many common utility functions used in calibrations of savings and income taxation models, including the simple example function in equation (1). Conditions $\mathcal{S}'_\theta \geq 0$ and $\mathcal{Z}'_\theta \geq 0$ are single crossing conditions for savings and earnings, while other conditions intuitively relate to the concavity of preferences.

The sufficiency conditions for LED systems are more restrictive. Beyond the single-crossing conditions $\mathcal{S}'_\theta \geq 0$ and $\mathcal{Z}'_\theta \geq 0$, they place a constraint on the extent of local preference heterogeneity for savings, as compared with preference heterogeneity in earnings. In words, the preference for savings must not increase "too quickly" across types, or else the second-order condition for earnings may be violated. The intuition for this result can be seen from the definition of the potentially optimal LED system. If the marginal rate of substitution for saving, $\mathcal{S}$, increases very quickly with income at some point in the allocation, then the savings tax rate $\tau_s(z)$ must rise very quickly with z at that point, by equation (47). Since the savings tax rate $\tau_s(z)$ applies to total savings (including inframarginal savings), this increase in $\tau_s(z)$ must be offset by a sharp decrease in $T_z(z)$ at the same point in the distribution, by equation (48). Yet a sufficiently steep decrease in $T_z(z)$ will cause the second-order condition for earnings choice—holding fixed savings choice—to be violated.

Step 3: Global maxima. Having presented conditions under which the bundle $(c^*(\theta), s^*(\theta), z^*(\theta))$ assigned to type θ is a local optimum under the candidate SN and LED tax systems, we now present a set of regularity conditions ensuring that these local optima are also global optima.

Proposition B.2. *Assume single-crossing conditions for earnings and savings ($\mathcal{Z}'_\theta \geq 0$ and $\mathcal{S}'_\theta \geq 0$), that preferences are weakly separable ($U''_{cz} = 0$ and $U''_{sz} = 0$), and that commodities c and s are weak complements ($U''_{cs} \geq 0$). If $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, local optima correspond to global optima when:*

1. *$\mathcal{T}$ is a SN system, and we have that for all $s > s^*(\theta)$ and θ, $\frac{-U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} > \frac{-T''_{ss}(s)}{1+T'_s(s)}$.*
2. *$\mathcal{T}$ is a LED system, and we have that*

(a) for all $s < s^*(\theta)$ *and* θ, $\frac{-U''_{cc}(c(s,\theta),s,z^*(\theta);\theta)}{U'_c(c(s,\theta),s,z^*(\theta);\theta)} > \frac{1}{1+\tau_s(z^*(\theta))} \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s-T'_z(z^*(\theta))}$,

(b) for all $s > s^*(\theta)$ *and* θ, $\frac{-U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} > \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s-T'_z(z^*(\theta))}$,

where $c(s,\theta) := z^*(\theta) - s - \mathcal{T}(s, z^*(\theta))$

In essence, global optimality is ensured under the following assumptions. First, higher types θ derive higher gains from working and allocating those gains to consumption or savings — generalized single-crossing conditions. Second, additive separability of consumption and savings from labor. Third, the utility function U is sufficiently concave in consumption and savings.

For the case of SN tax systems, condition 1 imposes a particular concavity requirement with respect to savings. For the case of LED tax systems, condition 2 imposes particular concavity requirements with respect to both consumption and savings. Notably, these concavity conditions need only be checked when earnings are fixed at each type's assigned earnings level $z^*(\theta)$.

We can naturally apply this result to the candidate SN tax system defined in equations (45) and (46), and to the candidate LED tax system defined in equations (47) and (48). Because these candidate tax systems are defined in terms of individuals' preferences and optimal allocations, we can then express conditions 1 and 2 fully in terms of individuals' preferences and optimal allocations.

B.3 Structural characterization of s'_{inc} and s'_{het}

In economies with preference heterogeneity, budget heterogeneity, and auxiliary choices, individuals solve

$$\begin{aligned} &\max_{c,s,z,\chi} U(c, \phi_s(s,z,\chi;\theta), \phi_z(s,z,\chi;\theta), \chi;\theta) \ s.t.\ c \le B(s,z,\chi;\theta) - \mathcal{T}(s,z) \\ \iff &\max_z \left\{ \max_s \left[\max_\chi U(B(s,z,\chi;\theta) - \mathcal{T}(s,z), \phi_s(s,z,\chi;\theta), \phi_z(s,z,\chi;\theta), \chi;\theta) \right] \right\}. \end{aligned} \quad (52)$$

We denote $\chi(s,z;\theta)$ the solution to the inner problem, $s(z;\theta)$ the solution to the intermediate problem, and $z(\theta)$ the solution to the outer problem. We assume that $\chi(s,z;\theta)$ and $s(z;\theta)$ are interior solutions that satisfy the first-order conditions of these problems, and we maintain the assumption that $z(\theta)$ is strictly increasing to denote $\vartheta(z)$ the type that chooses earnings z.

In this setting, we decompose across-income heterogeneity in $s(z) := s(z;\vartheta(z))$ between $s'_{inc}(z) := \frac{\partial s(z;\vartheta(z'))}{\partial z}|_{z'=z}$ and $s'_{het}(z) := \frac{\partial s(z';\vartheta(z))}{\partial z}|_{z'=z}$ as follows:

Proposition B.3. *In economies with preference heterogeneity, budget heterogeneity, and auxiliary choices, sufficient statistics* $s'_{inc}(z)$ *and* $s'_{het}(z)$ *are given by*

$$s'_{inc}(z) = -\frac{\mathcal{N}^1_{inc}(z) + \mathcal{N}^2_{inc}(z)}{\mathcal{D}^1(z) + \mathcal{D}^2(z)} \quad (53)$$

$$s'_{het}(z) = -\frac{\mathcal{N}^1_{het}(z) + \mathcal{N}^2_{het}(z)}{\mathcal{D}^1(z) + \mathcal{D}^2(z)} \quad (54)$$

where terms in the numerators and denominators are

$$\mathcal{N}^1_{inc} := \mathcal{K}_c\Big[B'_z + B'_\chi \frac{\partial \chi}{\partial z} - \mathcal{T}'_z\Big] + \mathcal{K}_s\Big[\frac{\partial \phi_s}{\partial z} + \frac{\partial \phi_s}{\partial \chi}\frac{\partial \chi}{\partial z}\Big] + \mathcal{K}_z\Big[\frac{\partial \phi_z}{\partial z} + \frac{\partial \phi_z}{\partial \chi}\frac{\partial \chi}{\partial z}\Big] + \mathcal{K}_\chi \frac{\partial \chi}{\partial z} \quad (55)$$

$$\mathcal{N}^2_{inc} := U'_c\Big[B''_{sz} + B''_{s\chi} \frac{\partial \chi}{\partial z} - \mathcal{T}''_{sz}\Big] + U'_s\Big[\frac{\partial^2 \phi_s}{\partial s \partial z} + \frac{\partial^2 \phi_s}{\partial s \partial \chi}\frac{\partial \chi}{\partial z}\Big] + U'_z\Big[\frac{\partial^2 \phi_z}{\partial s \partial z} + \frac{\partial^2 \phi_z}{\partial s \partial \chi}\frac{\partial \chi}{\partial z}\Big] \quad (56)$$

$$\mathcal{N}^1_{het} := \mathcal{K}_c B'_\chi \frac{\partial \chi}{\partial \theta} + \mathcal{K}_s\Big[\frac{\partial \phi_s}{\partial \chi}\frac{\partial \chi}{\partial \theta} + \frac{\partial \phi_s}{\partial \theta}\Big] + \mathcal{K}_z\Big[\frac{\partial \phi_z}{\partial \chi}\frac{\partial \chi}{\partial \theta} + \frac{\partial \phi_z}{\partial \theta}\Big] + \mathcal{K}_\chi \frac{\partial \chi}{\partial \theta} + \mathcal{K}_\theta \quad (57)$$

$$\mathcal{N}^2_{het} := U'_c B''_{s\chi} \frac{\partial \chi}{\partial \theta} + U'_s\Big[\frac{\partial^2 \phi_s}{\partial s \partial \chi}\frac{\partial \chi}{\partial \theta} + \frac{\partial^2 \phi_s}{\partial s \partial \theta}\Big] + U'_z\Big[\frac{\partial^2 \phi_z}{\partial s \partial \chi}\frac{\partial \chi}{\partial \theta} + \frac{\partial^2 \phi_z}{\partial s \partial \theta}\Big] \quad (58)$$

$$\mathcal{D}^1 := \mathcal{K}_c\Big[B'_s + B'_\chi \frac{\partial \chi}{\partial s} - \mathcal{T}'_s\Big] + \mathcal{K}_s\Big[\frac{\partial \phi_s}{\partial s} + \frac{\partial \phi_s}{\partial \chi}\frac{\partial \chi}{\partial s}\Big] + \mathcal{K}_z\Big[\frac{\partial \phi_z}{\partial s} + \frac{\partial \phi_z}{\partial \chi}\frac{\partial \chi}{\partial s}\Big] + \mathcal{K}_\chi \frac{\partial \chi}{\partial s} \quad (59)$$

$$\mathcal{D}^2 := U'_c\Big[B''_{ss} + B''_{s\chi} \frac{\partial \chi}{\partial s} - \mathcal{T}''_{ss}\Big] + U'_s\Big[\frac{\partial^2 \phi_s}{(\partial s)^2} + \frac{\partial^2 \phi_s}{\partial s \partial \chi}\frac{\partial \chi}{\partial s}\Big] + U'_z\Big[\frac{\partial^2 \phi_z}{(\partial s)^2} + \frac{\partial^2 \phi_z}{\partial s \partial \chi}\frac{\partial \chi}{\partial s}\Big] \quad (60)$$

with all quantities being evaluated at z, $s(z)$, $\vartheta(z)$, $\chi(z) := \chi(s(z)), z; \vartheta(z))$, *as well as* $c(z) := B(s(z), z, \chi(z); \vartheta(z)) - \mathcal{T}(s(z), z)$, *and where*

$$\mathcal{K}_c := U''_{cc}\left(B'_s - \mathcal{T}'_s\right) + U''_{cs}\frac{\partial \phi_s}{\partial s} + U''_{cz}\frac{\partial \phi_z}{\partial s} \quad (61)$$

$$\mathcal{K}_s := U''_{cs}\left(B'_s - \mathcal{T}'_s\right) + U''_{ss}\frac{\partial \phi_s}{\partial s} + U''_{sz}\frac{\partial \phi_z}{\partial s} \quad (62)$$

$$\mathcal{K}_z := U''_{cz}\left(B'_s - \mathcal{T}'_s\right) + U''_{sz}\frac{\partial \phi_s}{\partial s} + U''_{zz}\frac{\partial \phi_z}{\partial s} \quad (63)$$

$$\mathcal{K}_\chi := U''_{c\chi}\left(B'_s - \mathcal{T}'_s\right) + U''_{s\chi}\frac{\partial \phi_s}{\partial s} + U''_{z\chi}\frac{\partial \phi_z}{\partial s} \quad (64)$$

$$\mathcal{K}_\theta := U''_{c\theta}\left(B'_s - \mathcal{T}'_s\right) + U''_{s\theta}\frac{\partial \phi_s}{\partial s} + U''_{z\theta}\frac{\partial \phi_z}{\partial s}. \quad (65)$$

Numerators of $s'_{inc}(z)$ and $s'_{het}(z)$ are different as they capture direct changes in s, coming from either a change in z or a change in $\vartheta(z)$. Denominators are the same because they capture processes of circular adjustments induced by direct changes in s.

In a simple setting like example (1) with additively separable utility, a separable tax system and preference heterogeneity for s only, the only non-zero term in the numerator of s'_{inc} would be proportional to $\mathcal{K}_c$ capturing changes in the marginal utility of c from changes in z, and the only non-zero term in the numerator of s'_{inc} would be proportional to $\mathcal{K}_\theta$ capturing changes in marginal utility of s from changes in θ.

B.4 Optimal Taxes on s in Simple Tax Systems

We present optimal savings tax formulas for simple tax systems, which characterize the optimal savings tax schedule for *any* given earnings tax schedule—including a potentially suboptimal one. These formulas are derived assuming unidimensional heterogeneity and are similar to those presented in Proposition 4, where heterogeneity is allowed to be multidimensional.

Proposition B.4. *Consider a given (and potentially suboptimal) earnings tax schedule $T_z(z)$, suppose that SL, SN and LED systems verify Assumption 3, and suppose that in the SN system s is strictly monotonic (increasing or decreasing) in z. At each bundle (c, s, z) chosen by a type θ, these systems satisfy the following optimality conditions for taxes on s:*

$$SL : \frac{\tau_s}{1+\tau_s}\int_{x=z_{min}}^{z_{max}} s(x)\zeta_{s|z}^c(x)dH_z(x) = \int_{x=z_{min}}^{z_{max}}\left\{s\left(x\right)\left(1-\hat{g}(x)\right) - \frac{T_z'(x)+s_{inc}'(x)\,\tau_s}{1-T_z'\left(x\right)}x\zeta_z^c(x)\,s_{inc}'(x)\right\}dH_z(x) \tag{66}$$

$$SN : \frac{T_s'(s)}{1+T_s'(s)}s\zeta_{s|z}^c(z)h_z(z) = s'\left(z\right)\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x) - \frac{T_z'(z)+s_{inc}'(z)T_s'(s)}{1-T_z'(z)}z\zeta_z^c(z)s_{inc}'(z)h_z(z) \tag{67}$$

$$LED : \frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'(z)\tau_s(z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(z)\,s\,h_z(z) + \int_{x\geq z}\frac{\tau_s(x)}{1+\tau_s(x)}s(x)\zeta_{s|z}^c(x)\,dH_z(x) \tag{68}$$

$$= \int_{x\geq z}\left\{\left(1-\hat{g}(x)\right)s\left(x\right) - \frac{T_z'\left(x\right)+\tau_s'\left(x\right)s\left(x\right)+s_{inc}'(x)\tau_s(x)}{1-T_z'\left(x\right)-\tau_s'\left(x\right)s\left(x\right)}x\zeta_z^c(x)s_{inc}'(x)\right\}\,dH_z(x).$$

These optimal savings tax formulas are all different, reflecting differences between the savings tax instruments that we consider, yet they share common elements. First, the preference heterogeneity term $s_{het}'\left(z\right)$ no longer appears in the formulas. The intuition is that outside of the full optimum, it may still be desirable to tax savings in the absence of preference heterogeneity, implying that optimality may clash with Pareto efficiency when the earnings tax is suboptimal. Second, $s_{inc}'\left(z\right)$ is a key sufficient statistic for optimal savings tax schedules. Indeed, by Lemma 1, a larger $s_{inc}'\left(z\right)$ means that savings tax reforms impose higher distortions on earnings and thus generally calls for lower savings tax rate.

B.5 Optimal Taxes on z in Simple Tax Systems

We now present optimal earnings tax formulas for simple tax systems.

Proposition B.5. *Consider a given (and potentially suboptimal) earnings tax schedule $T_z(z)$ and suppose that SL, SN and LED systems verify Assumption 3, and suppose that in the SN system s is strictly monotonic (increasing or decreasing) in z. At each bundle (c, s, z) chosen by a type θ, these systems satisfy the following optimality conditions for taxes on z:*

$$SL : \frac{T_z'\left(z\right)}{1-T_z'\left(z\right)} = \frac{1}{z\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x) - s_{inc}'(z)\frac{\tau_s}{1-T_z'\left(z\right)} \tag{69}$$

$$SN : \frac{T_z'\left(z\right)}{1-T_z'\left(z\right)} = \frac{1}{z\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x) - s_{inc}'(z)\frac{T_s'\left(s\right)}{1-T_z'\left(z\right)} \tag{70}$$

$$LED : \frac{T_z'\left(z\right)+\tau_s'\left(z\right)s}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s} = \frac{1}{z\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x) - s_{inc}'(z)\frac{\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}. \tag{71}$$

These conditions pinning down the optimal schedule of marginal earnings tax rates are a direct application of equation (19) presented in Theorem 2 for smooth tax systems. While formulas for SL and SN tax systems look almost identical to the general condition, the formula for LED tax system looks a bit different. This difference only reflects the fact that, for a LED tax system, the marginal earnings tax rate is given by $\mathcal{T}'_z(s,z) = T'_z(z) + \tau'_s(z)\, s(z)$, accounting for the earnings-dependent nature of savings taxes.

B.6 Optimal Taxes on z with Multidimensional Heterogeneity

Proposition B.6. *Consider given (and potentially suboptimal) SL, SN, and LED savings tax schedule, and assume that under each simple tax system individuals first-order and second-order conditions strictly hold. Then, at each bundle (c^0, s^0, z^0) chosen by a type θ^0, marginal tax rates on z in SL/SN/LED systems must satisfy the following optimality conditions:*

$$SL : \frac{T'_z(z^0)}{1 - T'_z(z^0)} \mathbb{E}\left[\zeta^c_z(s,z) \middle| z^0\right] = \frac{1}{z^0 h_z(z^0)} \int_{z \geq z^0} \left\{\mathbb{E}\left[1 - \hat{g}(s,z)\,|z\right]\right\} dH_z(z) - \mathbb{E}\left[s'_{inc}(s,z) \frac{\tau_s}{1 - T'_z(z)} \zeta^c_z(s,z) \middle| z^0\right] \tag{72}$$

$$SN : \frac{T'_z(z^0)}{1 - T'_z(z^0)} \mathbb{E}\left[\zeta^c_z(s,z) \middle| z^0\right] = \frac{1}{z^0 h_z(z^0)} \int_{z \geq z^0} \left\{\mathbb{E}\left[1 - \hat{g}(s,z)\,|z\right]\right\} dH_z(z) - \mathbb{E}\left[s'_{inc}(s,z) \frac{T'_s(s)}{1 - T'_z(z)} \zeta^c_z(s,z) \middle| z^0\right] \tag{73}$$

$$LED : \mathbb{E}\left[\frac{T'_z(z) + \tau'_s(z)\, s}{1 - T'_z(z) - \tau'_s(z)\, s} \zeta^c_z(s,z) \middle| z^0\right] = \frac{1}{z^0 h_z(z^0)} \int_{z \geq z^0} \left\{\mathbb{E}\left[1 - \hat{g}(s,z)\,|z\right]\right\} dH_z(z) - \mathbb{E}\left[s'_{inc}(s,z) \frac{\tau_s(z)}{1 - T'_z(z) - \tau'_s(z)\, s} \zeta^c_z(s,z) \middle| z^0\right] \tag{74}$$

These conditions are similar to those presented above for optimal marginal earnings tax rates under unidimensional heterogeneity (Proposition B.5). Indeed, Lemma 1 still applies such that the previous derivations carry over when adding an expectation with respect to savings. Proofs are thus omitted.

B.7 Equivalences with Tax Systems Involving Gross Period-2 Savings

Suppose that there are two periods, and set $1 + r = 1/p$. In period 1 the individual earns z, consumes c, and pays income taxes $T_1(z)$. In period 2 the individual realizes *gross pre-tax savings* $s_g = (z - c - T_1(z))(1 + r)$ and pays income taxes $T_2(s_g, z)$. The realized savings s are given by $s_g - T_2(s_g, z)$. The total tax paid in "period-1 dollars" is given by $T_1(z) + T_2(s_g, z)/(1 + r)$. The individual maximizes $U(c, s, z)$ subject to the constraint

$$s \leq (z - c - T_1(z))(1 + r) - T_2(s_g, z)$$
$$\Leftrightarrow c + \frac{s}{1 + r} \leq z - T_1(z) - \frac{T_2((z - c - T_1(z))(1 + r), z)}{1 + r}.$$

In our baseline formulation with period-1 tax function $\mathcal{T}(s,z)$, individuals choose s and z to maximize $U(z-s-\mathcal{T}(s,z),s,z;\theta)$. To convert from the formulation with period-2 taxes to our baseline formulation, define a function $\tilde{s}_g(s,z)$ implicitly to satisfy the equation

$$\tilde{s}_g - T_2(\tilde{s}_g, z) = s$$

Note that $\tilde{s}_g$ is generally uniquely defined if we have a system with monotonic realized savings s. Then, the equivalence in tax schedules is given by

$$\mathcal{T}_s'(s,z) = \frac{1}{1+r}\frac{\partial}{\partial s_g}T_2(s_g,z)|_{s_g=\tilde{s}_g}\frac{\partial}{\partial s}\tilde{s}_g \tag{75}$$

and $\mathcal{T}_z' = T_z'$. equation (75) simply computes how a marginal change in s changes the tax burden in terms of period-1 units of money, and the division by $1+r$ is to convert to period-1 units. Now differentiating the definition of $\tilde{s}_g$ gives

$$\frac{\partial}{\partial s}\tilde{s}_g - \frac{\partial}{\partial s_g}T_2(s_g,z)\frac{\partial}{\partial s}\tilde{s}_g = 1$$

and thus

$$\frac{\partial}{\partial s}\tilde{s}_g = \frac{1}{1-\frac{\partial}{\partial s_g}T_2(s_g,z)}$$

from which it follows that

$$\mathcal{T}_s'(s,z) = \frac{1}{1+r}\frac{\frac{\partial}{\partial s_g}T_2(s_g,z)|_{s_g=\tilde{s}_g}}{1-\frac{\partial}{\partial s_g}T_2(s_g,z)|_{s_g=\tilde{s}_g}}. \tag{76}$$

We can also start with a schedule $\mathcal{T}$ and converts it to the two-period tax schedule. Now if s is the realized savings, we can define gross savings in period 2 as $s_g = s + \mathcal{T}(z,s)(1+r) - \mathcal{T}(z,0)$, and we define the function $\tilde{s}(s_g,z)$ to satisfy

$$s_g = \tilde{s} + (1+r)\left(\mathcal{T}(\tilde{s},z) - \mathcal{T}(0,z)\right).$$

Then,

$$\begin{aligned}\frac{\partial}{\partial s_g}T_2(s_g,z) &= (1+r)\mathcal{T}_s'(\tilde{s},z)\frac{\partial}{\partial s_g}\tilde{s} \\ &= \frac{(1+r)\mathcal{T}_s'(\tilde{s},z)}{1+(1+r)\mathcal{T}_s'(\tilde{s},z)}\end{aligned} \tag{77}$$

B.7.1 Separable tax systems (SN).

Now if T_2 is a function of s_g only (a separable tax system), then s_g will be a function of s only, and thus $\mathcal{T}_s'$ will only depend on s. Conversely, note that if $\mathcal{T}$ is a separable system, so that $\mathcal{T}_s'$ does not depend on z, then (77) implies that $\frac{\partial}{\partial s_g}T_2(s_g,z)$ does not depend on z either. Thus, separability is a property preserved under these transformations.

Now if we start with a separable $\mathcal{T}$, then T_2 is given by

$$T_2'(s_g) = (1+r)\frac{\frac{\partial}{\partial s}\mathcal{T}_s'(s))|_{s=\tilde{s}}}{1+\frac{\partial}{\partial s}\mathcal{T}_s'(s))|_{s=\tilde{s}}}$$

where $\tilde{s}$ is the value that solves $s_g = \tilde{s} + \mathcal{T}(\tilde{s})$.

B.7.2 Linear tax systems (LED and SL).

If $T_2 = s_g\,\tau(z)$, a linear earnings-dependent system, then $s = s_g(1-\tau(z))$ and $s_g = \frac{s}{1-\tau(z)}$. Moreover, $\frac{\partial}{\partial s}s_g = \frac{1}{1-\tau(z)}$, and thus we have that

$$\mathcal{T}_s' = \frac{1}{1+r}\frac{\tau(z)}{1-\tau(z)}$$

which again implies that we have a linear earnings-dependent system with rate $\tilde{\tau}(z) = \frac{1}{1+r}\frac{\tau(z)}{1-\tau(z)}$.

Conversely, if we start with a LED system $\mathcal{T}$ with $\mathcal{T}_s' = \tau(z)$, then

$$\frac{\partial}{\partial s_g}T_2(s_g, z) = (1+r)\frac{\tau(z)}{1+\tau(z)}.$$

When the tax rates τ are not functions of z, the calculations above show that the conversions preserve not just linearity, but also independence of z.

C Proofs

C.1 Proof of Lemma B.1 (Monotonicity with Preference Heterogeneity)

We show by contradiction that the extended Spence-Mirrlees condition implies that type $\theta_2 > \theta_1$ chooses earnings $z(\theta_2) > z(\theta_1)$. Note that, by Assumption 2, $c(\theta)$, $s(\theta)$, and $z(\theta)$ are smooth functions of θ in the optimal incentive-compatible allocation, and that by Assumption 1 utility U is twice continuously differentiable.

Assume (without loss of generality) that there is an open set $(\theta_1, \theta_2) \in \Theta$ where $z(\theta)$ is decreasing with θ such that $z(\theta_2) < z(\theta_1)$.[33] Then,

[33] Since we assume no bunching in income, meaning that types θ for which $z'(\theta) = 0$ are of measure zero, this inequality has to be strict with $\theta_2 > \theta_1$.

$$
\begin{aligned}
&U\left(c(\theta_2), s(\theta_2), z(\theta_2); \theta_2\right) - U\left(c(\theta_1), s(\theta_1), z(\theta_1); \theta_2\right) \\
&= \int_{\theta=\theta_1}^{\theta_2} \left[\frac{dU\left(c(\theta), s(\theta), z(\theta); \theta_2\right)}{d\theta} \right] d\theta \\
&= \int_{\theta=\theta_1}^{\theta_2} \left[U_c'\left(c(\theta), s(\theta), z(\theta); \theta_2\right) c'(\theta) + U_s'\left(c(\theta), s(\theta), z(\theta); \theta_2\right) s'(\theta) + U_z'\left(c(\theta), s(\theta), z(\theta); \theta_2\right) z'(\theta) \right] d\theta \\
&= \int_{\theta=\theta_1}^{\theta_2} U_c'\left(c(\theta), s(\theta), z(\theta); \theta_2\right) \left[c'(\theta) + \mathcal{S}\left(c(\theta), s(\theta), z(\theta); \theta_2\right) s'(\theta) + \mathcal{Z}\left(c(\theta), s(\theta), z(\theta); \theta_2\right) z'(\theta) \right] d\theta
\end{aligned} \tag{78}
$$

Now, for each $\theta \in (\theta_1, \theta_2)$ the first-order condition implied by incentive compatibility implies that, at point $(c(\theta), s(\theta), z(\theta))$,

$$
\begin{aligned}
&U_c'\left(c(\theta), s(\theta), z(\theta); \theta\right) c'(\theta) + U_s'\left(c(\theta), s(\theta), z(\theta); \theta\right) s'(\theta) + U_z'\left(c(\theta), s(\theta), z(\theta); \theta\right) z'(\theta) = 0 \\
&\iff c'(\theta) + \mathcal{S}\left(c(\theta), s(\theta), z(\theta); \theta\right) s'(\theta) + \mathcal{Z}\left(c(\theta), s(\theta), z(\theta); \theta\right) z'(\theta) = 0.
\end{aligned} \tag{79}
$$

When $z'(\theta) \neq 0$, the extended Spence-Mirrlees condition states that for any θ',

$$
\begin{aligned}
&\mathcal{S}_\theta'\left(c(\theta), s(\theta), z(\theta); \theta'\right) \frac{s'(\theta)}{z'(\theta)} + \mathcal{Z}_\theta'\left(c(\theta), s(\theta), z(\theta); \theta'\right) > 0 \\
&\iff \mathcal{S}_\theta'\left(c(\theta), s(\theta), z(\theta); \theta'\right) s'(\theta) + \mathcal{Z}_\theta'\left(c(\theta), s(\theta), z(\theta); \theta'\right) z'(\theta) < 0
\end{aligned} \tag{80}
$$

where the last inequality is reversed because $z(\theta)$ is decreasing in $\theta \in (\theta_1, \theta_2)$, meaning $z'(\theta) < 0$. This implies that with $\theta_2 > \theta$

$$
c'(\theta) + \mathcal{S}\left(c(\theta), s(\theta), z(\theta); \theta_2\right) s'(\theta) + \mathcal{Z}\left(c(\theta), s(\theta), z(\theta); \theta_2\right) z'(\theta) < 0. \tag{81}
$$

Since $U_c' > 0$, and since we assume no bunching on income (meaning that types θ for which $z'(\theta) = 0$ are of measure zero), this means that the integral above is negative, and thus that

$$
U\left(c(\theta_2), s(\theta_2), z(\theta_2); \theta_2\right) < U\left(c(\theta_1), s(\theta_1), z(\theta_1); \theta_2\right). \tag{82}
$$

This is a contradiction with the fact that type θ_2 (strictly) prefers its allocation $(c(\theta_2), s(\theta_2), z(\theta_2))$ in the optimal incentive-compatible allocation, which concludes the proof.

C.2 Proof of Theorem 1 (Implementation with a Smooth Tax System)

In the appendix, we adopt the notation that individual's allocations in the optimal mechanism are labeled with a "star"; i.e., $(c^*(\theta), s^*(\theta), z^*(\theta))$. We construct a smooth tax system that implements the optimal incentive-compatible allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ by introducing penalties for deviations away from these allocations. This proof relies on Lemma C.1 and Lemma C.2, which we derive at the end of this subsection. Throughout, we adopt $p \equiv 1$ to economize on notations.

With unidimensional heterogeneity, type θ belongs to the compact space $\Theta = [\theta_{min}, \theta_{max}]$. Moreover, there is always a mapping $s^*(z)$ that denotes the savings level associated with earnings level $z = z^*(\theta)$ at the optimal incentive-compatible allocation. We consider without loss of generality the case in which $s(z)$ is strictly increasing; the proof can be adapted to cases with non-monotonic $s(z)$.

Let $z_{max} := z^*(\theta_{max})$ and $z_{min} := z^*(\theta_{min})$ denote the maximal and minimal, respectively, earnings levels in the allocation. Let $s_{max} := \max_z s^*(z)$ and $s_{min} := \min_z s^*(z)$ denote the maximal and minimal savings levels.

Step 1: Defining the smooth tax system. We start from a separable and smooth tax system $T_s(s) + T_z(z)$ that satisfies type-specific feasibility and individuals' first-order conditions at the optimal incentive-compatible allocation. We then add quadratic penalty terms parametrized by k for deviations from this allocation. For a given deviation, this allows to make the penalty arbitrarily large and enables us to make the individuals' optimization problems locally concave around the optimal incentive-compatible allocation. The other terms that we add are there to guarantee the smoothness of the penalized tax system $\mathcal{T}(s, z; k)$ at the boundaries of the set of optimal allocations.

Formally, $\mathcal{T}_k = \mathcal{T}(s, z; k)$ is defined by:

1. $T_s(s_{min}) = 0$ and $T_z(z_{min}) = z^*(\theta_{min}) - c^*(\theta_{min}) - s^*(\theta_{min})$
2. $\forall z \in [\, z_{min}\, ;\, z_{max}\,], T_z'(z) = \mathcal{Z}(c^*(\theta_z), s^*(\theta_z), z^*(\theta_z); \theta_z) + 1$ with θ_z such that $z = z^*(\theta_z)$
3. $\forall s \in [\, s_{min}\, ;\, s_{max}\,], T_s'(s) = \mathcal{S}(c^*(\theta_s), s^*(\theta_s), z^*(\theta_s); \theta_s) - 1$ with θ_s such that $s = s^*(\theta_s)$
4. $$\mathcal{T}(s, z; k) = \begin{cases} T_s(s) + T_z(z) + k(s - s^*(z))^2 & \text{if } z_{min} \le z \le z_{max}, \\ & s_{min} \le s \le s_{max} \\ T_s(s_{min}) + T_z(z) + k(s - s^*(z))^2 + T_s'(s_{min})(s - s_{min}) & \text{if } z_{min} \le z \le z_{max}, s < s_{min} \\ T_s(s_{max}) + T_z(z) + k(s - s^*(z))^2 + T_s'(s_{max})(s - s_{max}) & \text{if } z_{min} \le z \le z_{max}, s > s_{max} \\ T_s(s) + T_z(z_{min}) + k(s - s_{min})^2 + k(z - z_{min})^2 & \text{if } z < z_{min}, s_{min} \le s \le s_{max} \\ \qquad + T_z'(z_{min})(z - z_{min}) & \\ T_s(s_{min}) + T_z(z_{min}) + k(s - s_{min})^2 + k(z - z_{min})^2 & \text{if } z < z_{min}, s < s_{min} \\ \qquad + T_z'(z_{min})(z - z_{min}) + T_s'(s_{min})(s - s_{min}) & \\ T_s(s_{max}) + T_z(z_{min}) + k(s - s_{min})^2 + k(z - z_{min})^2 & \text{if } z < z_{min}, s > s_{max} \\ \qquad + T_z'(z_{min})(z - z_{min}) + T_s'(s_{max})(s - s_{max}) & \\ T_s(s) + T_z(z_{max}) + k(s - s_{max})^2 + k(z - z_{max})^2 & \text{if } z > z_{max}, s_{min} \le s \le s_{max} \\ \qquad + T_z'(z_{max})(z - z_{max}) & \\ T_s(s_{max}) + T_z(z_{max}) + k(s - s_{max})^2 + k(z - z_{max})^2 & \text{if } z > z_{max}, s > s_{max} \\ \qquad + T_z'(z_{max})(z - z_{max}) + T_s'(s_{max})(s - s_{max}) & \\ T_s(s_{min}) + T_z(z_{max}) + k(s - s_{max})^2 + k(z - z_{max})^2 & \text{if } z > z_{max}, s < s_{min} \\ \qquad + T_z'(z_{max})(z - z_{max}) + T_s'(s_{min})(s - s_{min}) & \end{cases}$$

Assumptions 1 and 2 guarantee that the separable tax system $T_s(s) + T_z(z)$ is smooth, i.e., a twice continuously differentiable function. Our construction of the penalized tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$ guarantees that it inherits this smoothness property.

Step 2: Local maxima for sufficiently large k. For a given type θ, we show that the bundle $(c^*(\theta), s^*(\theta), z^*(\theta))$ is a local optimum under the tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$ for sufficiently large k. To do so, we first establish that type-specific feasibility is satisfied together with the first-order conditions of type θ's maximization problem. We then show that for sufficiently large k, second-order conditions are also satisfied implying that the intended bundle is a local maximum.

The previous definition of the tax system implies

$$\mathcal{T}'_z(s^*(\theta), z^*(\theta); k) = T'_z(z^*(\theta)) = \mathcal{Z}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) + 1$$
$$\mathcal{T}'_s(s^*(\theta), z^*(\theta); k) = T'_s(s^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$$

meaning type-specific feasibility is satisfied by Lemma C.1 (see below).

Now, defining

$$V(s, z; \theta, k) := U(z - s - \mathcal{T}(s, z; k), s, z; \theta), \tag{83}$$

the first-order conditions for type θ's choice of savings s and earnings z are

$$V'_s(s, z; \theta, k) = -(1 + \mathcal{T}'_s(s, z; k))U'_c(z - s - \mathcal{T}(s, z; k), s, z; \theta) + U'_s(z - s - \mathcal{T}(s, z; k), s, z; \theta) = 0$$
$$V'_z(s, z; \theta, k) = (1 - \mathcal{T}'_z(s, z; k))U'_c(z - s - \mathcal{T}(s, z; k), s, z; \theta) + U'_z(z - s - \mathcal{T}(s, z; k), s, z; \theta) = 0$$

and they are by construction satisfied at $(s^*(\theta), z^*(\theta))$ for each type θ.

Using Lemma C.2 (see below), second-order conditions at $(s^*(\theta), z^*(\theta))$ are

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \leq 0 \tag{84}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \leq 0 \tag{85}$$

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta$$
$$+ \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg] \leq 0 \tag{86}$$

where we denote $s^{*\prime}(z^*) := \frac{s^{*\prime}(\theta)}{z^{*\prime}(\theta)}$.

Here, U, $\mathcal{S}$, and $\mathcal{Z}$ are smooth functions, implying that their derivatives are continuous functions over compact spaces and are thus bounded. Now, by definition of $\mathcal{T}_k = \mathcal{T}(s, z; k)$, we have $\mathcal{T}''_{sz} = -2ks^{*\prime}(z)$ which is negative for any $k \geq 0$ and increasing in magnitude with k.

Noting $U'_c > 0$ and $s^{*\prime}(z) > 0$, this implies that V''_{ss} and V''_{zz} must be negative for sufficiently large k, thanks to the last term, since the other terms are bounded and do not depend on k. By the same reasoning, under the extended Spence-Mirrlees single-crossing assumption that $\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta > 0$, we also have that $(V''_{sz})^2 - V''_{ss}V''_{zz}$ must be negative for sufficiently large k.

This shows that for a given type θ, there exists a k_0 such that for all $k \geq k_0$ the allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ is a local optimum to type θ's maximization problem under the smooth penalized tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$.

Step 3: Global maxima for sufficiently large k. Let $s_{\mathcal{T}_k}(\theta)$ and $z_{\mathcal{T}_k}(\theta)$ denote the level of savings and earnings, respectively, that a type θ chooses given a smooth penalized tax system $\mathcal{T}_k$. To prove implementability of optimal incentive-compatible allocations, we show that there exists a sufficiently large k such that for all θ, $s_{\mathcal{T}_k}(\theta) = s^*(\theta)$ and $z_{\mathcal{T}_k}(\theta) = z^*(\theta)$.

Let's proceed by contradiction, and suppose that it is not the case. Then, there exists an infinite sequence of types θ_k, choosing savings $s_{\mathcal{T}_k}(\theta_k) \neq s^*(\theta_k)$ and earnings $z_{\mathcal{T}_k}(\theta_k) \neq z^*(\theta_k)$ under tax system $\mathcal{T}_k$, and enjoying utility gains from this "deviation" to allocation $(s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$.

First, the fact that we impose quadratic penalties for earnings choices outside of $[\, z_{min}\, ;\, z_{max}\,]$ guarantees that for any $\varepsilon > 0$, there exists k_1, such that for all $k \geq k_1$ and for all types θ, individuals' earnings choices belong to the compact set $[\, z_{min} - \varepsilon\, ;\, z_{max} + \varepsilon\,]$. Indeed, starting from a given earnings level $z > z_{max} + \varepsilon$, the utility change associated with an earnings change is $[(1 - \mathcal{T}_z')\, U_c' + U_z']\, dz$. By construction, the marginal earnings tax rate in this region is $\mathcal{T}_z' = 2k(z - z_{max}) + T_z'(z_{max})$. Noting that $U_c' > 0$, $U_z' < 0$, and that the type space is compact, we can make for all individuals the utility change from a decrease in earnings arbitrarily positive for sufficiently large k. This shows that all individuals choose earnings $z \leq z_{max} + \varepsilon$ for sufficiently large k. Symmetrically, we can show that all individuals choose earnings $z \geq z_{min} - \varepsilon$ for sufficiently large k.

Second, since earnings shape individuals' disposable incomes, the fact that earnings belong to a compact set for sufficiently large k implies that individuals' allocation choices also belong to a compact set. Indeed, for sufficiently large k, individuals' allocation choices must belong to the set of (c, s, z) such that $c \geq 0$, $s \geq 0$, $z \in [\, z_{min} - \varepsilon\, ;\, z_{max} + \varepsilon\,]$, and $c + s \leq z - \mathcal{T}(s, z; k)$ where the tax function is smooth and finite. These constraints make the space of allocations compact.

As a result, the infinite sequence $(\theta_k, s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$ belongs to a compact space of allocations and types, it is thus bounded. By the Bolzano-Weierstrass theorem, this means that there exists a convergent subsequence $\left(\theta_j, s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{\theta}, \hat{s}, \hat{z})$. Since all types θ_j belong to $[\, \theta_{min}\, ;\, \theta_{max}\,]$, we know that the limit type $\hat{\theta}$ must belong to $[\, \theta_{min}\, ;\, \theta_{max}\,]$. Now, from the previous paragraph, individuals' earnings choices have to be arbitrarily close to $[\, z_{min}\, ;\, z_{max}\,]$ as the penalty grows. This implies that the limit $\hat{z}$ must belong to $[\, z_{min}\, ;\, z_{max}\,]$.

Next, we establish that the limit $\hat{s}$ must be such that $\hat{s} = s^*(\hat{z})$. First fix an earnings level $z \in [\, z_{min}\, ;\, z_{max}\,]$. Then, starting from a savings level $s \neq s^*(z)$, the utility change associated with a savings change is $[-(1 + \mathcal{T}_s')\, U_c' + U_s']\, ds$. Assuming without loss of generality that s belongs to $[\, s_{min}\, ;\, s_{max}\,]$, the marginal savings tax rate in this region is $\mathcal{T}_s' = T_s'(s) + 2k(s - s^*(z))$. Noting that $U_c' > 0$, and that U_s' is bounded, we can make the utility gains from a savings change towards $s^*(z)$ arbitrarily large for sufficiently large k. As a result, for any $\varepsilon > 0$, there exists k_2 such that for all $k \geq k_2$, type $\hat{\theta}$ chooses savings $s \in [\, s^*(z) - \varepsilon\, ;\, s^*(z) + \varepsilon\,]$ for a fixed z.[34] Since type $\hat{\theta}$'s savings choice can be made arbitrarily close to $s^*(z)$ for sufficiently large k, we must have at the limit $s = s^*(z)$. Now, because earnings z converge to $\hat{z}$ and the function $s^*(z)$ is by assumption continuous, we must have at the limit $\hat{s} = s^*(\hat{z})$.

We have thus established that the limit $(\hat{\theta}, \hat{s}, \hat{z})$ is such that $\hat{\theta} \in [\, \theta_{min}\, ;\, \theta_{max}\,]$, $\hat{z} \in [\, z_{min}\, ;\, z_{max}\,]$, and $\hat{s} = s^*(\hat{z})$. This means that the limit allocation $(\hat{c}, \hat{s}, \hat{z})$ belongs to the set of optimal incentive-compatible allocations. Given our continuity and monotonicity assumptions, this implies that it is the optimal allocation of some type θ and it has to be by definition that of type $\hat{\theta}$. Hence, $(\hat{c}, \hat{s}, \hat{z}) = (c^*(\hat{\theta}), s^*(\hat{\theta}), z^*(\hat{\theta}))$.

To complete the proof and find a contradiction, fix a value $k^\dagger$ that is large enough such that

[34] A way to see this is that the marginal rate of substitution between consumption and savings $\mathcal{S}$ is continuous on a compact space and thus bounded, whereas the marginal tax rate parametrized by k can be made arbitrarily large. As a result, individuals' first-order conditions can never hold for sufficiently large k, while we can similarly exclude corner solutions for sufficiently large k.

second-order conditions hold for type $\hat{\theta}$ at allocation $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ under tax system $\mathcal{T}_{k^\dagger}$ – this $k^\dagger$ exists by step 2. This implies that there exists an open set N containing $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ such that $V(s, z; \hat{\theta}, k^\dagger)$ is strictly concave over $(s, z) \in N$.

Now, consider types θ^j with $j \geq k^\dagger$. Since these individuals belong to the previously defined subsequence, they prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_j$. Because penalties are increasingly large and $j \geq k^\dagger$, this implies that types θ^j also prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_{k^\dagger}$.

Yet, by continuity, the function $V\left(s, z; \theta_j, k^\dagger\right)$ gets arbitrarily close to the function $V(s, z; \hat{\theta}, k^\dagger)$ for sufficiently large j since $\theta_j \to \hat{\theta}$. For the same reason, $(s^*(\theta_j), z^*(\theta_j)) \to (s^*(\hat{\theta}), z^*(\hat{\theta}))$. Moreover, by definition $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{s}, \hat{z})$. As a result, for any open set $N' \subsetneq N$ containing $(s^*(\hat{\theta}), z^*(\hat{\theta}))$, there exists a $j^\dagger \geq k^\dagger$ such that $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$ and such that both $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ and $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ belong to the set N'.

Since $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$, it has a unique optimum on N'. By definition of $\mathcal{T}_{k^\dagger}$, type $\theta_{j^\dagger}$'s first-order conditions are satisfied at $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$. Hence, $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ is type $\theta_{j^\dagger}$'s maximum under the tax system $\mathcal{T}_{k^\dagger}$. This contradicts the fact established above that type $\theta_{j^\dagger}$ prefers $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ to allocation $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ under tax system $\mathcal{T}_{k^\dagger}$, which completes the proof.

Lemma for type-specific feasibility.

Lemma C.1. *A smooth tax system $\mathcal{T}$ satisfies type-specific feasibility over the compact type space $[\,\theta_{min}\,;\,\theta_{max}\,]$ if it satisfies the following conditions:*

1. $\mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) = z^*(\theta_{min}) - c^*(\theta_{min}) - s^*(\theta_{min})$
2. $\mathcal{T}'_z(s^*(\theta), z^*(\theta)) = \mathcal{Z}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) + 1$
3. $\mathcal{T}'_s(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$

Proof. Consider the type-specific feasible tax system $T^*_\theta(\theta) = z^*(\theta) - s^*(\theta) - c^*(\theta)$, and note that the lemma amounts to showing that $T^*_\theta(\theta) = \mathcal{T}(s^*(\theta), z^*(\theta))$ for all θ. To that end, note that the first-order condition for truthful reporting of θ under the optimal mechanism implies

$$U'_c \cdot (z'(\theta) - s'(\theta) - T^{*\prime}_\theta(\theta)) + U'_s \cdot s'(\theta) + U'_z \cdot z'(\theta) = 0,$$

with derivatives evaluated at the optimal allocation. This can be rearranged as

$$\begin{aligned} T^{*\prime}_\theta(\theta) &= \left(\frac{U'_s}{U'_c} - 1\right) s'(\theta) + \left(\frac{U'_z}{U'_c} + 1\right) z'(\theta) \\ &= \mathcal{T}'_s(s^*(\theta)) s^{*\prime}(\theta) + \mathcal{T}'_z(z^*(\theta)) z^{*\prime}(\theta). \end{aligned}$$

It thus follows that

$$\begin{aligned} \mathcal{T}(s^*(\theta), z^*(\theta)) - \mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) &= \int_{\vartheta=\theta_{min}}^{\vartheta=\theta} \left(\mathcal{T}'_s(s^*(\vartheta)) s^{*\prime}(\vartheta) + \mathcal{T}'_z(z^*(\vartheta)) z^{*\prime}(\vartheta)\right) d\vartheta \\ &= T^*_\theta(\theta) - T^*_\theta(\theta_{min}). \end{aligned}$$

Since $\mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) = T^*_\theta(\theta_{min})$, this implies that $\mathcal{T}(s^*(\theta), z^*(\theta)) = T^*_\theta(\theta)$ for all θ. The smooth tax system $\mathcal{T}$ therefore satisfies type-specific feasibility. □

Lemma on second-order conditions.

Lemma C.2. *Consider a smooth tax system $\mathcal{T}$ satisfying the conditions in Lemma C.1 and define*

$$V(s, z; \theta) := U(z - s - \mathcal{T}(s, z), s, z; \theta). \tag{87}$$

When evaluated at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$, we show that

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \tag{88}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \tag{89}$$

$$\begin{aligned}\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[&\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta \\ &+ \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg]\end{aligned} \tag{90}$$

where we denote $s^{*\prime}(z^*) := \frac{s^{*\prime}(\theta)}{z^{*\prime}(\theta)}$.

Proof. The first-order derivatives are

$$\begin{aligned}V'_s(s, z; \theta) &= -(1 + \mathcal{T}'_s(s, z))U'_c(z - s - \mathcal{T}(s, z), s, z; \theta) + U'_s(z - s - \mathcal{T}(s, z), s, z; \theta) \\ V'_z(s, z; \theta) &= (1 - \mathcal{T}'_z(s, z))U'_c(z - s - \mathcal{T}(s, z), s, z; \theta) + U'_z(z - s - \mathcal{T}(s, z), s, z; \theta).\end{aligned}$$

The second-order derivatives are

$$V''_{ss}(s, z; \theta) = -\mathcal{T}''_{ss}U'_c - (1 + \mathcal{T}'_s)\left(-(1 + \mathcal{T}'_s)U''_{cc} + U''_{cs}\right) - (1 + \mathcal{T}'_s)U''_{cs} + U''_{ss} \tag{91}$$

$$V''_{zz}(s, z; \theta) = -\mathcal{T}''_{zz}U'_c + (1 - \mathcal{T}'_z)\left((1 - \mathcal{T}'_z)U''_{cc} + U''_{cz}\right) + (1 - \mathcal{T}'_z)U''_{cz} + U''_{zz}. \tag{92}$$

To obtain the first result of the Lemma, we compute $\mathcal{T}''_{ss}$ by differentiating both sides of $\mathcal{T}'_s(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$ with respect to θ:

$$\begin{aligned}\mathcal{T}''_{ss}s^{*\prime}(\theta) + \mathcal{T}''_{sz}z^{*\prime}(\theta) &= \frac{d}{d\theta}\mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) \\ &= \mathcal{S}'_c c^{*\prime}(\theta) + \mathcal{S}'_s s^{*\prime}(\theta) + \mathcal{S}'_z z^{*\prime}(\theta) + \mathcal{S}'_\theta,\end{aligned}$$

plugging in $c^{*\prime}(\theta) = (1 - \mathcal{T}'_z)\, z^{*\prime}(\theta) - (1 + \mathcal{T}'_s)\, s^{*\prime}(\theta)$ and denoting $s^{*\prime}(z^*) := s^{*\prime}(\theta)/z^{*\prime}(\theta)$. The previous expression can be rearranged as

$$\mathcal{T}''_{ss} = \mathcal{S}'_c\frac{1 - \mathcal{T}'_z}{s^{*\prime}(z^*)} - \mathcal{S}'_c(1 + \mathcal{T}'_s) + \mathcal{S}'_s + \frac{\mathcal{S}'_z}{s^{*\prime}(z^*)} + \frac{\mathcal{S}'_\theta}{s^{*\prime}(\theta)} - \frac{\mathcal{T}''_{sz}}{s^{*\prime}(z^*)}. \tag{93}$$

Moreover, we differentiate the definition $\mathcal{S} := \frac{U_s'}{U_c'}$ to express the derivative of $\mathcal{S}$ with respect to c as

$$\begin{aligned}\mathcal{S}_c'(c^*(\theta), s^*(\theta), z^*(\theta);\theta) &= \frac{U_c'U_{sc}'' - U_s'U_{cc}''}{(U_c')^2}\\ &= \frac{1}{U_c'}\left(-\frac{U_s'}{U_c'}U_{cc}'' + U_{sc}''\right)\\ &= \frac{1}{U_c'}\left(-(1+\mathcal{T}_s')U_{cc}'' + U_{sc}''\right) \qquad (94)\end{aligned}$$

and the derivative of $\mathcal{S}$ with respect to s as

$$\begin{aligned}\mathcal{S}_s'(c^*(\theta), s^*(\theta), z^*(\theta);\theta) &= \frac{U_c'U_{ss}'' - U_s'U_{cs}''}{(U_c')^2}\\ &= \frac{1}{U_c'}\left(-\frac{U_s'}{U_c'}U_{cs}'' + U_{ss}''\right)\\ &= \frac{1}{U_c'}\left(-(1+\mathcal{T}_s')U_{cs}'' + U_{ss}''\right). \qquad (95)\end{aligned}$$

Substituting equations (93), (94) and (95) into (91), we have

$$\begin{aligned}V_{ss}''\left(s^*\left(\theta\right), z^*\left(\theta\right);\theta\right) &= -U_c'\cdot\left(\mathcal{S}_c'\frac{1-\mathcal{T}_z'}{s^{*\prime}(z)} - \mathcal{S}_c'(1+\mathcal{T}_s') + \mathcal{S}_s' + \frac{\mathcal{S}_z'}{s^{*\prime}(z)} + \frac{\mathcal{S}_\theta'}{s^{*\prime}(\theta)} - \frac{\mathcal{T}_{sz}''}{s^{*\prime}(z)}\right) - (1+\mathcal{T}_s')U_s'\mathcal{S}_c' + U_c'\mathcal{S}_s'\\ &= -U_c'\cdot\left(\frac{1-T_z'}{s^{*\prime}(z)}\mathcal{S}_c' + \frac{1}{s^{*\prime}(z)}\mathcal{S}_z' + \frac{1}{s^{*\prime}(\theta)}\mathcal{S}_\theta' - \frac{\mathcal{T}_{sz}''}{s^{*\prime}(z)}\right)\\ &= \frac{U_z'}{s^{*\prime}(z)}\mathcal{S}_c' - \frac{U_c'}{s^{*\prime}(z^*)}\mathcal{S}_z' - \frac{U_c'}{s^{*\prime}(\theta)}\mathcal{S}_\theta' + \frac{U_c'}{s^{*\prime}(z^*)}\mathcal{T}_{sz}'' \qquad (96)\end{aligned}$$

where we have used $U_z' = -U_c'\left(1-\mathcal{T}_z'\right)$ in the last line.

Similarly, we can obtain the second result of the Lemma by writing $\mathcal{T}_{zz}''$ as

$$\mathcal{T}_{zz}'' = \mathcal{Z}_c'\left(1-\mathcal{T}_z'\right) - \mathcal{Z}_c'\left(1+\mathcal{T}_s'\right)s^{*\prime}(z^*) + \mathcal{Z}_s's^{*\prime}(z^*) + \mathcal{Z}_z' + \frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} - \mathcal{T}_{sz}''s^{*\prime}(z^*). \qquad (97)$$

Using

$$\mathcal{Z}_c' = \frac{1}{U_c'}\left(U_{cz}'' + \left(1-\mathcal{T}_z'\right)U_{cc}''\right)$$

as well as

$$\mathcal{Z}_z' = \frac{1}{U_c'}\left(U_{zz}'' + \left(1-\mathcal{T}_z'\right)U_{cz}''\right)$$

we get

$$V_{zz}''\left(s^*\left(\theta\right), z^*\left(\theta\right);\theta\right) = U_s's^{*\prime}(z^*)\mathcal{Z}_c' - U_c's^{*\prime}(z^*)\mathcal{Z}_s' - U_c'\frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} + U_c'\mathcal{T}_{sz}''s^{*\prime}(z^*). \qquad (98)$$

Finally, to obtain the third result of the Lemma, we must compute $\left(V_{sz}''\right)^2 - V_{ss}''V_{zz}''$. Note that the first-order condition $V_s'(s^*(\theta), z^*(\theta);\theta) = 0$ holds at every θ by construction. Differentiating with respect to θ we get

$$\frac{d}{d\theta}V_s'(s^*(\theta), z^*(\theta); \theta) = V_{ss}''s^{*\prime}(\theta) + V_{sz}''z^{*\prime}(\theta) + V_{s\theta}'' = 0 \tag{99}$$

which we can rearrange as

$$-V_{sz}'' = V_{ss}''s^{*\prime}(z^*) + \frac{V_{s\theta}''}{z^{*\prime}(\theta)}. \tag{100}$$

Similarly, by totally differentiating the first-order condition $V_z'(s^*(\theta), z^*(\theta); \theta) = 0$ and rearranging we find

$$-V_{sz}'' = \frac{V_{zz}''}{s^{*\prime}(z^*)} + \frac{V_{z\theta}''}{s^{*\prime}(\theta)}. \tag{101}$$

Writing $(V_{sz}'')^2$ as the product of the right-hand sides of equations (100) and (101) yields

$$\begin{aligned}\left(V_{sz}''\right)^2 =& \left(V_{ss}''s^{*\prime}(z) + \frac{V_{s\theta}''}{z^{*\prime}(\theta)}\right)\left(\frac{V_{zz}''}{s^{*\prime}(z)} + \frac{V_{z\theta}''}{s^{*\prime}(\theta)}\right) \\ =& V_{ss}''V_{zz}'' + \frac{1}{z^{*\prime}(\theta)}V_{ss}''V_{z\theta}'' + \frac{1}{s^{*\prime}(\theta)}V_{zz}''V_{s\theta}'' + \frac{1}{s^{*\prime}(\theta)z^{*\prime}(\theta)}V_{s\theta}''V_{z\theta}''.\end{aligned} \tag{102}$$

Now from the definition $V(s, z; \theta) := U(z - s - \mathcal{T}(s, z), s, z; \theta)$, we can compute

$$\begin{aligned}V_{s\theta}''(s, z; \theta) &= -\left(1 + \mathcal{T}_s'(s, z)\right)U_{c\theta}'' + U_{s\theta}'' \\ V_{z\theta}''(s, z; \theta) &= \left(1 - \mathcal{T}_z'(s, z)\right)U_{c\theta}'' + U_{z\theta}''\end{aligned}$$

and use the fact that at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ we have

$$\begin{aligned}\mathcal{S}_\theta' &= \frac{1}{U_c'}\left(U_{s\theta}'' - \left(1 + \mathcal{T}_s'\right)U_{c\theta}''\right) \\ \mathcal{Z}_\theta' &= \frac{1}{U_c'}\left(U_{z\theta}'' + \left(1 - \mathcal{T}_z'\right)U_{c\theta}''\right)\end{aligned}$$

to obtain

$$V_{s\theta}''(s^*(\theta), z^*(\theta); \theta) = U_c'\mathcal{S}_\theta' \tag{103}$$

$$V_{z\theta}''(s^*(\theta), z^*(\theta); \theta) = U_c'\mathcal{Z}_\theta'. \tag{104}$$

Substituting these into equation (102) and rearranging, we have

$$\left(V_{sz}''\right)^2 - V_{ss}''V_{zz}'' = \frac{1}{z^{*\prime}(\theta)}V_{ss}''U_c'\mathcal{Z}_\theta' + \frac{1}{s^{*\prime}(\theta)}V_{zz}''U_c'\mathcal{S}_\theta' + \frac{1}{s^{*\prime}(\theta)z^{*\prime}(\theta)}\left(U_c'\right)^2\mathcal{S}_\theta'\mathcal{Z}_\theta'. \tag{105}$$

Expanding V''_{ss} from equation (96), and V''_{zz} from equation (98) yields after simplification

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\left[\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta \right.$$
$$\left. + \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\right],$$

which gives the third result of the Lemma above. □

C.3 Proof of Proposition B.1 & B.2 (Implementation with Simple Tax Systems)

C.3.1 Proof of Proposition B.1

SN tax system. The sufficient conditions for local optimality under the candidate SN tax system follow directly from Lemma C.2 which computes individuals' second-order conditions (SOCs) at the optimal incentive-compatible allocation under a general tax system $\mathcal{T}(s,z)$. Indeed, individuals' SOCs are satisfied if equations (88), (89), and (90) are negative under the SN tax system. Since the cross-partial derivative $\mathcal{T}''_{sz}$ is zero for a SN tax system, it is easy to verify that conditions (49) and (50) on the derivatives of $\mathcal{S}$ and $\mathcal{Z}$, combined with monotonicity ($s^{*\prime}(\theta) > 0$, $s^{*\prime}(z) > 0$) and Assumption 1 on the derivatives of U, jointly imply that each of these three equations is the sum of negative terms. This implies that individuals' SOCs are satisfied at the optimal incentive-compatible allocation under the candidate SN tax system.

LED tax system. To derive sufficient conditions for local optimality under the candidate LED tax system, we begin from results obtained in the derivations of Lemma C.2 which computes individuals' SOCs at the optimal incentive-compatible allocation. We consider the requirements $V''_{ss} < 0$, $V''_{zz} < 0$, and $V''_{ss}V''_{zz} > (V''_{sz})^2$ in turn.

First, to show that V''_{ss} is negative, note that under a LED tax system, $\mathcal{T}''_{ss} = 0$. Therefore, using the fact that under the candidate LED tax system we have $1 + \mathcal{T}'_s = \frac{U'_s}{U'_c}$ at the optimal incentive-compatible allocation, the general expression for V''_{ss} given in equation (91) reduces to

$$V''_{ss}(s^*(\theta), z^*(\theta); \theta) = \left(\frac{U'_s}{U'_c}\right)^2 U''_{cc} - 2\frac{U'_s}{U'_c}U''_{cs} + U''_{ss}.$$

Therefore when utility is additively separable in c and s (implying $U''_{cs} = 0$), the concavity of preferences ($U''_{cc} \leq 0$ and $U''_{ss} \leq 0$) guarantees that this expression is negative.

Second, to show that V''_{zz} is negative, note that under the candidate LED tax system defined in equations (47) and (48) we have

$$\mathcal{T}''_{sz}(s,z) = \tau'_s(z).$$

We can thus find an expression for $\tau'_s(z)$ at any point in the allocation in question by totally differentiating equation (47) with respect to θ:

$$\begin{aligned}\tau_s'\left(z^*(\theta)\right) z^{*\prime}(\theta) &= \frac{d}{d\theta}\Big[\mathcal{S}\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\Big] \\ &= \frac{d}{d\theta}\Big[\mathcal{S}\left(z^*(\theta) - s^*(\theta) - \mathcal{T}\left(s^*(\theta), z^*(\theta)\right), s^*(\theta), z^*(\theta); \theta\right)\Big] \\ &= \mathcal{S}_c' \cdot \left[(1-\mathcal{T}_z')z^{*\prime}(\theta) - \left(1+\mathcal{T}_s'\right) s^{*\prime}(\theta)\right] + \mathcal{S}_s' s^{*\prime}(\theta) + \mathcal{S}_z' z^{*\prime}(\theta) + \mathcal{S}_\theta',\end{aligned}$$

which yields

$$\tau_s'\left(z^*(\theta)\right) = \mathcal{S}_c' \cdot \left(1-\mathcal{T}_z'\right) - \mathcal{S}_c' \cdot \left(1+\mathcal{T}_s'\right) s^{*\prime}(z^*) + \mathcal{S}_s' \cdot s^{*\prime}(z^*) + \mathcal{S}_z' + \frac{\mathcal{S}_\theta'}{z^{*\prime}(\theta)}.$$

Substituting this into the expression for V_{zz}'' in (98), we have

$$\begin{aligned}V_{zz}''\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) &= U_s' s^{*\prime}(z^*)\mathcal{Z}_c' - U_c' s^{*\prime}(z^*)\mathcal{Z}_s' - U_c' \frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} \\ &\quad + U_c' s^{*\prime}(z^*)\left[\mathcal{S}_c' \cdot \left(1-\mathcal{T}_z'\right) - \mathcal{S}_c' \cdot \left(1+\mathcal{T}_s'\right) s^{*\prime}(z^*) + \mathcal{S}_s' \cdot s^{*\prime}(z^*) + \mathcal{S}_z' + \frac{\mathcal{S}_\theta'}{z^{*\prime}(\theta)}\right].\end{aligned} \tag{106}$$

Now employing the assumption that utility is separable in c, s, and z, (implying both $U_{cz}'' = 0$ and $U_{cs}'' = 0$) we have

$$\begin{aligned}U_s'\mathcal{Z}_c' + U_c'\mathcal{S}_c'(1-\mathcal{T}_z') &= U_s'\mathcal{Z}_c' - U_z'\mathcal{S}_c' \\ &= U_s' \frac{U_c'U_{cz}'' - U_z'U_{cc}''}{\left(U_c'\right)^2} - U_z' \frac{U_c'U_{cs}'' - U_s'U_{cc}''}{\left(U_c'\right)^2} \\ &= 0.\end{aligned}$$

Substituting this result into equation (106), and noting that $\mathcal{Z}_s' = \mathcal{S}_z' = 0$ by separability, yields

$$V_{zz}''(s^*(\theta), z^*(\theta); \theta) = \left(s^{*\prime}(z^*)\right)^2\left[U_c'\mathcal{S}_s' - U_s'\mathcal{S}_c'\cdot\right] - \frac{U_c'}{z^{*\prime}(\theta)}\left[\mathcal{Z}_\theta' - s^{*\prime}(z^*)\mathcal{S}_\theta'\right]. \tag{107}$$

Again employing separability, we have

$$U_c'\mathcal{S}_s' - U_s'\mathcal{S}_c' = U_c' \frac{U_c'U_{ss}'' - U_s'U_{cs}''}{\left(U_c'\right)^2} - U_s' \frac{U_c'U_{cs}'' - U_s'U_{cc}''}{\left(U_c'\right)^2} = U_{ss}'' + \left(\frac{U_s'}{U_c'}\right)^2 U_{cc}'' \leq 0,$$

implying that the first term on the right-hand side of equation (107) is negative. The condition $\mathcal{Z}_\theta' - s^{*\prime}(z^*)\mathcal{S}_\theta' \geq 0$ from (51) in the Proposition then implies equation (107) (and thus V_{zz}'') is negative.

Third, to show $V''_{ss}V''_{zz} > (V''_{sz})^2$, we proceed from equation (90) in Lemma C.2:

$$\begin{aligned}
&\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} \\
&= \frac{U'_c}{s^{*\prime}(\theta)}\left[\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta + \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\right] \\
&= \left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta \\
&+ \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta U'_c\mathcal{T}''_{sz} + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta\left(U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{z^{*\prime}(\theta)} + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz}\right).
\end{aligned}$$

Recognizing that the last bracket term is exactly the expression for V''_{zz} given in Lemma C.2, this gives

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta U'_c\mathcal{T}''_{sz} + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta V''_{zz}.$$

Using the previous expression derived for $\mathcal{T}''_{sz} = \tau'_s$, and the fact that separability ensures $\mathcal{S}'_z = 0$, we obtain after simplification

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = -\frac{(U'_c)^2}{s^{*\prime}(\theta)z^{*\prime}(\theta)}\mathcal{Z}'_\theta\left[s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right) - \mathcal{S}'_\theta\right] + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta V''_{zz}.$$

We have already shown that V''_{zz} is negative. Thus the conditions $\mathcal{S}'_\theta \geq 0$ and $\mathcal{S}'_\theta \leq s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right)$ from (51) in the Proposition imply that both terms on the right-hand side are negative, implying that all second-order conditions hold.

C.3.2 Proof of Proposition B.2

We begin with a more general statement, and then derive Proposition B.2 as a corollary. For a fixed type θ, let $c(z,\theta)$ and $s(z,\theta)$ be its preferred consumption and savings choices at earnings z, given the budget constraint induced by $\mathcal{T}(s,z)$.

Lemma C.3. *Suppose that $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, where $z^*(\theta)$ is increasing. Individuals' local optima correspond to their global optima when*

1. *$\mathcal{Z} = \frac{U'_z(c,s,z;\theta)}{U'_c(c,s,z;\theta)}$ and $\mathcal{S} = \frac{U'_s(c,s,z;\theta)}{U'_c(c,s,z;\theta)}$ are strictly increasing in θ for all (c,s,z).*
2. *For any two types θ and θ', we cannot have both*

$$\begin{aligned}
&U'_c\Big(c^*(\theta), s^*(\theta), z^*(\theta);\theta\Big)\sigma_c\Big(s^*(\theta), z^*(\theta)\Big) + U'_z\Big(c^*(\theta), s^*(\theta), z;\theta\Big) \\
&< U'_c\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)\sigma_c\Big(s\left(z^*(\theta),\theta'\right), z^*(\theta)\Big) \\
&+ U'_z\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)
\end{aligned} \tag{108}$$

and

$$\begin{aligned} &U_s'\Big(c^*(\theta), s^*(\theta), z^*(\theta); \theta\Big)\sigma_c\Big(s^*(\theta), z^*(\theta)\Big) + U_z'\Big(c^*(\theta), s^*(\theta), z; \theta\Big)\Big) \\ &< U_s'\Big(c\left(z^*(\theta), \theta'\right), s\left(z^*(\theta), \theta'\right), z^*(\theta); \theta\Big)\sigma_s\Big(s\left(z^*(\theta), \theta'\right), z^*(\theta)\Big) \\ &+ U_z'\Big(c\left(z^*(\theta), \theta'\right), s\left(z^*(\theta), \theta'\right), z^*(\theta); \theta \end{aligned} \tag{109}$$

where $\sigma_c(s,z) := 1 - \mathcal{T}_z'(s,z)$ *and* $\sigma_s(s,z) := \frac{1-\mathcal{T}_z'(s,z)}{1+\mathcal{T}_s'(s,z)}$.

Condition 1 corresponds to single-crossing assumptions for earnings and savings. Condition 2 is a requirement that if type θ preserves its assigned earnings level $z^*(\theta)$, but chooses some other consumption level s (corresponding to a level that some other type θ' would choose if forced to choose earnings level $z^*(\theta)$), then at this alternative consumption bundle, type θ cannot have both higher marginal utility from increasing its savings through one more unit of work *and* increasing its consumption through one more unit of work. Generally, this condition will hold as long as U is sufficiently concave in consumption and savings when type θ chooses earnings level $z^*(\theta)$.

Proof. To prove that individuals' local optima are global optima, we want to show that for any given type θ^*, utility decreases when moving from allocation $(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*))$ to allocation $(c(z,\theta^*), s(z,\theta^*), z)$.

The first step is to compute type θ^*'s utility change. The envelope theorem applied to savings choices $s(z,\theta^*)$ implies

$$\begin{aligned} &\frac{d}{dz}U\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right) \\ &= U_c'\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right)\sigma_c\left(s(z,\theta^*), z\right) + U_z'\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right) \end{aligned}$$

where $\sigma_c(s,z) = 1 - \mathcal{T}_z'(s,z)$. Note that, as established by Milgrom and Segal (2002), these equalities hold as long as U is differentiable in z (holding s and c fixed)—differentiability of $c(z,\theta^*)$ or $s(z,\theta^*)$ is actually not required.

Similarly, the envelope theorem applied to consumption choices $c(z,\theta^*)$ implies

$$\begin{aligned} &\frac{d}{dz}U\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right) \\ &= U_s'\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right)\sigma_s\left(s(z,\theta^*), z\right) + U_z'\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right) \end{aligned} \tag{110}$$

where $\sigma_s(s,z) = \frac{1-\mathcal{T}_z'(s,z)}{1+\mathcal{T}_s'(s,z)}$.

Therefore, type θ^*'s utility change when moving from allocation $(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*))$ to allocation $(c(z,\theta^*), s(z,\theta^*), z)$ is

$$\begin{aligned} &U\left(c(z,\theta^*), s(z,\theta^*), z; \theta^*\right) - U\left(c(z^*(\theta^*),\theta^*), s(z^*(\theta^*),\theta^*), z^*(\theta^*); \theta^*\right) \\ &= \int_{x=z^*(\theta^*)}^{x=z}\Big[\min\left\{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)\sigma_c\left(s(x,\theta^*), x\right), U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)\sigma_s\left(s(x,\theta^*), x\right)\right\} \\ &\qquad + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta^*\right)\Big]dx \end{aligned} \tag{111}$$

where the $\min$ operator is introduced without loss of generality given that both terms are equal.

The second step is to show that under our assumptions, type θ^*'s utility change in equation (111) is negative. To do so, let θ_x be the type that chooses earnings x. Then, by definition, type θ_x's utility is maximal at earnings x, implying both

$$U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \sigma_c\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) = 0$$
$$U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \sigma_s\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) = 0$$

such that

$$\max\left\{U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \sigma_c\left(s^*(\theta_x), x\right), U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \sigma_s\left(s^*(\theta_x), x\right)\right\} + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) = 0. \tag{112}$$

Now, by condition 2, we either have[35]

$$U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \sigma_c\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \geq U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \sigma_c\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)$$

or

$$U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \sigma_s\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \geq U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \sigma_s\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)$$

implying that

$$\begin{aligned}&\max\left\{U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \sigma_c\left(s^*(\theta_x), x\right), U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \sigma_s\left(s^*(\theta_x), x\right)\right\} \\ &+ U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \\ &\geq \min\left\{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \sigma_c\left(s(x,\theta^*), x\right), U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \sigma_s\left(s(x,\theta^*), x\right)\right\} \\ &+ U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right).\end{aligned} \tag{113}$$

But since the maximum is zero, this minimum has to be negative. Hence, we have either

$$U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \sigma_c\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \leq 0$$
$$\iff \frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)}{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right)$$

or

$$U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \sigma_s\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \leq 0$$
$$\iff \frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)}{U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right).$$

Suppose that $z > z^*(\theta^*)$ such that $x > z^*(\theta^*)$; the case $z < z^*(\theta^*)$ follows identically. For

[35]Not having $\{a < c \quad and \quad b < c\}$ means having $\{a \geq c \quad \text{or} \quad b \geq d\}$, which implies $\max(a,b) \geq \min(c,d)$.

any $x > z^*(\theta^*)$, the monotonicity of the earnings function means that $\theta_x > \theta^*$. Then, by the single-crossing conditions for $\mathcal{Z} = \frac{U_z'}{U_c'}$ and $\mathcal{S} = \frac{U_s'}{U_c'}$, this means that we have either[36]

$$\frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)}{U_c'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right)$$

or

$$\frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)}{U_s'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right)$$

implying that for any $x > z^*(\theta^*)$,

$$\begin{aligned} &\min\left\{U_c'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\sigma_c\left(s(x,\theta^*), x\right), U_s'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\sigma_s\left(s(x,\theta^*), x\right)\right\} \\ &+ U_z'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right) \leq 0. \end{aligned} \tag{114}$$

As a result, the right hand-side of equation (111) is an integral of negative terms, which shows that

$$U\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right) - U\left(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*);\theta^*\right) \leq 0. \tag{115}$$

The case with $z < z^*(\theta^*)$ follows identically, proving Lemma C.3. □

Proof of Proposition B.2

We now derive Proposition B.2 as a consequence of Lemma C.3 by deriving assumptions under which condition 2 is met for SN and LED tax systems.

SN systems. First, suppose that $s < s^*(\theta)$, then $c > c^*(\theta)$. Noting that $\sigma_c = 1 - T_z'\left(z^*\left(\theta\right)\right)$ is not a function of s, we can use $U_{cc}'' \leq 0$ and $U_{cs}'' \geq 0$ to obtain

$$U_c'\left(c^*(\theta), s^*(\theta), z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right) \geq U_c'\left(c, s, z^*(\theta);\theta\right)\sigma_c\left(s, z^*(\theta)\right).$$

Further relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we obtain

$$\begin{aligned} &U_c'\left(c^*(\theta), s^*(\theta), z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right) + U_z'\left(c^*(\theta), s^*(\theta), z^*(\theta);\theta\right) \\ &\geq U_c'\left(c, s, z^*(\theta);\theta\right)\sigma_c\left(s, z^*(\theta)\right) + U_z'\left(c, s, z^*(\theta);\theta\right). \end{aligned}$$

Conversely, suppose that $s > s^*(\theta)$, then $c < c^*(\theta)$. We have

$$\begin{aligned} &\frac{d}{ds}\left[\frac{U_s'(z - T_z\left(z\right) - s - T_s\left(s\right), s, z^*(\theta);\theta)}{1 + T_s'(s)}\right] \\ &= -U_{cs}'' + \frac{1}{(1 + T_s'(s))}\left[U_{ss}'' - U_s'\frac{T_{ss}''(s)}{1 + T_s'(s)}\right]. \end{aligned}$$

The condition that $\frac{U_{ss}''(c(s,\theta),s,z^*(\theta);\theta)}{U_s'(c(s,\theta),s,z^*(\theta);\theta)} < \frac{T_{ss}''(s)}{1+T_s'(s)}$, together with $U_{cs}'' > 0$, implies that $\frac{U_s'(c(s,\theta),s,z^*(\theta);\theta)}{1+T_s'(s)}$

[36]Note that having both $\mathcal{Z}$ and $\mathcal{S}$ increasing in θ also implies that $\frac{\mathcal{Z}}{\mathcal{S}} = \frac{U_z'}{U_s'}$ is increasing in θ.

is decreasing in s and thus that

$$\frac{U_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta)}{1 + T_s'(s^*(\theta))} \geq \frac{U_s'(c, s, z^*(\theta); \theta)}{1 + T_s'(s)}.$$

Further relying on the fact that $U''_{cz} = 0$ and $U''_{sz} = 0$, and that $\mathcal{T}_s' = T_z'(z)$ is independent of s, we obtain

$$\begin{aligned}
&U_s'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right) \sigma_s\left(s^*(\theta), z^*(\theta)\right) + U_z'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right) \\
&\geq U_s'\left(c, s, z^*(\theta); \theta\right) \sigma_s\left(s, z^*(\theta)\right) + U_z'\left(c, s, z^*(\theta); \theta\right).
\end{aligned}$$

LED systems. First, consider a type θ' choosing earnings $z = z^*(\theta) > z^*(\theta')$. We have

$$\begin{aligned}
&\frac{d}{ds}\left[U_c'\left(z - s - \tau_s\left(z^*(\theta)\right)s - T_z\left(z^*(\theta)\right), s, z^*(\theta); \theta\right)\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right)\right] \\
&= U''_{cs}\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) - U''_{cc}\left(1 + \tau_s(z^*(\theta))\right)\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) - U_c'\tau_s'\left(z^*(\theta)\right).
\end{aligned}$$

The first term is negative because $U''_{cs} \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z^*(\theta) > z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') < s^*(\theta)$. In this case, condition *(2a)* of the proposition implies that the remaining terms are negative such that

$$U_c'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right)\sigma_c\left(s, z^*(\theta)\right)$$

is increasing in s for $s < s^*(\theta)$, where $\sigma_c\left(s, z^*(\theta)\right) = 1 - T_z'(z) - \tau_s'(z)s$. As a result,

$$\begin{aligned}
&U_c'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right) \\
&\geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right)
\end{aligned}$$

and thus relying on the fact that $U''_{cz} = 0$ and $U''_{sz} = 0$, we have

$$\begin{aligned}
&U_c'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right) + U_z'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right) \\
&\geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) + U_z'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right).
\end{aligned}$$

Second, consider a type θ' choosing $z = z^*(\theta) < z^*(\theta')$. We have

$$\begin{aligned}
&\frac{d}{ds}\left[U_s'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right)\frac{1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s}{1 + \tau_s(z)}\right] \\
&= -U''_{cs}\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) + U''_{ss}\frac{1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s}{1 + \tau_s(z)} + U_s'\frac{\tau_s'\left(z^*(\theta)\right)}{1 + \tau_s(z)}.
\end{aligned}$$

The first term is negative because $U''_{cs} \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z = z^*(\theta) < z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') > s^*(\theta)$. Hence, condition *(2b)* of the proposition implies that the remaining terms are negative such that

$$U_s'\left(z-s-\tau_s(z^*(\theta))s-T(z^*(\theta)),s,z^*(\theta);\theta\right)\sigma_s\left(s,z^*(\theta)\right)$$

is decreasing in s for $s>s^*(z)$, where $\sigma_s\left(s,z^*(\theta)\right)=\frac{1-T_z'(z)-\tau_s'(z)s}{1+\tau_s(z)}$. This ensures that

$$\begin{aligned}&U_s'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta),z^*(\theta)\right)\\&\geq U_s'\left(c(z^*(\theta),\theta'),s(z^*(\theta),\theta'),z^*(\theta);\theta\right)\sigma_c\left(s(z^*(\theta),\theta'),z^*(\theta)\right)\end{aligned}$$

and thus, relying on the fact that $U_{cz}''=0$ and $U_{sz}''=0$, we have

$$\begin{aligned}&U_s'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta),z^*(\theta)\right)+U_z'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\\&\geq U_s'\left(c(z^*(\theta),\theta'),s(z^*(\theta),\theta'),z^*(\theta);\theta\right)\sigma_c\left(s(z^*(\theta),\theta'),z^*(\theta)\right)+U_z'\left(c(z^*(\theta),\theta'),s(z^*(\theta),\theta'),z^*(\theta);\theta\right).\end{aligned}$$

C.4 Proof of Proposition 2 (Measurement of Causal Income Effects)

Here, we derive that the different expressions of the sufficient statistic $s_{inc}'(z)$ can be measured empirically.

Case 1. If individuals' preferences are weakly separable between the utility of consumption $u(.)$ and the disutility to work $k(.)$, type θ's problem is written as

$$\max_{c,s,z}\ u\left(c,s;\theta\right)-k\left(z/w(\theta)\right)\ s.t.\ c\leq z-ps-\mathcal{T}\left(s,z\right),$$

meaning that conditional on earnings z, savings $s\left(z;\theta\right)$ is defined as the solution to

$$-\left(p+\mathcal{T}_s'\left(s,z\right)\right)u_c'\left(z-ps-\mathcal{T}\left(s,z\right),s;\theta\right)+u_s'\left(z-ps-\mathcal{T}\left(s,z\right),s;\theta\right)=0.$$

Differentiating this equation with respect to savings s and earnings z yields

$$\frac{\partial s}{\partial z}=-\frac{\left[-\mathcal{T}_{sz}''u_c'-\left(p+\mathcal{T}_s'\right)\left(1-\mathcal{T}_z'\right)u_{cc}''+\left(1-\mathcal{T}_z'\right)u_{cs}''\right]}{\left[-\mathcal{T}_{ss}''u_c'+\left(p+\mathcal{T}_s'\right)^2u_{cc}''-2\left(p+\mathcal{T}_s'\left(s,z\right)\right)u_{cs}''+u_{ss}''\right]}.$$

Differentiating this equation with respect to savings s and disposable income y yields

$$\frac{\partial s}{\partial y}=-\frac{\left[-\left(p+\mathcal{T}_s'\right)u_{cc}''+u_{cs}''\right]}{\left[-\mathcal{T}_{ss}''u_c'+\left(p+\mathcal{T}_s'\right)^2u_{cc}''-2\left(p+\mathcal{T}_s'\left(s,z\right)\right)u_{cs}''+u_{ss}''\right]}.$$

Hence, if $\mathcal{T}_{sz}''=0$, we get

$$s_{inc}'\left(z\right):=\frac{\partial s\left(z;\theta\right)}{\partial z}=\left(1-\mathcal{T}_z'\right)\frac{\partial s}{\partial y}=\left(1-\mathcal{T}_z'\right)\frac{\eta_{s|z}\left(z(\theta)\right)}{1+\mathcal{T}_s'},$$

where the last equality follows from the definition of $\eta_{s|z}\left(z(\theta)\right)$. The intuition behind this result is that with separable preferences, savings s depend on earnings z only through disposable income

$y = z - ps - \mathcal{T}(s, z)$.

Case 2. If individuals' wage rates w and hours h are observable, and earnings z are given by $z = w \cdot h$, we can infer s'_{inc} from changes in wages through

$$\begin{aligned}
\frac{\partial s}{\partial w} &= \frac{\partial s(w \cdot h; \theta)}{\partial w} = \frac{\partial s(z; \theta)}{\partial z}\left(1 + \frac{\partial h}{\partial w}\right) \\
\iff \frac{\partial s(z; \theta)}{\partial z} &= \frac{\frac{\partial s}{\partial w}}{1 + \frac{\partial h}{\partial w}} = s\frac{\frac{w}{s}\frac{\partial s}{\partial w}}{w + h\frac{w}{h}\frac{\partial h}{\partial w}} \\
\iff s'_{inc}(z) &= s(z)\frac{\xi^s_w(z)}{w(z) + h(z)\xi^h_w(z)}
\end{aligned}$$

where $\xi^s_w(z) \equiv \frac{w(z)}{s(z)}\frac{\partial s(z)}{\partial w(z)}$ is individuals' elasticity of savings with respect to their wage rate, and $\xi^h_w(z) \equiv \frac{w(z)}{h(z)}\frac{\partial h(z)}{\partial w(z)}$ is individuals' elasticity of hours with respect to their wage rate.

Case 3. Otherwise, if we can measure the elasticity of savings s and earnings z upon a compensated change in the marginal earnings tax rate $\mathcal{T}'_z$, respectively denoted $\chi^c_s := -\frac{1-\mathcal{T}'_z}{s}\frac{\partial s}{\partial \mathcal{T}'_z}$ and $\zeta^c_z := -\frac{1-\mathcal{T}'_z}{z}\frac{\partial z}{\partial \mathcal{T}'_z}$, we then have

$$\begin{aligned}
\frac{\partial s}{\partial \mathcal{T}'_z} &= \frac{\partial s(z; \theta)}{\partial z}\frac{\partial z}{\mathcal{T}'_z} \\
\iff \left(-\frac{s}{1-\mathcal{T}'_z}\chi^c_s\right) &= s'_{inc}(z)\left(-\frac{z}{1-\mathcal{T}'_z}\zeta^c_z\right) \\
\iff s'_{inc}(z) &= \frac{s(z)}{z}\frac{\chi^c_s(z)}{\zeta^c_z(z)}.
\end{aligned}$$

C.5 Proof of Lemma 1 (Earnings Responses to Taxes on s)

Throughout the paper, we characterize earnings responses to (different) savings tax reforms using generalizations of Lemma 1 in Saez (2002). The robust insight in all cases is that a $\Delta\tau$ increase in the marginal tax rate on s induces the same earnings changes (through substitution effects) as a $s'_{inc}(z)\Delta\tau$ increase in earnings tax rate. This is what appears in the body of the text as Lemma 1.

In our proofs we use a version that pertains to reforms that have an LED, SL, or SN structure. For example, a reform with LED structure adds a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0, z^0 + \Delta z\right]$. Note that the reform itself has an LED structure, but it can applied to any nonlinear tax system, not just one with an LED structure. The results below allow for multidimensional heterogeneity.

Let

$$V(\mathcal{T}(., z), z; \theta) = \max_s U(z - ps - \mathcal{T}(s, z), s, z; \theta)$$

be type θ's indirect utility function at earnings z.

LED reform. Consider a tax reform $\Delta\mathcal{T}_s$ that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$, that is:[37]

$$\Delta\mathcal{T}_s(s,z) = \begin{cases} 0 & if\ z \leq z^0 \\ \Delta\tau_s\,(z-z^0)\,s & if\ z \in [z^0, z^0+\Delta z] \\ \Delta\tau_s\,\Delta z\, s & if\ z \geq z^0+\Delta z \end{cases}$$

We now construct for each type θ a tax reform $\Delta\mathcal{T}_z^\theta$ that affects marginal earnings tax rates, and induces the same earnings response as the initial perturbation $\Delta\mathcal{T}_s$. We define this perturbation for each type θ such that at all earnings z,

$$V(\mathcal{T}(.,z)+\Delta\mathcal{T}_s(.,z),z;\theta) = V(\mathcal{T}(.,z)+\Delta\mathcal{T}_z^\theta(.,z),z;\theta).$$

Then, by construction, the perturbation $\Delta\mathcal{T}_z^\theta$ induces the same earnings response dz as the initial perturbation $\Delta\mathcal{T}_s$. Moreover, both tax reforms must induce the same utility change for type θ. To compute these utility changes, we make use of the envelope theorem.

For types θ with earnings $z(\theta) \in [z^0, z^0+\Delta z]$, this implies

$$\begin{aligned} &U_c'\,\Delta\tau_s\left(z-z^0\right)s\,(z;\theta) = U_c'\,\Delta\mathcal{T}_z^\theta(z) \\ \iff\ &\Delta\mathcal{T}_z^\theta(z) = \Delta\tau_s\left(z-z^0\right)s\,(z;\theta)\,. \end{aligned}$$

Differentiating both sides with respect to z and letting $\Delta z \to 0$, this implies that in the phase-in region, the reform induces the same earnings change as a small increase $s'_{inc}\,(z)\,\Delta\tau_s$ in the marginal earnings tax rate.

For types θ with earnings $z(\theta) \geq z^0+\Delta z$, this implies

$$\begin{aligned} &U_c'\,\Delta\tau_s\,\Delta z\,s\,(z;\theta) = U_c'\,\Delta\mathcal{T}_z^\theta(z) \\ \iff\ &\Delta\mathcal{T}_z^\theta(z) = \Delta\tau_s\,\Delta z\,s\,(z;\theta)\,. \end{aligned}$$

That is, above the phase-in region, the reform induces the same earnings changes as a $\Delta\tau_s\Delta z\,s\,(z)$ increase in tax liability combined with a $\Delta\tau_s\Delta z\,s'_{inc}\,(z)$ increase in the marginal earnings tax rate.

SL reform. Consider a tax reform $\Delta\mathcal{T}_s$ that consists in adding a linear tax rate $\Delta\tau_s$ on s for all individuals. This is a special case of a LED reform. As a result, we directly obtain that this reform induces the same earnings changes as a $\Delta\tau_s\,s\,(z)$ increase in tax liability combined with a $\Delta\tau_s\,s'_{inc}\,(z)$ increase in the marginal earnings tax rate.

[37] This reform, which is natural to consider for LED tax systems, allows us to derive a sufficient statistics characterization of the optimal smooth tax system (Theorem 2) without the requirement that $s(z)$ is monotonic. In contrast, if we were to rely on an increase in the marginal savings tax rates over a certain bandwidth of savings, which is natural to consider for SN tax systems, we would need further assumptions.

SN reform. Consider a tax reform $\Delta\mathcal{T}_s$ that consists in a small increase $\Delta\tau_s$ in the marginal tax rate on s in a bandwidth $\left[s^0, s^0+\Delta s\right]$, with $\Delta\tau_s$ much smaller than Δs:

$$\Delta\mathcal{T}_s(s,z) = \begin{cases} 0 & if\ s \leq s^0 \\ \Delta\tau_s(s-s^0) & if\ s \in [s^0, s^0+\Delta s] \\ \Delta\tau_s \Delta s & if\ s \geq s^0+\Delta s \end{cases}$$

We now construct for each type θ a perturbation of the earnings tax $\Delta\mathcal{T}_z^\theta$ that induces the same earnings response as the initial perturbation $\Delta\mathcal{T}_s$. Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(\mathcal{T}(.,z)+\Delta\mathcal{T}_s(.,z),z;\theta) = V(\mathcal{T}(.,z)+\Delta\mathcal{T}_z^\theta(.,z),z;\theta).$$

Then, by construction, the perturbation $\Delta\mathcal{T}_z^\theta$ induces the same earnings response dz as the initial perturbation $\Delta\mathcal{T}_s$. Moreover, both tax reforms must induce the same utility change for type θ. To compute these utility changes, we make use of the envelope theorem.

For types θ with $s(z;\theta) \in [s^0, s^0+\Delta s]$, this implies

$$\begin{aligned} &U_c'\Delta\tau_s\left(s\left(z;\theta\right)-s^0\right) = U_c'\Delta\mathcal{T}_z^\theta(z) \\ \iff\ &\Delta\mathcal{T}_z^\theta(z) = \left(s\left(z;\theta\right)-s^0\right)\Delta\tau_s. \end{aligned}$$

Differentiating both sides with respect to z and letting $\Delta s \to 0$, this implies that a small increase $\Delta\tau_s$ in the marginal tax rate on s induces the same earnings change as a small increase $s_{inc}'\left(z\right)\Delta\tau_s$ in the marginal earnings tax rate.

For types θ with $s(z;\theta) \geq s^0+\Delta s$, this implies

$$\begin{aligned} &U_c'\,\Delta\tau_s\,\Delta s = U_c'\,\Delta\mathcal{T}_z^\theta(z) \\ \iff\ &\Delta\mathcal{T}_z^\theta(z) = \Delta\tau_s\,\Delta s. \end{aligned}$$

Thus, a $\Delta\tau_s\,\Delta s$ lump-sum (savings) tax increase induces the same earnings change as a $\Delta\tau_s\,\Delta s$ lump-sum (earnings) tax increase.

C.6 Proof of Theorem 2 (Optimal Smooth Tax Systems)

When $z\left(\theta\right)$ is a strictly increasing function, we can define its inverse by $\vartheta\left(z\right)$. This allows us to define consumption of good c as $c(z) := c(z;\vartheta\left(z\right))$, consumption of good s as $s\left(z\right) := s\left(z;\vartheta\left(z\right)\right)$, and the planner's weights as $\alpha(z) := \alpha(\vartheta\left(z\right))$.

In this notation, the problem of the government is to maximize the Lagrangian

$$\mathcal{L} = \int_z \Big[\alpha(z)\,U\Big(c(z),s(z),z;\vartheta(z)\Big) + \lambda\Big(\mathcal{T}\left(s(z),z\right)-E\Big)\Big]\,dH_z(z), \tag{116}$$

where λ is the social marginal value of public funds, and the tax function implicitly enters individuals' utility through $c(z) = z - s(z) - \mathcal{T}\left(s(z),z\right)$.

C.6.1 Optimality Condition for Marginal Tax Rates on z

Reform. We consider a small reform at earnings level z^0 that consists in a small increase $\Delta\tau_z$ of the marginal tax rate on earnings in a small bandwidth Δz. Formally,

$$\Delta\mathcal{T}(s,z) = \begin{cases} 0 & if\ z \leq z^0 \\ \Delta\tau_z(z-z^0) & if\ z \in [z^0, z^0+\Delta z] \\ \Delta\tau_z\Delta z & if\ z \geq z^0+\Delta z \end{cases}$$

We characterize the impact of this reform on the government's objective function $\mathcal{L}$ as $\Delta z \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*:

$$\int_{z\geq z^0}\left(1-\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\vartheta(z)\right)\right)\Delta\tau_z\Delta z\, dH_z(z)$$

- *behavioral effects from changes in z:*[38]

$$-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)\Delta\tau_z\,\Delta z h_z(z^0) \\ -\int_{z\geq z^0}\mathcal{T}_z'\left(s\left(z\right),z\right)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\Delta\tau_z\Delta z\, dH_z(z)$$

- *behavioral effects from changes in s:*

$$-\mathcal{T}_s'\left(s(z^0),z^0\right)s_{inc}'(z^0)\left[\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)\Delta\tau_z\right]\Delta z h_z(z^0) \\ -\int_{z\geq z^0}\mathcal{T}_s'\left(s(z),z\right)\left[\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\right]\Delta\tau_z\Delta z\, dH_z(z)$$

Summing over these different effects yields the total impact of the reform

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta z} = \int_{z\geq z^0}\left(1-\hat{g}(z)\right)\Delta\tau_z dH_z(z) \\ -\left(\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)\Delta\tau_z\, h_z(z^0) \tag{117}$$

[38]Note that by definition elasticity concepts include all circularities and adjustments induced by tax reforms such that changes in z and s are given by

$$\begin{cases} dz = -\frac{z}{1-\mathcal{T}_z'}\zeta_z^c(z)\Delta\mathcal{T}_z'(s,z) - \frac{\eta_z(z)}{1-\mathcal{T}_z'}\Delta\mathcal{T}(s,z) \\ ds = -\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'}\Delta\mathcal{T}(s,z) + s_{inc}'(z)dz \end{cases}$$

where $\hat{g}(z)$ is the social marginal welfare weight augmented with income effects, given by

$$\hat{g}(z)=\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\vartheta(z)\right)-\frac{\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\mathcal{T}_s'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\eta_z(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}\eta_{s|z}(z).$$

Optimality. A direct implication is a sufficient statistics characterization of the optimal schedule of marginal tax rates on z. Indeed, at the optimum, the reform should have a zero impact on the government objective, $d\mathcal{L}=0$, meaning that at each earnings z^0 the optimal marginal earnings tax rate satisfies

$$\frac{\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}=\frac{1}{\zeta_z^c(z^0)}\frac{1}{z^0h_z(z^0)}\int_{z\geq z^0}\left(1-\hat{g}(z)\right)dH_z(z)-s_{inc}'(z^0)\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)} \tag{118}$$

which is the optimality condition in equation (19) presented in Theorem 2.

C.6.2 Optimality Condition for Marginal Tax Rates on s

Reform. We consider a small reform $\Delta\mathcal{T}_s$ that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0,z^0+\Delta z\right]$, that is:[39]

$$\Delta\mathcal{T}_s(s,z)=\begin{cases}0 & if\ z\leq z^0\\ \Delta\tau_s\left(z-z^0\right)s & if\ z\in[z^0,z^0+\Delta z]\\ \Delta\tau_s\,\Delta z\,s & if\ z\geq z^0+\Delta z\end{cases}$$

Let $s^0=s(z^0)$. We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta z\to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z\geq z^0}\left(1-\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\theta(z)\right)\right)\Delta\tau_s\Delta z\,s\left(z\right)\,dH_z(z) \tag{119}$$

[39] We use this reform to derive a sufficient statistics characterization of the optimal smooth tax system, without the requirement that $s(z)$ is monotonic. If we instead consider an increase in the marginal savings tax rates over a certain bandwidth of savings, which is natural to consider for SN tax systems, we need this extra assumption.

- *behavioral effects from changes in z:*[40]

$$
\begin{aligned}
&- \mathcal{T}_z'\left(s^0, z^0\right)\left[\frac{z^0 \zeta_z^c(z^0)}{1-\mathcal{T}_z'\left(s^0, z^0\right)}\,\Delta\tau_s s^0\right] h_z(z^0)\Delta z \\
&- \int_{z \geq z^0} \mathcal{T}_z'\left(s\left(z\right), z\right)\left[\frac{z\zeta_z^c(z) s_{inc}'(z)}{1-\mathcal{T}_z'\left(s\left(z\right), z\right)} + \frac{\eta_z(z) s(z)}{1-\mathcal{T}_z'\left(s\left(z\right), z\right)}\right]\Delta\tau_s \Delta z\, dH_z(z) \qquad (120)
\end{aligned}
$$

- *behavioral effects from changes in s:*

$$
\begin{aligned}
&- \mathcal{T}_s'\left(s^0, z^0\right) s_{inc}'(z^0)\left[\frac{z^0 \zeta_z^c(z^0)}{1-\mathcal{T}_z'\left(s^0, z^0\right)}\,\Delta\tau_s\, s^0\right] h_z(z^0)\Delta z \\
&- \int_{z \geq z^0} \mathcal{T}_s'\left(s(z), z\right)\left[\frac{\zeta_{s|z}^c(z) + \eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z), z\right)}\, s(z) + s_{inc}'(z)\left[\frac{z\zeta_z^c(z) s_{inc}'(z)}{1-\mathcal{T}_z'\left(s(z), z\right)} + \frac{\eta_z(z) s(z)}{1-\mathcal{T}_z'\left(s(z), z\right)}\right]\right]\Delta\tau_s \Delta z\, dH_z(z) \qquad (121)
\end{aligned}
$$

Summing over these different effects yields the total impact of the reform

$$
\begin{aligned}
&\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_s \Delta z} \\
&= -\frac{\mathcal{T}_z'\left(s^0, z^0\right) + s_{inc}'(z^0)\mathcal{T}_s'\left(s^0, z^0\right)}{1-\mathcal{T}_z'\left(s^0, z^0\right)} z^0 \zeta_z^c(z^0)\, s^0\, h_z(z^0) \qquad (122) \\
&+ \int_{z \geq z^0}\left\{\left(1-\hat{g}(z)\right) s\left(z\right) - \frac{\mathcal{T}_z'\left(s\left(z\right), z\right) + s_{inc}'(z)\mathcal{T}_s'\left(s(z), z\right)}{1-\mathcal{T}_z'\left(s\left(z\right), z\right)} z\zeta_z^c(z) s_{inc}'(z) - \frac{\mathcal{T}_s'\left(s(z), z\right)}{1+\mathcal{T}_s'\left(s(z), z\right)} s(z)\zeta_{s|z}^c(z)\right\} dH_z(z)
\end{aligned}
$$

where $\hat{g}(z)$ is the social marginal welfare weight augmented with income effects, given by

$$
\hat{g}(z) = \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right) - \frac{\mathcal{T}_z'\left(s\left(z\right), z\right) + s_{inc}'(z)\mathcal{T}_s'\left(s(z), z\right)}{1-\mathcal{T}_z'\left(s\left(z\right), z\right)}\eta_z(z) - \frac{\mathcal{T}_s'\left(s(z), z\right)}{1+\mathcal{T}_s'\left(s(z), z\right)}\eta_{s|z}(z).
$$

Optimality. A direct implication of this result is a sufficient statistics characterization of the optimal marginal tax rates on s. Indeed, at the optimum, the reform should have a zero impact on the government objective, $d\mathcal{L} = 0$, which implies that at each $s^0 = s\left(z^0\right)$ and earnings z^0, the optimal marginal tax rate on s satisfies

$$
\begin{aligned}
&\frac{\mathcal{T}_z'\left(s^0, z^0\right) + s_{inc}'(z^0)\mathcal{T}_z'\left(s^0, z^0\right)}{1-\mathcal{T}_s'\left(s^0, z^0\right)} z^0 \zeta_z^c(z^0)\, s^0\, h_z(z^0) \qquad (123) \\
&= \int_{z \geq z^0}\left\{\left(1-\hat{g}(z)\right) s\left(z\right) - \frac{\mathcal{T}_z'\left(s(z), z\right) + s_{inc}'(z)\mathcal{T}_z'\left(s(z), z\right)}{1-\mathcal{T}_z'\left(s(z), z\right)} z\zeta_z^c(z) s_{inc}'(z) - \frac{\mathcal{T}_s'\left(s(z), z\right)}{1+\mathcal{T}_s'\left(s(z), z\right)} s(z)\zeta_{s|z}^c(z)\right\} dH_z(z)
\end{aligned}
$$

[40] Applying Lemma 1, changes in z and s at earnings z^0 and above earnings z^0 are respectively

$$
\begin{cases} dz = -\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'}\,\Delta\tau_s\, s^0 \\ ds = s_{inc}'(z^0) dz \end{cases} \quad \text{and} \quad \begin{cases} dz = -\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'}\,\Delta\tau_s\Delta z\, s_{inc}'(z) - \frac{\eta_z(z)}{1-\mathcal{T}_z'}\,\Delta\tau_s\Delta z\, s\left(z\right) \\ ds = -\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'}\,\Delta\tau_s\Delta z - \frac{\eta_s(z)}{1+\mathcal{T}_s'}\Delta\tau_s\Delta z\, s\left(z\right) + s_{inc}'(z) dz \end{cases}
$$

Using the formula for the optimal schedule of marginal earnings tax rates in equation (118) to replace the term on the left-hand side, this formula can be rearranged as

$$s(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z) \tag{124}$$
$$=\int_{z\geq z^0}\left\{(1-\hat{g}(z))\,s(z)-\frac{\mathcal{T}_z'\left(s(z),z\right)+s_{inc}'(z)\mathcal{T}_z'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s(z),z\right)}z\zeta_z^c(z)s_{inc}'(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}s(z)\zeta_{s|z}^c(z)\right\}dH_z(z)$$

Differentiating both sides with respect to z^0 yields

$$s'(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z)-s^0(1-\hat{g}(z^0))h_z(z^0)-\frac{\mathcal{T}_s'\left(s^0,z^0\right)}{1+\mathcal{T}_s'\left(s^0,z^0\right)}s^0\zeta_{s|z}^c(z^0)h_z(z^0)$$
$$=-\left(1-\hat{g}(z^0)\right)s^0h_z(z^0)+s_{inc}'(z^0)\frac{\mathcal{T}_z'\left(s^0,z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s^0,z^0\right)}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)z^0h_z(z^0)$$

where both $s^0(1-\hat{g}(z^0))h_z(z^0)$ terms cancel out. Using equation (118) again, the last term is equal to $s_{inc}'(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z)$ such that we finally obtain

$$\frac{\mathcal{T}_s'\left(s^0,z^0\right)}{1+\mathcal{T}_s'\left(s^0,z^0\right)}s(z^0)\zeta_{s|z}^c(z^0)h_z(z^0)=\underbrace{\left[s'(z^0)-s_{inc}'(z^0)\right]}_{s_{het}'(z^0)}\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z), \tag{125}$$

which is the optimality condition in equation (20) presented in Theorem 2.

C.6.3 Pareto Efficiency Condition

We can combine formulas for optimal marginal tax rates on z and on s to obtain a characterization of Pareto efficiency. Indeed, leveraging the above optimal formula for marginal tax rates on s written in terms of $s_{het}'(z^0)$, and replacing the integral term by its value from the optimal formula for marginal earnings tax rates in equation (118) yields

$$\frac{\mathcal{T}_s'(s^0,z^0)}{1+T_s'(s^0,z^0)}s^0\zeta_{s|z}^c(z^0)=s_{het}'(z^0)\frac{\mathcal{T}_z'(z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)}{1-\mathcal{T}_z'(s^0,z^0)}z^0\zeta_z^c(z^0)$$

which is the Pareto-efficiency condition in equation (21) presented in Theorem (2).

C.7 Proof of Proposition B.3 (Structural characterization of s_{inc}' and s_{het}')

In economies with preference heterogeneity, budget heterogeneity, and auxiliary choices, $s(z;\theta)$ solves

$$\max_s U\Big(B\big(s,z,\chi(s,z;\theta);\theta\big)-\mathcal{T}(s,z),\phi_s\big(z,s,\chi(s,z;\theta);\theta\big),\phi_z\big(z,s,\chi(s,z;\theta);\theta\big),\chi(s,z;\theta);\theta\Big) \tag{126}$$

where $\chi(s,z;\theta)$ denotes utility-maximizing auxiliary choices. As a result, applying the envelope theorem to changes in χ, $s(z;\theta)$ is defined by the following first-order condition

$$U_c'(.)\big[B_s'(s(z;\theta),z,\chi(s(z;\theta),z;\theta);\theta)-\mathcal{T}_s'(s(z;\theta),z)\big] \\ +U_s'(.)\frac{\partial\phi_s(z,s,\chi(s,z;\theta);\theta)}{\partial s}\Big|_{s=s(z;\theta)}+U_z'(.)\frac{\partial\phi_z(z,s,\chi(s,z;\theta);\theta)}{\partial s}\Big|_{s=s(z;\theta)}=0. \quad (127)$$

Now, to compute $s_{inc}'=\frac{\partial s(z;\theta)}{\partial z}$, we differentiate this first-order condition with respect to z while holding θ fixed:

$$\Big[\underbrace{U_{cc}''(.)(B_s'-\mathcal{T}_s')+U_{cs}''(.)\frac{\partial\phi_s}{\partial s}+U_{cz}''(.)\frac{\partial\phi_z}{\partial s}}_{\mathcal{K}_c}\Big]\Big[B_s'\frac{\partial s(z;\theta)}{\partial z}+B_z'+B_\chi'\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)-\mathcal{T}_s'\frac{\partial s(z;\theta)}{\partial z}-\mathcal{T}_z'\Big]$$
$$+\Big[\underbrace{U_{cs}''(.)(B_s'-\mathcal{T}_s')+U_{ss}''(.)\frac{\partial\phi_s}{\partial s}+U_{sz}''(.)\frac{\partial\phi_z}{\partial s}}_{\mathcal{K}_s}\Big]\Big[\frac{\partial\phi_s}{\partial z}+\frac{\partial\phi_s}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\phi_s}{\partial\chi}\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)\Big]$$
$$+\Big[\underbrace{U_{cz}''(.)(B_s'-\mathcal{T}_s')+U_{sz}''(.)\frac{\partial\phi_s}{\partial s}+U_{zz}''(.)\frac{\partial\phi_z}{\partial s}}_{\mathcal{K}_z}\Big]\Big[\frac{\partial\phi_z}{\partial z}+\frac{\partial\phi_z}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\phi_z}{\partial\chi}\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)\Big]$$
$$+\Big[\underbrace{U_{c\chi}''(.)(B_s'-\mathcal{T}_s')+U_{s\chi}''(.)\frac{\partial\phi_s}{\partial s}+U_{z\chi}''(.)\frac{\partial\phi_z}{\partial s}}_{\mathcal{K}_\chi}\Big]\Big[\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big]$$
$$+U_c'\Big[B_{ss}''\frac{\partial s(z;\theta)}{\partial z}+B_{sz}''+B_{s\chi}''\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)-\mathcal{T}_{ss}''\frac{\partial s(z;\theta)}{\partial z}-\mathcal{T}_{sz}''\Big]$$
$$+U_s'\Big[\frac{\partial^2\phi_s}{\partial s\partial z}+\frac{\partial^2\phi_s}{(\partial s)^2}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial^2\phi_s}{\partial s\partial\chi}\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)\Big]$$
$$+U_z'\Big[\frac{\partial^2\phi_z}{\partial s\partial z}+\frac{\partial^2\phi_z}{(\partial s)^2}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial^2\phi_z}{\partial s\partial\chi}\Big(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial z}+\frac{\partial\chi}{\partial z}\Big)\Big]=0. \quad (128)$$

Rearranging terms yields

$$\frac{\partial s(z;\theta)}{\partial z}=-\frac{\mathcal{K}_c\left[B_z'+B_\chi'\frac{\partial\chi}{\partial z}-\mathcal{T}_z'\right]+\mathcal{K}_s\left[\frac{\partial\phi_s}{\partial z}+\frac{\partial\phi_s}{\partial\chi}\frac{\partial\chi}{\partial z}\right]+\mathcal{K}_z\left[\frac{\partial\phi_z}{\partial z}+\frac{\partial\phi_z}{\partial\chi}\frac{\partial\chi}{\partial z}\right]+\mathcal{K}_\chi\left[\frac{\partial\chi}{\partial z}\right]+...}{\mathcal{K}_c\left[B_s'+B_\chi'\frac{\partial\chi}{\partial s}-\mathcal{T}_s'\right]+\mathcal{K}_s\left[\frac{\partial\phi_s}{\partial s}+\frac{\partial\phi_s}{\partial\chi}\frac{\partial\chi}{\partial s}\right]+\mathcal{K}_z\left[\frac{\partial\phi_z}{\partial s}+\frac{\partial\phi_z}{\partial\chi}\frac{\partial\chi}{\partial s}\right]+\mathcal{K}_\chi\left[\frac{\partial\chi}{\partial s}\right]+...}$$
$$\frac{...+U_c'\left[B_{sz}''+B_{s\chi}''\frac{\partial\chi}{\partial z}-\mathcal{T}_{sz}''\right]+U_s'\left[\frac{\partial^2\phi_s}{\partial s\partial z}+\frac{\partial^2\phi_s}{\partial s\partial\chi}\frac{\partial\chi}{\partial z}\right]+U_z'\left[\frac{\partial^2\phi_z}{\partial s\partial z}+\frac{\partial^2\phi_z}{\partial s\partial\chi}\frac{\partial\chi}{\partial z}\right]}{...+U_c'\left[B_{ss}''+B_{s\chi}''\frac{\partial\chi}{\partial s}-\mathcal{T}_{ss}''\right]+U_s'\left[\frac{\partial^2\phi_s}{(\partial s)^2}+\frac{\partial^2\phi_s}{\partial s\partial\chi}\frac{\partial\chi}{\partial s}\right]+U_z'\left[\frac{\partial^2\phi_z}{(\partial s)^2}+\frac{\partial^2\phi_z}{\partial s\partial\chi}\frac{\partial\chi}{\partial s}\right]}. \quad (129)$$

Similarly, to compute $s_{het}'=\frac{\partial s(z;\theta)}{\partial\theta}$, we differentiate the first-order condition for $s(z;\theta)$ with respect to θ while holding z fixed:

$$\underbrace{\left[U''_{cc}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{cs}(.)\frac{\partial\phi_s}{\partial s}+U''_{cz}(.)\frac{\partial\phi_z}{\partial s}\right]}_{\mathcal{K}_c}\left[B'_s\frac{\partial s(z;\theta)}{\partial\theta}+B'_\chi\left(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial\chi}{\partial\theta}\right)-\mathcal{T}'_s\frac{\partial s(z;\theta)}{\partial\theta}\right]$$
$$+\underbrace{\left[U''_{cs}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{ss}(.)\frac{\partial\phi_s}{\partial s}+U''_{sz}(.)\frac{\partial\phi_z}{\partial s}\right]}_{\mathcal{K}_s}\left[\frac{\partial\phi_s}{\partial s}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial\phi_s}{\partial\chi}\left(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial\chi}{\partial\theta}\right)+\frac{\partial\phi_s}{\partial\theta}\right]$$
$$+\underbrace{\left[U''_{cz}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{sz}(.)\frac{\partial\phi_s}{\partial s}+U''_{zz}(.)\frac{\partial\phi_z}{\partial s}\right]}_{\mathcal{K}_z}\left[\frac{\partial\phi_z}{\partial s}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial\phi_z}{\partial\chi}\left(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial\chi}{\partial\theta}\right)+\frac{\partial\phi_z}{\partial\theta}\right]$$
$$+\underbrace{\left[U''_{c\chi}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{s\chi}(.)\frac{\partial\phi_s}{\partial s}+U''_{z\chi}(.)\frac{\partial\phi_z}{\partial s}\right]}_{\mathcal{K}_\chi}\left[\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial\chi}{\partial\theta}\right]$$
$$+\underbrace{\left[U''_{c\theta}(.)\left(B'_s-\mathcal{T}'_s\right)+U''_{s\theta}(.)\frac{\partial\phi_s}{\partial s}+U''_{z\theta}(.)\frac{\partial\phi_z}{\partial s}\right]}_{\mathcal{K}_\theta}$$
$$+U'_c\left[B''_{ss}\frac{\partial s(z;\theta)}{\partial\theta}+B''_{s\chi}\left(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial\chi}{\partial\theta}\right)-\mathcal{T}''_{ss}\frac{\partial s(z;\theta)}{\partial\theta}\right]$$
$$+U'_s\left[\frac{\partial^2\phi_s}{(\partial s)^2}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial^2\phi_s}{\partial s\partial\chi}\left(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial\chi}{\partial\theta}\right)+\frac{\partial^2\phi_s}{\partial s\partial\theta}\right]$$
$$+U'_z\left[\frac{\partial^2\phi_z}{(\partial s)^2}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial^2\phi_z}{\partial s\partial\chi}\left(\frac{\partial\chi}{\partial s}\frac{\partial s(z;\theta)}{\partial\theta}+\frac{\partial\chi}{\partial\theta}\right)+\frac{\partial^2\phi_z}{\partial s\partial\theta}\right]=0.$$

Rearranging terms yields

$$\frac{\partial s(z;\theta)}{\partial\theta}=-\frac{\mathcal{K}_c\left[B'_\chi\frac{\partial\chi}{\partial\theta}\right]+\mathcal{K}_s\left[\frac{\partial\phi_s}{\partial\chi}\frac{\partial\chi}{\partial\theta}+\frac{\partial\phi_s}{\partial\theta}\right]+\mathcal{K}_z\left[\frac{\partial\phi_z}{\partial\chi}\frac{\partial\chi}{\partial\theta}+\frac{\partial\phi_z}{\partial\theta}\right]+\mathcal{K}_\chi\left[\frac{\partial\chi}{\partial\theta}\right]+\mathcal{K}_\theta+...}{\mathcal{K}_c\left[B'_s+B'_\chi\frac{\partial\chi}{\partial s}-\mathcal{T}'_s\right]+\mathcal{K}_s\left[\frac{\partial\phi_s}{\partial s}+\frac{\partial\phi_s}{\partial\chi}\frac{\partial\chi}{\partial s}\right]+\mathcal{K}_z\left[\frac{\partial\phi_z}{\partial s}+\frac{\partial\phi_z}{\partial\chi}\frac{\partial\chi}{\partial s}\right]+\mathcal{K}_\chi\left[\frac{\partial\chi}{\partial s}\right]+...}$$
$$\frac{...+U'_c\left[B''_{s\chi}\frac{\partial\chi}{\partial\theta}\right]+U'_s\left[\frac{\partial^2\phi_s}{\partial s\partial\chi}\frac{\partial\chi}{\partial\theta}+\frac{\partial^2\phi_s}{\partial s\partial\theta}\right]+U'_z\left[\frac{\partial^2\phi_z}{\partial s\partial\chi}\frac{\partial\chi}{\partial\theta}+\frac{\partial^2\phi_z}{\partial s\partial\theta}\right]}{...+U'_c\left[B''_{ss}+B''_{s\chi}\frac{\partial\chi}{\partial s}-\mathcal{T}''_{ss}\right]+U'_s\left[\frac{\partial^2\phi_s}{(\partial s)^2}+\frac{\partial^2\phi_s}{\partial s\partial\chi}\frac{\partial\chi}{\partial s}\right]+U'_z\left[\frac{\partial^2\phi_z}{(\partial s)^2}+\frac{\partial^2\phi_z}{\partial s\partial\chi}\frac{\partial\chi}{\partial s}\right]}.\quad(130)$$

C.8 Proof of Propositions 3, B.4, and B.5 (Optimal Simple Tax Systems)

The derivation of optimal earnings tax formulas for simple tax systems parallels that of general smooth tax systems and the optimal formula for marginal earnings tax rates formula, equation (19), continues to hold. This proves Proposition B.5.

Moreover, the particular linear reforms considered in the sufficient statistics characterization of optimal marginal tax rates on s for general smooth tax systems $\mathcal{T}(s,z)$ are also available for LED tax systems. As a result, the derivation of optimal marginal tax rates on s in LED tax systems is identical to the derivation for general smooth tax systems, and the optimality formula in equation

(20) continues to hold. This, in turn, implies that the Pareto-efficiency condition in equation (21) also holds, thereby proving all sufficient statistics characterizations for LED tax systems.

In contrast, LED reforms of tax rates on s are not available under SL and SN tax systems, and we derive below sufficient statistics characterizations of optimal tax rates on s and Pareto-efficiency conditions in SL and SN tax systems.

C.8.1 SL tax system

SL tax reform. When the government uses a linear tax on s such that $\mathcal{T}(s,z) = \tau_s\, s + T_z(z)$, we consider a small reform of the linear tax rate τ_s that consists in a small increase $\Delta\tau_s$. For an individual with earnings z, this reform increases tax liability by $\Delta\tau_s\, s(z)$ and increases the marginal tax rate on s by $\Delta\tau_s$.

We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_z \left(1 - \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right)\right) \Delta\tau_s\, s\left(z\right)\, dH_z(z) \tag{131}$$

- *behavioral effects from changes in z:*[41]

$$-\int_z T_z'(z) \left[\frac{z\zeta_z^c(z)}{1 - T_z'\left(z\right)} \Delta\tau_s\, s_{inc}'(z) + \frac{\eta_z(z)}{1 - T_z'\left(z\right)} \Delta\tau_s\, s\left(z\right)\right] dH_z(z) \tag{133}$$

- *behavioral effects from changes in s:*

$$-\int_z \tau_s \left[\frac{s(z)\zeta_{s|z}^c(z)}{1+\tau_s}\Delta\tau_s + \frac{\eta_{s|z}(z)}{1+\tau_s}\Delta\tau_s\, s\left(z\right)\right] dH_z(z)$$
$$- \int_z \tau_s s_{inc}'(z) \left[\frac{z\zeta_z^c(z)}{1-T_z'} \Delta\tau_s\, s_{inc}'(z) + \frac{\eta_z(z)}{1-T_z'} \Delta\tau_s\, s\left(z\right)\right] dH_z(z) \tag{134}$$

Summing over these different effects yields the total impact of the reform

$$\frac{d\mathcal{L}}{\lambda} = \int_z \left\{ \left(1 - \hat{g}(z)\right) s\left(z\right) - \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1 - T_z'\left(z\right)} z\zeta_z^c(z)\, s_{inc}'(z) - \frac{\tau_s}{1+\tau_s} s(z)\zeta_{s|z}^c(z) \right\} \Delta\tau_s\, dH_z(z), \tag{135}$$

with social marginal welfare weights augmented with the fiscal impact of income effects given by

$$\hat{g}(z) = \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \theta(z)\right) + \frac{T_z'(z)}{1 - T_z'\left(z\right)} \eta_z(z) + \tau_s \left[\frac{\eta_{s|z}(z)}{1+\tau_s} + s_{inc}'(z)\frac{\eta_z(z)}{1-T_z'}\right].$$

[41] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz = -\frac{z\zeta_z^c(z)}{1-T_z'(z)}\,\Delta\tau_s\, s_{inc}'(z) - \frac{\eta_z(z)}{1-T_z'(z)}\,\Delta\tau_s\, s\left(z\right) \\ ds = -\frac{s(z)\zeta_{s|z}^c(z)}{1+\tau_s}\,\Delta\tau_s - \frac{\eta_{s|z}(z)}{1+\tau_s}\Delta\tau_s\, s\left(z\right) + s_{inc}'(z)dz \end{cases} \tag{132}$$

Optimal linear tax rate on s. A direct implication of this result is a sufficient statistics characterization of the optimal linear tax rate τ_s. Indeed, at the optimum, the reform should have a zero impact on the government objective, meaning that the optimal τ_s satisfies

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta^c_{s|z}(z)dH_z(z) = \int_z\left\{(1-\hat{g}(z))\ s\left(z\right) - \frac{T_z'(z)+\tau_s\, s_{inc}'(z)}{1-T_z'\left(z\right)}z\zeta^c_z(z)\, s_{inc}'(z)\right\}dH_z(z). \tag{136}$$

This is equation (69) in Proposition B.4, and it holds for any (potentially suboptimal) nonlinear earnings tax schedule $T_z\left(z\right)$.

Now, assume that the earnings tax schedule is optimal. Equation (118) applied to SL tax systems then implies that at each earnings z,

$$\frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'(z)} = \frac{1}{\zeta^c_z(z)}\frac{1}{zh_z(z)}\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x) \tag{137}$$

such that plugging in this expression to replace the last term, we obtain

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta^c_{s|z}(z)dH_z(z) = \int_z\left\{s\left(z\right)(1-\hat{g}(z))\right\}dH_z(z) - \int_z\left\{s_{inc}'(z)\int_{x\geq z}(1-\hat{g}(x))\,h_z(x)dx\right\}dz. \tag{138}$$

Defining $s_{inc}(z) \equiv \int_{x=0}^{z} s_{inc}'\left(x\right)dx$, we can integrate by parts the last term to re-express it as[42]

$$\int_z\left\{s_{inc}'(z)\int_{x\geq z}(1-\hat{g}(x))\,h_z(x)dx\right\}dz = \int_z\left\{s_{inc}(z)(1-\hat{g}(z))\,h_z(z)\right\}dz \tag{139}$$

to obtain

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta^c_{s|z}(z)dH_z(z) = \int_z\left\{\left[s\left(z\right)-s_{inc}(z)\right](1-\hat{g}(z))\right\}dH_z(z). \tag{140}$$

Note that here $\int_z \frac{s(z)}{1+\tau_s}\zeta^c_{s|z}(z)dH_z(z)$ is the aggregate population response to a change in τ_s. Defining $\overline{\zeta^c_{s|z}}$ as the aggregate elasticity of $\overline{s} := \int_z s(z)dH_z(z)$, we can rewrite this term as $\frac{\overline{s}}{1+\tau_s}\overline{\zeta^c_{s|z}}$ such that

$$\frac{\tau_s}{1+\tau_s} = \frac{1}{\overline{s}\overline{\zeta^c_{s|z}}}\int_z s_{het}\left(z\right)(1-\hat{g}(z))\,dH_z(z). \tag{141}$$

This is equation (23) in Proposition 3. Integrating by part the right-hand side, this formula is also equivalent to

$$\frac{\tau_s}{1+\tau_s} = \frac{1}{\overline{s}\overline{\zeta^c_{s|z}}}\int_z\left[s_{het}'\left(z\right)\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x)\right]dz. \tag{142}$$

[42]Define $\phi\left(z\right) = \int_{x=0}^{z} s_{inc}'\left(x\right)dx$ such that $\phi'\left(z\right) = s_{inc}'\left(z\right)$ and $\psi\left(z\right) = \int_{x=z}^{z_{max}}(1-\hat{g}(x))\,h_z(x)dx$ such that $\psi'\left(z\right) = -(1-\hat{g}(z))\,h_z(z)$, and apply

$$\int_{x=z}^{z_{max}}\left[\phi'\left(x\right)\psi\left(x\right)\right]dx = \left[\phi\left(z_{max}\right)\psi\left(z_{max}\right)-\phi\left(z\right)\psi\left(z\right)\right] - \int_{x=z}^{z_{max}}\left[\phi\left(x\right)\psi'\left(x\right)\right]dx.$$

Pareto efficiency for SL tax systems. To characterize Pareto efficiency, we combine tax reforms in a way that neutralizes all lump-sum changes in tax liability, thereby offsetting all utility changes.

We start with a small reform of the linear tax rate τ_s that consists in small increase $\Delta\tau_s$. At the bottom of the earnings distribution ($z = z_{min}$), the mechanical effect of the reform is an increase in tax liability by $s\left(z_{min}\right)\Delta\tau_s$. We thus adjust the earnings tax liability through a downward lump-sum shift by $s\left(z_{min}\right)\Delta\tau_s$ at all earnings levels. This joint reform has the following impact on the government objective

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=z_{min}}^{z_{max}} \left\{ \left[1-\hat{g}(z)\right]\ \left[s\left(z\right)-s\left(z_{min}\right)\right] - \frac{T_z'(z)+\tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z) \right\} \Delta\tau_s\, dH_z(z) \tag{143}$$

meaning that the lump-sum change in tax liability is nil at earnings $z = z_{min}$, but not at earnings $z \geq z_{min}$.

To cancel out lump-sum changes in tax liability at all earnings levels, we construct a sequence of earnings tax reforms. We discretize the range of earnings $[z_{min}, z_{max}]$ into N bins and consider reforms in the small earnings bandwidths $\Delta z = \frac{z_{max}-z_{min}}{N}$. We proceed by induction to derive a general formula:

- First, consider a decrease in the marginal earnings tax rate by $\Delta\tau_z = s'\left(z_{min}\right)\Delta\tau_s$ over the bandwidth $[z_{min}, z_{min}+\Delta z]$. In this bandwidth, this additional reform (i) cancels out lump-sum changes in tax liability to a first-order approximation since $\left[s\left(z_{min}+\Delta z\right)-s\left(z_{min}\right)\right]\Delta\tau_s \approx s'\left(z_{min}\right)\Delta z\Delta\tau_s$, and (ii) induces earnings responses through the change in marginal tax rates. Moreover, it also decreases the lump-sum tax liability on all individuals with earnings $z \geq z_{min}+\Delta z$ by $s'\left(z_{min}\right)\Delta z\Delta\tau_s$. The total impact of this sequence of reforms is then

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda} &= \int_{z=z_{min}+\Delta z}^{z_{max}} \left\{ \left[1-\hat{g}(z)\right]\ \left[s\left(z\right)-s\left(z_{min}\right)-s'\left(z_{min}\right)\Delta z\right] \right\} \Delta\tau_s\, dH_z(z) \\ &- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z)+\tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z) \right\} \Delta\tau_s\, dH_z(z) \\ &+ \int_{z=z_{min}}^{z_{min}+\Delta z} \frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'\left(z\right)} z\zeta_z^c(z)\left(s'\left(z_{min}\right)\Delta\tau_s\right)\ \Delta H_z(z).\end{aligned} \tag{144}$$

- Second, consider a decrease in the marginal earnings tax rate by $\Delta\tau_z = s'\left(z_{min}+\Delta z\right)\Delta\tau_s$ over the bandwidth $[z_{min}+\Delta z, z_{min}+2\Delta z]$. This again cancels out lump-sum changes in this bandwidth up to a first-order approximation since $\left[s\left(z_{min}+2\Delta z\right)-s\left(z_{min}\right)-s'\left(z_{min}\right)\Delta z\right] \approx s'\left(z_{min}+\Delta z\right)\Delta z$.

The total impact of this sequence of reforms is then

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda} &= \int_{z=z_{min}+2\Delta z}^{z_{max}} \left\{ [1-\hat{g}(z)] \left[s(z) - s(z_{min}) - s'(z_{min})\Delta z - s'(z_{min}+\Delta z)\Delta z \right] \right\} \Delta\tau_s \, dH_z(z) \\ &- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z)+\tau_s \, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} \Delta\tau_s \, dH_z(z) \\ &+ \int_{z=z_{min}}^{z_{min}+\Delta z} \frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) \left(s'(z_{min})\Delta\tau_s \right) \, dH_z(z) \\ &+ \int_{z=z_{min}+\Delta z}^{z_{min}+2\Delta z} \frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) \left(s'(z_{min}+\Delta z)\Delta\tau_s \right) \, dH_z(z). \end{aligned} \quad (145)$$

- Iterating over to step k, in which we consider a decrease in the marginal earnings tax rate by $\Delta\tau_z = s'\left(z_{min} + (k-1)\frac{\Delta z}{N}\right)\Delta\tau_s$ over the bandwidth $[z_{min} + (k-1)\frac{\Delta z}{N}, z_{min} + k\frac{\Delta z}{N}]$. The total impact of this sequence of reforms is then

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda} &= \int_{z=z_{min}+k\frac{\Delta z}{N}}^{z_{max}} \left\{ [1-\hat{g}(z)] \left[s(z) - s(z_{min}) - \frac{\Delta z}{N} \left[\sum_{p=0}^{k-1} s'\left(z_{min} + p\frac{\Delta z}{N} \right) \right] \right] \right\} \Delta\tau_s \, dH_z(z) \\ &- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z)+\tau_s \, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} \Delta\tau_s \, dH_z(z) \\ &+ \sum_{p=0}^{k-1} \int_{z=z_{min}+p\frac{\Delta z}{N}}^{z_{min}+(p+1)\frac{\Delta z}{N}} \frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'\left(z_{min} + p\frac{\Delta z}{N} \right) \Delta\tau_s \, dH_z(z). \end{aligned} \quad (146)$$

- Pushing the iteration forward until $k = N$, the first integral disappears (integration over an empty set) such that the total impact of this sequence of reforms is given by

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda} &= -\int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z)+\tau_s \, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} \Delta\tau_s \, dH_z(z) \\ &+ \sum_{p=0}^{N-1} \int_{z=z_{min}+p\frac{\Delta z}{N}}^{z_{min}+(p+1)\frac{\Delta z}{N}} \frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'\left(z_{min} + p\frac{\Delta z}{N} \right) \Delta\tau_s \, dH_z(z). \end{aligned} \quad (147)$$

Let's now compute the last term at the limit $N \to \infty$. Denoting $z^p := z_{min} + p\frac{\Delta z}{N}$, we have

$$\begin{aligned} &\sum_{p=0}^{N-1} \int_{z=z^p}^{z^p+\frac{\Delta z}{N}} \frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'(z^p) \Delta\tau_s \, dH_z(z) \\ &\approx \sum_{p=0}^{N-1} \frac{T_z'(z^p)+s_{inc}'(z^p)\tau_s}{1-T_z'(z^p)} (z^p)\, \zeta_z^c(z^p)\, s'(z^p)\, \Delta\tau_s h_z(z^p) \frac{\Delta z}{N} \\ &\underset{N\to\infty}{\longrightarrow} \int_{z=z_{min}}^{z_{max}} \frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z)\, s'(z)\, \Delta\tau_s \, h_z(z) \, dz \end{aligned} \quad (148)$$

where the last line follows from the (Riemann) definition of the integral in terms of Riemann sums. Hence, the total impact of this sequence of reforms is at the limit given by

$$\frac{d\mathcal{L}}{\lambda} = -\int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1 - T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} \Delta\tau_s\, h_z(z) dz + \int_{z=z_{min}}^{z_{max}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1 - T_z'(z)} z\zeta_z^c(z)\, s'(z)\, \Delta\tau_s\, h_z(z)\, dz. \quad (149)$$

By construction, the sequence of reforms we have constructed does not affect individuals' utility, and only affects tax revenue through the expression above. When the impact of this reform is non-zero, the type of sequence of reforms we consider delivers a Pareto improvement over the existing tax system. Characterizing a Pareto-efficient SL tax system as one that cannot be reformed in a Pareto-improving way yields the following Pareto-efficiency formula

$$\frac{\tau_s}{1+\tau_s} \int_z s(z) \zeta_{s|z}^c(z) h_z(z) dz = \int_z \underbrace{\left[s'(z) - s_{inc}'(z)\right]}_{s_{het}'(z)} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1 - T_z'(z)} z\zeta_z^c(z)\, h_z(z)\, dz, \quad (150)$$

which is equation (27) in Proposition 3.

C.8.2 SN tax systems

SN tax reform. When the government uses a SN tax system such that $\mathcal{T}(s,z) = T_s(s) + T_z(z)$, we consider a small reform of the tax on s at $s^0 = s(\theta^0)$ that consists in a small increase $\Delta\tau_s$ of the marginal tax rate on s in a small bandwidth Δs. Formally,

$$\Delta\mathcal{T}(s,z) = \begin{cases} 0 & if\ s \le s^0 \\ \Delta\tau_s(s - s^0) & if\ s \in [s^0, s^0 + \Delta s] \\ \Delta\tau_s \Delta s & if\ s \ge s^0 + \Delta s \end{cases}$$

Assuming there exists a strictly increasing mapping between z and s, we denote z^0 the earnings level such that $s^0 = s(z^0)$.[43] We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta s \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z \ge z^0} \left(1 - \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right)\right) \Delta\tau_s \Delta s\, dH_z(z)$$

[43]Our sufficient statistic characterization of optimal SN tax systems fundamentally relies on monotonicity of the function $s(z)$. Hence, it is also valid if we assume a strictly decreasing mapping $s(z)$. Moreover, it can be extended to weakly monotonic $s(z)$ (i.e., non-decreasing or non-increasing) with slight modifications.

- *behavioral effects from changes in* z:[44]

$$-\mathcal{T}_z'(s^0,z^0)\left[\frac{z^0}{1-\mathcal{T}_z'(s^0,z^0)}\zeta_z^c(z^0)\, s_{inc}'(z)\Delta\tau_s\right]\Delta s\,\frac{h_z(z^0)}{s'(z^0)}-\int_{z\geq z^0}\mathcal{T}_z'\left(s,z\right)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s,z\right)}\Delta\tau_s\Delta s\, dH_z(z)$$

- *behavioral effects from changes in* s*:*

$$-\mathcal{T}_s'\left(s^0,z^0\right)\left[\frac{s^0}{1+\mathcal{T}_s'\left(s^0,z^0\right)}\zeta_{s|z}^c(z^0)\Delta\tau_s+s_{inc}'(z^0)\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)\, s_{inc}'(z^0)\Delta\tau_s\right]\Delta s\,\frac{h_z(z^0)}{s'(z^0)}$$
$$-\int_{z\geq z^0}\mathcal{T}_s'\left(s,z\right)\left[\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s,z\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s,z\right)}\right]\Delta\tau_s\Delta s\, dH_z(z).$$

Summing over these different effects yields the total impact of the reform

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta s}=s'\left(z^0\right)\int_{z\geq z^0}\left(1-\hat{g}(z)\right)\Delta\tau_s\, dH_z(z)-\Bigg\{\mathcal{T}_s'\left(s^0,z^0\right)\frac{s^0}{1+\mathcal{T}_s'\left(s^0,z^0\right)}\zeta_{s|z}^c(z^0)$$
$$+\left[\mathcal{T}_z'(s^0,z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)s_{inc}'(z^0)\Bigg\}\Delta\tau_s\, h_z(z^0). \quad (152)$$

Optimal nonlinear tax rate on s**.** A direct implication of this result is a sufficient statistics characterization of the optimal marginal tax rates on s. Indeed, at the optimum, the reform should have a zero impact on the government objective, $d\mathcal{L}=0$, which implies that at each $s^0=s\left(z^0\right)$ the optimal marginal tax rate on s satisfies

$$\frac{\mathcal{T}_s'(s^0,z^0)}{1+\mathcal{T}_s'(s^0,z^0)}s^0\zeta_{s|z}^c(z^0)h_z(z^0)$$
$$=s'\left(z^0\right)\int_{z\geq z^0}\left(1-\hat{g}(z)\right)dH_z(z)-s_{inc}'(z^0)\frac{\mathcal{T}_z'(s^0,z^0)+s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)}{1-\mathcal{T}_z'(s^0,z^0)}z^0\zeta_z^c(z^0)h_z(z^0) \quad (153)$$

which is equation (67) in Proposition B.4, recognizing that $\mathcal{T}_z'(s,z)=T_z'(z)$ and $\mathcal{T}_s'(s,z)=T_s'(s)$. This characterization holds for any (potentially suboptimal) nonlinear earnings tax schedule $T_z(z)$.

Now, further assume that the earnings tax schedule is optimal. Equation (118) applied to SN

[44] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz=-\frac{z}{1-T_z'}\zeta_z^c(z)\Delta T_z^{\theta\prime}-\frac{\eta_z(z)}{1-T_z'}\Delta T_z^{\theta} \\ ds=-\frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z)\Delta T_s'-\frac{\eta_{s|z}(z)}{1+T_s'}\Delta T_s+s_{inc}'(z)dz \end{cases} \quad (151)$$

where T_z^{θ} is a $s_{inc}'(z)\Delta\tau_s$ increase in the marginal earnings tax rate when $s\in[s^0,s^0+\Delta s]$, and a $\Delta\tau_s\Delta s$ increase in tax liability when $s\geq s^0+\Delta s$. Moreover, the mass of individuals in the bandwidth is $\Delta s\, h_s(s(z^0))=\Delta s\,\frac{h_z(z^0)}{s'(z^0)}$.

tax systems then implies that at each earnings z^0,

$$\frac{T_z'(z^0) + s_{inc}'(z^0) T_s'(s(z^0))}{1 - T_z'(z^0)} = \frac{1}{\zeta_z^c(z^0)} \frac{1}{z^0 h_z(z^0)} \int_{z \geq z^0} (1 - \hat{g}(z)) \, dH_z(z). \tag{154}$$

Using this expression to substitute the last term in the formula for optimal marginal tax rates on s yields

$$\frac{T_s'(s^0)}{1 + T_s'(s^0)} s^0 \zeta_{s|z}^c(z^0) h_z(z^0) = \underbrace{\left[s'(z^0) - s_{inc}'(z^0)\right]}_{s_{het}'(z^0)} \int_{z \geq z^0} (1 - \hat{g}(z)) \, dH_z(z)$$

which is equation (25) in Proposition 3.

Pareto efficiency for SN tax systems. We can combine formulas for optimal marginal tax rates on s and z to obtain a characterization of Pareto efficiency. Indeed, leveraging the previous optimal formula for marginal tax rates on s written in terms of $s_{het}'(z^0)$, and replacing the integral term by its value given from the optimal formula for marginal earnings tax rates yields

$$\frac{\mathcal{T}_s'(s^0, z^0)}{1 + T_s'(s^0, z^0)} s^0 \zeta_{s|z}^c(z^0) = s_{het}'(z^0) \frac{\mathcal{T}_z'(z^0) + s_{inc}'(z^0) \mathcal{T}_s'(s^0, z^0)}{1 - \mathcal{T}_z'(s^0, z^0)} z^0 \zeta_z^c(z^0)$$

which is the Pareto-efficiency condition (28) presented in Proposition 3, recognizing that $\mathcal{T}_z'(s, z) = T_z'(z)$ and $\mathcal{T}_s'(s, z) = T_s'(s)$.

C.9 Proof of Proposition 4 (Multidimensional Heterogeneity)

We characterize in Proposition 4 optimal tax rates on s for each type of simple tax system in the presence of multidimensional heterogeneity. These formulas take the actual earnings tax schedule as given, be they optimally set or not, and extend the results derived in the unidimensional case. Crucially, we are able to provide similar characterizations because Lemma 1 still holds in the presence of multidimensional heterogeneity.

C.9.1 Separable linear (SL) tax system

Consider a reform that consists in a $\Delta\tau_s$ increase in the linear tax rate τ_s. For all individuals, this triggers an increase in tax liability by $s\,\Delta\tau_s$ and an increase in the marginal tax rate on s by $\Delta\tau_s$, which by Lemma 1 produces earnings responses equivalent to an increase in the marginal earnings tax rate by $s_{inc}'\Delta\tau_s$.

We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*

$$\int_z \int_s \Big[(1 - g(s, z))\, s\, \Delta\tau_s\Big] h(s, z)\, ds dz$$
$$= \int_z \mathbb{E}\left[(1 - g(s, z))\, s | z\right] \, \Delta\tau_s h_z(z)\, dz \tag{155}$$

- *behavioral effects from changes in* z[45]

$$\int_z T_z'(z)\left\{\int_s\left(-\frac{z}{1-T_z'(z)}\zeta_z^c(s,z)\,s_{inc}'(s,z)\,\Delta\tau_s-\frac{\eta_z(s,z)}{1-T_z'(z)}s\,\Delta\tau_s\right)h(s,z)\,ds\right\}dz$$
$$=-\int_z\frac{T_z'(z)}{1-T_z'(z)}\left\{\mathbb{E}\left[z\zeta_z^c(s,z)s_{inc}'(s,z)+\eta_z(s,z)s|z\right]\right\}\Delta\tau_s h_z(z)\,dz \tag{157}$$

- *behavioral effects from changes in* s

$$\tau_s\int_z\int_s\Bigg\{-\frac{s}{1+\tau_s}\zeta_{s|z}^c(s,z)\,\Delta\tau_s-\frac{\eta_{s|z}(s,z)}{1+\tau_s}\,s\,\Delta\tau_s$$
$$+s_{inc}'(s,z)\left(-\frac{z}{1-T_z'(z)}\zeta_z^c(s,z)\,s_{inc}'(s,z)\,\Delta\tau_s-\frac{\eta_z(s,z)}{1-T_z'(z)}s\,\Delta\tau_s\right)\Bigg\}h(s,z)\,dsdz$$
$$=-\tau_s\int_z\Bigg\{\frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta_{s|z}^c(s,z)+\eta_{s|z}(s,z)\,s|z\right] \tag{158}$$
$$+\frac{1}{1-T_z'(z)}\left(\mathbb{E}\left[z\zeta_z^c(s,z)\,\left(s_{inc}'(s,z)\right)^2+\eta_z(s,z)\,s\,s_{inc}'(s,z)\,|z\right]\right)\Bigg\}\Delta\tau_s h_z(z)\,dz$$

such that the total impact of the reform on the government objective is

$$\frac{d\mathcal{L}}{\Delta\tau_s}=\int_z\mathbb{E}\left[(1-g(s,z))\,s|z\right]\,h_z(z)\,dz \tag{159}$$
$$-\int_z\frac{T_z'(z)}{1-T_z'(z)}\left\{\mathbb{E}\left[z\zeta_z^c(s,z)s_{inc}'(s,z)+\eta_z(s,z)\,s|z\right]\right\}h_z(z)\,dz$$
$$-\tau_s\int_z\Bigg\{\frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta_{s|z}^c(s,z)+\eta_{s|z}(s,z)\,s|z\right]$$
$$+\frac{1}{1-T_z'(z)}\left(\mathbb{E}\left[z\zeta_z^c(s,z)\,\left(s_{inc}'(s,z)\right)^2+\eta_z(s,z)\,s\,s_{inc}'(s,z)\,|z\right]\right)\Bigg\}h_z(z)\,dz.$$

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z)=g(s,z)+\frac{T_z'(z)}{1-T_z'(z)}\eta_z(s,z)+\frac{\tau_s}{1+\tau_s}\eta_{s|z}(s,z)+\frac{\tau_s}{1-T_z'(z)}\eta_z(s,z)s_{inc}'(s,z) \tag{160}$$

[45] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz=-\frac{z\zeta_z^c(s,z)}{1-T_z'(z)}\,\Delta\tau_s\,s_{inc}'(s,z)-\frac{\eta_z(s,z)}{1-T_z'(z)}\,\Delta\tau_s\,s\\ ds=-\frac{s(z)\zeta_{s|z}^c(s,z)}{1+\tau_s}\,\Delta\tau_s-\frac{\eta_{s|z}(,sz)}{1+\tau_s}\Delta\tau_s\,s+s_{inc}'(s,z)dz\end{cases} \tag{156}$$

we finally get

$$\frac{d\mathcal{L}}{\Delta\tau_s} = \int_z \mathbb{E}\left[(1-\hat{g}(s,z))\,s|z\right]\,h_z(z)\,dz - \int_z \frac{T_z'(z)}{1-T_z'(z)}\left\{\mathbb{E}\left[z\zeta_z^c(s,z)s_{inc}'(s,z)\,|z\right]\right\}h_z(z)\,dz$$
$$-\tau_s\int_z\left\{\frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta_{s|z}^c(s,z)|z\right]+\frac{1}{1-T_z'(z)}\left(\mathbb{E}\left[z\zeta_z^c(s,z)\,\left(s_{inc}'(s,z)\right)^2|z\right]\right)\right\}h_z(z)\,dz. \tag{161}$$

Characterizing the optimal linear tax rate τ_s through $d\mathcal{L}=0$, it satisfies

$$\frac{\tau_s}{1+\tau_s}\int_z\left\{\mathbb{E}\left[s\zeta_{s|z}^c(s,z)|z\right]\right\}dH_z(z) \tag{162}$$
$$=\int_z\left\{\mathbb{E}\left[(1-\hat{g}(s,z))\,s|z\right]-\mathbb{E}\left[\frac{T_z'(z)+s_{inc}'(s,z)\,\tau_s}{1-T_z'(z)}z\zeta_z^c(s,z)s_{inc}'(s,z)\,|z\right]\right\}dH_z(z)$$

which is equation (30) in Proposition 4.

C.9.2 Separable nonlinear (SN) tax system

Consider a reform that consists in a small $\delta\tau_s$ increase in the marginal tax rate on s across the bandwidth $\left[s^0, s^0+\Delta s\right]$. For all individuals with savings above s^0, this triggers a $\Delta s\,\Delta\tau_s$ increase in tax liability. For individuals at s^0, this triggers a $\Delta\tau_s$ increase in the marginal tax rate on s – which by Lemma 1 produces earnings responses equivalent to a $s_{inc}'\Delta\tau_s$ increase in the marginal earnings tax rate.

We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta s\to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*

$$\int_{s\geq s^0}\int_z\left\{(1-g(s,z))\,\Delta s\Delta\tau_s\right\}h(s,z)\,dsdz \tag{163}$$
$$=\int_z\left\{\mathbb{E}\left[1-g(s,z)\,|z,s\geq s^0\right]\right\}\Delta s\Delta\tau_s h_z(z)\,dz$$

- *behavioral effects from changes in* z[46]

$$
\begin{aligned}
&-\int_z T_z'(z)\left\{\frac{z}{1-T_z'}\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)\Delta\tau_s\right\}\Delta s\, h\left(s^0,z\right)dz \qquad (165)\\
&-\int_{s\geq s^0}\int_z T_z'(z)\left\{\frac{\eta_z(s,z)}{1-T_z'(z)}\Delta\tau_s\Delta s\right\}h(s,z)\,dsdz\\
&=-\int_z\frac{T_z'(z)}{1-T_z'}\left\{z\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)+\mathbb{E}\left[\eta_z(s,z)s|z,s\geq s^0\right]\right\}\Delta\tau_s\Delta s\, h_z(z)\,dz
\end{aligned}
$$

- *behavioral effects from changes in* s

$$
-T_s'\left(s^0\right)\int_z\left\{\frac{s^0}{1+T_s'\left(s^0\right)}\zeta_{s|z}^c(s^0,z)\,\Delta\tau_s+s_{inc}'\left(s^0,z\right)\frac{z}{1-T_z'(z)}\zeta_z^c(s^0,z)\,s_{inc}'\left(s^0,z\right)\Delta\tau_s\right\}\Delta s\, h\left(s^0,z\right)dz \qquad (166)
$$

$$
\begin{aligned}
&-\int_{s\geq s^0}\int_z\left\{T_s'(s)\left(\frac{\eta_{s|z}(s,z)}{1+T_s'(s)}\Delta s\Delta\tau_s+s_{inc}'(s,z)\frac{\eta_z(s,z)}{1-T_z'(z)}\Delta s\Delta\tau_s\right)\right\}h(s,z)\,dsdz\\
&=-\int_z\left\{\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c(s^0,z)+\frac{T_s'\left(s^0\right)}{1-T_z'(z)}s_{inc}'\left(s^0,z\right)^2 z\zeta_z^c(s^0,z)\right\}\Delta\tau_s\Delta s\, h_z(z)\,dz \qquad (167)\\
&-\int_z\left\{\mathbb{E}\left[\frac{T_s'(s)}{1+T_s'(s)}\eta_{s|z}(s,z)\middle|z,s\geq s^0\right]+\mathbb{E}\left[s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\eta_z(s,z)\middle|z,s\geq s^0\right]\right\}\Delta s\Delta\tau_s h_z(z)\,dz
\end{aligned}
$$

such that the total impact of the reform on the government objective is

$$
\begin{aligned}
\frac{d\mathcal{L}}{\Delta s\Delta\tau_s}=&\int_z\left\{\mathbb{E}\left[1-g(s,z)\,|z,s\geq s^0\right]\right\}dH_z(z) \qquad (168)\\
&-\int_z\frac{T_z'(z)}{1-T_z'}\left\{z\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)+\mathbb{E}\left[\eta_z(s,z)|z,s\geq s^0\right]\right\}dH_z(z)\\
&-\int_z\left\{\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c(s^0,z)+\frac{T_s'\left(s^0\right)}{1-T_z'(z)}s_{inc}'\left(s^0,z\right)^2 z\zeta_z^c(s^0,z)\right\}dH_z(z)\\
&-\int_z\left\{\mathbb{E}\left[\frac{T_s'(s)}{1+T_s'(s)}\eta_{s|z}(s,z)\middle|z,s\geq s^0\right]+\mathbb{E}\left[s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\eta_z(s,z)\middle|z,s\geq s^0\right]\right\}dH_z(z).
\end{aligned}
$$

[46] Applying Lemma 1, changes in z and s are here given by

$$
\begin{cases}
dz=-\frac{z}{1-T_z'}\zeta_z^c(z)\delta T_z^{\theta\prime}-\frac{\eta_z(z)}{1-T_z'}\delta T_z^\theta\\
ds=-\frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z)\delta T_s'-\frac{\eta_{s|z}(z)}{1+T_s'}\delta T_s+s_{inc}'(z)dz
\end{cases} \qquad (164)
$$

where the reform δT_z^θ is a $s_{inc}'(s,z)\delta\tau_s$ increase in the marginal earnings tax rate when $s\in[s^0,s^0+ds]$, and a $\delta\tau_s\delta s$ increase in tax liability when $s\geq s^0+\delta s$.

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z) = g(s,z) + \frac{T_z'(z)}{1-T_z'(z)}\eta_z(s,z) + \frac{T_s'(s)}{1+T_s'(s)}\eta_{s|z}(s,z) + s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\eta_z(s,z) \quad (169)$$

we finally get

$$\frac{d\mathcal{L}}{\Delta s \Delta\tau_s} = \int_z \left\{ \mathbb{E}\left[1-\hat{g}(s,z)\,|z, s\geq s^0\right] \right\} dH_z(z) - \int_z \left\{ \frac{T_z'(z)}{1-T_z'} z\zeta_z^c\left(s^0,z\right) s_{inc}'\left(s^0,z\right) \right\} dH_z(z)$$
$$- \int_z \left\{ \frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)} s^0\zeta_{s|z}^c(s^0,z) + \frac{T_s'\left(s^0\right)}{1-T_z'(z)} s_{inc}'\left(s^0,z\right)^2 z\zeta_z^c(s^0,z) \right\} dH_z(z)\,. \quad (170)$$

Characterizing the optimal marginal tax rate on s, through $\frac{d\mathcal{L}}{\Delta s\Delta\tau_s} = 0$, it satisfies at each savings s^0,

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)} \int_z \left\{ s^0\zeta_{s|z}^c(s^0,z) \right\} dH_z(z) \quad (171)$$
$$= \int_z \left\{ \mathbb{E}\left[1-\hat{g}(s,z)\,|z,s\geq s^0\right] \right\} dH_z(z) - \int_z \left\{ \frac{T_z'(z) + s_{inc}'\left(s^0,z\right) T_s'\left(s^0\right)}{1-T_z'(z)} z\zeta_z^c\left(s^0,z\right) s_{inc}'\left(s^0,z\right) \right\} dH_z(z)$$

which is equation (73) in Proposition B.5.

C.9.3 Linear earnings-dependent (LED) tax system

Consider a reform that consists in a $\Delta\tau_s\Delta z$ increase in $\tau_s(z)$, the linear earnings-dependent tax rate on s, phased-in across the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$.[47]

For all individuals with earnings above $z^0+\Delta z$, this triggers an increase in the linear tax rate by $\Delta\tau_s\Delta z$ meaning that the marginal tax rate on s increases by the same magnitude, which—by Lemma 1—triggers earnings responses equivalent to those induced by a $s_{inc}'\Delta\tau_s\Delta z$ increase in the marginal earnings tax rate, and so individuals' tax liability increases by $s\,\Delta\tau_s\Delta z$.

For individuals in the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$, the only direct effect of the reform is to induce earnings responses which by Lemma 1 are equivalent to an increase in the marginal earnings tax rate given by $s\,\Delta\tau_s$.

We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $\Delta z \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*

$$\int_{z\geq z^0}\int_s \left\{ (1-g(s,z))\,\Delta z\Delta\tau_s s \right\} h(s,z)\,ds dz$$
$$= \int_{z\geq z^0} \left\{ E_s\left[(1-g(s,z))\,s \middle| z \right] \Delta z \Delta\tau_s \right\} h_z(z)\,dz \quad (172)$$

[47] To avoid any ambiguity, we here use d for integration and δ for attributes of the reform we consider.

- *behavioral effects from changes in* z[48]

$$
\begin{aligned}
& - \int_s \left(T_z'\left(z^0\right) + \tau_s'\left(z^0\right) s\right) \left\{ \frac{z^0 \zeta_z^c(s,z^0)}{1 - T_z'\left(z^0\right) - \tau_s'\left(z^0\right) s} \, s \, \Delta\tau_s \right\} \Delta z \, h\left(s, z^0\right) ds \\
& - \int_{z \geq z^0} \int_s \left(T_z'(z) + \tau_s'(z) s\right) \left\{ \frac{z \zeta_z^c(s,z)}{1 - T_z'(z) - \tau_s'(z) s} s_{inc}' \Delta z \Delta\tau_s + \frac{\eta_z(s,z)}{1 - T_z'(z) - \tau_s'(z) s} \, s \, \Delta z \Delta\tau_s \right\} h(s,z) \, ds \\
& = -\mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z) s}{1 - T_z'(z) - \tau_s'(z) s} z \zeta_z^c(s,z) s \middle| z = z^0 \right] \Delta z \Delta\tau_s \, h_z\left(z^0\right) \qquad (173) \\
& - \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z) s}{1 - T_z'(z) - \tau_s'(z) s} \left(z \zeta_z^c(s,z) s_{inc}'(s,z) + s \, \eta_z(s,z) \right) \middle| z \right] \right\} \Delta z \Delta\tau_s h_z(z) \, dz
\end{aligned}
$$

- *behavioral effects from changes in* s

$$
\begin{aligned}
& - \tau_s\left(z^0\right) \int_s s_{inc}'\left(s, z^0\right) \left\{ \frac{z^0 \zeta_z^c(s,z^0)}{1 - T_z'\left(z^0\right) - \tau_s'\left(z^0\right) s} \, s \, \Delta\tau_s \right\} \Delta z \, h\left(s, z^0\right) ds \\
& - \int_{z \geq z^0} \int_s \tau_s(z) \left\{ \frac{s \zeta_{s|z}^c(s,z)}{1 + \tau_s(z)} \Delta z \Delta\tau_s + s_{inc}'(s,z) \frac{z \zeta_z^c(s,z)}{1 - T_z'(z) - \tau_s'(z) s} s_{inc}'(s,z) \Delta z \Delta\tau_s \right\} h(s,z) \, dz \\
& - \int_{z \geq z^0} \int_s \left\{ \tau_s(z) \left(\frac{\eta_{s|z}(s,z)}{1 + \tau_s(z)} \, s \, \Delta z \Delta\tau_s + s_{inc}'(s,z) \frac{\eta_z(s,z)}{1 - T_z'(z) - \tau_s'(z) s} \, s \, \Delta z \Delta\tau_s \right) \right\} h(s,z) \, ds dz \\
& = -\tau_s\left(z^0\right) \mathbb{E}\left[s_{inc}'(s,z) \frac{z \zeta_z^c(s,z) s}{1 - T_z'(z) - \tau_s'(z) s} \middle| z = z^0 \right] \Delta z \Delta\tau_s \, h_z\left(z^0\right) \qquad (174) \\
& - \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{\tau_s(z)}{1 + \tau_s(z)} s \zeta_{s|z}^c(s,z) \middle| z \right] + \mathbb{E}\left[\frac{\tau_s(z)}{1 - T_z'(z) - \tau_s'(z) s} z \zeta_z^c(s,z) s_{inc}'(s,z)^2 \right] \right\} \Delta z \Delta\tau_s \, h_z(z) \, dz \\
& - \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{\tau_s(z)}{1 + \tau_s(z)} s \eta_{s|z}(s,z) \middle| z \right] + \mathbb{E}\left[\frac{\tau_s(z)}{1 - T_z'(z) - \tau_s'(z) s} s \eta_z(s,z) s_{inc}'(s,z) \middle| z \right] \right\} \Delta z \Delta\tau_s \, h_z(z) \, dz
\end{aligned}
$$

such that the total impact of the reform on the government objective is

[48] Applying Lemma 1, changes in z and s at earnings z^0 and above earnings z^0 are respectively

$$
\begin{cases} dz = -\frac{z^0 \zeta_z^c(s,z^0)}{1 - T_z'} \Delta\tau_s \, s \\ ds = s_{inc}'(s,z^0) dz \end{cases} \quad \text{and} \quad \begin{cases} dz = -\frac{z \zeta_z^c(s,z)}{1 - T_z'} \Delta\tau_s \Delta z \, s_{inc}'(s,z) - \frac{\eta_z(s,z)}{1 - T_z'} \Delta\tau_s \Delta z \, s \\ ds = -\frac{s \zeta_{s|z}^c(s,z)}{1 + T_s'} \Delta\tau_s \Delta z - \frac{\eta_s(s,z)}{1 + T_s'} \Delta\tau_s \Delta z \, s + s_{inc}'(s,z) dz \end{cases}
$$

$$
\begin{aligned}
\frac{d\mathcal{L}}{\Delta s \Delta \tau_s} = & \int_{z \geq z^0} \left\{ E_s \left[(1 - g(s,z))\, s \middle| z \right] \right\} h_z(z)\, dz \\
& - \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z)\, s}{1 - T_z'(z) - \tau_s'(z)\, s} z \zeta_z^c(s,z) s \middle| z = z^0 \right] h_z\left(z^0\right) \\
& - \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z)\, s}{1 - T_z'(z) - \tau_s'(z)\, s} \left(z \zeta_z^c(s,z) s_{inc}'(s,z) + s\, \eta_z(s,z) \right) \middle| z \right] \right\} h_z(z)\, dz \\
& - \mathbb{E}\left[\frac{s_{inc}'(s,z)\, \tau_s(z)}{1 - T_z'(z) - \tau_s'(z)\, s} z \zeta_z^c(s,z) s \middle| z = z^0 \right] \delta z \delta \tau_s\, h_z\left(z^0\right) \\
& - \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{\tau_s(z)}{1 + \tau_s(z)} s \zeta_{s|z}^c(s,z) \middle| z \right] + \mathbb{E}\left[\frac{s_{inc}'(s,z)\, \tau_s(z)}{1 - T_z'(z) - \tau_s'(z)\, s} z \zeta_z^c(s,z) s_{inc}'(s,z) \middle| z \right] \right\} h_z(z)\, dz \\
& - \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{\tau_s(z)}{1 + \tau_s(z)} s \eta_{s|z}(s,z) \middle| z \right] + \mathbb{E}\left[\frac{s_{inc}'(s,z)\, \tau_s(z)}{1 - T_z'(z) - \tau_s'(z)\, s} s \eta_z(s,z) \middle| z \right] \right\} h_z(z)\, dz.
\end{aligned} \tag{175}
$$

Redefining augmented social marginal welfare weights as

$$
\hat{g}(s,z) = g(s,z) + \frac{T_z'(z) + \tau_s'(z)\, s + s_{inc}'(s,z) \tau_s(z)}{1 - T_z'(z) - \tau_s'(z)\, s} \eta_z(s,z) + \frac{\tau_s(z)}{1 + \tau_s(z)} \eta_{s|z}(s,z) \tag{176}
$$

we finally get

$$
\begin{aligned}
\frac{d\mathcal{L}}{\Delta s \Delta \tau_s} = & \int_{z \geq z^0} \mathbb{E}\left[(1 - \hat{g}(s,z))\, s \middle| z \right] h_z(z)\, dz \\
& - \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z)\, s + s_{inc}'(s,z)\, \tau_s(z)}{1 - T_z'(z) - \tau_s'(z)\, s} z \zeta_z^c(s,z) s \middle| z = z^0 \right] h_z\left(z^0\right) \\
& - \int_{z \geq z^0} \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z)\, s + s_{inc}'(s,z)\, \tau_s(z)}{1 - T_z'(z) - \tau_s'(z)\, s} z \zeta_z^c(s,z) s_{inc}'(s,z) \Big) \middle| z \right] h_z(z)\, dz \\
& - \int_{z \geq z^0} \mathbb{E}\left[\frac{\tau_s(z)}{1 + \tau_s(z)} s \zeta_{s|z}^c(s,z) \middle| z \right] h_z(z)\, dz
\end{aligned} \tag{177}
$$

Characterizing the optimal linear earnings-dependent tax rate $\tau_s(.)$ through $d\mathcal{L} = 0$, it satisfies at each earnings z^0

$$
\begin{aligned}
& \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z)\, s + s_{inc}'(s,z)\, \tau_s(z)}{1 - T_z'(z) - \tau_s'(z)\, s} z \zeta_z^c(s,z) s \middle| z = z^0 \right] h_z\left(z^0\right) + \int_{z \geq z^0} \mathbb{E}\left[\frac{\tau_s(z)}{1 + \tau_s(z)} s \zeta_{s|z}^c(s,z) \middle| z \right] h_z(z)\, dz \\
& = \int_{z \geq z^0} \mathbb{E}\left[(1 - \hat{g}(s,z))\, s \middle| z \right] h_z(z)\, dz - \int_{z \geq z^0} \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z)\, s + s_{inc}'(s,z)\, \tau_s(z)}{1 - T_z'(z) - \tau_s'(z)\, s} z \zeta_z^c(s,z) s_{inc}'(s,z) \middle| z \right] h_z(z)\, d
\end{aligned} \tag{178}
$$

which is equation (74) in Proposition B.5.

C.10 Proof of Proposition 5 (Multiple Goods)

C.10.1 Setting and definitions

The problem of the government is to maximize the following Lagrangian

$$\mathcal{L} = \int_z \left\{ \alpha(z)\, U\Big(z - \mathcal{T}(\boldsymbol{s}(z), z) - \sum_{i=1}^{n} s_i(z), \boldsymbol{s}(z), z; \vartheta\,(z)\Big) + \lambda \mathcal{T}(\boldsymbol{s}(z), z) - E \right\} dH_z(z) \quad (179)$$

where we use the fact that $z\,(\theta)$ is a bijective mapping to denote $\vartheta\,(z)$ its inverse, and to define Pareto weights $\alpha(z) := \alpha(\vartheta(z))$ and the vector of n consumption goods as $\boldsymbol{s}\,(z) := \boldsymbol{s}\,(z; \vartheta\,(z))$.

In this setting, we express optimal tax formulas in terms of the following elasticity concepts that measure consumption responses of s_i and s_j to changes in $\mathcal{T}'_{s_i}$:

$$\zeta^c_{s_i|z}\,(z(\theta)) := -\frac{1 + \mathcal{T}'_{s_i}\,(\boldsymbol{s}\,(z;\theta)\,, z)}{s_i\,(z;\theta)} \frac{\partial s_i\,(z;\theta)}{\partial \mathcal{T}'_{s_i}\,(\boldsymbol{s}\,(z;\theta)\,, z)}\Big|_{z=z(\theta)} \quad (180)$$

$$\xi^c_{s_{j,i}|z}\,(z(\theta)) := \frac{\mathcal{T}'_{s_i}\,(\boldsymbol{s}\,(z;\theta)\,, z)}{s_j\,(z;\theta)} \frac{\partial s_j\,(z;\theta)}{\partial \mathcal{T}'_{s_i}\,(\boldsymbol{s}\,(z;\theta)\,, z)}\Big|_{z=z(\theta)} \quad (181)$$

and in terms of the following statistics,

$$s'_{i,inc}(z(\theta)) := \frac{\partial s_i(z;\theta)}{\partial z}\Big|_{z=z(\theta)} \quad (182)$$

$$\hat{g}(z(\theta)) := \left[\alpha(z)\frac{U'_c(z)}{\lambda} - \left(\mathcal{T}'_z\,(\boldsymbol{s}(z), z) + \sum_{i=1}^{n} s'_{i,inc}(z)\mathcal{T}'_{s_i}\,(\boldsymbol{s}(z), z)\right)\frac{\partial z(.)}{\partial \mathcal{T}} - \sum_{i=1}^{n} \mathcal{T}'_{s_i}\,(\boldsymbol{s}(z), z)\,\frac{\partial s_i(.)}{\partial \mathcal{T}}\right]\Big|_{z=z(\theta)}. \quad (183)$$

C.10.2 Optimal marginal tax rates on earnings z

We consider a small reform at earnings level z^0 that consists in a small increase $\Delta\tau_z$ of the marginal earnings tax rate $\mathcal{T}'_z$ in a small bandwidth Δz. The impact of this reform on the Lagrangian as $\Delta z \to 0$ is

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_z \Delta z} = \int_{x\geq z^0}\Big(1 - \alpha(x)\frac{U'_c(x)}{\lambda}\Big) dH_z(x) \quad (184)$$

$$+ \mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} h_z(z^0) + \int_{x\geq z_0} \mathcal{T}'_z\,(\boldsymbol{s}(x), x)\,\frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} dH_z(x)$$

$$+ \sum_{i=1}^{n} \mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right) s'_{i,inc}(z^0)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} h_z(z^0)$$

$$+ \int_{x\geq z_0}\sum_{i=1}^{n} \mathcal{T}'_{s_i}\,(\boldsymbol{s}(x), x)\left[\frac{\partial s_i(.)}{\partial \mathcal{T}}\Big|_{z=x} + s'_{i,inc}(x)\frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x}\right] dH_z(x). \quad (185)$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Plugging in social marginal welfare weights augmented with the fiscal impacts of income effects $\hat{g}(z)$, we obtain

$$-\left[\mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right) + \sum_{i=1}^{n} \mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right) s'_{i,inc}(z^0)\right] \frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} h_z(z_0) = \int_{x \geq z_0} \Big(1 - \hat{g}(x)\Big) dH_z(x). \tag{186}$$

C.10.3 Optimal marginal tax rates on good i

We consider a small reform at earnings level z^0 that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s_i for all individuals with earnings z above z^0, phased-in over the earnings bandwidth $\left[z^0, z^0 + \Delta z\right]$. In the bandwidth $\left[z^0, z^0 + \Delta z\right]$, this reform induces labor supply distortions on earnings z. At earnings $z \geq z^0 + \Delta z$, this reform induces (a) substitution effects away from s_i, (b) labor supply distortions on earnings z, and, new to this setting, (c) cross-effects on the consumption of goods s_{-i}.[49]

The impact of this reform on the Lagrangian as $\Delta z \to 0$ is

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_s\Delta z} = \int_{x=z^0}^{z_{max}} \Big(1 - \alpha(x)\frac{U'_c(x)}{\lambda}\Big) s_i(x) dH_z(x) \tag{187}$$
$$+ \mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right) \frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} s_i(z^0) h_z(z^0) + \int_{x=z^0}^{z_{max}} \mathcal{T}'_z\left(\boldsymbol{s}(x), x\right) \left[\frac{\partial z(.)}{\partial \mathcal{T}'_z} s'_{i,inc}(x) + \frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} s_i(x)\right] dH_z(z)$$
$$+ \sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0), z^0\right) \left[s'_{j,inc}(z^0)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} s_i(z^0)\right] h_z(z^0)$$
$$+ \int_{x=z^0}^{z_{max}} \sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(\boldsymbol{s}(x), x\right) \left\{\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=x} + \frac{\partial s_j(.)}{\partial \mathcal{T}}\Big|_{z=x} s_i(x) + s'_{j,inc}(x) \left[\frac{\partial z(.)}{\partial \mathcal{T}'_z} s'_{i,inc}(x) + \frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} s_i(x)\right]\right\} dH_z(z)$$

where $\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}$ capture cross-effects for all $j \neq i$.

We characterize optimal taxes through $d\mathcal{L} = 0$. Plugging in social marginal welfare weights augmented with the fiscal impacts of income effects $\hat{g}(x)$, we obtain

$$-\left[\mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right) + \sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0), z^0\right) s'_{j,inc}(z^0)\right] \frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} s_i(z^0) h_z(z^0) = \int_{x=z^0}^{z_{max}} \Big(1 - \hat{g}(x)\Big) s_i(x) dH_z(z)$$
$$+ \int_{x=z^0}^{z_{max}} \left\{\left[\mathcal{T}'_z\left(\boldsymbol{s}(x), x\right) + \sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(\boldsymbol{s}(x), x\right) s'_{j,inc}(x)\right] \frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=x} s'_{i,inc}(x) + \sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(\boldsymbol{s}(x), x\right) \frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=x}\right\} dH_z(z). \tag{188}$$

[49] Applying Lemma 1, which still holds in this setting, changes in z and s_j at earnings z^0 and above earnings z^0 are respectively

$$\begin{cases} dz = \frac{\partial z(.)}{\partial \mathcal{T}'_z} \Delta\tau_s \, s_i(z^0) \\ ds_j = s'_{j,inc}(z^0) dz \end{cases} \quad \text{and} \quad \begin{cases} dz = \frac{\partial z(.)}{\partial \mathcal{T}'_z} \Delta\tau_s\Delta z \, s'_{i,inc}(z) + \frac{\partial z(.)}{\partial \mathcal{T}} \Delta\tau_s\Delta z \, s_i(z) \\ ds_j = \frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}} \Delta\tau_s\Delta z + \frac{\partial s_j(.)}{\partial \mathcal{T}} \Delta\tau_s\Delta z \, s_i(z) + s'_{j,inc}(z) dz \end{cases}$$

C.10.4 Deriving Proposition 5

For any good i, we combine the optimality condition for marginal tax rates on earnings z with the one for marginal tax rates on good i to obtain

$$s_i(z^0)\int_{x=z_0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x)=\int_{x=z^0}^{z_{max}}\Big(1-\hat{g}(x)\Big)s_i(x)dH_z(z)$$
$$+\int_{x=z^0}^{z_{max}}\left[\left[\mathcal{T}_z'\left(\boldsymbol{s}(x),x\right)+\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(x),x\right)s_{j,inc}'(x)\right]\frac{\partial z(.)}{\partial\mathcal{T}_z'}\Big|_{z=x}s_{i,inc}'(x)+\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(x),x\right)\frac{\partial s_j(.)}{\partial\mathcal{T}_{s_i}'}\Big|_{z=x}\right]dH_z(z) \tag{189}$$

such that differentiating with respect to earnings z^0 gives after simplification

$$s_i'(z^0)\int_{x=z_0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x) \tag{190}$$
$$=-\left[\left[\mathcal{T}_z'\left(\boldsymbol{s}(z^0),z^0\right)+\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)s_{j,inc}'(z^0)\right]\frac{\partial z(.)}{\partial\mathcal{T}_z'}\Big|_{z=z^0}s_{i,inc}'(z^0)+\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)\frac{\partial s_j(.)}{\partial\mathcal{T}_{s_i}'}\Big|_{z=z^0}\right]h_z(z^0)$$

Making use of the optimality condition for marginal earnings tax rates, we can substitute the first term on the right-hand side to obtain

$$-\sum_{j=1}^{n}\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)\frac{\partial s_j(.)}{\partial\mathcal{T}_{s_i}'}\Big|_{z=z^0}=\underbrace{\left[s_i'(z^0)-s_{i,inc}'(z^0)\right]}_{s_{i,het}(z^0)}\frac{1}{h_z(z^0)}\int_{x=z_0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x). \tag{191}$$

Isolating the term relative to $\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)$ on the left-hand side yields the following optimal tax formula in terms of $s_{i,het}'$

$$-\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)\frac{\partial s_i(.)}{\partial\mathcal{T}_{s_i}'}\Big|_{z=z^0}=\frac{1}{h_z(z^0)}s_{i,het}'(z^0)\int_{x=z^0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x)+\sum_{j\neq i}\mathcal{T}_{s_j}'\left(z^0\right)\frac{\partial s_j(.)}{\partial\mathcal{T}_{s_i}'}\Big|_{z=z^0} \tag{192}$$

where $\frac{\partial s_j(.)}{\partial\mathcal{T}_{s_i}'}$ capture cross-effects for all $j\neq i$.

We can rewrite this optimality condition in terms of the compensated elasticity $\zeta_{s_i|z}^c$ and the cross elasticity $\xi_{s_{j,i}|z}^c$ to finally obtain

$$\frac{\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)}{1+\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)}=s_{i,het}'(z^0)\frac{1}{s_i(z^0)\zeta_{s_i|z}^c(z^0)}\frac{1}{h_z(z^0)}\int_{z=z^0}^{z_{max}}\Big(1-\hat{g}(z)\Big)dH_z(z) \tag{193}$$
$$+\sum_{j\neq i}\frac{\mathcal{T}_{s_j}'\left(\boldsymbol{s}(z^0),z^0\right)}{\mathcal{T}_{s_i}'\left(\boldsymbol{s}(z^0),z^0\right)}\frac{s_j(z^0)\xi_{s_{j,i}|z}^c(z^0)}{s_i(z^0)\zeta_{s_i|z}^c(z^0)}$$

which is the first condition stated in Proposition 5.

To derive the second condition stated in Proposition 5, we substitute the first term on the right-hand side using the optimality condition for marginal tax rates on earnings z to directly obtain

$$\frac{\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0),z^0\right)}{1+\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0),z^0\right)}=s'_{i,het}(z^0)\frac{\mathcal{T}'_z\left(\boldsymbol{s}(z^0),z^0\right)+\sum_{j=1}^n\mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0),z^0\right)s'_{j,inc}(z^0)}{1-\mathcal{T}'_z\left(\boldsymbol{s}(z^0),z^0\right)}\frac{z^0\zeta^c_z(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)}$$
$$+\sum_{j\neq i}\frac{\mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0),z^0\right)}{\mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0),z^0\right)}\frac{s_j(z^0)\xi^c_{s_{j,i}|z}(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)}. \tag{194}$$

This completes the proof of Proposition 5.

C.11 Proof of Proposition 6 (Bequest Taxation and Behavioral Biases)

C.11.1 Setting

We here provide a sufficient statistics characterization of a smooth tax system $\mathcal{T}(s,z)$ under the following additively separable representation of individuals' preferences

$$U\big(c,s,z;\theta\big)=u\left(c;\theta\right)-k\left(z;\theta\right)+\beta\left(\theta\right)v\left(s;\theta\right),$$

and for a utilitarian government that maximizes

$$\int_\theta\big[U\left(c\left(\theta\right),s\left(\theta\right),z\left(\theta\right);\theta\right)+\nu\left(\theta\right)v\left(s\left(\theta\right);\theta\right)\big]dF\left(\theta\right), \tag{195}$$

where $\nu(\theta)$ parametrizes the degree of misalignment on the valuation of s.

Using the mapping between types θ and earnings z, the Lagrangian of the problem is written as

$$\mathcal{L}=\int_z\Big[U\left(c\left(z\right),s\left(z\right),z;\vartheta\left(z\right)\right)+\nu\left(z\right)v\left(s\left(z\right);\vartheta\left(z\right)\right)+\lambda\left(\mathcal{T}\left(s,z\right)-E\right)\Big]dH_z\left(z\right). \tag{196}$$

As before, we derive optimal tax formulas by considering reforms of marginal tax rates on z and s. Thanks to the additively separable representation of preferences, there are no income effects on labor supply choices. As a result, the only substantial change is that savings changes now lead to changes in social welfare proportional to the degree of misalignment.

C.11.2 Optimal marginal tax rates on z.

A small reform at earnings z^0 that consists in a small increase $\Delta\tau_z$ of the marginal earnings tax rate in a small bandwidth Δz has the following effect as $\Delta z\to 0$,

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_z\Delta z}=\int_{z^0}^{z_{max}}\left(1-\hat{g}(z)\right)dH_z(z)$$
$$-\left(\mathcal{T}'_z\left(s\left(z^0\right),z^0\right)+s'_{inc}(z^0)\Big(\mathcal{T}'_s\left(s(z^0),z^0\right)+\nu\left(z^0\right)\frac{v'\left(s\left(z^0\right)\right)}{\lambda}\Big)\right)\frac{z^0}{1-\mathcal{T}'_z\left(s\left(z^0\right),z^0\right)}\zeta^c_z(z^0)\,h_z(z^0).$$

In this context, social marginal welfare weights augmented with income effects $\hat{g}(z)$ are equal to

$$\hat{g}(z) = \frac{u'(c(z))}{\lambda} + \Big(\mathcal{T}_s'(s(z),z) + \nu(z)\frac{v'(s(z))}{\lambda}\Big)\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'(s(z),z)}$$

and we can use individuals' first-order condition for s, $(1+\mathcal{T}_s')\,u'(c) = \beta v'(s)$, to express the misalignment wedge in terms of the social marginal welfare weights $g(z) := \frac{u'(c(z))}{\lambda}$ as

$$\nu(z)\frac{v'(s(z))}{\lambda} = \frac{\nu(z)}{\beta(z)}g(z)\left(1+\mathcal{T}_s'\right).$$

The optimal schedule of marginal earnings tax rates is thus characterized by

$$\begin{aligned}\frac{\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)} &= \frac{1}{\zeta_z^c(z^0)}\frac{1}{z^0h_z(z^0)}\int_{z=z^0}^{z_{max}}\left(1-\hat{g}(z)\right)dH_z(z)\\ &\quad - s_{inc}'(z^0)\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)} - s_{inc}'(z^0)\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\frac{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}.\end{aligned}$$

C.11.3 Optimal marginal tax rates on s.

A small reform at earnings level z^0 that consists in adding a linear tax rate $\Delta\tau_s\Delta z$ on s for all individuals with earnings z above z^0, phased-in over the earnings bandwidth $\left[z^0, z^0+\Delta z\right]$, has the following effect as $\Delta s \to 0$,

$$\begin{aligned}&\frac{1}{\lambda}\frac{d\mathcal{L}}{\Delta\tau_s\Delta z}\\ &= -\left[\mathcal{T}_z'\left(s^0,z^0\right) + s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s^0,z^0\right) + \nu\left(z^0\right)\frac{v'\left(s\left(z^0\right)\right)}{\lambda}\Big)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)\,s^0\,h_z(z^0)\\ &+ \int_{z\geq z^0}\left\{\left(1-\hat{g}(z)\right)s(z) - \left[\mathcal{T}_s'(s(z),z) + \nu(z)\frac{v'(s(z))}{\lambda}\right]\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'(s(z),z)}\right\}dH_z(z)\\ &- \int_{z\geq z^0}\left\{\left[\mathcal{T}_z'(s(z),z) + s_{inc}'(z)\Big(\mathcal{T}_s'(s(z),z) + \nu(z)\frac{v'(s(z))}{\lambda}\Big)\right]\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'(s(z),z)}s_{inc}'(z)\right\}dH_z(z).\end{aligned}$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Replacing the misalignment wedge by its expression in terms of social marginal welfare weights $g(z)$, we obtain that the optimal schedule of marginal tax rates on s is characterized by

$$\begin{aligned}&\left[\mathcal{T}_z'\left(s^0,z^0\right) + s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s^0,z^0\right) + \frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z)\left(1+\mathcal{T}_s'\left(s^0,z^0\right)\right)\Big)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)\,s^0\,h_z(z^0)\\ &= \int_{z\geq z^0}\left\{\left(1-\hat{g}(z)\right)s(z) - \left[\mathcal{T}_s'(s(z),z) + \frac{\nu(z)}{\beta(z)}g(z)\left(1+\mathcal{T}_s'(s(z),z)\right)\right]\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'(s(z),z)}\right\}dH_z(z)\\ &- \int_{z\geq z^0}\left\{\left[\mathcal{T}_z'(s(z),z) + s_{inc}'(z)\Big(\mathcal{T}_s'(s(z),z) + \frac{\nu(z)}{\beta(z)}g(z)\left(1+\mathcal{T}_s'(s(z),z)\right)\Big)\right]\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'}s_{inc}'(z)\right\}dH_z(z).\end{aligned}$$

C.11.4 Deriving Proposition 6

Combining optimality conditions for marginal tax rates on z and s yields

$$s(z^0)\int_{z=z^0}^{z_{max}}(1-\hat{g}(z))\,dH_z(z)=\int_{z\geq z^0}(1-\hat{g}(z))\,s\left(z\right)dH_z(z) \tag{197}$$
$$-\int_{z\geq z^0}\left\{\left[\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\left(\mathcal{T}_s'\left(s(z),z\right)+\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\right)\right]\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'}s_{inc}'(z)\right\}dH_z(z)$$
$$-\int_{z\geq z^0}\left\{\left[\mathcal{T}_s'\left(s(z),z\right)+\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left(1+\mathcal{T}_s'\left(s(z),z\right)\right)\right]\frac{s(z)\zeta_{s|z}^c(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}\right\}dH_z(z).$$

Differentiating with respect to z^0, we obtain after simplification

$$s'(z^0)\int_{z=z^0}^{z_{max}}(1-\hat{g}(z))\,dH_z(z) \tag{198}$$
$$=\left\{\left[\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\Big)\right]\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'}s_{inc}'(z^0)\right\}h_z(z^0)$$
$$+\left\{\left[\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\right]\frac{s(z^0)\zeta_{s|z}^c(z^0)}{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}\right\}h_z(z^0).$$

Substituting the first term on the right-hand side by its expression from the optimality condition for marginal tax rates on z, and rearranging we obtain

$$\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)=\underbrace{\left[s'(z^0)-s_{inc}'(z^0)\right]}_{s_{het}'(z^0)}\frac{1}{s(z^0)\zeta_{s|z}^c(z^0)}\frac{1}{h_z(z^0)}\int_{z=z^0}^{z_{max}}(1-\hat{g}(z))\,dH_z(z) \tag{199}$$

which is the first optimality condition in Proposition 6.

Conversely, substituting the term on the left-hand side by its expression from the optimality condition for marginal tax rates on z, and rearranging we obtain

$$\left[\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\right]\frac{s(z^0)\zeta_{s|z}^c(z^0)}{1+\mathcal{T}_s'\left(s(z^0),z^0\right)} \tag{200}$$
$$=s_{het}'(z^0)\left[\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\Big(\mathcal{T}_s'\left(s(z^0),z^0\right)+\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left(1+\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\Big)\right]\frac{z^0\zeta_z^c(z^0)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}$$

which is the second optimality condition in Proposition 6.

C.12 Proof of Proposition 7 (Multidimensional Range with Heterogeneous Prices)

C.12.1 Setting

We consider heterogeneous marginal rates of transformation or "prices" $p(z,\theta)$ between c and s, and a two-part tax structure, where a person must pay a tax $T_1(z)$ in units of c and a tax $T_2(s,z)$ in units of s. In particular, we consider simple tax systems of the SN type, where the tax on s is nonlinear but independent of earnings z such that $T_2(s,z) = T_2(s)$, and of the LED type, where the tax on s is linear but earnings-dependent such that $T_2(s,z) = \tau_s(z)\, s$.

In this setting, we can write type θ's problem as

$$\max_{c,s,z} U(c,s,z;\theta) \text{ s.t. } c + p(z,\theta)s \leq z - T_1(z) - p(z,\theta)T_2(s,z) \tag{201}$$

$$\iff \max_{z} \left\{ \max_{s} U\Big(z - T_1(z) - p(z,\theta)\left(s + T_2(s,z)\right), s, z; \theta\Big) \right\} \tag{202}$$

where the inner problem leads to consumption choices $c(z;\theta)$ and $s(z;\theta)$, and the outer problem leads to an earnings choice $z(\theta)$. Assuming $z(\theta)$ continues to be a bijective mapping, we again denote $\vartheta(z)$ its inverse. This allows us to define $s(z) := s(z;\vartheta(z))$, $p(z) := p(z(\vartheta(z));\vartheta(z))$ and to formulate the problem in terms of observable earnings z.[50]

Let λ_1 and λ_2 be the marginal values of public funds associated with the resource constraints

$$\int_z T_1(z) dH_z(z) \geq E_1 \tag{203}$$

$$\int_z T_2(s(z),z) dH_z(z) \geq E_2. \tag{204}$$

The problem of the government is to maximize the Lagrangian

$$\mathcal{L} = \int_z \Big\{ \alpha(z)\, U\Big(z - T_1(z) - p(z)\big(s(z) + T_2(s(z),z)\big), s(z), z; \vartheta(z)\Big) + \lambda_1 T_1(z) + \lambda_2 T_2(s(z),z) - E_1 - E_2 \Big\} dH_z(z). \tag{205}$$

C.12.2 Adapting Lemma 1

Lemma C.4. *For an type $\theta = \vartheta(z)$, we have that:*
(1a) *a small increase $\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ generates the same earnings change as a small increase $p(z)\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_1}{\partial z}$.*
(1b) *a small increase $\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ generates the same earnings change as a small increase $p(z)s'_{inc}(z)\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_1}{\partial z}$.*
(2) *a small increase ΔT in the T_2 tax liability faced by type $\theta = \vartheta(z)$ generates the same earnings change as a small increase $p(z)\Delta T$ in the T_1 tax liability.*

[50] When taking derivatives, the presence of these two arguments is implicit. For instance, a total derivative corresponds to $\frac{dp}{dz} := \frac{\partial p}{\partial z} + \frac{\partial p}{\partial \theta}\frac{\partial \theta}{\partial z}$, whereas a partial derivative $\frac{\partial p}{\partial z}$ represents variation in only the first argument.

Proof. We first derive an abstract characterization that we then apply to different tax reforms.

Let type θ indirect utility function at earnings z be

$$V\left(T_1(z), T_2(., z), z; \theta\right) := \max_s U\Big(z - T_1(z) - p(z,\theta)\left(s + T_2(s,z)\right), s, z; \theta\Big). \tag{206}$$

Consider a small reform $\Delta T_2(s,z)$ of T_2, and construct for each type θ a perturbation $\Delta T_1^\theta(z)$ of T_1 that induces the same earnings response as the initial perturbation. Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(T_1(z) + \Delta T_1^\theta(z), T_2(., z), z; \theta) = V(T_1(z), T_2(., z) + \Delta T_2(., z), z; \theta). \tag{207}$$

Then, by construction, the perturbation $\Delta T_1^\theta(z)$ induces the same earnings response dz as the initial perturbation $\Delta T_2(., z)$. Moreover, both tax reforms must induce the same utility change for type θ. Applying the envelope theorem yields

$$-U_c'(z;\theta) \cdot \Delta T_1^\theta(z) = -U_c'(z;\theta)\, p(z,\theta) \cdot \Delta T_2\left(s(z;\theta), z\right) \tag{208}$$

such that finally, the perturbation $\Delta T_1^\theta(z)$ is

$$\Delta T_1^\theta(z) = p(z,\theta) \cdot \Delta T_2\left(s(z;\theta), z\right). \tag{209}$$

and we can now apply this abstract characterization to different tax reforms.

(1a) Consider a small increase $\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ over a small bandwidth of income $\left[z^0, z^0 + \Delta z\right]$. Then, for any type θ such that $z(\theta) \in \left[z^0, z^0 + \Delta z\right]$, we have $\Delta T_2\left(s(z;\theta), z\right) = \Delta\tau_z\left(z - z^0\right)$ such that $\Delta T_1^\theta(z) = p(z,\theta)\Delta\tau_z\left(z - z^0\right)$ and differentiating with respect to z we get

$$\Big(\Delta T_1^\theta(z)\Big)' = \frac{\partial p(z,\theta)}{\partial z}\Delta\tau_z\left(z - z^0\right) + p(z,\theta)\Delta\tau_z. \tag{210}$$

At the limit $\Delta z \to 0$ such that $z \to z^0$, a small increase $\Delta\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ generates the same earnings change as a small increase $p(z)\Delta\tau_z$ in the marginal tax rate $T_1'(z)$.

(1b) Consider a small increase $\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ over a small bandwidth of savings $\left[s^0, s^0 + \Delta s\right]$. Then, for any type θ such that $s(\theta) \in \left[s^0, s^0 + \Delta s\right]$, we have $\Delta T_2\left(s(z;\theta), z\right) = \Delta\tau_s\left(s(z;\theta) - s^0\right)$ such that $\Delta T_1^\theta(z) = p(z,\theta)\Delta\tau_z\left(s(z;\theta) - s^0\right)$ and differentiating with respect to z we get

$$\Big(\Delta T_1^\theta(z)\Big)' = \frac{\partial p(z,\theta)}{\partial z}\Delta\tau_z\left(s(z;\theta) - s^0\right) + p(z,\theta)\Delta\tau_z s_{inc}'(z). \tag{211}$$

At the limit $\Delta s \to 0$ such that $s \to s^0$, a small increase $\Delta\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ generates the same earnings change as a small increase $p(z)s_{inc}'(z)\Delta\tau_z$ in the marginal tax rate $T_1'(z)$.

(2) Consider a small lump-sum increase ΔT in the T_2 tax liability for a type θ who earns z, we then have $\Delta T_1^\theta(z) = p(z,\theta)\Delta T$ such that the equivalent reform is no longer a lump-sum increase. Hence, a small increase ΔT in the T_2 tax liability faced by a type $\vartheta(z)$ generates the same earnings change as a small increase $p(z)\Delta T$ in the T_1 tax liability. □

C.12.3 Marginal values of public funds

An important prerequisite to derive optimality conditions is to pin down the marginal values of public funds λ_1 and λ_2. At the optimum, λ_1 and λ_2 are pinned down by optimally setting the tax level T_1 and T_2. Characterizing the impact of lump-sum changes in tax liabilities yields the following two equations that can be solved for λ_1 and λ_2:

$$\int_{x=z_{min}}^{z_{max}} \left\{ -\alpha(x)U_c'(x) + \lambda_1 + \left(\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z} + s_{inc}'(x)\lambda_2 \frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1} + \lambda_2 \frac{\partial T_2}{\partial s} \frac{\partial s(.)}{\partial T_1} \right\} dH_z(x) = 0 \tag{212}$$

$$\int_{x=z_{min}}^{z_{max}} \left\{ -\alpha(x)p(x)U_c'(x) + \lambda_2 + \left(\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z} + s_{inc}'(x)\lambda_2 \frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_2} + \lambda_2 \frac{\partial T_2}{\partial s} \frac{\partial s(.)}{\partial T_2} \right\} dH_z(x) = 0 \tag{213}$$

where $z(.)$ and $s(.)$ denote, with a slight abuse of notation, the earnings and savings choices, and all partial derivatives are evaluated at earnings x.

Renormalizing these equations by λ_1, we can use the fact that by Lemma C.4, $\frac{\partial z(.)}{\partial T_2} = \frac{\partial z(.)}{\partial T_1} p(z) + \frac{\partial z(.)}{\partial T_1'} \frac{\partial p}{\partial z}$ and that $\frac{\partial s(.)}{\partial T_2} = \frac{\partial s(.)}{\partial T_1} p(z)$ to obtain

$$\int_{x=z_{min}}^{z_{max}} \left\{ 1 - \left[\alpha(x)\frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1} \right] \right\} dH_z(z) = 0 \tag{214}$$

$$\int_{x=z_{min}}^{z_{max}} \left\{ \frac{\lambda_2}{\lambda_1} - p(x) \left[\alpha(x)\frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1} \right] \right. \tag{215}$$

$$\left. + \left(T_1'(z) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1'}\frac{\partial p}{\partial z} \right\} dH_z(x) = 0.$$

At any given earnings x, defining social marginal welfare weights augmented with the fiscal impact of income effects $\hat{g}(x)$ and the fiscal impacts of the novel substitution effects $\varphi(x)$ as respectively

$$\hat{g}(x) := \alpha(x)\frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1} \tag{216}$$

$$\varphi(x) := \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s} \right) \frac{\partial z(.)}{\partial T_1'}\frac{\partial p}{\partial z} \tag{217}$$

where all partial derivatives are evaluated at x, we finally obtain

$$\overline{\hat{g}} := \int_{x=z_{min}}^{z_{max}} \hat{g}(x)\, dH_z(x) = 1 \tag{218}$$

$$\overline{\hat{g}p - \varphi} := \int_{x=z_{min}}^{z_{max}} \Big(\hat{g}(x)p(x) - \varphi(x) \Big)\, dH_z(x) = \frac{\lambda_2}{\lambda_1}. \tag{219}$$

C.12.4 Optimal tax rates on z

We consider a small reform at earnings level z^0 that consists in a small increase $\Delta\tau_z$ of the marginal earnings tax rate $T_1'(z)$ in a small bandwidth Δz. The impact on the Lagrangian is as $\Delta z \to 0$,

$$\begin{aligned}\frac{d\mathcal{L}}{\Delta\tau_z \Delta z} &= \int_{x\geq z^0} \Big(\lambda_1 - \alpha(x)U_c'(x)\Big) dH_z(x) \\ &+ \left[\lambda_1 T_1'\left(z^0\right) + \lambda_2 \frac{\partial T_2}{\partial z}\Big|_{z=z^0}\right] \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} h_z(z^0) \\ &+ \int_{x\geq z^0} \left[\lambda_1 T_1'\left(x\right) + \lambda_2 \frac{\partial T_2}{\partial z}\right] \frac{\partial z(.)}{\partial T_1} dH_z(x) \\ &+ \lambda_2 \frac{\partial T_2}{\partial s}\Big|_{z=z^0} s_{inc}'(z^0) \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} h_z(z^0) \\ &+ \int_{x\geq z^0} \lambda_2 \frac{\partial T_2}{\partial s} \left[\frac{\partial s(.)}{\partial T_1} + s_{inc}'(x) \frac{\partial z(.)}{\partial T_1}\right] dH_z(x).\end{aligned} \tag{220}$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Renormalizing everything by λ_1, plugging in social marginal welfare weights augmented with income effects $\hat{g}(x)$, we obtain the following optimality condition for marginal earnings tax rates at each earnings z^0

$$-\left[T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z}\Big|_{z^0} + s_{inc}'(z^0)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\right] \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} = \frac{1}{h_z(z^0)} \int_{x\geq z^0} \Big[1 - \hat{g}(x)\Big] dH_z(x). \tag{221}$$

C.12.5 Optimal tax rates on s

SN tax system. We consider a small reform at $s^0 = s(z^0)$ that consists in a small increase $\Delta\tau_s$ of $\frac{\partial T_2}{\partial s}$, the marginal tax rate on s, in a small bandwidth Δs. Using Lemma 2, we characterize the impact of the reform on the Lagrangian as $\Delta s \to 0$

$$\begin{aligned}\frac{d\mathcal{L}}{\Delta\tau_s \Delta s} &= \int_{x\geq z^0} \Big(\lambda_2 - \alpha(x)p(x)U_c'(x)\Big) dH_z(x) \\ &+ \left[\lambda_1 T_1'(z^0) + \lambda_2 \frac{\partial T_2}{\partial z}\Big|_{z=z^0}\right] \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} s_{inc}'(z^0) p(z^0) \frac{h_z(z^0)}{s'(z^0)} \\ &+ \int_{x\geq z^0} \left[\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z}\right] \left(\frac{\partial z(.)}{\partial T_1} p(x) + \frac{\partial z(.)}{\partial T_1'\left(x\right)} \frac{\partial p}{\partial z}\right) dH_z(x) \\ &+ \lambda_2 \frac{\partial T_2}{\partial s}\Big|_{z=z^0} \left[\frac{\partial s(.)}{\partial \left(\frac{\partial T_2}{\partial s}\Big|_{z=z^0}\right)} + s_{inc}'(z^0) \frac{\partial z(.)}{\partial T_1'(z^0)} s_{inc}'(z^0) p(z^0)\right] \frac{h_z(z^0)}{s'(z^0)} \\ &+ \int_{x\geq z^0} \lambda_2 \frac{\partial T_2}{\partial s} \left[\frac{\partial s(.)}{\partial T_2} + s_{inc}'(x) \left(\frac{\partial z(.)}{\partial T_1} p(x) + \frac{\partial z(.)}{\partial T_1'\left(x\right)} \frac{\partial p}{\partial z}\right)\right] dH_z(x).\end{aligned} \tag{222}$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Renormalizing by λ_1 and using $\frac{\partial s(.)}{\partial T_2} = \frac{\partial s(.)}{\partial T_1} p(x)$, we can plug in $\hat{g}(x)$ and $\varphi(x)$ to obtain the following optimality condition for marginal tax rates on s at each savings $s^0 = s(z^0)$:

$$
\begin{aligned}
&-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}h_z(z^0) = s'(z^0)\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\right\}dH_z(x) \qquad (223)\\
&+\left[T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z}\Big|_{z^0} + s'_{inc}(z^0)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\right]\frac{\partial z(.)}{\partial T_1'(z^0)}s'_{inc}(z^0)p(z^0)h_z(z^0)
\end{aligned}
$$

LED tax system. We consider a small reform at $s^0 = s(z^0)$ that consists in a small increase $\Delta\tau_s$ of the linear savings tax rate $\tau_s(z)$ phased in over the earnings bandwidth $[z^0, z^0 + \Delta z]$. Using Lemma (2), we characterize the impact of the reform on the Lagrangian as $\Delta z \to 0$

$$
\begin{aligned}
&\frac{d\mathcal{L}}{\Delta\tau_s\Delta z} = \int_{x\geq z^0}\Big(\lambda_2 - \alpha(x)p(x)U_c'(x)\Big)s(x)\,dH_z(x) \qquad (224)\\
&+\left(\lambda_1 T_1'(z^0) + \lambda_2\tau_s'(z^0)s(z^0)\right)\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s(z^0)\,h_z(z^0)\\
&+\int_{x\geq z^0}\left(\lambda_1 T_1'(x) + \lambda_2\tau_s'(z^0)s(z^0)\right)\left[\frac{\partial z(.)}{\partial T_1}p(x)s(x) + \frac{\partial z(.)}{\partial T_1'(x)}\left(\frac{\partial p}{\partial z}s(x) + p(x)s'_{inc}(x)\right)\right]dH_z(x)\\
&+\lambda_2\tau_s(z^0)s'_{inc}(z^0)\left[\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s(z^0)\right]h_z(z^0)\\
&+\int_{x\geq z^0}\lambda_2\tau_s(x)\left[\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)} + \frac{\partial s(.)}{\partial T_1}p(x)s(x) + s'_{inc}(x)\left[\frac{\partial z(.)}{\partial T_1}p(x)s(x) + \frac{\partial z(.)}{\partial T_1'(x)}\left(\frac{\partial p}{\partial z}s(x) + p(x)s'_{inc}(x)\right)\right]\right]dH_z
\end{aligned}
$$

since the reform triggers for individuals at z^0 changes in earnings z equivalent to those induced by a $p(z)\,\Delta\tau_s\,s(z)$ increase in $T_1'(z^0)$, and for individuals above z^0 an increase in tax liability equivalent to a $p(z)\,\Delta\tau_s\,\Delta z\,s(z)$ increase in T_1 and a change in marginal earnings tax rates equivalent to a $\left(\frac{\partial p}{\partial z}\,s(z) + p(z)\,s'_{inc}(z)\right)\Delta\tau_s\,\Delta z$ increase in $T_1'(z)$, in addition to the $\Delta\tau_s\Delta z$ increase in the linear tax rate on s.

We characterize optimal taxes through $d\mathcal{L} = 0$. Renormalizing by λ_1, we can plug in $\hat{g}(x)$ and $\varphi(x)$ to obtain the following optimality condition for linear earnings-dependent tax rates on s at each earnings z^0

$$
\begin{aligned}
&-\left(T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\tau_s'(z^0)s(z^0) + \frac{\lambda_2}{\lambda_1}s'_{inc}(z^0)\tau_s(z^0)\right)\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s(z^0)\,h_z(z^0)\\
&= \int_{x\geq z^0}\left\{\left(\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\right)s(x) + \frac{\lambda_2}{\lambda_1}\tau_s(x)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)}\right\}dH_z(x) \qquad (225)\\
&+\int_{x\geq z^0}\left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\tau_s'(x)s(x) + \frac{\lambda_2}{\lambda_1}s'_{inc}(x)\tau_s(x)\right)\frac{\partial z(.)}{\partial T_1'(x)}p(x)s'_{inc}(x)\,dH_z(x)
\end{aligned}
$$

C.12.6 Deriving Proposition 7

SN tax system. A two-part SN tax system $\{T_1(z), T_2(s)\}$ thus satisfies two optimality conditions: the optimality condition in equation (221) for $T_1'(z)$ and the optimality condition in equation (223) for $T_2'(s)$. Combining these two conditions, we get that at each earnings z^0, the optimal SN tax system satisfies

$$-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} = \frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\right\}dH_z(x) - p(z^0)\frac{s'_{inc}(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]dH_z(x) \tag{226}$$

Adding and subtracting $p(z^0)\frac{s'(z^0)}{h_z(z^0)}\int_{x=z^0}^{z_{max}}\Big[1-\hat{g}(x)\Big]dH_z(x)$ yields

$$\begin{aligned}-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} &= p(z^0)\frac{s'(z^0) - s'_{inc}(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]dH_z(x) \\ &+ \frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\right\}dH_z(x) - p(z^0)\frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]dH_z(x).\end{aligned} \tag{227}$$

Defining $\zeta^c_{s|z}(z) = -\frac{1+p\frac{\partial T_2}{\partial s}\Big|_{z^0}}{s}\frac{\partial s(.)}{p\,\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}$ such that $\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} = -\frac{p\,s}{1+p\frac{\partial T_2}{\partial s}\Big|_{z^0}}\zeta^c_{s|z}(z)$, we get[51]

$$\begin{aligned}&\frac{\overline{\hat{g}p-\varphi}\frac{\partial T_2}{\partial s}\Big|_{z^0}}{1+p(z^0)\frac{\partial T_2}{\partial s}\Big|_{z^0}} \\ &= \frac{1}{s(z^0)\zeta^c_{s|z}(z^0)}\frac{1}{h_z(z^0)}\left\{s'_{het}(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]dH_z(x) + \frac{s'(z^0)}{p(z^0)}\Big[\Psi(z^0)+\Phi(z^0)\Big]\right\}\end{aligned} \tag{228}$$

[51] With homogeneous p, a SN savings tax levied in period-1 dollars $T_s(s)$ is simply equal to $T_s(s) = pT_2(s)$. As a result, this elasticity definition ensures that $\zeta^c_{s|z}(z)$ coincides with the elasticity concept introduced before:

$$\zeta^c_{s|z}(z) = -\frac{1+T_s'(s)}{s}\frac{\partial s(.)}{\partial T_s'(s)} = -\frac{1+pT_2'(s)}{s}\frac{\partial s(.)}{p\partial T_2'(s)}.$$

where we use $\overline{\hat{g}p - \varphi} = \frac{\lambda_2}{\lambda_1}$ and $\overline{\hat{g}(x)} = 1$ to obtain the additional terms

$$
\begin{aligned}
\Psi(z^0) &:= \int_{z\geq z^0} \left[\overline{\hat{g}p} - \hat{g}(z)p(z)\right] dH_z(z) - p(z^0)\int_{z=z^0}^{z_{max}} \left[\overline{\hat{g}} - \hat{g}(z)\right] dH_z(z) \qquad (229)\\
&= \int_{z\geq z^0} \left[\int_{x=z_{min}}^{z_{max}} \hat{g}(x)p(x)dH_z(x) - \hat{g}(z)p(z)\right] dH_z(z) - p(z^0)\int_{z\geq z^0} \left[\int_{x=z_{min}}^{z_{max}} \hat{g}(x)dH_z(x) - \hat{g}(z)\right] dH_z(z)\\
&= \left(1 - H_z(z^0)\right)\int_{x=z_{min}}^{z_{max}} \hat{g}(x)p(x)dH_z(x) - \int_{z\geq z^0} \hat{g}(z)p(z)dH_z(z)\\
&\qquad - p(z^0)\left(1 - H_z(z^0)\right)\int_{x=z_{min}}^{z_{max}} \hat{g}(x)dH_z(x) - p(z^0)\int_{z\geq z^0} \hat{g}(z)dH_z(z)\\
&= \left(1 - H_z(z^0)\right)\int_{x=z_{min}}^{z_{max}} \hat{g}(x)\Big(p(x) - p(z^0)\Big)dH_z(x) - \int_{x\geq z^0} \hat{g}(x)\Big(p(x) - p(z^0)\Big)dH_z(x)\\
&= \Big(1 - H_z(z^0)\Big)\int_{x\leq z^0} \hat{g}(x)\Big(p(x) - p(z^0)\Big)dH_z(z) + H_z(z^0)\int_{x\geq z^0} \hat{g}(x)\Big(p(z^0) - p(x)\Big)dH_z(x) \qquad (230)
\end{aligned}
$$

$$
\Phi(z^0) := \int_{x\geq z^0} \left[\varphi(x) - \overline{\varphi(x)}\right] dH_z(x) \qquad (231)
$$

which proves the optimal formula for SN tax systems in Proposition 7.

LED tax system. A two-part LED tax system $\{T_1(z), \tau_s(z)s\}$ thus satisfies two optimality conditions: the optimality condition in equation (221) for $T_1'(z)$ and the optimality condition in equation (225) for $\tau_s(z)$. Combining these two conditions, we get that at each earnings z^0 the optimal LED tax system satisfies

$$
\begin{aligned}
&p(z^0)s(z^0)\int_{x\geq z^0} \left[1 - \hat{g}(x)\right] dH_z(x)\\
&= \int_{x\geq z^0} \left\{\Big(\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\Big)s(x) + \frac{\lambda_2}{\lambda_1}\tau_s(x)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)}\right\} dH_z(x)\\
&+ \int_{x\geq z^0} \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\tau_s'(x)s(x) + \frac{\lambda_2}{\lambda_1}s_{inc}'(x)\tau_s(x)\right)\frac{\partial z(.)}{\partial T_1'(x)}p(x)s_{inc}'(x)\, dH_z(x).
\end{aligned}
$$

Differentiating with respect to z^0 yields

$$
\begin{aligned}
&\Big(p'(z^0)s(z^0) + p(z^0)s'(z^0)\Big)\int_{x\geq z^0} \left[1 - \hat{g}(x)\right] dH_z(x) - p(z^0)s(z^0)\Big[1 - \hat{g}(z^0)\Big] h_z(z^0)\\
&= -\left\{\Big(\frac{\lambda_2}{\lambda_1} - \hat{g}(z^0)p(z^0) + \varphi(z^0)\Big)s(z^0) + \frac{\lambda_2}{\lambda_1}\tau_s(z^0)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}\right\} h_z(z^0)\\
&- \left(T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\tau_s'(z^0)s(z^0) + \frac{\lambda_2}{\lambda_1}s_{inc}'(z^0)\tau_s(z^0)\right)\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s_{inc}'(z^0)\, h_z(z^0).
\end{aligned}
$$

Using the optimality condition in equation (221) for $T_1'(z)$, the last term is equal to $p(z^0)s_{inc}'(z^0)\int_{x\geq z^0}\left[1-\hat{g}(x)\right]\, dH_z(x$ at the optimum such that

$$
\begin{aligned}
&-\frac{\lambda_2}{\lambda_1}\tau_s(z^0)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}h_z(z^0)\\
&=p(z^0)s_{het}'(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)+p'(z^0)s(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)\\
&+\left\{\frac{\lambda_2}{\lambda_1}-\Big(\hat{g}(z^0)p(z^0)-\varphi(z^0)\Big)-p(z^0)\Big[1-\hat{g}(z^0)\Big]\right\}s(z^0)h_z(z^0).
\end{aligned}
$$

We can now plug in the elasticity $\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}=-\frac{p(z^0)s(z^0)}{1+p(z^0)\frac{\partial T_2}{\partial s}\Big|_{z^0}}\zeta_{s|z}^c(z^0)$ with $\frac{\partial T_2}{\partial s}\Big|_{z^0}=\tau_s(z^0)$ and use the fact that $\overline{\hat{g}p-\varphi}=\frac{\lambda_2}{\lambda_1}$ and $\overline{\hat{g}}=1$ to obtain

$$
\begin{aligned}
&\overline{\hat{g}p-\varphi}\frac{\tau_s(z^0)}{1+p(z^0)\tau_s(z^0)} \qquad (232)\\
&=\frac{1}{s(z^0)\zeta_{s|z}^c(z^0)}\frac{1}{h_z(z^0)}\left\{s_{het}'(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)+\frac{p'(z^0)}{p(z^0)}s(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)\right\}\\
&+\frac{1}{p(z^0)}\frac{1}{\zeta_{s|z}^c(z^0)}\left\{\left[\overline{\hat{g}p}-p(z^0)\overline{\hat{g}}\right]-\left[\overline{\varphi}-\varphi(z^0)\right]\right\}
\end{aligned}
$$

which proves the optimal formula for LED tax systems in Proposition 7.

D Details on the Empirical Application

This appendix describes the details underlying the numerical results presented in Section 7. In Section D.1, we describe how we calibrate a baseline two-period, unidimensional model of the U.S. economy, which we use to compute the simple savings tax schedules that are consistent with the prevailing income tax, i.e., that satisfy the Pareto-efficiency formulas in Proposition 3. These are reported in Figure II. In Section D.2, we describe how we extend this exercise to calibrate the optimal simple savings tax systems in the presence of multidimensional heterogeneity as in Proposition 4, assuming that redistributive preferences and other sufficient statistics are the same as in the baseline calibration. In Section D.3, we describe how we instead extend the baseline exercise to allow for heterogeneous rates of return, with an efficiency-based rationale for taxing those with access to high returns, as in Proposition 7. Results for these extensions are reported in Figure III.Throughout this exercise, we make two assumptions for tractability: We assume that preferences are weakly separable as described in Proposition 2, so that the income effect on savings, $\eta_{s|z}(z)$ can be identified from $s_{inc}'(z)$, and we assume that income effects on labor supply ($\eta_z(z)$) are negligible.

For comparability with the literature on wealth taxation, we express all savings tax rates in terms of "period-2" taxes on gross savings, so that a marginal savings tax rate of 0.1 indicates that if an individual's total wealth at retirement increases by $1, then they must pay an additional $0.10 in taxes when they retire.[52]

[52]Notationally, we write this translation as in Appendix B.7, with s_1 and s_g denoting gross savings before

The LaTeX source code underlying this document—which can be viewed in the accompanying replication files—uses equation labels that match those in the Matlab simulation code.

D.1 Baseline Calibration with Unidimensional Heterogeneity

We first calibrate a simplified version of the U.S. economy with unidimensional heterogeneity. This calibration has two periods, with the first period corresponding to working life and the second to retirement. We assume these periods are separated by 20 years, with a risk-free annual rate of return of 3.8% per year between period 1 and period 2 (see Fagereng et al. (2020), Table 3).

D.1.1 Joint Distribution of Earnings and Savings, and the Status Quo Income Tax

We calibrate the joint distribution of earnings and savings using the Distributional National Accounts micro-files of Piketty et al. (2018), henceforth PSZ. We use individual measures of pretax labor income (*plinc*) and net personal wealth (*hweal*) as well as the age category (20 to 44 years old, 45 to 64, and above 65) and household information. We discretize the income distribution into percentiles by age group, and we partition the top percentile into the top 0.01%, the top 0.1% (excluding the top 0.01%), and the rest of the top percentile. Our measure of annualized earnings during work life z at the n-th percentile is constructed by averaging *plinc* at the n-th percentile across those aged 20 to 44 and those aged 45 to 64. For married households, we use the average earnings of the couple and assign both members of the couple to the same percentile of income. For households with one member above 65 years old, we keep only the younger spouse in the sample. We drop the bottom 2% of observations with non-positive labor income; these individuals have positive average income from other sources, suggesting they are not representative of the zero-ability types which would correspond to $z = 0$ in our model.

Our measure of gross retirement savings per year worked, which we denote s_g in the notation of Appendix B.7, at each labor income quantile is constructed by projecting forward to age 65 the average wealth we observe in the 45 to 65 age category. We project forward by assuming that individuals within each percentile save the same share of post-tax income while young and middle-aged.[53] For married households, we take household wealth to be the average wealth of its members. We then normalize the total wealth at retirement by the number of working years ($65 - 25 = 40$) so that z and s_g are in comparable units measured per working year. This yields a monotonic profile of savings across earnings z, and pins down the cross-sectional variation in gross savings $s_g'(z)$.

taxes, measured in period-1 and period-2 dollars, respectively, and $T_2(s_g, z)$ denoting the savings tax function in period 2. Appendix B.7 demonstrates that the simplicity structure of a tax system (SL, SN, and LED) is preserved when translating between $\mathcal{T}(s, z)$ and $\{T_1(z), T_2(s_g, z)\}$. In the accompanying code replication files, all savings taxes are computed in terms of $\mathcal{T}(s, z)$, but marginal tax rates are converted into $\frac{\partial T_2(s_g,z)}{\partial s_g}$ when plotted in figures.

[53] Specifically, we construct a representative working agent for each income percentile in each age category: a "young" agent of age 35 (in the 20 to 44 age category, where we assume work begins at age 25), and a "middle-aged" agent of age 55 (in the 45 to 64 age category). We assume wealth at middle age is the result of the sum of 20 years' worth of savings while young, with returns compounded for an average of $55 - 35 = 20$ years, and 10 years of saving during middle age, compounded for an average of 5 years, with a constant share of post-tax income saved in the age range.

We convert this discrete distribution of labor income and savings into a smooth distribution with 1000 gridpoints with equal log-spacing, to ensure a smooth marginal tax function that converges to a fixed point when we iterate using the first-order conditions from our propositions. This conversion is performed using the smoothing spline fit in Matlab, with a smoothing parameter of 0.9 and the scale normalization setting set to "on." Measures of savings are noisy at low incomes, which also have outlier values of $\ln(z)$ after the logarithmic transformation used for our savings fit. To avoid having those percentiles generate a strong pull on the fit, we fit the log of savings to $\ln(z+k)$, where a larger k reduces the extent to which the low incomes are outliers. Our baseline uses $k = \$20,000$.

We construct the status quo income tax function by comparing gross income to the PSZ measure *diinc* ("extended disposable income") of post-tax income $z - T_1(z)$. We use the median value within each pre-tax income percentile, constructing a smoothed profile of disposable income y by fitting log *diinc* to log *plinc*, with the same setting described above. In the DINA files, total disposable income *diinc* exceeds total labor income *plinc*, reflecting non-labor factors of production in the economy and the taxes on them. For internal consistency, we apply a lump-sum adjustment so that total y and z are equal, although our results are not sensitive to this adjustment. We then calibrate the smooth marginal income tax rate schedule as $1 - \frac{dy}{dz}$. We treat Social Security as a fixed amount of forced savings, which are added to net-of-tax disposable savings to arrive at our total measure of net savings s. [54]

D.1.2 Status Quo Savings Tax Rates in the United States

We are interested in comparing our results to the profile of status quo effective tax rates on savings in the U.S. Constructing such a schedule presents several difficulties, however. There are many different types of taxes which apply to savings in the U.S., including capital gains taxes (which differ depending on the length of asset ownership), ordinary income taxes, and property taxes. Moreover, effective tax rates depend on assumptions about incidence, about which there is substantial disagreement.

We use a simple approach to construct an approximation of the U.S. savings tax based on the composition of savings portfolios across the income distribution. Bricker et al. (2019) use the Survey of Consumer Finances to construct a decomposition of saving types by asset ownership percentile; we summarize the analogous decomposition by income percentile in Figure A2 below. We then construct a savings tax rate at each income level based on the asset-weighted average of the tax rates that apply to each asset class.

For comparison to our results, the savings tax rate of interest is the distortion between work-life consumption and savings. Therefore savings which are subject to labor income taxes but no further taxes, such as a Roth IRA, should be understood as being subject to zero savings tax. We similarly classify traditional IRAs and pension plans as being subject to zero taxes, since they are also subject only to ordinary income taxes. We therefore treat assets in the "Financial (retirement)" category as subject to zero savings tax. We assume "Financial (transaction)" assets, which include checking and savings accounts, represent liquidity needs and similarly do not count toward taxed savings. We

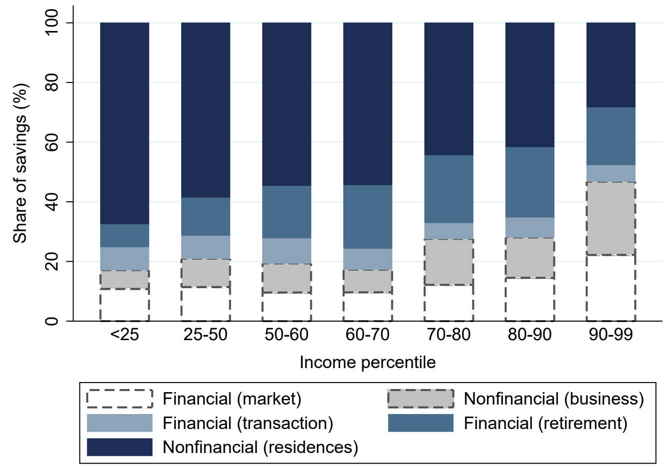
[54] The amount is computed as follows, using the SSA Fact Sheet: Retired workers receive on average \$1,514 per month from social security, which is $12 \times 1,514 = \$18,168$ annually. Through the lens of our two-period model, these benefits are received over an average retirement length of 20 years, and stem from contributions paid over 40 working years. We therefore approximate this as forced savings at the time of retirement of \$9000 per working year.

view property taxes on "Nonfinancial (residences)" savings as a tax that is incident on renters, and thus a component of imputed rent, which is paid regardless of whether the asset is owned by the user, so we also assume the tax rate on these savings is 0%. Therefore we view only the dotted-outline asset classes "Financial (market)" and "Nonfinancial (business)" as subject to savings taxes, in the form of capital gains. We do not know what share of these holdings represent gains, as opposed to the original contributions. To be conservative, we treat the entire asset classes as though they were subject to capital gains taxes at the time of retirement.

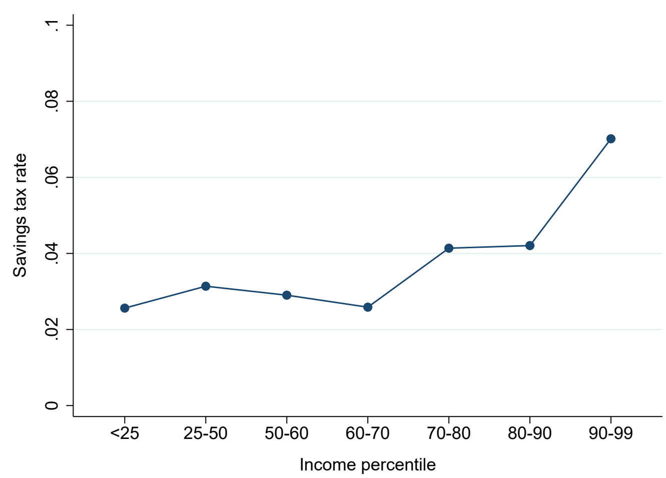
We treat this savings tax rate profile as a schedule of *average* tax rates on one's savings portfolio at each point in the income distribution. We smooth this schedule of average rates using the spline fit procedure described above, and apply that average tax rate to the calibrated level of gross savings at each point in the income distribution to reach a calibrated schedule of total savings taxes paid. We then compute the schedule of marginal rates that would give rise to that nonlinear profile of average tax rates; this schedule is plotted as the "U.S. Status quo" savings tax, e.g., in Figure II.

Figure A2: Calibration of Savings Tax Rates Across Incomes in the U.S.

(a) Decomposition of Savings Types: Bricker et al. (2019)



(b) Calibrated Savings Tax Rates in the United States, by Income Percentile



Notes: This figure illustrates the calibration of savings tax rates in the U.S. across the income distribution. Panel (a) plots the composition of asset types in individuals' portfolios across the income distribution, reported by Bricker et al. (2019). Panel (b) plots the implied weighted average savings tax rate in each bin. See Appendix D.1.2 for details.

D.1.3 Measures of s'_{inc}

A key input for our sufficient statistics is the marginal propensity to save out of earned income, $s'_{inc}(z) := \frac{\partial s(z)}{\partial z}|_{\theta=\theta(z)}$, which relates changes in the amount of net-of-tax savings at the time of retirement to changes in the amount of pre-tax earnings z. We draw from two sources of empirical data to calibrate our marginal propensities to consume (or save), translated into measures of $s'_{inc}(z)$. These results are plotted in Figure I.

Norwegian estimates from Fagereng et al. (2021). Fagereng et al. (2021) estimate marginal propensities to consume (MPC) across the earnings distribution using information on lottery prizes linked with administrative data in Norway. They find that individuals' consumption peaks during the year in which the prize is won, before gradually reverting to their previous consumption level. Over a 5-year horizon, they estimate winners consume close to 90% of the tax-exempt lottery prize; see the "consumption" panels in Fagereng et al. (2021) Figure 2. This translates into an MPC of 0.9, and thus a marginal propensity to save of 0.1. Under the assumption that preferences are weakly separable with respect to the disutility of labor supply, this is also the marginal propensity to save out of net earned income from labor supply. (See Proposition 2.)

They find little evidence of variation in MPCs across income levels which implies

$$\frac{\partial c(z)}{\partial(z - T_1(z))} = 0.9$$

and recognizing that individuals' budget constraint is $s_1(z) = z - T_1(z) - c(z)$, we get

$$\frac{\partial s_1(z)}{\partial(z - T_1(z))} = 1 - \frac{\partial c(z)}{\partial(z - T_1(z))} = 0.1.$$

The identity $s = (s_1 - T_s(s))(1+r)$ implies that $\frac{\partial s}{\partial s_1} = \frac{1}{\frac{1}{1+r} + T_s'(s)}$, and thus that the local causal effect of *pre-tax* income z on *net* savings s satisfies

$$\begin{aligned} s'_{inc}(z) &= \frac{\partial s_1(z)}{\partial(z - T_1(z))} \cdot \frac{\partial s}{\partial s_1} \cdot \frac{\partial(z - T_1(z))}{\partial z} \\ &= 0.1 \cdot \frac{1 - T_1'(z)}{\frac{1}{1+r} + T_s'(s(z))}. \end{aligned} \tag{233}$$

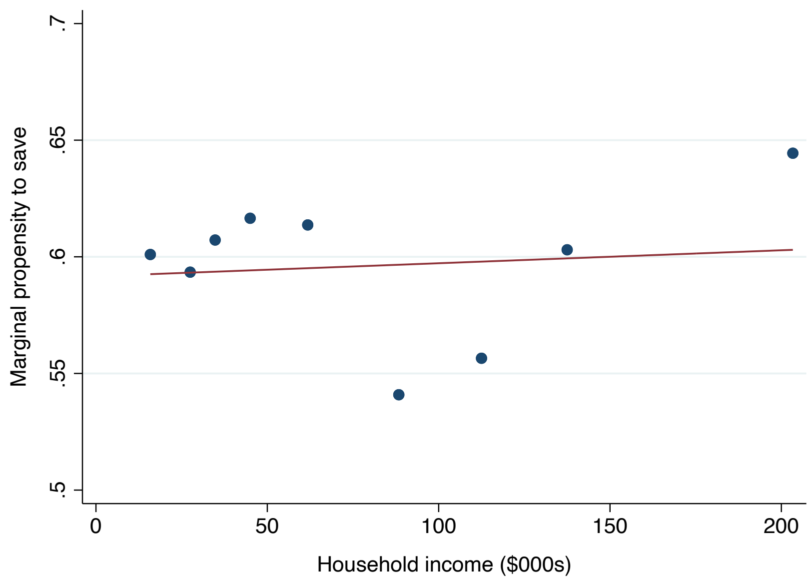
We can then use our calibrated U.S. tax schedule to obtain a profile of $s'_{inc}(z)$, under the key assumption that U.S. households have similar MPCs as Norwegian households. This profile is plotted in Figure I.

U.S. estimates from a new AmeriSpeak survey. We conducted a probability-based survey of the American population in the spring of 2021, which asked the following question:

> Imagine that you or someone else in your household gets a raise such that over the next five years, your household's income is $1,000 higher each year than what you expected. How much of this would your household spend, and how much would your household save over each of the next five years? (For purposes of this question, consider paying off debt, such as reducing your mortgage, a form of saving.) If no one in your household is going to be employed for most of the next five years, please write "N/A."
>
> Spend an extra $ ______ per year
>
> Save an extra $ ______ per year

Answers to this question provide information about individuals' reported marginal propensity to consume (MPC) and marginal propensity to save (MPS) out of a small and persistent change in *earned* income – in contrast to empirical estimates based on lottery winnings which measure MPC and MPS out of a one-time *windfall* income gain. Our survey sample consisted of 1,703 respondents who reported an average marginal propensity to save of 0.60 in the year of the raise.[55] We also requested information on household income in the survey, so we can observe marginal propensity to save across earnings levels, plotted in Figure A3. Marginal propensities to save appear quite stable across income levels, a finding that is consistent with the results of Fagereng et al. (2021).

Figure A3: Marginal propensity to save across household income (own survey)



Notes: Marginal propensities to save are computed from the answers to our survey question. They are computed as the ratio between the amount respondents report they would save and the amount of the raise.

Since our survey question asks about consumption and spending within each year, we interpret these estimates as short-run responses. Fagereng et al. (2021) show that positive income shocks are followed by consumption responses that can last up to 5 years. We use their impulse-response profile to convert these 1-year MPS into a 5-year MPS, which we interpret as a total effect on savings before returns. To do so, we use the fact that they report a 1-year MPC of 0.52 and a 5-year MPC of 0.90; we therefore compute our long run marginal propensity to save as

$$MPS_{5y} = MPS_{1y} \cdot \frac{1 - 0.90}{1 - 0.52} = 0.60 \cdot 0.208 = 0.125.$$

[55] This average is computed using the sample weights provided AmeriSpeak; the unweighted average is 0.59.

Because our survey question asked about a change in pre-tax income, we do not need to multiply by $1 - T_1'(z)$ as in equation (233); we just divide by $\frac{1}{1+r} + T_s'(s(z))$ to reach our measure of $s_{inc}'(z)$. This results in an estimate somewhat higher than that obtained by Fagereng et al. (2021) for Norway, plotted in Figure I. We use this as the baseline measure of $s_{inc}'(z)$ for our simulations, and the difference between the cross-sectional slope $s'(z)$ and $s_{inc}'(z)$ provides our estimate of the key statistic for preference heterogeneity, $s_{het}'(z)$, which is also plotted in Figure I.

Comparison to Golosov et al. (2013). Golosov et al. (2013) also study preference heterogeneity, providing a useful point of comparison. In their baseline calibration, they assume individuals' preferences are Constant-Relative-Risk-Aversion

$$U(c,s,l) = \frac{\alpha(w)}{1+\alpha(w)} \ln c + \frac{1}{1+\alpha(w)} \ln s - \frac{1}{\sigma}(l)^{\sigma},$$

where l is the labor supply of an individual with hourly wage w such that earnings are given by $z = wl$. The risk-aversion parameter is set to $\gamma = 1$, the isoelastic disulity from labor effort is such that $\sigma = 3$, and the taste parameter is given by

$$\alpha(w) = 1.0526\,(w)^{-0.0036}.$$

In other words, the taste parameter varies from 1.0433 for individuals in the bottom quintile of the earnings distribution (mean hourly wage of \$12.35, in 1992 dollars) to 1.0406 for individuals in the top quintile of the earnings distribution (mean hourly wage of \$25.39, in 1992 dollars). This means that this taste parameter is almost constant with income around an average of $\overline{\alpha} = 1.042$.

To illustrate how little preference heterogeneity this implies, we compute the s_{inc}' and s_{het}' implied by their calibration. Individuals' savings choices follow from maximizing $U(c, s, \frac{z}{w})$ subject to the budget constraint $c \leq z - \frac{1}{R}s - \mathcal{T}(s,z)$. This implies

$$s = \frac{z - \mathcal{T}(s,z)}{1/R + \alpha(1/R + \mathcal{T}_s')}$$

such that, neglecting the (potential) curvature of the tax function $\mathcal{T}'' \approx 0$, we can decompose the variation of savings s across earnings z as

$$\underbrace{\frac{ds}{dz}}_{s'(z)} = \underbrace{\frac{1 - \mathcal{T}_z'}{1/R + \alpha(1/R + \mathcal{T}_s') + \mathcal{T}_s'}}_{s_{inc}'(z)} + \underbrace{\frac{-(1/R + \mathcal{T}_s')}{1/R + \alpha(1/R + \mathcal{T}_s') + \mathcal{T}_s'} \frac{d\alpha}{dz} s}_{s_{het}'(z)}.$$

To obtain an approximation of $s_{het}'(z)$ in their setting, we use the fact that Golosov et al. (2013) report in their simulation results that individuals with an annual income $z = \$100,000$ have an hourly wage $w = \$40$ while those with an annual income $z = \$150,000$ have an hourly wage $w = \$62.5$. We can thus approximate $\frac{d\alpha}{dz} = \frac{\alpha(62.5) - \alpha(40)}{150{,}000 - 100{,}000} = \frac{1.0370 - 1.0387}{50{,}000} = -34 * 10^{-9}$. For $\mathcal{T}_z'$, we assume a linear income tax rate $\tau_z = 0.3$, for $R = 2.1$ we use our real interest rate of 3.8% compounded over 20 years), and for $\mathcal{T}_s'$ we assume a linear income tax rate $\tau_s = 0.01$ which we show below (see equation (235)) to be consistent with a linear tax of 4% on capital gains (the approximate average in Figure A2b).

This gives a constant $s'_{inc} = \frac{1-0.3}{1/2.1+1.042*(1/2.1+0.01)} = 0.71$, which is much higher than our estimate. Leveraging the fact that s'_{inc} is constant, we can also infer that at an annual income of $\$125,000$, the annual amount of savings available for consumption in period 2 (including compounded interest) is approximately equal to $s = s'_{inc} * \$125,000 = 0.71 * 125,000 = \$88,750$. Thus, $s'_{het} = \frac{1/2.1+0.02}{1/2.1+1.042*(1/2.1+0.02)+0.02} * (34 * 10^{-9}) * 88,750 = 0.0015$.[56]

These values for s'_{inc} and s'_{het} imply that in the calibration of Golosov et al. (2013), preference heterogeneity is substantially smaller than our estimate of across-income heterogeneity, as it only explains $\frac{s'_{het}}{s'_{het}+s'_{inc}} = \frac{0.0015}{0.71+0.0015} = 0.2\%$ of the variation in savings between individuals earning $\$100,000$ annually and those earning $\$150,000$.

D.1.4 Savings elasticity

For purposes of calibration, we assume that the income-conditional compensated elasticity of savings is constant across earnings, $\zeta^c_{s|z}(z) = \bar{\zeta}^c_{s|z}$. We follow Golosov et al. (2013) in drawing on the literature estimating the intertemporal elasticity of substitution (IES), and reporting results for a range of values. To motivate these values, we describe here how we can translate from the IES to a compensated elasticity $\zeta^c_{s|z}$ in the case of a representative agent.

The IES is defined as the elasticity of the growth rate of consumption with respect to the net price of consumption. We assume consumption is smoothed during retirement, so that retirement consumption is proportional to the net stock of savings s, and thus the elasticity of the growth rate of consumption (with respect to a tax change) is the same as the elasticity of the ratio of s to work-life consumption c. We consider a change in the price of retirement consumption induced by a small reform to a SL system like the one described in Table I with a constant linear tax rate τ_s, in which case the net-of-tax price of retirement savings is $\frac{R}{1+R\tau_s}$. (This can be found using the relationship $(s_1 - \tau_s s)R = s$ and solving for $\frac{ds}{ds_1} = -\frac{ds}{dc}$.) We can therefore write

$$
\begin{aligned}
IES &= \frac{d\ln(s/c)}{d\ln(\frac{R}{1+R\tau_s})} \\
&= -\frac{d\ln(s/c)}{d\ln(1+R\tau_s)} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{d\ln c}{d\ln(1+R\tau_s)} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{dc}{d\ln(1+R\tau_s)}\frac{1}{c} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{ds}{d\ln(1+R\tau_s)}\frac{dc}{ds}\frac{1}{c}.
\end{aligned}
$$

[56]More specifically, we postulate $s'_{het} \ll s'_{inc}$ to infer $s(z) = s'_{inc} \cdot z$ and then compute s'_{het}. Since we obtain a value that verifies $s'_{het} \ll s'_{inc}$, this reasoning is consistent and proves that $s'_{het} \ll s'_{inc}$. Put differently, even if we assume $s'_{het} \approx s'_{inc}$ which implies that $s(z) = 2s'_{inc} \cdot z$, we still obtain $s'_{het} \ll s'_{inc}$.

Substituting for $\frac{dc}{ds} = \frac{1+R\tau_s}{R}$, we then obtain

$$
\begin{aligned}
IES &= -\frac{d\ln s}{d\ln(1+R\tau_s)} - \frac{d\ln s}{d\ln(1+R\tau_s)}\frac{1+R\tau_s}{R}\frac{s}{c} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{d\ln s}{d\ln(1+R\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{d\ln(1+\tau_s)}{d\ln(1+R\tau_s)}\frac{d\ln s}{d\ln(1+\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\left(\frac{d(1+R\tau_s)}{d\tau_s}\right)^{-1}\frac{1+R\tau_s}{1+\tau_s}\frac{d\ln s}{d\ln(1+\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{1+R\tau_s}{R(1+\tau_s)}\frac{d\ln s}{d\ln(1+\tau_s)} \\
\implies \quad \frac{d\ln s}{d\ln(1+\tau_s)} &= -\frac{IES}{\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{1+R\tau_s}{R(1+\tau_s)}}. \qquad (234)
\end{aligned}
$$

Using a value of $s/c = 0.67$ (the population average in our calibrated two-period economy), and using the values $R = 2.1$ (from our real interest rate of 3.8% compounded over 20 years) and $\tau_s = 0.01$ (corresponding to a linear tax of 4% on capital gains, the approximate average in Figure A2b), we find[57]

$$\frac{d\ln s}{d\ln(1+\tau_s)} = -\frac{IES}{0.64}.$$

Treating this as the population estimate of $\frac{d\ln \bar{s}}{d\ln(1+\tau_s)}$, we can then compute the value of the elasticity $\bar{\zeta}^c_{s|z}$ that is consistent with this estimate. From the proof of the optimal SL tax system (see Appendix C.8.1, equation (134)), the response of aggregate savings $\bar{s}$ to a change in the separable

[57] A linear tax rate τ^{cg} on capital gains $(R-1)\,s_1$ leads to net savings $s = s_1(1+(R-1)(1-\tau^{cg}))$. Similarly, a period-1 linear tax τ_s on net savings s leads to net savings $s = (s_1 - \tau_s s)\,R \iff s = \frac{s_1 R}{1+\tau_s R}$. As a result,

$$
\begin{aligned}
s_1(1+(R-1)(1-\tau^{cg})) &= \frac{s_1 R}{1+\tau_s R} \\
\iff 1+\tau_s R &= \frac{R}{1+(R-1)(1-\tau^{cg})} \\
\iff \tau_s &= \frac{1}{1+(R-1)(1-\tau^{cg})} - \frac{1}{R}. \qquad (235)
\end{aligned}
$$

linear tax rate τ_s (measured in period-1 dollars, as distinct from $\tau_{s,2}$) is:

$$\frac{d\bar{s}}{d\tau_s} = -\int_z \left\{ \frac{1}{1+\tau_s} \left(s(z)\bar{\zeta}^c_{s|z} + \eta_{s|z}(z)s(z) \right) + \frac{s'_{inc}(z)}{1-T'_z(z)} \left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z) \right) \right\} dH_z(z)$$

$$\frac{d\bar{s}}{d\tau_s}\frac{1+\tau_s}{1} = -\bar{\zeta}^c_{s|z}\bar{s} - \int_z \left\{ \eta_{s|z}(z)s(z) + s'_{inc}(z) \frac{1+\tau_s}{1-T'_z(z)} \left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z) \right) \right\} dH_z(z)$$

$$\underbrace{\frac{d\bar{s}}{d\tau_s}\frac{1+\tau_s}{\bar{s}}}_{\frac{d\ln\bar{s}}{d\ln(1+\tau_s)}} = -\bar{\zeta}^c_{s|z} - \int_z \left\{ \eta_{s|z}(z)\frac{s(z)}{\bar{s}} + \frac{s'_{inc}(z)}{\bar{s}} \frac{1+\tau_s}{1-T'_z(z)} \left(z\,\zeta^c_z(z)s'_{inc}(z) + s(z)\,\eta_z(z) \right) \right\} dH_z(z)$$

$$\bar{\zeta}^c_{s|z} = -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \mathbb{E}\left[\eta_{s|z}(z)\frac{s(z)}{\bar{s}} + \frac{s'_{inc}(z)}{\bar{s}} \frac{1+\tau_s}{1-T'_z(z)} \left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z) \right) \right]$$

This could be computed directly if we had an independent estimate of the income-conditional income effect $\eta_{s|z}$. We instead invoke our assumptions of weak separability and a separable tax system, implying $\eta_{s|z}(z) = s'_{inc}(z)\frac{1+T'_s(s(z))}{1-T'_z(z)}$ (see Proposition 2), and negligible income effects on earnings, to write

$$\begin{aligned} \bar{\zeta}^c_{s|z} &= -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \mathbb{E}\left[\frac{1+T'_s(z)}{1-T'_z(z)} \frac{s'_{inc}(z)}{\bar{s}} \left(s(z) + z\bar{\zeta}^c_z s'_{inc}(z) \right) \right] \\ &= -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \frac{1}{\bar{s}} \cdot \mathbb{E}\left[\frac{1+T'_s(z)}{1-T'_z(z)} s'_{inc}(z) \left(s(z) + z\bar{\zeta}^c_z s'_{inc}(z) \right) \right]. \end{aligned} \tag{236}$$

In our calibration, the value of the second term is 0.38, suggesting a translation of $\bar{\zeta}^c_{s|z} \approx IES/0.64 - 0.38$. Thus a value of $IES = 1$, the baseline in Golosov et al. (2013), suggests an elasticity of $\bar{\zeta}^c_{s|z} = 1.2$. We use a baseline value of $\bar{\zeta}^c_{s|z} = 1$. IES values of 0.5 and 2 (the "low" and "high" values considered in Golosov et al. (2013)) suggest savings elasticities of $\bar{\zeta}^c_{s|z} = 0.4$ and $\bar{\zeta}^c_{s|z} = 2.7$. This is a wide range; values of savings elasticities below $\bar{\zeta}^c_{s|z} = 0.6$ in particular suggest that consistency with the status quo income tax requires a savings tax that is extreme or non-convergent.[58] We report results for alternative values of $\bar{\zeta}^c_{s|z} = 0.7$, $\bar{\zeta}^c_{s|z} = 2$, and $\bar{\zeta}^c_{s|z} = 3$.

D.2 Simulations of Optimal Savings Taxes with Multidimensional Heterogeneity

We now extend the above calibration to accommodate multidimensional heterogeneity, which we use to apply the formulas derived in Proposition 4. In the multidimensional setting, we do not have Pareto-efficiency formulas like those for unidimensional setting, because in the presence of income-conditional savings heterogeneity, Pareto-improving reforms are not generally available. Therefore, we use the formulas in Proposition 4 to compute the *optimal* schedule of savings tax rates for each type of simple tax system. In order to isolate and illustrate the effects of multidimensional

[58] Intuitively, as the savings elasticity becomes low, one's level of savings becomes a reliable signal of underlying ability, and more of the total redistribution in the tax system should be carried out through the savings tax, rather than the income tax. Thus for sufficiently low $\bar{\zeta}^c_{s|z}$, the status quo income tax cannot be Pareto efficient.

heterogeneity, we hold fixed the sufficient statistics used in the unidimensional setting. We also hold fixed the distributional preferences of the policy maker. The Pareto-efficiency computations above are equivalent to computing the optimal tax under "inverse optimum" welfare weights that would rationalize the status quo income tax. We compute these welfare weights explicitly, as described below, assuming that they vary with earnings, but not with savings conditional on earnings. We then use those inverse optimum weights for the optimal tax calculations.

D.2.1 Inverse Optimum Approach

We assume that income effects on labor supply are negligible, so that $\eta_z \approx 0$, which simplifies the computation of $\hat{g}(z)$ from equation (18) to

$$\hat{g}(z) = g(z) + \left(\frac{\mathcal{T}_s'}{1+\mathcal{T}_s'}\right)\eta_{s|z}(z). \tag{237}$$

We also assume that preferences are weakly separable so that, as shown in Proposition 2, we have

$$\eta_{s|z}(z) = s_{inc}'(z)\,\frac{1+T_s'(s(z))}{1-T_z'(z)}. \tag{238}$$

Because equations (237) and (238) depend on the marginal savings tax rates $\mathcal{T}_s'$, we must impose an assumption about how they adjust when we recompute the savings tax. We assume that the welfare weights $g(z)$ remain proportional to those calibrated using the inverse optimum procedure described above but we rescale to preserve the normalization $\int_z \hat{g}(z)dH_z(z) = 1$.[59] In equation (238), after computing $\eta_{s|z}(z)$ from the baseline calibration of $s_{inc}'(z)$, we assume that $\eta_{s|z}(z)$ remains stable when savings taxes are recomputed.

The inverse optimum computes the social marginal welfare weights (SMWW) consistent with existing tax policy (Bourguignon and Spadaro, 2012; Lockwood and Weinzierl, 2016). This exercise is typically performed using labor income taxes. Our setting presents a complication, as we have both a status quo income tax and savings tax, which need not produce a consistent set of weights. We compute weights assuming that the status quo schedule of earnings tax rates is optimal, for consistency with the Pareto-efficiency formulas above. Since the status quo savings tax rates also appear in this calculation, we must choose whether to use the status quo rates, or the rates that would counterfactually be optimal. In practice, results are insensitive to this latter issue; for consistency with the "inverse optimum" motivation, we use the Pareto-efficient set of SN tax rates.

Under these assumptions, we can compute the inverse optimum social marginal welfare weights

[59] Specifically, letting $g^0(z)$ denote our baseline welfare weights, we set $g(z) = \kappa g^0(z)$, where

$$\kappa = \frac{1 - \int_z \left(\frac{\mathcal{T}_s'}{1+\mathcal{T}_s'}\right)\eta_{s|z}(z)dH_z(z)}{\int_z g^0(z)dH_z(z)}. \tag{239}$$

at each earnings z by inverting the optimal tax rate condition,

$$\frac{T_z'(z)}{1-T_z'(z)} = \frac{1}{\zeta_z^c(z)\,z}\frac{1}{h_z(z)}\int_{x=z}^{z_{max}}(1-\hat{g}(x))\,dH_z(x) - s_{inc}'(z)\frac{T_s'(s(z))}{1-T_z'(z)} \tag{240}$$

$$\iff \int_{x=z}^{z_{max}}(1-\hat{g}(x))\,dH_z(x) = \zeta_z^c(z)\,z\,h_z(z)\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)}, \tag{241}$$

where the right-hand side term can be identified from the data. Differentiating with respect to z yields the expression we use to implement this computation numerically,

$$\hat{g}(z) = 1 + \frac{1}{h_z(z)}\cdot\frac{d}{dz}\left[\zeta_z^c(z)\,z\,h_z(z)\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)}\right]. \tag{242}$$

Using the fact that augmented social marginal welfare weights are defined as

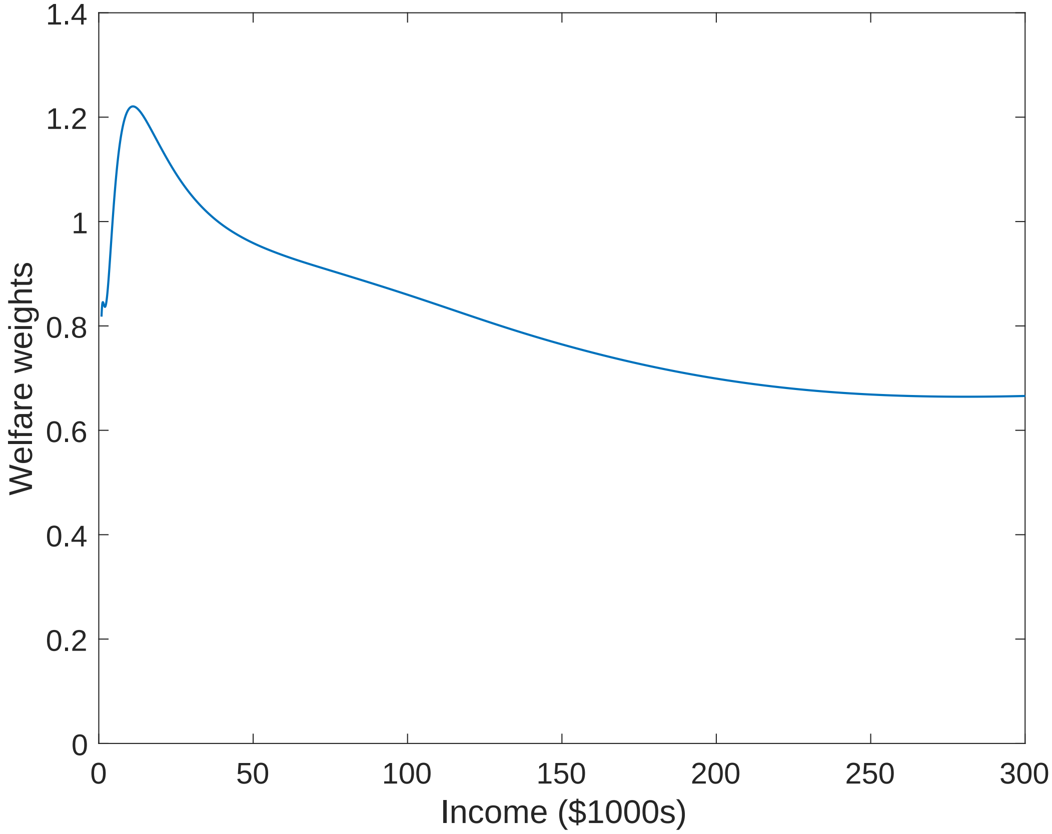
$$\hat{g}(z) := g(z) + T_z'(z)\,\frac{\eta_z(z)}{1-T_z'(z)} + T_s'(s(z))\left(\frac{\eta_{s|z}(z)}{1+T_s'(s(z))} + s_{inc}'(z)\frac{\eta_z(z)}{1-T_z'(z)}\right), \tag{243}$$

and assuming preferences are weakly separable, such that by Proposition 2 we have $s_{inc}'(z) = \frac{1-T_z'(z)}{1+T_s'(s(z))}\eta_{s|z}(z)$, inverse optimum weights $g(z)$ are obtained from $\hat{g}(z)$ as follows:

$$g(z) = \hat{g}(z) - s_{inc}'(z)\left(\frac{T_s'(s(z))}{1-T_z'(z)}\right). \tag{244}$$

Figure A4 plots our estimated profile of inverse optimum weights.

Figure A4: Schedule of Inverse Optimum Social Welfare Weights in the U.S.



Notes: This figure plots the schedule of inverse optimum welfare weights that would rationalize the U.S. income tax schedule. These weights are computed under the assumption that the savings tax is the Pareto-efficient SN schedule reported in Figure II.

D.2.2 Calibration Details

To extend our calibrated two-period model economy to a multidimensional setting, we retain the same discretized grid of incomes as in the unidimensional case, using the calibration described in Appendix D.1. At each income, we now allow for heterogeneous levels of savings. Specifically, using the same measure of gross savings described in Appendix D.1, we now use a calibration with four different levels of savings at each level of income, each representing a quartile of the income-conditional savings distribution. Across the income distribution, we assume savings within each quartile are a constant ratio of the income-conditional average level of saving. These ratios are 15%, 40%, 70% and 280% of the income-conditional average savings level; they are calibrated to reflect the average ratios across percentiles 50 to 100 in the PSZ data. We calibrate these ratios excluding the bottom portion of the distribution because the average level of saving is very low in the bottom half, resulting in noisily measured ratios.

To calibrate the savings income effect $\eta_{s|z}(s, z)$, we assume that the income elasticity of savings is constant within earnings and equal to its unidimensional counterpart, implying that $\eta_{s|z}(s,z) = \frac{s}{\overline{s(z)}}\eta_{s|z}(z)$, where $\overline{s(z)} := \mathbb{E}\left[s\middle|z\right]$ denotes the average savings level at earnings z, and that similarly, $s'_{inc}(s,z) = \frac{s}{\overline{s(z)}}s'_{inc}(z)$. Using these expressions, we can adapt equation (239)—the scaling factor necessary to ensure that $\hat{g}(s,z)$ integrates to one when recomputing savings taxes—to this setting:

$$\kappa = \frac{1 - \int_z \int_s \left(\frac{\mathcal{T}'_s}{1+\mathcal{T}'_s}\right) \eta_{s|z}(s,z)\, h(s,z) ds dz}{\int_z g^0(z) dH_z(z)}.$$

Letting $\mathcal{T}'_s = \tau$ in the SL case and $\mathcal{T}'_s = \tau_s(z)$ in the LED case, we have

$$\kappa = \frac{1 - \int_z \frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\, dH_z(z)}{\int_z g^0(z) dH_z(z)}, \tag{245}$$

and with $\mathcal{T}'_s = T'_s(s)$ in the SN case such that

$$\kappa = \frac{1 - \int_s \frac{T'_s(s)}{1+T'_s(s)} \int_z \eta_{s|z}(s,z)\, h(s,z) dz ds}{\int_z g^0(z) dH_z(z)}. \tag{246}$$

D.2.3 Separable linear (SL) tax system

The optimal savings tax formula with multidimensional heterogeneity (Proposition 4) is

$$\begin{aligned}
&\frac{\tau_s}{1+\tau_s}\int_z \left\{\mathbb{E}\left[s\zeta^c_{s|z}(s,z)\middle|z\right]\right\} dH_z(z) \qquad (247)\\
&= \int_z \left\{\mathbb{E}\left[(1-\hat{g}(s,z))\, s\middle|z\right] - \mathbb{E}\left[\frac{T'_z(z) + s'_{inc}(s,z)\,\tau_s}{1-T'_z(z)} z\zeta^c_z(s,z)\, s'_{inc}(s,z)\middle|z\right]\right\} dH_z(z).
\end{aligned}$$

Under the aforementioned assumptions, expanding $\hat{g}(s,z)$, replacing $s'_{inc}(s,z)$ and $\eta_{s|z}(s,z)$ by their values, and assuming η_z is negligible gives

$$\begin{aligned}
&\frac{\tau_s}{1+\tau_s}\int_z \left\{\overline{s(z)}\zeta^c_{s|z}\right\} dH_z(z) \qquad (248)\\
&= \int_z \left\{\mathbb{E}\left[\left(1 - g(z) - \tau_s \frac{\eta_{s|z}(z)}{1+\tau_s}\frac{s}{\overline{s(z)}}\right) s\middle|z\right] - \frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[T'_z(z)\frac{s}{\overline{s(z)}} + s'_{inc}(z)\,\tau_s\left(\frac{s}{\overline{s(z)}}\right)^2\middle|z\right]\right\} dH_z(z)
\end{aligned}$$

which after rearranging yields

$$\begin{aligned}
&\frac{\tau_s}{1+\tau_s}\zeta^c_{s|z}\int_z \overline{s(z)} dH_z(z) \qquad (249)\\
&= \int_z \left\{(1-g(z))\,\overline{s(z)} - \frac{\tau_s}{1+\tau_s}\frac{\eta_{s|z}(z)}{\overline{s(z)}}\mathbb{E}\left[s^2\middle|z\right] - \frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[\frac{T'_z(z)}{\overline{s(z)}}s + \frac{s'_{inc}(z)\,\tau_s}{\overline{s(z)}^2}s^2\middle|z\right]\right\} dH_z(z).
\end{aligned}$$

We can now use $\mathbb{E}\left[s^2\middle|z\right]=\mathbb{E}\left[\left(s-\overline{s\left(z\right)}\right)^2+2s\overline{s\left(z\right)}-\overline{s\left(z\right)}^2\middle|z\right]=\mathbb{V}\left(s|z\right)+\overline{s\left(z\right)}^2$ to obtain

$$\frac{\tau_s}{1+\tau_s}\zeta^c_{s|z}\int_z\overline{s\left(z\right)}dH_z\left(z\right) \tag{250}$$
$$=\int_z\left\{\left[1-g(z)-\frac{\tau_s\eta_{s|z}\left(z\right)}{1+\tau_s}\left(1+\frac{\mathbb{V}\left(s|z\right)}{\overline{s\left(z\right)}^2}\right)\right]\overline{s\left(z\right)}-\frac{z\zeta^c_z s'_{inc}\left(z\right)}{1-T'_z\left(z\right)}\left[T'_z\left(z\right)+s'_{inc}\left(z\right)\tau_s\left(1+\frac{\mathbb{V}\left(s|z\right)}{\overline{s\left(z\right)}^2}\right)\right]\right\}dH_z\left(z\right)$$

which we can finally rewrite as

$$\frac{\tau_s}{1+\tau_s}\int_z\overline{s\left(z\right)}\zeta^c_{s|z}dH_z\left(z\right) \tag{251}$$
$$=\int_z\left\{\left(1-g(z)-\frac{\tau_s}{1+\tau_s}\eta_{s|z}\left(z\right)\right)\overline{s\left(z\right)}-\frac{T'_z\left(z\right)+s'_{inc}\left(z\right)\tau_s}{1-T'_z\left(z\right)}z\zeta^c_z s'_{inc}\left(z\right)\right\}dH_z\left(z\right)$$
$$-\int_z\left\{\underbrace{\frac{\mathbb{V}\left(s|z\right)}{\overline{s\left(z\right)}^2}}_{\geq 0}\underbrace{\left(\frac{\tau_s}{1+\tau_s}\eta_{s|z}\left(z\right)\overline{s\left(z\right)}+\frac{s'_{inc}\left(z\right)\tau_s}{1-T'_z\left(z\right)}z\zeta^c_z s'_{inc}\left(z\right)\right)}_{\geq 0}\right\}dH_z\left(z\right).$$

The first two lines correspond to the optimal savings tax formula under unidimensional heterogeneity (Proposition B.4) and the last line captures the effect of multidimensional heterogeneity through $\mathbb{V}\left(s|z\right)$. Multidimensional heterogeneity adds a corrective term which is unambiguously negative and thus prescribes a lower linear savings tax rate.

D.2.4 Separable nonlinear (SN) tax system

At any given savings level s^0, the optimal savings tax formula with multidimensional heterogeneity (Proposition 4) is

$$\frac{T'_s\left(s^0\right)}{1+T'_s\left(s^0\right)}\int_z\left\{s^0\zeta^c_{s|z}(s^0,z)\right\}h\left(s^0,z\right)dz=\int_z\left\{\mathbb{E}\left[1-\hat{g}\left(s,z\right)\middle|z,s\geq s^0\right]\right\}h_z\left(z\right)dz \tag{252}$$
$$-\int_z\left\{\frac{T'_z\left(z\right)+s'_{inc}\left(s^0,z\right)T'_s\left(s^0\right)}{1-T'_z\left(z\right)}z\zeta^c_z(s^0,z)s'_{inc}\left(s^0,z\right)\right\}h\left(s^0,z\right)dz.$$

Under the aforementioned assumptions, expanding $\hat{g}\left(s,z\right)$ and assuming η_z is negligible gives

$$\frac{T'_s\left(s^0\right)}{1+T'_s\left(s^0\right)}s^0\zeta^c_{s|z}\int_z h\left(s^0,z\right)dz=\int_{s\geq s^0}\left\{\int_z\left[1-g(z)-T'_s\left(s\right)\frac{\eta_{s|z}\left(s,z\right)}{1+T'_s\left(s\right)}\right]h\left(s,z\right)dz\right\}ds \tag{253}$$
$$-\int_z\left[\frac{T'_z\left(z\right)+s'_{inc}\left(s^0,z\right)T'_s\left(s^0\right)}{1-T'_z\left(z\right)}z\zeta^c_z s'_{inc}\left(s^0,z\right)\right]h\left(s^0,z\right)dz$$

or equivalently, expressing this as a function of the savings density $h_s\left(s\right)=\int_z h\left(s,z\right)dz$,

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c h_s\left(s^0\right)=\int_{s\geq s^0}\left\{\mathbb{E}\left[1-g(z)-T_s'\left(s\right)\frac{\eta_{s|z}\left(s,z\right)}{1+T_s'\left(s\right)}\Big|s\right]\right\}h_s\left(s\right)ds - \mathbb{E}\left[\frac{T_z'\left(z\right)+s_{inc}'\left(s,z\right)T_s'\left(s\right)}{1-T_z'\left(z\right)}z\zeta_z^c s_{inc}'\left(s,z\right)\Big|s=s^0\right]h_s\left(s^0\right) \tag{254}$$

where the expectations operator denotes integration with respect to earnings conditional on savings.

For implementation, we assume that at each point in the income continuum, there are M different equal-sized saver bins (e.g., bottom-, middle-, and top-third of savers), indexed by $m = 1,\ldots,M$. Thus we can write $s_m(z)$ as the savings map for saver bin m at each income, with $s_m'(z)$ the cross-sectional savings profile within each saver-bin. Then the income density in each saver-bin is $h_{z,m}(z)=h(z)/M$, since the bins are equally sized conditional on income. The savings density among saver-bin m is therefore $h_{s,m}(s)=h_{z,m}(z)/s_m'(z)$, and we have $H(s)=\sum_{m=1}^{M}\int_{s=0}^{\infty}h_{s,m}(s)ds$, and $h_s(s)=\sum_{m=1}^{M}h_{s,m}(s)$. And the savings-conditional average of some $x(s,z)$ is $\mathbb{E}[x(s,z)|s]=\frac{\sum_{m=1}^{M}x(s_m,z)h_{s,m}(s)}{h_s(s)}$.

To better picture the link with the unidimensional formula in equation (67), let us also rewrite the latter as a function of the savings density $h_s\left(s\right)$—implicitly defining $z\left(s\right)$ as the earnings level of individuals with savings s—this yields

$$\frac{T_s'(s^0)}{1+T_s'(s^0)}s^0\zeta_{s|z}^c h_s\left(s^0\right)=\int_{s\geq s^0}\left\{1-g\left(z(s)\right)-\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}\left(z\left(s\right)\right)\right\}h_s\left(s\right)ds - \frac{T_z'(z\left(s^0\right))+s_{inc}'(z\left(s^0\right))T_s'(s^0)}{1-T_z'(z\left(s^0\right))}z\left(s^0\right)\zeta_z^c s_{inc}'\left(z(s^0)\right)h_s\left(s^0\right). \tag{255}$$

While it is clear that the multidimensional formula extends the unidimensional formula, determining the impact of multidimensional heterogeneity on tax rates is more analytically difficult and we thus rely on numerical simulations.

D.2.5 Linear earnings dependent (LED) tax system

At earnings z^0, the optimal LED savings tax formula in the presence of multidimensional heterogeneity (Proposition 4) is

$$\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s\Big|z=z^0\right]h_z\left(z^0\right)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\zeta_{s|z}^c(s,z)\Big|z\right]h_z\left(z\right)dz$$
$$=\int_{z\geq z^0}\mathbb{E}\Big[\left(1-\hat{g}\left(s,z\right)\right)s\Big|z\Big]h_z\left(z\right)dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s+s_{inc}'\left(s,z\right)\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s_{inc}'\left(s,z\right)\Big|z\right]h_z\left(z\right)d \tag{256}$$

which proves particularly cumbersome to use in numerical simulations, even under the aforementioned assumptions. To obtain an expression that is more easily implementable numerically, we further assume that the earnings tax is optimal (see Proposition B.6) such that

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c(s,z)\Big|z^0\right]h_z(z^0)=\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)|z\right]\Big\}h_z(z)\,dz. \tag{257}$$

Now, observing that $s=\overline{s(z^0)}+s-\overline{s(z^0)}$, we can rewrite the first term of the optimal savings tax formula as

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c s\Big|z^0\right]=\overline{s(z^0)}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\Big|z^0\right] \tag{258}$$
$$+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right].$$

Plugging this back into the optimal savings tax formula and using the optimal earnings tax formula, this implies that

$$\overline{s(z^0)}\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)|z\right]\Big\}h_z(z)\,dz \tag{259}$$
$$+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right]h_z(z^0)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c\Big|z\right]h_z(z)\,dz$$
$$=\int_{z\geq z^0}\mathbb{E}\Big[(1-\hat{g}(s,z))s\Big|z\Big]h_z(z)\,dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c s_{inc}'(s,z)\Big|z\right]h_z(z)\,dz.$$

Differentiating with respect to z^0 then yields

$$\frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)|z\right]\Big\}h_z(z)\,dz-\overline{s(z^0)}\,\mathbb{E}\left[1-\hat{g}(s,z)|z^0\right]h_z(z^0) \tag{260}$$
$$+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right]h_z(z^0)\right)-\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c\Big|z^0\right]h_z(z^0)$$
$$=-\mathbb{E}\Big[(1-\hat{g}(s,z))s\Big|z^0\Big]h_z(z^0)+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c s_{inc}'(s,z)\Big|z^0\right]h_z(z^0).$$

Rearranging gives

$$\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c+\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c s_{inc}'(s,z)\Big|z^0\right]h_z(z^0) \tag{261}$$
$$=\frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)|z\right]\Big\}h_z(z)\,dz-\mathbb{E}\Big[(\hat{g}(s,z))\left(s-\overline{s(z^0)}\right)\Big|z^0\Big]h_z(z^0)$$
$$+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}z\zeta_z^c\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right]h_z(z^0)\right).$$

Now, with $s'_{inc}(s,z) = \frac{s}{s(z)} s'_{inc}(z)$ as well as $\eta_{s|z}(s,z) = \frac{s}{s(z)}\eta_{s|z}(z)$ and $\hat{g}(s,z) = g(z) + \frac{\tau_s(z)}{1+\tau_s(z)}\frac{s}{s(z)}\eta_{s|z}(z)$, we get

$$\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\zeta^c_{s|z} + \frac{T'_z(z)+\tau'_s(z)s+\frac{s}{s(z)}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s} z\zeta^c_z \frac{s}{s(z)} s'_{inc}(z)\Big| z^0\right] h_z(z^0) \quad (262)$$
$$= \frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\frac{s}{s(z)}\eta_{s|z}(z)\Big| z\right]\right\} h_z(z)\,dz$$
$$-\mathbb{E}\left[\left(g(z)+\frac{\tau_s(z)}{1+\tau_s(z)}\frac{s}{s(z)}\eta_{s|z}(z)\right)\left(s-\overline{s(z^*)}\right)\Big| z^0\right] h_z(z^0)$$
$$+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)s+\frac{s}{s(z)}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s} z\zeta^c_z\left(s-\overline{s(z^0)}\right)\Big| z=z^0\right] h_z(z^0)\right)$$

which simplifies to the following exact formula

$$\frac{\tau_s(z^0)}{1+\tau_s(z^0)}\overline{s(z^0)}\zeta^c_{s|z}h_z(z^0) + z^0\zeta^c_z\frac{s'_{inc}(z^0)}{s(z^0)}\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)s+\frac{s}{s(z)}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s} s\Big| z^0\right] h_z(z^0) \quad (263)$$
$$= \frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\left\{1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right\} h_z(z)\,dz - \frac{\tau_s(z^0)}{1+\tau_s(z^0)}\eta_{s|z}(z^0)\frac{\mathbb{V}\left[s\big|z^0\right]}{s(z^0)}h_z(z^0)$$
$$+\frac{d}{dz^0}\left(z^0\zeta^c_z\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)s+\frac{s}{s(z)}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}\left(s-\overline{s(z^0)}\right)\Big| z=z^0\right] h_z(z^0)\right)$$

using $\mathbb{E}\left[s\left(s-\overline{s(z^0)}\right)\Big| z^0\right] = \mathbb{E}\left[s^2\Big| z^0\right] - \left(\overline{s(z^0)}\right)^2 = \mathbb{V}\left[s\Big| z^0\right]$.

Since the marginal tax rate on earnings $T'_z(z)+\tau'_s(z)s$ features savings s, it is hard to further simplify this formula while retaining an exact characterization. To further simplify this expression, we disregard this dependence by setting $s = \overline{s(z^0)}$ in marginal earnings tax rates. We believe that these formulas are informative in that they converge to exact expressions as the linear earnings dependent savings tax rate tends to a simple linear savings tax rate—that is $\tau'_s(z) = 0$ for all z. Moreover, although these approximations are not unbiased in that they provide an upper bound on the linear-earnings dependent savings tax rate, these upper bounds are tight as the approximation only amounts to assuming $\tau'_s(z^0)\,\mathbb{V}\left(s|z^0\right)$ is negligible.

We thus use

$$\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)s+\frac{s}{s(z)}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s} s\Big| z^0\right]$$
$$\approx \overline{s(z^0)}\left[\frac{T'_z(z^0)+\tau'_s(z^0)\overline{s(z^0)}}{1-T'_z(z^0)-\tau'_s(z^0)\overline{s(z^0)}} + \frac{s'_{inc}(z^0)\tau_s(z^0)}{1-T'_z(z^0)-\tau'_s(z^0)\overline{s(z^0)}}\left(1+\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s(z^0)}^2}\right)\right]$$

as well as

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+\frac{s}{s(z)}s_{inc}'(z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}\left(s-\overline{s\left(z^0\right)}\right)\Big|z=z^0\right]$$
$$\approx\overline{s\left(z^0\right)}\left[\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}^2}\right]$$

to finally obtain

$$\begin{aligned}&\frac{\tau_s\left(z^0\right)}{1+\tau_s\left(z^0\right)}\overline{s\left(z^0\right)}\zeta_{s|z}^c h_z\left(z^0\right)+s_{inc}'\left(z^0\right)\frac{T_z'\left(z^0\right)+\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}+s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}z^0\zeta_z^c h_z\left(z^0\right)\\
&=\frac{d\left(\overline{s\left(z^0\right)}\right)}{dz^0}\int_{z\geq z^0}\left\{1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right\}h_z(z)\,dz \qquad (264)\\
&-z^0\zeta_z^c s_{inc}'\left(z^0\right)\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}^2}h_z\left(z^0\right)-\frac{\tau_s\left(z^0\right)}{1+\tau_s\left(z^0\right)}\eta_{s|z}\left(z^0\right)\frac{\mathbb{V}\left(s\middle|z^0\right)}{\overline{s\left(z^0\right)}}h_z\left(z^0\right)\\
&+\frac{d}{dz^0}\left(z^0\zeta_z^c\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}}h_z\left(z^0\right)\right).\end{aligned}$$

As an element of comparison, a similar derivation under unidimensional heterogeneity combining the optimal LED savings tax formula (Proposition B.4) and the optimal earnings tax formula (Proposition B.5) yields the following unidimensional analogue

$$\begin{aligned}&\frac{\tau_s(z^0)}{1+\tau_s(z^0)}s(z^0)\zeta_{s|z}^c(z^0)\,h_z(z^0)+z^0\zeta_z^c(z^0)s_{inc}'(z^0)\frac{T_z'\left(z^0\right)+\tau_s'\left(z^0\right)s\left(z^0\right)+s_{inc}'(z^0)\tau_s(z^0)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)s\left(z^0\right)}h_z(z^0)\qquad(265)\\
&=s'(z^0)\int_{z\geq z^0}\left(1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right)dH_z(z).\end{aligned}$$

Multidimensional heterogeneity thus adds new terms related to $\mathbb{V}\left(s|z^0\right)$ that naturally wash out under unidimensional heterogeneity. The two terms on the third line of equation (264) are clearly negative and push for lower savings tax rates in the presence of multidimensional heterogeneity. The term on the fourth line cannot be signed unambiguously. In our calibration, it appears to be negative at low earnings but positive at high earnings. However, its order magnitude is so small (around 10^{-4}) that it does not meaningfully affects the optimal LED savings tax rate and can thus be neglected. As a result, we also get in this case that taking multidimensional heterogeneity into account calls for lower tax rates.

D.3 Simulations of Optimal Savings Taxes with Heterogeneous Returns

For the extension to the case with efficiency arbitrage effects, considered in Section 6.3, we now compute the optimal savings tax rates using the formulas derived in Proposition 7, again using the same set of inverse optimum welfare weights derived above.

These results are reported in the bottom two panels of Figure III, which display schedules of LED and SN savings tax rates computed under the assumption that (i) individuals with different income levels differ in their private rates of return, and that (ii) the savings tax is levied in period-2 dollars. We compute the tax schedules that satisfy the equations for the optimal tax conditions in Proposition 7. As in the case of multidimensional heterogeneity, we hold fixed the schedule of marginal social welfare weights $g(z)$ proportional to those which rationalize the status quo income tax in our baseline inverse optimum calculation. Building on the findings of Fagereng et al. (2020), we follow Gerritsen et al. (2020) in assuming that rates of return rise by 1.4% from the bottom to the top of the income distribution. We linearly interpolate this difference across income percentiles, centered on our 3.8% baseline rate of return.

Maintaining our assumptions of negligible labor supply income effects and weakly separable preferences, equation (216) simplifies to

$$\hat{g}(x) := g(x) + \frac{\lambda_2}{\lambda_1} \frac{T_2'(s)}{1 + pT_2'(s)} \eta_{s|z}(z) \tag{266}$$

for an SN system. To ensure that $\hat{g}(z)$ still integrates to one, the rescaling factor in equation (239) now becomes

$$\kappa = \frac{1 - \int_z \left(\frac{\lambda_2}{\lambda_1} \frac{T_2'(s)}{1+pT_2'(s)} \right) \eta_{s|z}(z) dH_z(z)}{\int_z g^0(z) dH_z(z)}. \tag{267}$$

Similarly, equation (217) simplifies to

$$\varphi(x) = -\left(T_1'(x) + s_{inc}'(x) \frac{\lambda_2}{\lambda_1} T_2'(s) \right) \left(\zeta_z^c(x) \frac{x}{1 - T_1'(x)} \right) \frac{\partial p}{\partial z}. \tag{268}$$

For an LED system we can replace $T_2'(s)$ with $\tau_s(z)$ in the previous formulas.