Online Appendix: Using Bid Rotation and Incumbency to Detect Collusion: A Regression Discontinuity Approach

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Abstract

This Online Appendix to "Using Bid Rotation and Incumbency to Detect Collusion: A Regression Discontinuity Approach" provides examples, robustness checks and proofs. Section OA presents examples referred to in the main text. Section OB provides empirical results omitted from the main text, as well as robustness checks. Section OC collects all proofs.

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OA Examples

OA.1 An example of non-smooth demand.

Consider a complete information auction with an incumbent I and an entrant E with respective known costs $c_I < c_E$. Assume that bidding cost k is zero.

Lemma OA.1 (non-smooth demand)**.** *In any efficient equilibrium in weakly undominated strategies, the incumbent wins with bid c_E with probability 1. The density of the entrant's bid below c_E is 0. The density of the entrant's bids above c_E is strictly positive and bounded away from 0. Specifically, for all $\epsilon > 0$, the incumbent's demand D_I satisfies $\frac{D_I(c_E+\epsilon)-1}{\epsilon} \leq -\frac{1}{c_E+\epsilon-c_I}$.*

Proof. In an efficient equilibrium in weakly undominated strategies, the incumbent cannot bid above c_E with positive probability: the entrant's optimal bid would win with positive probability.

In turn, the entrant cannot bid below c_E. This implies that the incumbent's optimal bid is c_E. Optimality of c_E implies that for any $\epsilon > 0$,

$$D_I(c_E+\epsilon)(c_E+\epsilon-c_I) \leq D_I(c_E)(c_E-c_I) = c_E - c_I \iff \frac{D_I(c_E+\epsilon)-1}{\epsilon} \leq -\frac{1}{c_E+\epsilon-c_I}.$$

□

OA.2 A collusive Markov perfect equilibrium

We now describe an environment and an MPE which satisfy our assumptions, including positive participation costs, but nevertheless supports collusive behavior and fails to pass our tests.

Two bidders $i \in \{1, 2\}$ compete for contracts. Bidder 1 has a publicly observable cost $c_H > 0$ at even periods, and a publicly observable cost $c_L \in (0, c_H)$ at odd periods. Bidder 2 has i.i.d. costs, equal to c_L with probability $q > 50\%$, and equal to c_H with probability $1-q$. Bidder 2's cost is her private information. Auctions have reserve price $r = 1 > c_H$. Bid preparation cost k is small, satisfying $k < \min\{(1-q)(r-c_H), (c_H-c_L)\}$. Let $\hat{c} = qc_L + (1-q)c_H$ be bidder 2's expected cost. We assume that $\delta < 1$ is sufficiently large, so that $c_H > \max\{r(1-\delta) + (c_L+k)\delta, (r-k)(1-\delta) + \delta\hat{c}\}$.

For simplicity, we expand the bidding space to deal with tied bids. For every bid b, we add bid b^-, equal in value to b, but such that $b^- \prec b$. We also consider a degenerate case where the impact of the state on costs is vanishingly small.[1] The state θ_t keeps track of:

- Is time period $t \in \mathbb{N}$ even or odd (i.e. $t \mod 2$)?
- Has any bidder won the auction both at times $2t$ and $2t+1$ the past?
- Who has won the auction in the last period?

The collusive equilibrium we construct is as follows. If at any point in the past a bidder has won the auction in consecutive even and odd periods, or no player won an auction, players bid according to a static Nash equilibrium.[2] If this is not the case, then:

- If $t \mod 2 = 0$, bidder 1 participates, and bids r; bidder 2 participates only if her cost is c_L, and bids r^-.
- If $t \mod 2 = 1$, and bidder 1 won in the previous period, then only bidder 2 participates, and bids r.
- If $t \mod 2 = 1$, and bidder 2 won in the previous period, then only bidder 1 participates, and bids r.

One can check that, when discount factor δ high enough, this is a Markov perfect equilibrium. Furthermore, bidder 2 is the only close winner, and conditional on being a close winner, has an expected 1-period backlog equal to $1-q$. Bidder 1 is the only close loser, and conditional on being a close loser, bidder 1 has an expected 1-period backlog equal to $q > 1-q$.

We note that bidding behavior under this MPE is sensitive. Indeed, both bidders' expected continuation payoff fall discretely if the winning bid changes from $b_w \le r$ to $b_w > r$ (i.e., to $b_w = \emptyset$).

OB Further Empirics

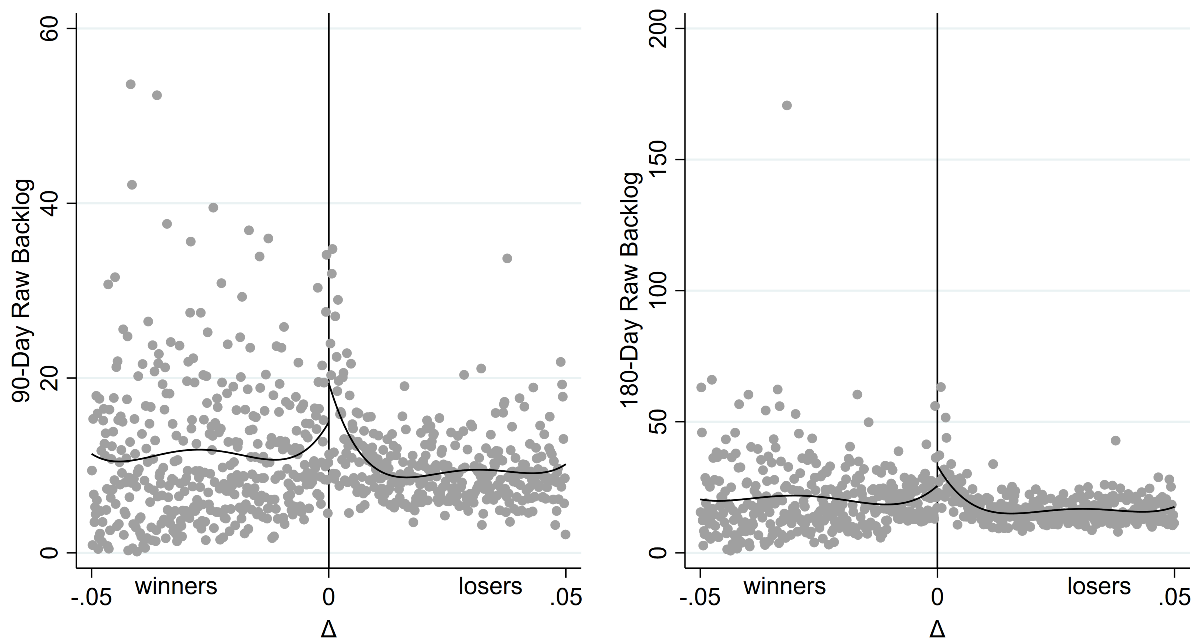
In this section, we first present the binned scatter plots corresponding to the regression results in Tables 5, 6 and 7 of Section 5. We next present a series of results that show robustness of

[1]This information can be made payoff relevant in different ways, for instance by shifting costs slightly as a function of the state.

[2]For t even or odd, the stage game has a Nash equilibrium in which bidder 1 randomizes between entering or not, and bidder 2 enters with probability 1 if her cost is c_L (earning profits $c_H - c_L$), and enters with probability 0 if her cost is c_H.

the results that we report in Section 5.2. In particular, we present the regression discontinuity estimates when we partition the sample of bids into two depending on whether or not the winning bid of the auction is above or below the median. This alternative way of partitioning equates the sample sizes across the two groups. We next report the results from using an alternative way of standardizing the backlog so that it is measurable with respect to the information of bidder i, $h_{i,t}$. We also report the results when we limit our sample to the municipalities that use public reserve prices for their auctions. Finally, we report findings for the entire sample of auctions for which we have data.

Omitted binned scatter plots for Table 5. Figure OB.1 plots the binned scatter plots of 90-day and 180-day raw backlog that correspond to columns (2) and (4) of Table 5. The discontinuities at 0 are quite modest.



Note: The curves in the figure correspond to 4th order (global) polynomial approximations of the conditional means.

Figure OB.1: Binned Scatter Plot for 90-Day and 180-Day Raw Backlog: Municipal Auctions from Japan.

Omitted binned scatter plots for Table 6. Figure OB.2 displays binned scatter plots of 90-day and 180-day raw backlog corresponding to the regression results reported in columns

(2) and (4) of Table 6. The top panels correspond to the results for the high bid sample (Panel (A)) and the bottom panels correspond to the low bid sample (Panel (B)). The left two panels plot the raw 90-day backlog against Δ and the right two panels plot the raw 180-day backlog against Δ. There is a modest discontinuity in the binned averages at $\Delta = 0$ in the top panels. In contrast, the graphs in the bottom panels, corresponding to columns (2) and (4) of Panel (B), do not exhibit any clear discontinuities at $\Delta = 0$.

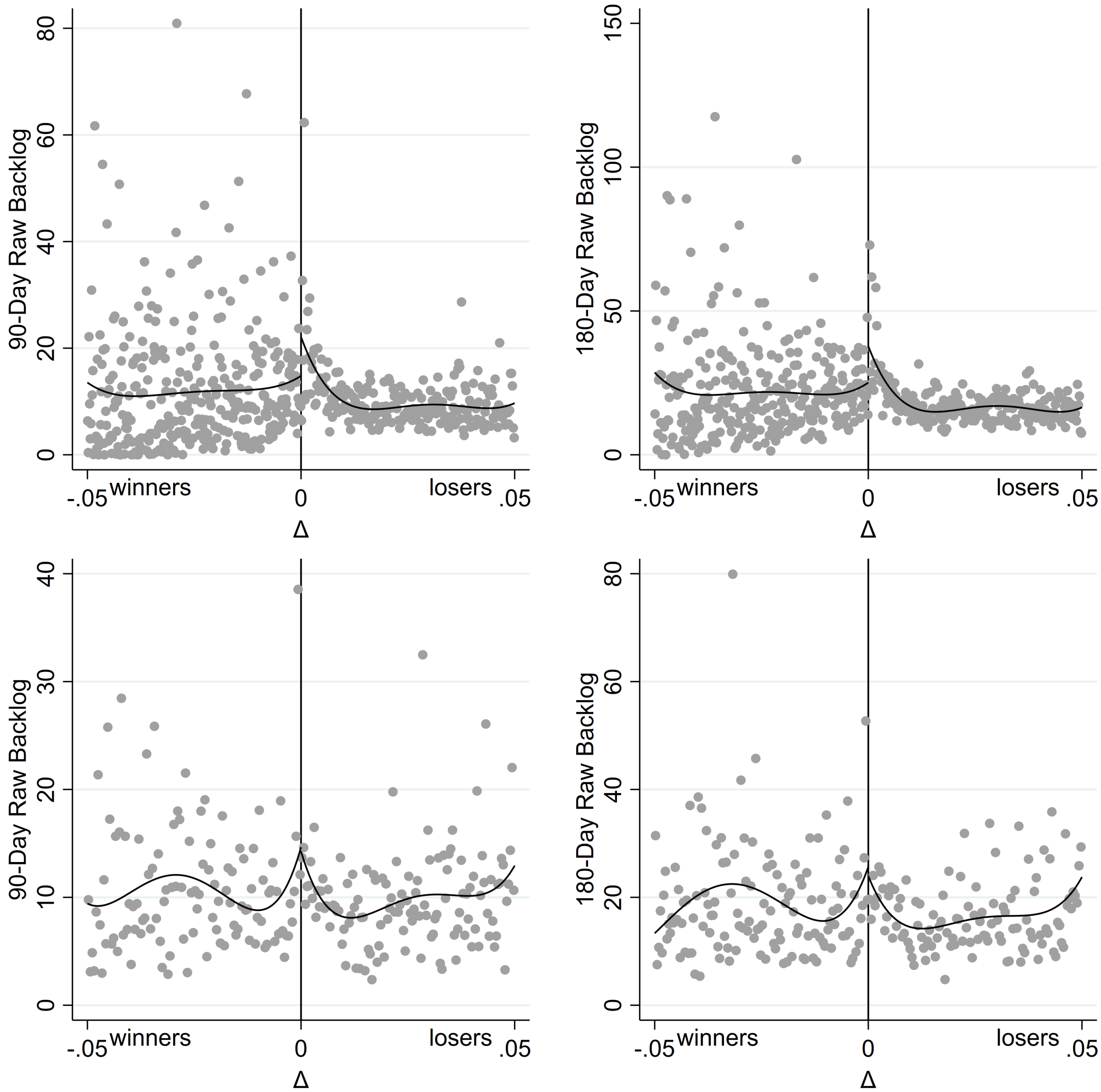
Omitted binned scatter plots for Table 7. Figures OB.3, OB.4 and OB.5 are the binned scatter plots corresponding to Table 7. In all of the panels, the horizontal axis corresponds to values $\Delta^2_{i,t} \equiv b_{i,t} - \min\{b_{j,t}, \text{s.t. } j \neq i \text{ and } j \text{ loses}\}$ for losing bidders i. A small negative value of $\Delta^2_{i,t}$ corresponds to a bid that is second lowest, but close to being third lowest. A small positive value of $\Delta^2_{i,t}$ corresponds to a bid that was higher than, but close to the second lowest bid.

The panels in Figure OB.3 are the binned scatter plots that correspond to columns (1) and (3) of Table 7. The panels in Figure OB.4 correspond to columns (2) and (4) of Table 7. The panels in Figure OB.5 correspond to column (5). The top panels of each figure correspond to the sample of bids that are above the municipal median. The bottom panels correspond to the sample of bids that are below the median. Unlike our results for marginal winners and marginal losers, the figures do not show any discontinuities around $\Delta^2_{i,t} = 0$.

Partitioning auctions by the winning bid. In our main analysis, we partition the sample of bids according to whether or not the bids are above or below the median winning bid. This results in the sample sizes of the two partitions to be unequal. In order to show that our results are not driven by differences in sample sizes, we consider an alternative partitioning in which we divide bids according to whether or not the winning bid of the auction is above or below the median winning bid.[3] This partitioning results in the same number of auctions in the two groups, and hence, roughly the same number of bids.

Table OB.1 reports the results. The top panel corresponds to the sample of bids submitted in auctions in which the winning bid is higher than the median. We find that the

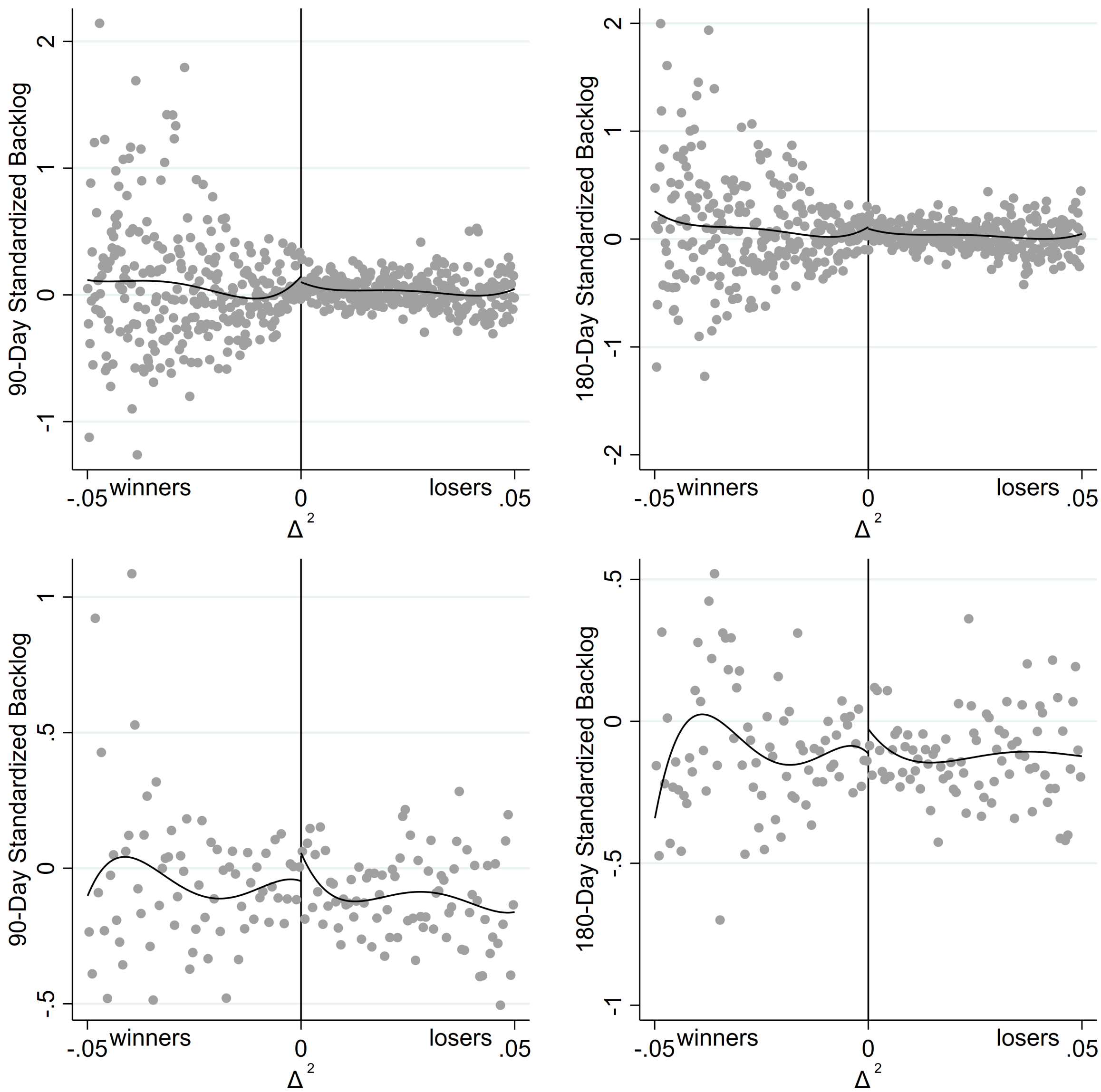
[3] In particular, for each bid, we consider the winning bid of the auction. We partition the sample of bids depending on whether or not the winning bid is above or below the median.



Note: Top panels correspond to Panel (A) of Table 6 and bottom panels correspond to Panel (B) of Table 6. Left panels correspond to 90-day raw backlog and the right panels correspond to 180-day standardized backlog.

Figure OB.2: Binned Scatter Plot for 90-Day and 180-Day Raw Backlog: Municipal Auctions from Japan – High vs. Low bids.

estimate of β is statistically significant for all five regressions. The bottom panel corresponds to the sample of bids submitted in auctions in which the winning bid is below the median. We find that in Panel (B), none of the estimates of β are statistically significant at the 5%. Note that the sample sizes in Panel (A) and (B) are roughly equal. The results of Table

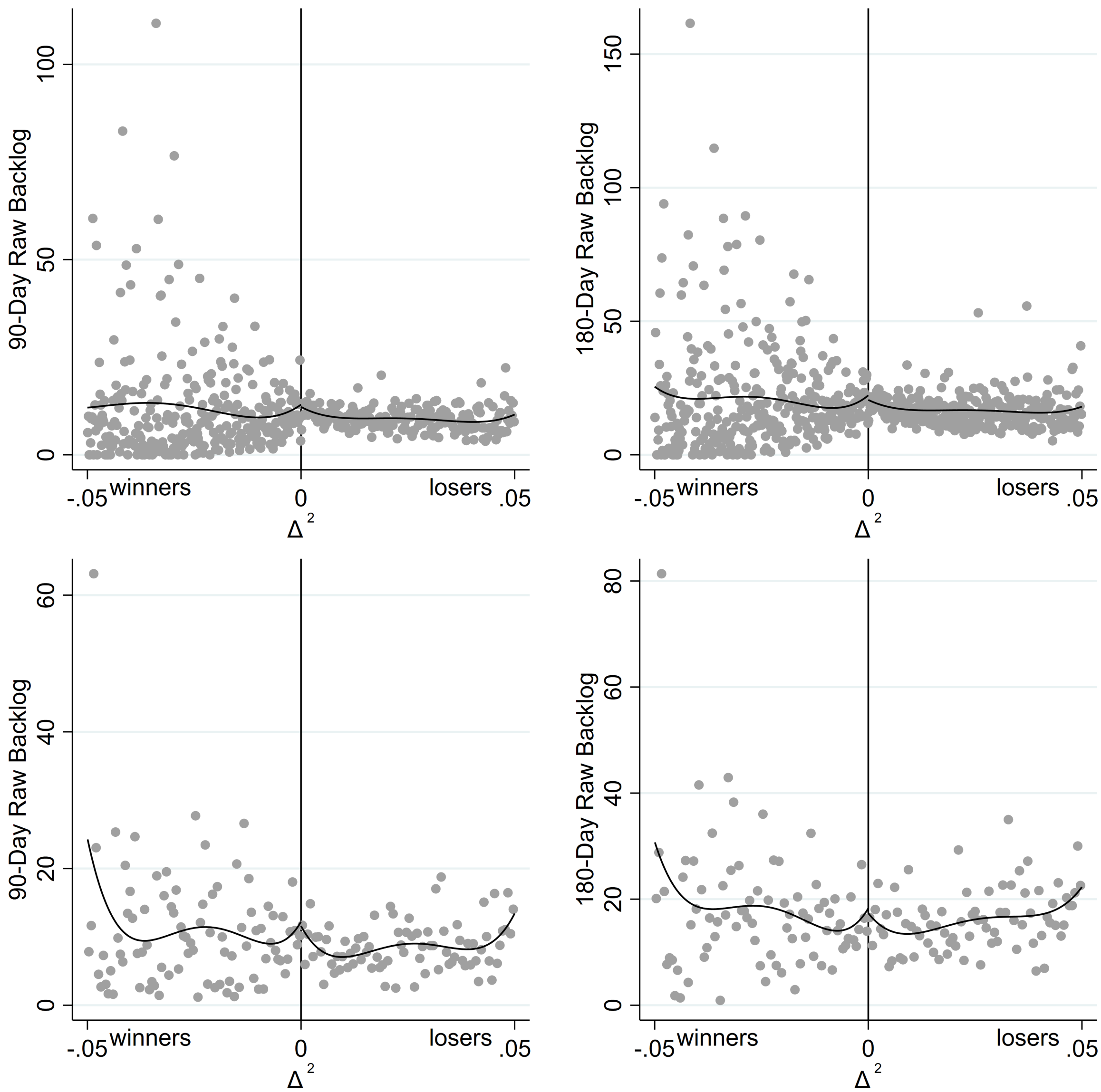


Note: Top panels correspond to columns (1) and (3) of Panel (A) of Table 7. Bottom panels correspond to columns (1) and (3) of Panel (B) of Table 7.

Figure OB.3: Binned Scatter Plot for Standarized Backlog with Respect to Δ^2: Municipal Auctions from Japan.

OB.1 suggests that sample sizes are not driving our results in the main text.

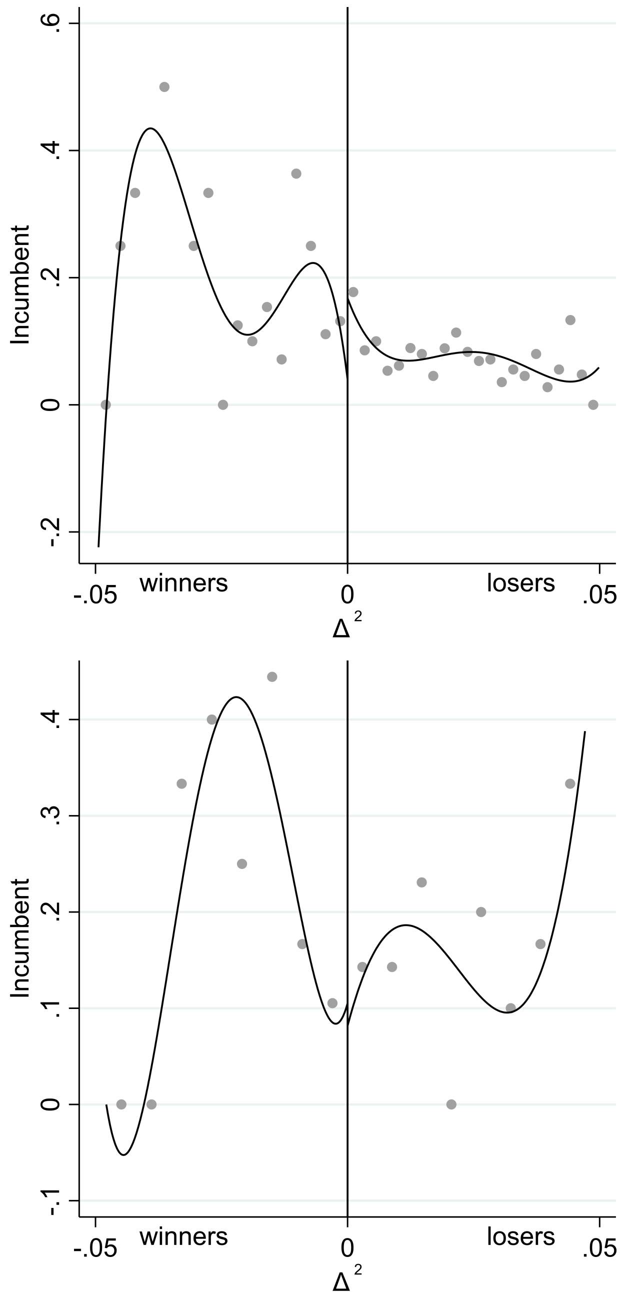
Alternative standardization of backlog. In the main specification, we define the standardized backlog by subtracting the within-firm mean from the raw backlog and then dividing



Note: Top panels correspond to columns (2) and (4) of Panel (A) of Table 7. Bottom panels correspond to columns (2) and (4) of Panel (B) of Table 7.

Figure OB.4: Binned Scatter Plot for Raw Backlog with Respect to Δ^2: Municipal Auctions from Japan.

it by the within-firm standard error. Strictly speaking, standardized backlog defined this way is not measurable with respect to bidders' information at the time of bidding, as required by the theory. In order to define an outcome variable that is perfectly consistent with the theory, we consider an alternative standardization of backlog in which we use the mean and



Note: The top panel corresponds to column (5), Panel (A) of Table 7. The bottom panel corresponds to column (5), Panel (B) of Table 7.

Figure OB.5: Binned Scatter Plot for Incumbent with Respect to Δ^2: Municipal Auctions from Japan.

	(1)	(2)	(3)	(4)	(5)
	90-Day Backlog		180-Day Backlog		Incumbent
	Standardized	Raw	Standardized	Raw	
Panel (A) : Above Median					
$\hat{\beta}$	0.249***	7.155**	0.221***	13.205***	−0.277**
	(0.048)	(3.239)	(0.046)	(4.708)	(0.110)
h	0.019	0.014	0.025	0.013	0.031
Obs.	28,650	30,666	28,665	30,666	1,058
Panel (B) : Below Median					
$\hat{\beta}$	−0.058	−0.290	−0.022	−1.278	−0.268*
	(0.065)	(1.983)	(0.064)	(2.957)	(0.143)
h	0.021	0.027	0.021	0.024	0.027
Obs.	30,739	33,100	30,770	33,100	1,032

Note: We partition auctions into two depending on whether or not the winning bid is above or below the median. Panel (A) corresponds to the bids of auctions that are above the median. Panel (B) corresponds to the bids of auctions that are below the median. Standard errors are clustered at the auction level and reported in parenthesis. *, **, and *** respectively denote significance at the 10%, 5%, and 1% levels.

Table OB.1: Partitioning Sample by Auctions: Municipal Auctions from Japan.

standard error of the rolling backlog as follows. Let

$$\mu'_{x_{i,t}^{B_k}} = \frac{1}{N_{i,t}} \sum_{\tau<t} x_{i,\tau}^{B_k},$$

$$\sigma'_{x_{i,t}^{B_k}} = \sqrt{\frac{1}{N_{i,t}} \sum_{\tau<t} (x_{i,\tau}^{B_k} - \mu'_{x_{i,t}^{B_k}})^2},$$

where $N_{i,t}$ is the number of auctions that firm i participates before auction t, $\mu'_{x_{i,t}^{B_k}}$ is the average of firm i's backlog up to auction t, and $\sigma'_{x_{i,t}^{B_k}}$ is the standard deviation of firm i's backlog up to auction t. We define standardized backlog as

$$x_{i,t}^{\overline{B}_k} = \frac{x_{i,t}^{B_k} - \mu'_{x_{i,t}^{B_k}}}{\sigma'_{x_{i,t}^{B_k}}}.$$

The difference between this definition and the one in the main text is that we now consider only auctions that take place before auction t in the summation ($\tau < t$). Note that the new definition of standardized backlog is measurable with respect to bidders' information at the time of bidding.

We estimate β using a local linear regression as follows:

$$\widehat{\beta} = \widehat{b_0^+} - \widehat{b_0^-}, \text{ with}$$
$$(\widehat{b_0^+}, \widehat{b_1^+}) = \arg\min \textstyle\sum_{i,t}^{T} (x_{i,t}^{B_k} - b_0^+ - b_1^+ \Delta_{i,t})^2 K\left(\frac{\Delta_{i,t}}{h_n}\right) \mathbf{1}_{\{\Delta_{i,t}>0\}\cap\{x_{i,t}^{B_k}\neq 0\}},$$
$$(\widehat{b_0^-}, \widehat{b_1^-}) = \arg\min \textstyle\sum_{i,t}^{T} (x_{i,t}^{B_k} - b_0^- - b_1^- \Delta_{i,t})^2 K\left(\frac{\Delta_{i,t}}{h_n}\right) \mathbf{1}_{\{\Delta_{i,t}<0\}\cap\{x_{i,t}^{B_k}\neq 0\}}.$$

Note that we condition our regression discontinuity estimate on the event $\{x_{i,t}^{B_k} \neq 0\}$. Because this event is measurable with respect to firms' information, Corollary 2 holds.

	(1) 90-Day Rolling Backlog	(2) 180-Day Rolling Backlog
Panel (A) : Above Median		
$\hat{\beta}$	0.599***	0.354***
	(0.076)	(0.069)
h	0.020	0.018
Obs.	20,343	26,449
Panel (B) : Below Median		
$\hat{\beta}$	0.198*	0.100
	(0.116)	(0.102)
h	0.026	0.025
Obs.	7,596	9,180

Note: Panel (A) corresponds to the sample of bids above the median winning bid. Panel (B) corresponds to the sample of bids below the median. Standard errors are clustered at the auction level and reported in parenthesis. The table also reports the bandwidth h used for the estimation. *, **, and *** respectively denote significance at the 10%, 5%, and 1% levels.

Table OB.2: Alternative Standardization of Backlog: Municipal Auctions from Japan.

Table OB.2 reports the results. Panel (A) corresponds to the sample of bids above the median winning bid and Panel (B) corresponds to the sample of bids below the median. The estimates for Panel (A) are statistically significant at the 5% level while the estimates in Panel (B) are not. The results of Table OB.2 are similar to the results we report in column (2) and (4) of Table 6.

Results for the sample of auctions with a public reserve price. We now report the results of our analysis when we restrict the sample to auctions let by municipalities using public reserve prices. Table OB.3 reports the results. Panel (A) of Table OB.3 reports estimation results for the set of bids above the municipal median.[4] Although the estimate of β is not statistically significant for the 90-day raw backlog in column (2), we find statistically significant differences between marginal losers and marginal winners for other measures of backlog in columns (1), (3), and (4). These results are qualitatively similar to those reported in Table 6. The results overall strongly suggest that there are non-competitive auctions among the sample of public reserve auctions in which a close winner submits a high bid.

In Panel (B), we report the results for the set of low bids. We find that there are no statistically significant differences between the marginal winner and the marginal loser for this subset, implying that we cannot reject the null of competition.

All municipalities. We now discuss the results of our tests when we include auctions from Japanese municipalities that we drop in our main analysis. There are a total of 109 municipalities for which we have auction data. Recall that, in order to construct the dataset used in Section 5.2, we drop municipalities for which the distribution of Δ has a missing mass at 0 (71 municipalities) and those for which the distribution of Δ has a point mass at exactly 0 (22 municipalities).

Figure OB.6 plots the histograms of $\Delta_{i,t}$ for auctions let by the municipalities with missing mass in the distribution of $\Delta_{i,t}$ at 0 (first row) and for those let by municipalities with a mass in the distribution of $\Delta_{i,t}$ at exactly zero (second row). The left two panels correspond to the histogram for all of the auctions let by each of the groups of municipalities. The middle and right panels correspond to the histogram for bids below the municipal median (middle panel) and above the municipal median (right panel).

[4]As before, we compute the median winning bid for each municipality and divide the sample according to whether or not a bid is above or below the municipal median.

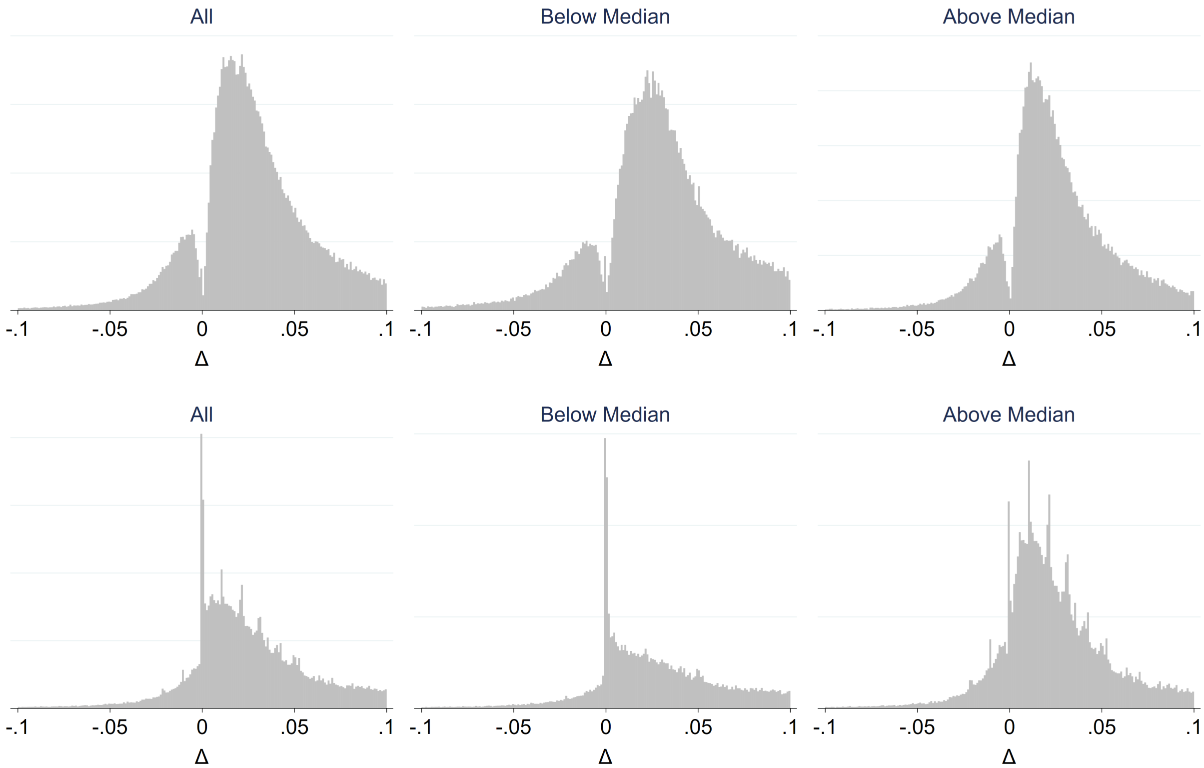
	(1)	(2)	(3)	(4)	(5)
	90-Day Backlog		180-Day Backlog		Incumbent
	Standardized	Raw	Standardized	Raw	
Panel (A) : Above Median					
$\hat{\beta}$	0.242**	12.375	0.239**	24.760**	−0.025
	(0.117)	(8.449)	(0.120)	(11.607)	(0.276)
h	0.006	0.007	0.006	0.006	0.009
Obs.	10,665	11,432	10,674	11,432	464
Panel (B) : Below Median					
$\hat{\beta}$	−0.002	−1.895	−0.012	−1.166	−0.306
	(0.136)	(6.476)	(0.131)	(8.432)	(0.441)
h	0.022	0.021	0.018	0.019	0.017
Obs.	2,675	2,883	2,676	2,883	61

Note: Panel (A) corresponds to the sample of bids above the median. Panel (B) corresponds to the sample of bids below the median. Standard errors are clustered at the auction level and reported in parenthesis. The table also reports the bandwidth h used for the estimation. *, **, and *** respectively denote significance at the 10%, 5%, and 1% levels.

Table OB.3: Restricting the Sample to Municipalities with Public Reserve Price.

The missing mass in the distribution of $\Delta_{i,t}$, apparent in the top panels, has previously been documented in Chassang et al. (2020). In that paper, we show that this distinctive pattern in the distribution of $\Delta_{i,t}$ is inconsistent with competitive bidding under fairly general conditions. Because our previous paper specifically focuses on the implications of these patterns, we opted to exclude these municipalities in our baseline analysis.

The distributions of $\Delta_{i,t}$ in the bottom panels have spikes at zero which are the result of binding price floors. Price floors can result in multiple bidders bidding exactly at the price floor. Note that because the spikes are generated by price floors, and because multiple bids at the price floor typically imply that the winning bid of the auction is low, the spike is very pronounced for the middle panel, but mostly disappears in the right panel. The summary statistics of the auctions for each of the groups are reported in Table OB.4. Column (1) corresponds to the sample statistics for municipalities with a missing mass at zero, column (2) corresponds to the sample statistics for those with a mass at 0, and column (3) corresponds to the sample statistics for the baseline sample used in Section 5.



Note: The top panels correspond to auctions from 71 municipalities with missing mass in the distribution of $\Delta_{i,t}$ at zero. The bottom panels correspond to auctions from 22 municipalities with a mass in the distribution of $\Delta_{i,t}$ at exactly zero. The left panels correspond to all auctions let by each of these groups, the middle panels condition on the winning bids to be below the municipality median and the right panels condition on the winning bids to be above the median.

Figure OB.6: Histogram of $\Delta_{i,t}$: Municipal Auctions from Japan.

We now report the regression discontinuity results for all of the auctions in our sample. Panel (A) of Table OB.5 reports the regression discontinuity estimates for bids above the median winning bid. Panel (B) of Table OB.5 reports the estimates for bids below the median winning bid. Focusing on Panel (A), we find that marginal losing bidders have about 0.086 higher 90-day standardized backlog (column (1)) and about 3.2 million yen more in terms of 90-day backlog (column (2)) than marginal winners. The estimates are statistically significant at the 5% level. Similarly, we find that marginal losing bidders have higher standardized and raw 180-day backlog (column (3), (4)) than marginal winners, and are less likely to be an incumbent (column (5)) than marginal winners. The coefficients are all statistically significant at the 1% level. These findings lead us to reject the null hypothesis of competition for this sample.

	(1) Sample with Missing Mass (71 munis)		(2) Sample with Mass at 0 (22 munis)		(3) Baseline Sample (16 munis)	
	Mean	Std.	Mean	Std.	Mean	Std.
Panel A: By Auction						
Reserve (Mil. Yen)	24.03	104.39	20.92	101.08	22.26	77.14
Winning Bid (Mil. Yen)	22.60	97.64	19.09	95.63	20.71	71.78
Win Bid/Reserve	0.940	0.073	0.911	0.078	0.926	0.083
# of Bids	7.44	3.78	8.00	4.64	6.80	4.21
Incumbent	0.064	0.244	0.043	0.202	0.044	0.204
Obs.	44,993		54,153		11,207	
Panel B: By Bidder						
# of Participation	23.80	60.13	32.65	74.45	22.56	45.93
# of Wins	3.20	8.26	4.23	10.17	3.32	6.97
Raw Backlog (90-Day)	4.02	21.15	4.75	19.19	4.11	17.16
Raw Backlog (180-Day)	6.58	35.09	7.18	26.30	6.45	22.85
Obs.	14,065		11,605		3,377	

Note: Column (1) reports summary statistics for the sample of auctions with missing mass in the distribution of $\Delta_{i,t}$ at zero (71 municipalities). Column (2) reports summary statistics for the sample with mass at exactly zero (22 municipalities). Column (3) reports sample statistics for the sample used in Section 5.

Table OB.4: Summary Statistics by Auctions and Bidders: All Municipalities.

The bottom panel of Table OB.5 reports the results for bids below the median. While the regression discontinuity estimate is statistically significant at the 5% level in columns (2), (4), and (5), the estimated differences between marginal winners and losers are smaller than in Panel (A). The results suggest the existence of some collusive bidding among this sample, but likely to a lesser extent than the sample in Panel (A). Overall, the results of Table OB.5 suggest that the null of competitive bidding is strongly rejected for the sample of high bids, but that the evidence is less strong for the sample of low winning bids. This is consistent with the expectation that there would be more collusion among auctions with high winning bids than among those with low winning bids.

	(1)	(2)	(3)	(4)	(5)
	90-Day Backlog		180-Day Backlog		Incumbent
	Standardized	Raw	Standardized	Raw	
Panel (A) : Above Median					
$\hat{\beta}$	0.086***	3.218**	0.117***	6.506***	−0.341***
	(0.019)	(1.279)	(0.019)	(2.175)	(0.052)
h	0.012	0.013	0.013	0.009	0.011
Obs.	530,966	558,152	531,357	558,152	19,929
Panel (B) : Below Median					
$\hat{\beta}$	−0.026	1.484**	−0.012	2.366**	−0.091***
	(0.016)	(0.701)	(0.018)	(1.168)	(0.034)
h	0.013	0.015	0.011	0.014	0.012
Obs.	177,304	188,032	177,457	188,032	6,864

In addition to the auctions used in the baseline analysis, we include auctions from 70 municipalities with missing mass in the distribution of $\Delta_{i,t}$ at zero and those from 18 municipalities with mass in the distribution of $\Delta_{i,t}$ at exactly zero. Panel (A) corresponds to the sample of bids above the median. Panel (B) corresponds to the sample of bids below the median. Standard errors are clustered at the auction level and reported in parenthesis. The forcing variable is Δ^1. The table also reports the bandwidth used for the estimation. *, **, and *** respectively denote significance at the 10%, 5%, and 1% levels.

Table OB.5: Regression Discontinuity Estimates: All Municipalities.

OC Proofs

OC.1 Proofs for Section 3

Proof of Lemma 1. We show that for all $\eta > 0$, there exists $\epsilon > 0$ small enough such that for all histories $h_{i,t}$,

$$\left| \mathsf{prob}(i \text{ wins} \mid h_{i,t} \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| \leq \epsilon) - \frac{1}{2} \right| \leq \eta.$$

By assumption, $D_i'(b_i|h_i)$ is continuous in $b_i \in [0,1]$ and strictly negative for all histories $h_i = (\theta, z_i)$. Since there are finitely many histories (θ, z_i), it follows that there exists $\nu > 0$ such that $D_i'(b_i|h_{i,t}) \leq -\nu$ for all b_i and all histories $h_{i,t}$. In addition, for all $\widehat{\eta} > 0$, there exists ϵ small enough that for all $\widehat{b}_i \in [b_i - \epsilon, b_i + \epsilon]$, $|D_i'(\widehat{b}_i|h_{i,t}) - D_i'(b_i|h_{i,t})| \leq \widehat{\eta}$.

This implies that for ϵ small

$$\begin{aligned}\left|\mathsf{prob}(i \text{ wins} \mid h_{i,t} \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| \leq \epsilon) - \frac{1}{2}\right| &= \left|\frac{D_i(b_{i,t}|h_{i,t}) - D_i(b_{i,t}+\epsilon|h_{i,t})}{D_i(b_{i,t}-\epsilon|h_{i,t}) - D_i(b_{i,t}+\epsilon|h_{i,t})} - \frac{1}{2}\right| \\ &\leq \left|\frac{-\epsilon D_i'(b_{i,t}|h_{i,t}) - \epsilon\widehat{\eta}}{-2\epsilon D_i'(b_{i,t}|h_{i,t}) + 2\epsilon\widehat{\eta}} - \frac{1}{2}\right| \\ &\leq \left|\frac{\nu - \widehat{\eta}}{2\nu + 2\widehat{\eta}} - \frac{1}{2}\right|.\end{aligned}$$

Lemma 1 follows by taking $\widehat{\eta}$ small enough. ■

Proof of Corrolary 1. Note that, for each $x \in X$,

$$\begin{aligned}\mathsf{prob}\left(x_i = x \mid \Delta_{i,t} \in (-\epsilon, 0)\right) &= \mathsf{prob}(x_{i,t} = x | i \text{ wins and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \\ &= \mathsf{prob}(x_{i,t} = x || b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \frac{\mathsf{prob}(i \text{ wins} \mid x_{i,t} = x \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon)}{\mathsf{prob}(i \text{ wins } || b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon)}\end{aligned}$$

Similarly,

$$\begin{aligned}\mathsf{prob}\left(x_i = x \mid \Delta_{i,t} \in (0, \epsilon)\right) &= \mathsf{prob}(x_{i,t} = x | i \text{ loses and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \\ &= \mathsf{prob}(x_{i,t} = x || b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \frac{\mathsf{prob}(i \text{ loses} \mid x_{i,t} = x \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon)}{\mathsf{prob}(i \text{ loses } || b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon)}\end{aligned}$$

By Lemma 1, we have that

$$\begin{aligned}\lim_{\epsilon \searrow 0} \mathsf{prob}(i \text{ wins} \mid x_{i,t} = x \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) &= \lim_{\epsilon \searrow 0} \mathsf{prob}(i \text{ wins } || b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) = \frac{1}{2}, \\ \lim_{\epsilon \searrow 0} \mathsf{prob}(i \text{ loses} \mid x_{i,t} = x \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) &= \lim_{\epsilon \searrow 0} \mathsf{prob}(i \text{ loses } || b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) = \frac{1}{2}.\end{aligned}$$

Hence, for each $x \in X$,

$$\lim_{\epsilon \searrow 0} \left|\mathsf{prob}\left(x_i = x \mid \Delta_{i,t} \in (-\epsilon, 0)\right) - \mathsf{prob}\left(x_i = x \mid \Delta_{i,t} \in (0, \epsilon)\right)\right| = 0.$$

Since X is finite, for all $\eta > 0$ there exists $\epsilon > 0$ small enough such that for all $x \in X$,

$$\left|\mathsf{prob}\left(x_i = x \mid \Delta_{i,t} \in (0, \epsilon)\right) - \mathsf{prob}\left(x_{i,t} = x \mid \Delta_{i,t} \in (-\epsilon, 0)\right)\right| < \eta.$$

This completes the proof. ■

OC.2 Proofs for Section 4

We now establish Proposition 1. Throughout this section we consider an environment $\mathcal{E}$ and an MPE σ that is competitively enforced. We begin by establishing two intermediary lemmas. Recall that continuation value $V_i(\zeta_i, b_w|h_i)$ does not depend on winning bid b_w when bidder i wins. Hence, we suppress the dependency of V_i on b_w when $\zeta_i = 1$.

Lemma OC.1 (minimum demand)**.** *There exists $\nu > 0$ such that for every history $h_i = (\theta, z_i)$ and bid $b_i \in [0,1]$ in the support of $\sigma_{i|h_i}$, $D_i(b_i|h_i) \geq \nu$. In addition,*

$$b_i - c_i + \delta \mathbb{E}_\sigma \left[V_i(1|h_i) - V_i(0, \wedge \mathbf{b}_{-i}|h_i) \,|\, h_i, b_i \prec \wedge \mathbf{b}_{-i} \right] \geq k.$$

Proof. Since firm i chooses to participate, it must be that

$$\begin{aligned}
\mathbb{E}_\sigma \left[\mathbf{1}_{b_i \prec \wedge \mathbf{b}_{-i}} (b_i - c_i + \delta V_i(1|h_i)) + \mathbf{1}_{b_i \succ \wedge \mathbf{b}_{-i}} \delta V_i(0, \wedge \mathbf{b}_{-i}|h_i) \,|\, h_i \right] - k &\geq \mathbb{E}_\sigma \left[\delta V_i(0, \wedge \mathbf{b}_{-i}|h_i) \,|\, h_i \right] \\
\iff \mathbb{E}_\sigma \left[\mathbf{1}_{b_i \prec \wedge \mathbf{b}_{-i}} (b_i - c_i + \delta V_i(1|h_i) - \delta V_i(0, \wedge \mathbf{b}_{-i}|h_i)) \,|\, h_i \right] &\geq k \\
\iff D_i(b_i|h_i) \left(b_i - c_i + \delta \mathbb{E}_\sigma \left[V_i(1|h_i) - V_i(0, \wedge \mathbf{b}_{-i}|h_i) \,|\, h_i, b_i \prec \wedge \mathbf{b}_{-i} \right] \right) &\geq k
\end{aligned}$$

Since $D_i \geq 0$, it must be that both left-hand side factors are strictly positive. In addition, since continuation values are bounded by some constant $\overline{V}$, it follows that $D_i(b_i|h_i) \geq k/(1 + 2\overline{V})$. Similarly, since demand is bounded above by 1, we have that

$$b_i - c_i + \delta \mathbb{E}_\sigma \left[V_i(1|h_i) - V_i(0, \wedge \mathbf{b}_{-i}|h_i) \,|\, h_i, b_i \prec \wedge \mathbf{b}_{-i} \right] \geq k.$$

This concludes the proof. □

Lemma OC.2 (continuous demand)**.** *For every history $h_i = (\theta, z_i)$, residual demand $D_i(b_i|h_i)$ is continuous in b_i over $(0, 1)$.*

Proof. The proof is by contradiction. Assume that demand $D_i(\cdot|h_i)$ is discontinuous at bid b_0. There must exist a bidder j and a history $h_j = (\theta, z_j)$ such that firm j bids $b_j = b_0$ with probability $q > 0$. By Lemma OC.1, bidder j must win with probability at least $\nu > 0$ when bidding b_0.

Consider a bidder l and a history $h_l = (\theta, z_l)$ such that history h_j has positive probability, and bidder l loses with positive probability against bidder j when bidder j bids b_0. Since the number of histories is finite, there exists $\nu_1 > 0$ such that at any such history h_l bidder j bids b_0 with positive probability ν_1.

Pick $\epsilon > 0$ and consider the payoff of bidder l bidding $b_l \in [b_0, b_0 + \epsilon)$. Bidder l gets payoff (excluding participation costs and payoffs upon non-participation)

$$U_l(b_l|h_l, c_l) = D(b_l|h_l)\left(b_l - c_l + \delta\mathbb{E}_\sigma[V_l(1|h_l) - V_l(0, \wedge\mathbf{b}_{-l}|h_l) \,|\, h_l, b_l \prec \wedge\mathbf{b}_{-l}]\right).$$

We know from Lemma OC.1 that

$$b_l - c_l + \delta\mathbb{E}_\sigma\left[V_l(1|h_l) - V_l(0, \wedge\mathbf{b}_{-l}|h_l) \,|\, h_l, b_l \prec \wedge\mathbf{b}_{-l}\right] \geq k.$$

Since bidding behavior is not sensitive, there exists a Lipcshitz constant $L > 0$ such that

$$\mathbb{E}_\sigma\left[V_l(0, \wedge\mathbf{b}_{-l}|h_l) \,|\, h_l, b_l - \epsilon \prec \wedge\mathbf{b}_{-l}\right] \leq \mathbb{E}_\sigma\left[V_l(0, \wedge\mathbf{b}_{-l}|h_l) \,|\, h_l, b_l \prec \wedge\mathbf{b}_{-l}\right] + \epsilon L.$$

Altogether, it follows that for every $\eta > 0$, there exists $\epsilon > 0$ small enough that

$$\begin{aligned} b_l - \epsilon - c_l + &\delta\mathbb{E}_\sigma\left[V_l(1|h_l) - V_l(0, \wedge\mathbf{b}_{-l}|h_l) \,|\, h_l, b_l - \epsilon \prec \wedge\mathbf{b}_{-l}\right] \\ &\geq b_l - c_l + \delta\mathbb{E}_\sigma\left[V_l(1|h_l) - V_l(0, \wedge\mathbf{b}_{-l}|h_l) \,|\, h_l, b_l \prec \wedge\mathbf{b}_{-l}\right] - \eta \geq k - \eta. \end{aligned}$$

Hence, it follows that by bidding $b_l - \epsilon$, bidder l gets a payoff

$$\begin{aligned} U_l(b_l - \epsilon|h_l, c_l) &= D(b_l - \epsilon|h_l)\left(b_l - \epsilon - c_l + \delta\mathbb{E}_\sigma[V_i(1|h_l) - V_i(0, \wedge\mathbf{b}_{-l}|h_l) \,|\, h_l, b_l - \epsilon \prec \wedge\mathbf{b}_{-l}]\right) \\ &\geq U_l(b_l|h_l, c_l) - \eta + \nu_1(k - \eta) \end{aligned}$$

Since ν_1 is fixed, it follows that for ϵ small enough $U_l(b_l - \epsilon|h_l, c_l) > U_l(b_l|h_l, c_l)$. Hence, there exists ϵ small such that bidder l does not bid in $[b_0, b_0 + \epsilon)$. Since there are only finite histories, this implies that there exists $\epsilon > 0$ such that no bidder l that loses against bidder j bidding b_0 bids in the range $[b_0, b_0 + \epsilon)$. Hence, bidder j would benefit from bidding $b_0 + \epsilon/2$ rather than b_0. This contradicts the assumption that σ is an MPE and concludes the proof. □

Proof of Proposition 1. Consider an environment $\mathcal{E}$ and an MPE σ that is competitively enforced. Fix a history $h_{i,t} = (\theta_t, z_{i,t})$ of firm i. Let $b_{i,t} < r = 1$ denote firm i's bid at this history when her costs are $c_{i,t}$. For any bid b, let $U_i(b|h_{i,t}, c_{i,t})$ denote i's payoff from bidding

b at history $h_{i,t}$ when her cost is $c_{i,t}$:

$$U_i(b|h_{i,t}, c_{i,t}) = \mathbb{E}_\sigma\left[\mathbf{1}_{\wedge \mathbf{b}_{-i,t} \succ b}(b - c_{i,t} + \delta V_i(1|h_{i,t}) + (1 - \mathbf{1}_{\wedge \mathbf{b}_{-i,t} \succ b})\delta V_i(0, \wedge \mathbf{b}_{-i,t}|h_{i,t}) \mid h_{i,t}\right] - k.$$

Since bid $b_{i,t}$ is optimal, for all $\epsilon > 0$ it must be that,

$$\begin{aligned} & U_i(b_{i,t}|h_{i,t}, c_{i,t}) \geq U_i(b_{i,t} + \epsilon|h_{i,t}, c_{i,t}) \\ \iff & (D_i(b_{i,t}|h_{i,t}) - D_i(b_{i,t} + \epsilon|h_{i,t}))(b_{i,t} - \kappa_{i,t}^{+\epsilon}) \geq D_i(b_{i,t} + \epsilon|h_{i,t}) \times \epsilon \end{aligned} \tag{O1}$$

where $\kappa_{i,t}^{+\epsilon} \equiv c_{i,t} - \delta \mathbb{E}^\sigma[V_i(1|h_{i,t}) - V_i(0, \wedge \mathbf{b}_{-i,t}|h_{i,t})|h_{i,t}, b_{i,t} + \epsilon \succ \wedge \mathbf{b}_{-i,t} \succ b_{i,t}]$. Since $D_i(\cdot|h_{i,t})$ is continuous at $b_{i,t}$ (Lemma OC.2), and since $D_i(b_{i,t}|h_{i,t}) > 0$ (Lemma OC.1) it must be that $b_{i,t} - \kappa_{i,t}^{+\epsilon} > 0$ for $\epsilon > 0$ small.

Similarly, for all $\epsilon > 0$ it must be that

$$\begin{aligned} & U_i(b_{i,t}|h_{i,t}, c_{i,t}) \geq U_i(b_{i,t} - \epsilon|h_{i,t}, c_{i,t}) \\ \iff & (D_i(b_{i,t} - \epsilon|h_{i,t}) - D_i(b_{i,t}|h_{i,t}))(b_{i,t} - \kappa_{i,t}^{+\epsilon}) \leq D_i(b_{i,t} - \epsilon|h_{i,t}) \times \epsilon \\ & \qquad - (D_i(b_{i,t} - \epsilon|h_{i,t}) - D_i(b_{i,t}|h_{i,t}))(\kappa_{i,t}^{+\epsilon} - \kappa_{i,t}^{-\epsilon}) \end{aligned} \tag{O2}$$

where $\kappa_{i,t}^{-\epsilon} \equiv c_{i,t} - \delta \mathbb{E}^\sigma[V_i(1|h_{i,t}) - V_i(0, \wedge \mathbf{b}_{-i,t}|h_{i,t})|h_{i,t}, b_{i,t} \succ \wedge \mathbf{b}_{-i,t} \succ b_{i,t} - \epsilon]$.

Using (O1) and (O2), together with $b_{i,t} - \kappa_{i,t}^{+\epsilon} > 0$, we have that

$$\begin{aligned} & \mathsf{prob}_\sigma(i \text{ wins} \mid h_{i,t} \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \\ & \quad = \frac{D_i(b_{i,t}|h_{i,t}) - D_i(b_{i,t} + \epsilon|h_{i,t})}{D_i(b_{i,t} - \epsilon|h_{i,t}) - D_i(b_{i,t} + \epsilon|h_{i,t})} \\ & \quad = \frac{D_i(b_{i,t}|h_{i,t}) - D_i(b_{i,t} + \epsilon|h_{i,t})}{D_i(b_{i,t} - \epsilon|h_{i,t}) - D_i(b_{i,t}|h_{i,t}) + D_i(b_{i,t}|h_{i,t}) - D_i(b_{i,t} + \epsilon|h_{i,t})} \\ & \quad \geq \frac{D_i(b_{i,t} + \epsilon|h_{i,t})}{D_i(b_{i,t} - \epsilon|h_{i,t}) - (D_i(b_{i,t} - \epsilon|h_{i,t}) - D_i(b_{i,t}|h_{i,t}))\frac{\kappa_{i,t}^{+\epsilon} - \kappa_{i,t}^{-\epsilon}}{\epsilon} + D_i(b_{i,t} + \epsilon|h_{i,t})}. \end{aligned} \tag{O3}$$

Since $D_i(\cdot|\theta_t, z_{i,t})$ is continuous on $[0, 1]$, it is uniformly continuous. Since there are finitely many (θ, z_i), for every $\gamma_D > 0$ there exists $\overline{\epsilon} > 0$ such that, for all i, θ, z_i and for all b, b' with $|b - b'| \leq 2\overline{\epsilon}$, $D_i(b|\theta, z_i) - D_i(b'|\theta, z_i) < \gamma_D$.

Moreover, since bidding behavior is not sensitive, and since there are finitely many $h_i = (\theta, z_i)$, there exists a Lipschitz constant $L > 0$ such that, for all $i, \theta, z_i, c_{i,t}$, $\kappa_{i,t}^{+\epsilon} - \kappa_{i,t}^{-\epsilon} \geq -2\epsilon L$.

Using (O3), for every $\gamma_D > 0$, there exists $\tilde{\epsilon} > 0$ such that, for all $\epsilon < \tilde{\epsilon}$,

$$\begin{aligned}
&\mathsf{prob}_\sigma(i \text{ wins} \mid h_{i,t} \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \\
\geq &\frac{D_i(b_{i,t} + \epsilon | h_{i,t})}{D_i(b_{i,t} + \epsilon | h_{i,t}) + \gamma_D + 2\gamma_D L + D_i(b_{i,t} + \epsilon | h_{i,t})} \\
\geq &\frac{\nu - \gamma_D}{\nu + \gamma_D 2L + \nu - \gamma_D},
\end{aligned} \tag{O4}$$

where the second inequality uses the inequality $D_i(b_{i,t} + \epsilon | h_{i,t}) \geq D_i(b_{i,t} | h_{i,t}) - \gamma_D \geq \nu - \gamma_D$ (Lemma OC.1). Picking γ_D small, we obtain that $\mathsf{prob}_\sigma(i \text{ wins} \mid h_{i,t} \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \geq 1/2 - \eta$. ■

Proof of Corollary 2. For each $\epsilon > 0$, let $\mathsf{prob}_\sigma(\cdot | \epsilon\text{-}\mathsf{close})$ denote the distribution over histories conditional on event ϵ-close. Then, for each $i \in N$ and each $\epsilon > 0$, the probability with which firm i wins an auction under σ conditional on event ϵ-close satisfies

$$\begin{aligned}
\mathsf{prob}_\sigma(i \text{ wins} | \epsilon\text{-}\mathsf{close}) = \mathbb{E}_{\mathcal{E},\sigma} &\left[\mathsf{prob}_\sigma(i \text{ wins} \mid h_{i,t} \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \middle| \epsilon\text{-}\mathsf{close} \right] \\
&\times \mathsf{prob}_\sigma(|b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon | \epsilon\text{-}\mathsf{close}).
\end{aligned} \tag{O5}$$

By Proposition 1, it follows that

$$\forall i \in N, \quad \liminf_{\epsilon \searrow 0} \mathbb{E}_{\mathcal{E},\sigma} \left[\mathsf{prob}_\sigma(i \text{ wins} \mid h_{i,t} \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \middle| \epsilon\text{-}\mathsf{close} \right] \geq \frac{1}{2}. \tag{O6}$$

Towards a contradiction, suppose that the result is not true. Hence, there exists a player j and a number $\eta > 0$ such that

$$\limsup_{\epsilon \searrow 0} \mathbb{E}_{\mathcal{E},\sigma} \left[\mathsf{prob}(j \text{ wins} \mid h_{j,t} \text{ and } |b_{j,t} - \wedge \mathbf{b}_{-j,t}| < \epsilon) \middle| \epsilon\text{-}\mathsf{close} \right] \geq \frac{1}{2} + \eta. \tag{O7}$$

Note that, for each $\epsilon > 0$, we have that

$$\begin{aligned}
&\sum_{i \in N} \mathsf{prob}_\sigma(i \text{ wins } | \epsilon\text{-}\mathsf{close}) = 1 \text{ and} \\
&\sum_{i \in N} \mathsf{prob}_\sigma(|b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon | \epsilon\text{-}\mathsf{close}) = \mathbb{E}_{\mathcal{E},\sigma}[|\{i \text{ s.t. } |b_{i,t} - \wedge \mathbf{b}_{-i,t} < \epsilon\}| \mid \epsilon\text{-}\mathsf{close}] \geq 2.
\end{aligned}$$

Using (O5), (O6) and (O7), we obtain that

$$
\begin{aligned}
1 = \limsup_{\epsilon \searrow 0} \sum_{i \in N} \mathsf{prob}_\sigma(i \text{ wins } |\epsilon\text{-}\mathsf{close}) \geq \frac{1}{2} \limsup_{\epsilon \searrow 0} \sum_{i \in N} & \mathsf{prob}_\sigma(|b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon | \epsilon\text{-}\mathsf{close}) \\
& + \eta \mathsf{prob}_\sigma(|b_{j,t} - \wedge \mathbf{b}_{-j,t}| < \epsilon | \epsilon\text{-}\mathsf{close}) \\
\geq 1 + \eta \limsup_{\epsilon \searrow 0} & \mathsf{prob}_\sigma(|b_{j,t} - \wedge \mathbf{b}_{-j,t}| < \epsilon | \epsilon\text{-}\mathsf{close}) > 1,
\end{aligned}
$$

a contradiction. ■

Proof of Corollary 3. Follows from the fact that Proposition 1 and Corollary 2 imply that Corollary 1 must hold whenever σ is an MPE that is competitively enforced. ■

Sample implications of Corollary 2. We now show that when the sample size is large, Corollary 2 must hold approximately under the sample distribution of bids and characteristics $\mathbf{b}, \mathbf{x}$.

Data consists of bids and observable characteristics $(\mathbf{b}_t, \mathbf{x}_t)_{t \in \{0, \cdots, T\}}$ for auctions happening at times $t \in \{0, \cdots, T\}$. Let $H = \{h_i\}$ be the set of histories corresponding to data $(\mathbf{b}_t, \mathbf{x}_t)$: i.e., for each data point $(b_{i,t}, x_{i,t})$, history $h_{i,t} \in H$ corresponds to the information that bidder i had at time t, prior to bidding. We denote by $\widehat{\mathsf{prob}}$ the sample joint distribution of bids and characteristics in $(\mathbf{b}_t, \mathbf{x}_t)$.

Definition OC.1. *We say that a set of histories H is adapted to the players' information if and only if the event $h_{i,t} \in H$ is measurable with respect to player i's information at time t, prior to bidding.*

A subset H can be thought of as a set of histories that satisfy a certain criteria defined by the analyst. Definition OC.1 states that H is adapted if it is possible to check whether $h_{i,t}$ satisfies the criteria needed for inclusion in H using only information available to bidder i at time t, prior to bidding. Consider, for example, the histories in which the bid is above a particular threshold. Because a bidder knows, at the time of bidding, that its bid will be above a given threshold, the set of histories in which a bid is above a given threshold is adapted. Consider next, the histories in which a particular bidder wins. Because a bidder does not know who will win the auction at the time of bidding, the set of histories in which a given bidder wins is not adapted.

As we now show, when Proposition 1 holds and the set of histories H is adapted, sample beliefs satisfy condition (6). This allows us to apply our tests to specific subsets of the data.

Given $\epsilon > 0$ and $x \in X$, we define $B_{x,\epsilon} \equiv \{(i,t) \text{ s.t. } x_{i,t} = x,\ |b_{i,t} - \wedge b_{-i,t}| \leq \epsilon\}$ the subsample of close bids such that the bidders characteristics x_i are equal to x. We denote by $B_\epsilon \equiv \{(i,t) \text{ s.t. } |b_{i,t} - \wedge b_{-i,t}| \leq \epsilon\}$ the sample of close bids. A bidder's sample probability of winning conditional on close bids and type x is denoted by $\widehat{P}_{x,\epsilon}$. Formally, we have,

$$\begin{aligned} \widehat{P}_{x,\epsilon} &\equiv \widehat{\mathsf{prob}}(i \text{ wins} \mid x_i = x,\ |b_i - \wedge b_{-i}| \leq \epsilon) \\ &= \frac{|\{(i,t) \in B_{x,\epsilon} \text{ s.t. } b_{i,t} \prec \wedge b_{-i,t}\}|}{|B_{x,\epsilon}|} \end{aligned} \tag{O8}$$

We make the following assumption about data.

Assumption OC.1. *There exists $\lambda > 0$ such that for all datasets of interest B, and all $x \in X$,*

$$\frac{\sum_{x' \in X \setminus x} |B_{x',\epsilon}|}{|B_{x,\epsilon}|} \leq \lambda$$

The following result holds:

Proposition OC.1 (winning is independent of bidder characteristics)**.** *Suppose H is adapted. For all $\eta > 0$, there exists $\epsilon > 0$ small enough such that with probability approaching 1 as $|B_\epsilon|$ goes to infinity,*

$$\forall x \in X, \quad \left| \widehat{P}_{x,\epsilon} - \frac{1}{2} \right| \leq \eta.$$

Proof. Take $\eta' > 0$ as given. We know from Proposition 1 that for epsilon small enough, for all histories $h_{i,t}$, $\mathsf{prob}(i \text{ wins} \mid h_{i,t} \text{ and } |b_{i,t} - \wedge \mathbf{b}_{-i,t}| < \epsilon) \geq 1/2 - \eta'$.

Fix $x \in X$. We show that with probability approaching 1 as $|B_\epsilon|$ goes to infinity, $\widehat{P}_{x,\epsilon} \geq \frac{1}{2} - 2\eta'$. Observe first that, by Assumption OC.1, when $|B_\epsilon|$ grows large, $|B_{x,\epsilon}|$ grows proportionally large:

$$\frac{|B_{x,\epsilon}|}{|B_\epsilon|} = 1 - \frac{\sum_{x' \in X \setminus x} |B_{x',\epsilon}|}{|B_{x,\epsilon}| + \sum_{x' \neq x} |B_{x',\epsilon}|} \geq 1 - \frac{\lambda}{1+\lambda}.$$

We denote by $\{t_1, \cdots, t_n\}$ auctions occurring at times t such that $(i,t) \in B_{x,\epsilon}$, ordered according to the timing of the auction. Since the number N of bidders is finite, n grows large proportionally with $|B_{x,\epsilon}|$. We define $C_k = \{i \in N \text{ s.t. } (i, t_k) \in B_{x,\epsilon}\}$. In equilibrium,

$$H_K \equiv \sum_{k=1}^{K} \sum_{i \in C_k} \mathbf{1}_{b_{i,t_k} \prec \wedge b_{-i,t_k}} - \mathsf{prob}_i(b_{i,t_k} \prec \wedge b_{-i,t_k} | i \in C_k)$$

is a martingale, provided set H is adapted. Indeed note that given the information I_K available at the time of bidding in auction K,

$$\begin{aligned}
\mathbb{E}\left[\sum_{i \in C_K} \mathbf{1}_{b_{i,t_K} \prec \wedge b_{-i,t_K}} \middle| I_K\right] &= \mathbb{E}\left[\sum_{i \in N} \mathbf{1}_{i \in C_K} \mathbf{1}_{b_{i,t_K} \prec \wedge b_{-i,t_K}} \middle| I_K\right] \\
&= \mathbb{E}\left[\mathbb{E}_{C_K}\left[\sum_{i \in N} \mathbf{1}_{i \in C_K} \mathbf{1}_{b_{i,t_K} \prec \wedge b_{-i,t_K}} \middle| I_K\right]\right] \\
&= \mathbb{E}\left[\sum_{i \in N} \mathbf{1}_{i \in C_K} \mathsf{prob}_i(\mathbf{1}_{b_{i,t_K} \prec \wedge b_{-i,t_K}} | i \in C_K) \middle| I_K\right] \\
&= \mathbb{E}\left[\sum_{i \in C_K} \mathsf{prob}_i(\mathbf{1}_{b_{i,t_K} \prec \wedge b_{-i,t_K}} | i \in C_K) \middle| I_K\right].
\end{aligned}$$

Using Proposition 1, this implies that

$$G_K \equiv \sum_{k=1}^{K} \sum_{i \in C_k} \mathbf{1}_{b_{i,t_k} \prec \wedge b_{-i,t_k}} - \frac{1}{2} + \eta'$$

is a submartingale with increments bounded by $|N|$ (the maximum number of bidders in an auction). It follows from the Azuma-Hoeffding Theorem that as n grows large, with probability approaching 1, $G_n \geq -\eta' n$. Since $n \leq |B_{x,\epsilon}|$, this implies that with probability approaching 1,

$$\widehat{P}_{x,\epsilon} \equiv \frac{1}{|B_{x,\epsilon}|} \sum_{k=1}^{n} \sum_{i \in C_k} \mathbf{1}_{b_{i,t_k} \prec \wedge b_{-i,t_k}} \geq \frac{1}{2} - 2\eta'.$$

Since X is finite, with probability approaching 1 as $|B_\epsilon|$ becomes large, we have that for all $x \in X$, $\widehat{P}_{x,\epsilon} \geq \frac{1}{2} - 2\eta'$. In addition, since $\sum_{x' \in X} |B_{x',\epsilon}| \widehat{P}_{x',\epsilon} = |\{(i,t) \in B_\epsilon \text{ s.t. } i \text{ wins }\}|$, it follows that

$$\frac{\sum_{x' \in X} |B_{x',\epsilon}| \widehat{P}_{x',\epsilon}}{\sum_{x' \in X} |B_{x',\epsilon}|} \leq \frac{1}{2}.$$

Hence, with probability approaching 1, we have that

$$|B_{x,\epsilon}|\widehat{P}_{x,\epsilon} \leq \frac{1}{2}|B_{x,\epsilon}| + \sum_{x'\in X\setminus x} |B_{x',\epsilon}| \left(\frac{1}{2} - \widehat{P}_{x',\epsilon}\right)$$
$$\Rightarrow \widehat{P}_{x,\epsilon} \leq \frac{1}{2} + 2\eta' \frac{\sum_{x'\in X\setminus x} |B_{x',\epsilon}|}{|B_{x,\epsilon}|} \leq \frac{1}{2} + 2\eta'\lambda.$$

Hence by selecting η' sufficiently small in the first place, it follows that for any $\eta > 0$, there exists ϵ such that as $|B_\epsilon|$ grows large, $|\widehat{P}_{x,\epsilon} - \frac{1}{2}| \leq \eta$ with probability 1. □

A corollary of Proposition OC.1 is that our regression discontinuity design remains valid: conditional on close bids, the sample distribution of covariates is independent of whether the bidder wins or loses the auction.

Corollary OC.1 (close winners and losers have similar characteristics)**.** *For all $\eta > 0$, there exists $\epsilon > 0$ small enough such that with probability approaching 1 as $|B_\epsilon|$ goes to infinity,*

$$\forall x \in X, \quad \left|\widehat{\textsf{prob}}(x_i = x \mid i \text{ wins }, |b_i - \wedge b_{-i}| \leq \epsilon) - \widehat{\textsf{prob}}(x_i = x \mid i \text{ loses }, |b_i - \wedge b_{-i}| \leq \epsilon)\right| \leq \eta.$$

Proof. Observe that

$$\widehat{\textsf{prob}}(x_i = x \mid i \text{ wins }, |b_i - \wedge b_{-i}| \leq \epsilon) = \widehat{\textsf{prob}}(x_i = x \mid |b_i - \wedge b_{-i}| \leq \epsilon) \frac{\widehat{\textsf{prob}}(i \text{ wins } \mid x_i = x, |b_i - \wedge b_{-i}| \leq \epsilon)}{\widehat{\textsf{prob}}(i \text{ wins } \mid |b_i - \wedge b_{-i}| \leq \epsilon)}$$
$$\widehat{\textsf{prob}}(x_i = x \mid i \text{ loses }, |b_i - \wedge b_{-i}| \leq \epsilon) = \widehat{\textsf{prob}}(x_i = x \mid |b_i - \wedge b_{-i}| \leq \epsilon) \frac{\widehat{\textsf{prob}}(i \text{ loses } \mid x_i = x, |b_i - \wedge b_{-i}| \leq \epsilon)}{\widehat{\textsf{prob}}(i \text{ loses } \mid |b_i - \wedge b_{-i}| \leq \epsilon)}.$$

Therefore,

$$\left|\widehat{\textsf{prob}}(x_i = x \mid i \text{ wins }, |b_i - \wedge b_{-i}| \leq \epsilon) - \widehat{\textsf{prob}}(x_i = x \mid i \text{ loses } |b_i - \wedge b_{-i}| \leq \epsilon)\right|$$
$$\leq \left|\frac{\widehat{\textsf{prob}}(i \text{ wins } \mid x_i = x, |b_i - \wedge b_{-i}| \leq \epsilon)}{\widehat{\textsf{prob}}(i \text{ wins } \mid |b_i - \wedge b_{-i}| \leq \epsilon)} - \frac{\widehat{\textsf{prob}}(i \text{ loses } \mid x_i = x, |b_i - \wedge b_{-i}| \leq \epsilon)}{\widehat{\textsf{prob}}(i \text{ loses } \mid |b_i - \wedge b_{-i}| \leq \epsilon)}\right|$$
$$= \left|\frac{\widehat{P}_{x,\epsilon}}{\sum_{x'\in X} \frac{|B_{x',\epsilon}|}{|B_\epsilon|}\widehat{P}_{x',\epsilon}} - \frac{1 - \widehat{P}_{x,\epsilon}}{1 - \sum_{x'\in X} \frac{|B_{x',\epsilon}|}{|B_\epsilon|}\widehat{P}_{x',\epsilon}}\right|.$$

It follows from Proposition OC.1 that for any $\eta' > 0$, there exists ϵ such that with probability 1 as $|B_\epsilon|$ grows large,

$$\frac{\widehat{P}_{x,\epsilon}}{\sum_{x'\in X}\frac{|B_{x',\epsilon}|}{|B_\epsilon|}\widehat{P}_{x',\epsilon}} - \frac{1-\widehat{P}_{x,\epsilon}}{1-\sum_{x'\in X}\frac{|B_{x',\epsilon}|}{|B_\epsilon|}\widehat{P}_{x',\epsilon}} \in \left[\frac{1/2-\eta'}{1/2+\eta'} - \frac{1/2+\eta'}{1/2-\eta'}, \frac{1/2+\eta'}{1/2-\eta'} - \frac{1/2-\eta'}{1/2+\eta'}\right].$$

By picking η' small enough, this implies that with probability approaching 1,

$$\left|\widehat{\mathsf{prob}}(x_i = x \mid i \text{ wins }, |b_i - \wedge b_{-i}| \leq \epsilon) - \widehat{\mathsf{prob}}(x_i = x \mid i \text{ loses }, |b_i - \wedge b_{-i}| \leq \epsilon)\right| \leq \eta.$$

□

References

Chassang, S., K. Kawai, J. Nakabayashi, and J. Ortner (2020): "Robust Screens for Non-Competitive Bidding in Procurement Auctions," .