Online Appendix to "Bargaining over Taxes and Entitlements in the Era of Unequal Growth"

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1 Quasilinear Case

1.1 Budget rules on private goods

Party's R Lagrangian for this problem at $t = 2$ is given by:

$$\mathcal{L} = c_{R,2} + \theta \ln(g_2) + \lambda \left[Y - c_{P,2} - c_{R,2} - g_2 \right] + \psi \left(c_{P,2} + \theta \ln(g_2) - \overline{c_P} - \ln(\overline{x}) \right)$$

The first-order and Kuhn-Tucker conditions party R are $c_{P,2}, c_{R,2}, g_2 \geq \overline{x}, \lambda, \psi \geq 0$ and

$$[c_{R,2}] \quad 1 - \lambda \leq 0.$$
$$(c_{R,2} - \bar{x}) \left[1 - \lambda \right] = 0. \tag{1}$$
$$[c_{P,2}] \quad \psi - \lambda \leq 0.$$
$$(c_{P,2} - \bar{x}) \left[\psi - \lambda \right] = 0. \tag{2}$$
$$[g_2] \quad \frac{\theta(1 + \psi)}{g_2} - \lambda \leq 0.$$
$$(g_2 - \bar{x}) \left[\frac{\theta(1 + \psi)}{g_2} - \lambda \right] = 0. \tag{3}$$
$$[RC] \quad \left[Y - c_{P,2} - c_{R,2} - g_2 \right] \geq 0.$$
$$\lambda \left[Y - c_{P,2} - c_{R,2} - g_2 \right] = 0. \tag{4}$$
$$[IRC] \quad \left[c_{P,2} + \ln(g_2) - \overline{c_P} - \ln(\overline{x}) \right] \geq 0.$$
$$\psi \left[c_{P,2} + \ln(g_2) - \overline{c_P} - \ln(\overline{x}) \right] = 0. \tag{5}$$

The solution to this problem depends on the status quo vector $\mathbf{s} = \{\overline{c_R}, \overline{c_P}\}$. Denote the solution by functions $\mathcal{C}_{i,R,2}(\mathbf{s})$ for private consumption of party i and $\mathcal{G}_{P,2}(\mathbf{s})$ for public goods.

First note that $\lambda > 0$, since $u(.)$ is increasing in its arguments. Also, since we assume $\bar{x}$ is relatively small relative to Y, $\mathcal{G}_{R,2}(\mathbf{s}) > \overline{x}$. We have four cases to consider:

- Case 1: $\psi = 0$. By eq. (2) we have that $\mathcal{C}_{P,R,2}(\mathbf{s}) = \overline{x}$. By eq. (1) and eq. (3) we have that $\mathcal{G}_{R,2}(\mathbf{s}) = \theta = g^D$. By eq. (4), we have that $\mathcal{C}_{R,R,2}(\mathbf{s}) = Y - \bar{x} - \theta$. By eq. 5, we see this case holds if and only if $\overline{c_P} < \bar{x} - \theta \ln \left(\frac{\theta}{\bar{x}} \right)$.

- Case 2: $\psi > 0$, $\mathcal{C}_{P,R,2}(\mathbf{s}) = \bar{x}$ and $\mathcal{C}_{R,R,2}(\mathbf{s}), \mathcal{G}_{R,2}(\mathbf{s}) > \bar{x}$. By eq. (5) we have that $\mathcal{G}_{R,2}(\mathbf{s}) = \bar{x}e^{\frac{\overline{c_P}-\bar{x}}{\theta}}$. By eq. (4), we have that $\mathcal{C}_{R,R,2}(\mathbf{s}) = Y - \bar{x} - e^{\frac{\overline{c_P}-\bar{x}}{\theta}}$. By eqs. (1)-(3) and the fact that $\mathcal{C}_{P,R,2}(\mathbf{s}) \geq \bar{x}$, this case holds if and only if $\bar{x} - \theta \ln\left(\frac{\theta}{\bar{x}}\right) \leq \overline{c_P} < \bar{x} + \theta \ln\left(\frac{2\theta}{\bar{x}}\right)$.

- Case 3: $\psi > 0$ and $\mathcal{C}_{i,R,2}(\mathbf{s}), \mathcal{G}_{R,2}(\mathbf{s}) > \bar{x}$. By eqs. (1)-(4), we have that $\mathcal{G}_{R,2}(\mathbf{s}) = 2\theta = g^*$. By eq. (5), we have that $\mathcal{C}_{P,R,2}(\mathbf{s}) = \overline{c_P} + \theta \ln\left(\frac{\bar{x}}{2\theta}\right)$. By eq. (4) we have that $\mathcal{C}_{R,R,2}(\mathbf{s}) = Y - 2\theta - \overline{c_P} - \theta \ln\left(\frac{\bar{x}}{2\theta}\right)$. This case holds if and only if $\bar{x} + \theta \ln\left(\frac{2\theta}{\bar{x}}\right) \leq \overline{c_P} \leq Y - 2\theta - \bar{x} - \theta \ln\left(\frac{\bar{x}}{2\theta}\right)$.

- Case 4: $\psi > 0$, and $\mathcal{C}_{R,R,2}(\mathbf{s}) = \bar{x}$, $\mathcal{C}_{P,R,2}(\mathbf{s}), \mathcal{G}_{R,2}(\mathbf{s}) > \bar{x}$. By eq. (4) and eq. (5), we have that $\mathcal{G}_{R,2}(\mathbf{s})$ solves

$$Y - \bar{x} - g_2 + \theta \ln g_2 - \overline{c_P} - \theta \ln(\bar{x}) = 0 \tag{6}$$

Note that by the implicit function theorem, $\mathcal{G}'_{R,2}(\mathbf{s}) = \frac{g_2}{\theta - g_2}$. Therefore, we have

$$\mathcal{G}_{R,2}(\mathbf{s}) = \begin{cases} \theta = g^D, & \text{if} \quad \overline{c_P} < \xi \\ \bar{x}e^{\frac{\overline{c_P}-\bar{x}}{\theta}}, & \text{if} \quad \overline{c_P} \in [\xi, \omega) \\ 2\theta = g^*, & \text{if} \quad \overline{c_P} \in [\omega, Y - 2\theta - \omega) \\ G_2(\overline{c_P}), & \text{if} \quad \overline{c_P} \geq Y - 2\theta - \omega \end{cases}$$

$$\mathcal{C}_{P,R,2}(\mathbf{s}) = \begin{cases} \bar{x}, & \text{if} \quad \overline{c_P} < \xi \\ \bar{x}, & \text{if} \quad \overline{c_P} \in [\xi, \omega) \\ \overline{c_P} + \theta \ln\left(\frac{\bar{x}}{2\theta}\right), & \text{if} \quad \overline{c_P} \in [\omega, Y - 2\theta - \omega) \\ Y - \bar{x} - G_2(\overline{c_P}), & \text{if} \quad \overline{c_P} \geq Y - 2\theta - \omega \end{cases}$$

where $\xi = \bar{x} - \theta \ln\left(\frac{\theta}{\bar{x}}\right)$, $\omega = \bar{x} + \theta \ln\left(\frac{2\theta}{\bar{x}}\right)$, $G_2(\overline{c_P})$ is given by eq. (6) and $\mathcal{C}_{R,R,2}(\mathbf{s})$ is given by eq. (4).

Now we follow back to period $t = 1$. It is convenient to first write continuation values. Let´s assume party R is in power at $t = 1$.

$$V_R(c_{P,1}) = \begin{cases} Y - \bar{x} - \theta + \theta \ln(\theta), & \text{if} \quad c_{P,1} < \xi \\ Y - \bar{x} - \bar{x}e^{\frac{c_{P,1}-\bar{x}}{\theta}} + \theta \ln(\theta) + \frac{c_{P,1}-\bar{x}}{\theta}, & \text{if} \quad c_{P,1} \in [\xi, \omega) \\ Y - c_{P,1} - \theta \ln(\bar{x}) - 2\theta + 2\theta \ln(2\theta), & \text{if} \quad c_{P,1} \in [\omega, Y - 2\theta - \omega) \\ \bar{x} + \theta \ln\left(G_2(c_{P,1})\right), & \text{if} \quad c_{P,1} \geq Y - 2\theta - \omega \end{cases}$$

and

$$W_R(c_{R,1}) = \begin{cases} \bar{x} + \theta \ln(\theta), & \text{if} \quad c_{R,1} < \xi \\ c_{R,1} + \theta \ln(\theta) + \frac{c_{R,1}-\bar{x}}{\theta}, & \text{if} \quad c_{R,1} \in [\xi, \omega) \\ c_{R,1} + \theta \ln(\bar{x}), & \text{if} \quad c_{R,1} \in [\omega, Y - 2\theta - \omega) \\ Y - \bar{x} - G_2(c_{R,1}) + \theta \ln\left(G_2(c_{R,1})\right), & \text{if} \quad c_{R,1} \geq Y - 2\theta - \omega \end{cases}$$

Note that the only relevant state for party R when in power is $c_{P,1}$ and for when out of power is $c_{R,1}$. Therefore, party's R Lagrangian for this problem at $t = 1$ when the acceptance constraint is not binding is given by:

$$\mathcal{L} = c_{R,1} + \theta \ln(g_1) + \beta \left[q V_R(c_{P,1}) + (1-q) W_R(c_{R,1})\right] + \lambda \left[Y - c_{R,1} - c_{P,1} - g_1\right]$$

The first-order and Kuhn-Tucker conditions party R are $c_{R,1}, c_{P,1}, g_1 \geq \bar{x}, \lambda, \psi \geq 0$ and

$$[c_{P,1}] \quad -\lambda + \beta q V_R'(c_{P,1}) \leq 0.$$
$$(c_{P,1} - \bar{x})\left[-\lambda + \beta q V_R'(c_{P,1})\right] = 0. \tag{7}$$
$$[c_{R,1}] \quad 1 - \lambda + \beta(1-q)W_R'(c_{R,1}) \leq 0.$$
$$(c_{R,1} - \bar{x})\left[1 - \lambda + \beta(1-q)W_R'(c_{R,1})\right] = 0. \tag{8}$$
$$[g_1] \quad \frac{\theta}{g_1} - \lambda \leq 0.$$
$$(g_1 - \bar{x})\left[\frac{\theta}{g_1} - \lambda\right] = 0. \tag{9}$$
$$[RC] \quad [Y - c_{R,1} - c_{P,1} - g_1] \geq 0.$$
$$\lambda[Y - c_{R,1} - c_{P,1} - g_1] = 0. \tag{10}$$

Since $u(.)$ is increasing in its arguments, $\lambda > 0$. Also, by eq. (8), we have that $\mathcal{C}_{P,1} = \bar{x}$. By eq. (7) and eq. (9), we have that

$$1 + \beta(1-q)W_R'(c_{R,1}) - \frac{\theta}{g_1} = 0 \tag{11}$$

Since $W_R'(c_{R,1}) = 0$ if $c_{R,1} < \xi$ and $W_R'(c_{R,1}) = 1$ otherwise, we have two cases to consider.

- Case 1: $c_{R,1} < \xi$. By eq. (11), we have that $g_1 = \theta = g^D$. By eq. (10), we have that $c_{R,1} = Y - \bar{x} - \theta$. For this case to hold, we would require $c_{R,1} = Y - \bar{x} - \theta < \xi = \bar{x} - \theta \ln\left(\frac{\theta}{\bar{x}}\right)$, which is a contradiction.

- Case 2: $c_{R,1} \geq \xi$. By eq. (11), we have that $g_1 = \frac{\theta}{1+\beta(1-q)} < \theta = g^D$. $c_{R,1} = Y - \bar{x} - \theta$ can be found by eq. (10).

Define g_1^c as the solution for the above problem, in which mandatory spending is on private goods.

1.2   Budget rules on public goods

Party's R Lagrangian for this problem at $t = 2$ is given by:

$$\mathcal{L} = c_{R,2} + \theta \ln(g_2) + \lambda\left[Y - c_{P,2} - c_{R,2} - g_2\right] + \psi\left(c_{P,2} + \theta \ln(g_2) - \bar{x} - \theta \ln(\bar{g})\right)$$

The first-order and Kuhn-Tucker conditions party R are $c_{P,2}, c_{R,2}, g_2 \geq \bar{x}, \lambda, \psi \geq 0$ and

$$[c_{R,2}] \quad 1 - \lambda \leq 0.$$
$$(c_{R,2} - \bar{x})[1 - \lambda] = 0. \tag{12}$$
$$[c_{P,2}] \quad \psi - \lambda \leq 0.$$
$$(c_{P,2} - \bar{x})[\psi - \lambda] = 0. \tag{13}$$
$$[g_2] \quad \frac{\theta(1 + \psi)}{g_2} - \lambda \leq 0.$$
$$(g_2 - \bar{x})\left[\frac{\theta(1 + \psi)}{g_2} - \lambda\right] = 0. \tag{14}$$
$$[RC] \quad [Y - c_{P,2} - c_{R,2} - g_2] \geq 0.$$
$$\lambda[Y - c_{P,2} - c_{R,2} - g_2] = 0. \tag{15}$$
$$[IRC] \quad [c_{P,2} + \theta \ln(g_2) - \bar{x} - \theta \ln(\bar{g})] \geq 0.$$
$$\psi[c_{P,2} + \theta \ln(g_2) - \bar{x} - \theta \ln(\bar{g})] = 0. \tag{16}$$

The solution to this problem depends on the status quo $\mathbf{s} = \{\bar{g}\}$. Denote the solution by functions $\mathcal{C}_{i,R,2}(\mathbf{s})$ for private consumption of party i and $\mathcal{G}_{P,2}(\mathbf{s})$ for public goods.

Since $u(.)$ is increasing in its arguments, $\lambda > 0$. We have four cases to consider:

- Case 1: $\psi = 0$. By eq. (13) we have that $\mathcal{C}_{P,R,2}(\mathbf{s}) = \bar{x}$. By eq. (12) and eq. (14) we have that $\mathcal{G}_{R,2}(\mathbf{s}) = \theta = g^D$. By eq. (15), we have that $\mathcal{C}_{R,R,2}(\mathbf{s}) = Y - \bar{x} - \theta$. By eq. 16, we see this case holds if and only if $\bar{g} < \theta = g^D$.

- Case 2: $\psi > 0$, $\mathcal{C}_{P,R,2}(\mathbf{s}) = \bar{x}$ and $\mathcal{C}_{R,R,2}(\mathbf{s}), \mathcal{G}_{R,2}(\mathbf{s}) > \bar{x}$. By eq. (16) we have that $\mathcal{G}_{R,2}(\mathbf{s}) = \bar{g}$. By eq. (15), we have that $\mathcal{C}_{R,R,2}(\mathbf{s}) = Y - \bar{x} - \bar{g}$. By eq. (12), eq. (14), eq. (2) and the fact that $\mathcal{C}_{P,R,2}(\mathbf{s}) \geq \bar{x}$, this case holds if and only if $g^D \leq \bar{g} \leq 2\theta = g^*$.

- Case 3: $\psi > 0$ and $\mathcal{C}_{i,R,2}(\mathbf{s}), \mathcal{G}_{R,2}(\mathbf{s}) > \bar{x}$. By eqs. (12)-(15), we have that $\mathcal{G}_{R,2}(\mathbf{s}) = 2\theta = g^*$. By eq. (16), we have that $\mathcal{C}_{P,R,2}(\mathbf{s}) = \bar{x} + \theta \ln\left(\frac{\bar{g}}{2\theta}\right)$. By eq. (15) we have that $\mathcal{C}_{R,R,2}(\mathbf{s}) = Y - \bar{x} - 2\theta - \theta \ln\left(\frac{\bar{g}}{2\theta}\right)$. This case holds if and only if $g^* < \bar{g} \leq \theta e^{\left(\frac{Y - 2(\bar{x} + \theta)}{\theta}\right)}$.

- Case 4: $\psi > 0$, and $\mathcal{C}_{R,R,2}(\mathbf{s}) = \bar{x}$, $\mathcal{C}_{P,R,2}(\mathbf{s}), \mathcal{G}_{R,2}(\mathbf{s}) > \bar{x}$. By eq. (15) and eq. (16), we have that $\mathcal{G}_{R,2}(\mathbf{s})$ solves

$$Y - \bar{x} - g_2 + \theta \ln g_2 - \bar{x} - \theta \ln \bar{g} = 0 \tag{17}$$

Note that by the implicit function theorem, $\mathcal{G}'_{R,2}(\mathbf{s}) = \frac{\theta}{\bar{g}} \frac{g_2}{\theta - g_2}$. Therefore, we have

$$
\mathcal{G}_{R,2}(\mathbf{s}) = \begin{cases}
\theta = g^D, & \text{if} \quad \bar{g} < g^D \\
\bar{g}, & \text{if} \quad \bar{g} \in [g^D, g^*) \\
2\theta = g^*, & \text{if} \quad \bar{g} \in \left[g^*, \theta e^{\left(\frac{Y - 2(\bar{x} + \theta)}{\theta}\right)}\right) \\
G_2(\bar{g}), & \text{if} \quad \bar{g} \geq, \theta e^{\left(\frac{Y - 2(\bar{x} + \theta)}{\theta}\right)}
\end{cases}
$$

$$
\mathcal{C}_{P,R,2}(\mathbf{s}) = \begin{cases}
\bar{x}, & \text{if} \quad \bar{g} < g^D \\
\bar{x}, & \text{if} \quad \bar{g} \in [g^D, g^*) \\
\bar{x} + \theta \ln\left(\frac{\bar{g}}{2\theta}\right), & \text{if} \quad \bar{g} \in \left[g^*, \theta \ln\left(\frac{Y - 2(\bar{x} + \theta)}{\theta}\right)\right) \\
Y - \bar{x} - G_2(\bar{g}), & \text{if} \quad \bar{g} \geq \theta \ln\left(\frac{Y - 2(\bar{x} + \theta)}{\theta}\right)
\end{cases}
$$

where $G_2(\mathbf{s})$ is given by eq. (17) and $\mathcal{C}_{R,R,2}(\mathbf{s})$ is given by eq. (15).

Now we follow back to period $t = 1$. It is convenient to first write continuation values:

$$
V_R(\mathbf{s}) = \begin{cases}
Y - \bar{x} - \theta + \theta \ln(\theta), & \text{if} \quad \overline{c_P} < g_i^D \\
Y - \bar{x} - \bar{g} + \theta \ln(\bar{g}), & \text{if} \quad \overline{c_P} \in [g_i^D, g^*) \\
Y - \bar{x} - \theta \ln(\bar{g}) - 2\theta (1 - \ln(2\theta)), & \text{if} \quad \overline{c_P} \in \left[g^*, \theta e^{\left(\frac{Y - 2(\bar{x} + \theta)}{\theta}\right)}\right) \\
\bar{x} + \theta \ln(G_2(\mathbf{s})), & \text{if} \quad \overline{c_P} \geq, \theta e^{\left(\frac{Y - 2(\bar{x} + \theta)}{\theta}\right)}
\end{cases}
$$

$$
W_R(\mathbf{s}) = \begin{cases}
\bar{x} + \theta \ln(\theta), & \text{if} \quad \overline{c_P} < g_i^D \\
\bar{x} + \theta \ln(\bar{g}), & \text{if} \quad \overline{c_P} \in [g_i^D, g^*) \\
\bar{x} + \theta \ln(\bar{g}), & \text{if} \quad \overline{c_P} \in \left[g^*, \theta e^{\left(\frac{Y - 2(\bar{x} + \theta)}{\theta}\right)}\right) \\
Y - \bar{x} - G_2(\mathbf{s}) + \theta \ln(G_2(\mathbf{s})), & \text{if} \quad \overline{c_P} \geq, \theta e^{\left(\frac{Y - 2(\bar{x} + \theta)}{\theta}\right)}
\end{cases}
$$

$$(18)$$

Party's R Lagrangian for this problem at $t = 1$ when the individual rationality constraint is not binding is given by:

$$
\mathcal{L} = c_{R,1} + \theta \ln(g_1) + \beta [q V_R(g_1) + (1 - q) W_R(g_1)] + \lambda [Y - c_{P,1} - c_{R,1} - g_1]
$$

The first-order and Kuhn-Tucker conditions party R are $c_{P,1}, c_{R,1}, g_1 \geq \bar{x}, \lambda, \psi \geq 0$ and

$$
\begin{aligned}
&[c_{R,1}] \quad 1 - \lambda \leq 0. \\
&\qquad (c_{R,1} - \bar{x}) [1 - \lambda] = 0.
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&[c_{P,1}] \quad -\lambda \leq 0. \\
&\qquad (c_{P,1} - \bar{x}) [-\lambda] = 0.
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&[g_1] \quad \frac{\theta}{g_1} - \lambda + \beta q V'_R(g_1) + \beta(1 - q) W'_R(g_1) \leq 0. \\
&\qquad (g_1 - \bar{x}) \left[\frac{\theta}{g_1} - \lambda + \beta q V'_R(g_1) + \beta(1 - q) W'_R(g_1)\right] = 0.
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&[RC] \quad [Y - c_{P,1} - c_{R,1} - g_1] \geq 0. \\
&\qquad \lambda [Y - c_{P,1} - c_{R,1} - g_1] = 0.
\end{aligned}
\tag{22}
$$

5

Since $u(.)$ is increasing in its arguments, $\lambda > 0$. Also, by eq. (20), we have that $\mathcal{C}_{P,R,1} = \bar{x}$. By eq. (19) and eq. (21), we have that

$$\frac{\theta}{g_1} + \beta q V_R'(g_1) + \beta(1-q)W_R'(g_1) - 1 = 0 \tag{23}$$

$W_R'(c_{R,1}) = 0$ if $g_1 < \theta$, $W_R'(g_1) = \frac{\theta}{g_1}$ otherwise, $V_R'(g_1) = 0$ if $g_1 < \theta$, $V_R'(g_1) = 0$ if $g_1 < \theta$, $V_R'(g_1) = \frac{\theta}{g_1} - 1$ if $g_1 \in [\theta, 2\theta)$, $V_R'(g_1) = -\frac{\theta}{g_1}$ if $g_1 \in \left[2\theta, \theta e^{\left(\frac{Y-2(\bar{x}+\theta)}{\theta}\right)}\right)$ and $V_R'(g_1) = \frac{\theta^2}{g_1(\theta G_2(g_1)}$ if $g_1 \geq \theta e^{\left(\frac{Y-2(\bar{x}+\theta)}{\theta}\right)}$ we have four cases to consider.

- Case 1: $g_1 < \theta$. By eq. (23), we have that $g_1 = \theta = g_i^D$, a contradiction.

- Case 2: $g_1 [\theta, 2\theta)$. By eq. (23), we have that $g_1 = \frac{(1+\beta)\theta}{1+\beta q}$. This case holds if and only if $p \geq \frac{\beta-1}{2\beta}$ which is always true, since $p \in [0,1]$ and $\beta \in [0,1]$.

- Case 3: $g_1 \in \left[2\theta, \theta e^{\left(\frac{Y-2(\bar{x}+\theta)}{\theta}\right)}\right)$. By eq. (23), we have that $g_1 = (1 + \beta(1-2q))\theta$. This case holds if and only if $p < \frac{\beta-1}{2\beta}$, which never holds.

- Case 4: $g_1 \geq \theta e^{\left(\frac{Y-2(\bar{x}+\theta)}{\theta}\right)}$. Eq. (23) and eq. (17) form a system of two non-linear equations that solve jointly g_1 and g_2.

Define g_1^g as the solution for the above problem, in which mandatory spending is on the public good.

2 Budget rules on private goods – concave case

The Lagrangian of the transformed problem of proposer P in the second period in the generic θ case is given by

$$\mathcal{L} = \ln(c_{P,2}) + \theta \ln(g_2) + \lambda\left[Y - c_{P,2} - c_{R,2} - g_2\right] + \psi\left(\ln(c_{R,2}) + \theta \ln(g_2) - \ln(\bar{c}_R) - \theta \ln(\bar{x})\right)$$

The first-order and Kuhn-Tucker conditions are $c_{P,2}, c_{R,2}, g_2 \geq \bar{x}$, $\lambda, \psi \geq 0$ and

$$[c_{P,2}] \quad \frac{1}{c_{P,2}} - \lambda \leq 0.$$

$$(c_{P,2} - \bar{x})\left[\frac{1}{c_{P,2}} - \lambda\right] = 0. \tag{24}$$

$$[c_{R,2}] \quad \frac{\psi}{c_{R,2}} - \lambda \leq 0.$$

$$(c_{R,2} - \bar{x})\left[\frac{\psi}{c_{R,2}} - \lambda\right] = 0. \tag{25}$$

$$[g_2] \quad \frac{(1+\psi)\theta}{g_2} - \lambda \leq 0.$$

$$(g_2 - \bar{x})\left[\frac{(1+\psi)\theta}{g_2} - \lambda\right] = 0. \tag{26}$$

$$[RC] \quad [Y - c_{P,2} - c_{R,2} - g_2] \geq 0.$$

$$\lambda\left[Y - c_{P,2} - c_{R,2} - g_2\right] = 0. \tag{27}$$

$$[IRC] \quad [\ln(c_{R,2}) + \theta \ln(g_2) - \ln(\overline{c_R}) - \theta \ln(\bar{x})] \geq 0.$$

$$\psi\left[\ln(c_{R,2}) + \theta \ln(g_2) - \ln(\overline{c_R}) - \theta \ln(\bar{x})\right] = 0. \tag{28}$$

The solution to this problem depends on the status quo vector $\mathbf{s} = \{\overline{c_R}, \overline{c_P}\}$. Denote the solution by functions $\mathcal{C}_{i,R,2}(\mathbf{s})$ for private consumption of party i, $\mathcal{G}_{P,2}(\mathbf{s})$ for public goods, and $\Psi_{P,2}(\mathbf{s})$ and $\mathcal{E}_{P,2}(\mathbf{s})$ for policy variables.

First note that $\lambda > 0$, since $u(.)$ is increasing in its arguments. We have four cases to consider:

- Case 1: $\psi = 0$. By eq. (25) we have that $\mathcal{C}_{R,P,2}(\mathbf{s}) = \bar{x}$. By eqs. (24)-(27), we have that $\mathcal{C}_{P,P,2}(\mathbf{s}) = \frac{Y-\bar{x}}{1+\theta}$ and $\mathcal{G}_{P,2}(\mathbf{s}) = \frac{\theta(Y-\bar{x})}{1+\theta}$. By eq. (28), this case holds if and only if $\overline{c_R} < \bar{x}\frac{\xi(Y-\bar{x})^\theta}{Y^\theta}$, where $\xi = \frac{(\theta Y)^\theta}{\bar{x}^\theta(1+\theta)^\theta}$. Using P's budget constraint to solve for entitlement transfers delivers $\mathcal{E}_{P,2}(\mathbf{s}) = \frac{\Delta_P - \bar{x}}{1+\theta} = e_P^D$, where $\Delta_P = y_R - \theta y_P$. The expression for $\Psi_{P,2}(\mathbf{s})$ can be obtained from the budget constraint of rich agents. In terms of policies, this case holds if and only if $\overline{\tau} > y_R - \bar{x}\frac{\xi(Y-\bar{x})^\theta}{Y^\theta}$.

- Case 2: $\psi > 0$, $\mathcal{C}_{R,P,2}(\mathbf{s}) = \bar{x}$ and $\mathcal{C}_{P,P,2}(\mathbf{s}), \mathcal{G}_{P,2}(\mathbf{s}) > \bar{x}$. By eq. (28) we have that $\mathcal{G}_{P,2}(\mathbf{s}) = \bar{x}\left(\frac{\overline{c_R}}{\bar{x}}\right)^{\frac{1}{\theta}}$. By eq. (27), we have that $\mathcal{C}_{P,P,2}(\mathbf{s}) = Y - \bar{x} - \bar{x}\left(\frac{\overline{c_R}}{\bar{x}}\right)^{\frac{1}{\theta}}$. By eqs. (24)-(26) and the fact that $\mathcal{C}_{R,P,2}(\mathbf{s}) = \bar{x}$, this case holds if and only if $y_R - \bar{x}\frac{\xi(Y-\bar{x})^\theta}{Y^\theta} \leq \overline{c_R} < \bar{x}\xi$. From P's budget constraint, we have that $\mathcal{E}_{P,2}(\mathbf{s}) = y_R - \bar{x}\left(1 + \left(\frac{\overline{c_R}}{\bar{x}}\right)\right)^{\frac{1}{\theta}}$. From the rich's budget constraint, $\Psi_{P,2}(\mathbf{s}) = \tau_P^D$. In terms of policies, this case holds if and only if $y_R - \bar{x}\xi \leq \overline{\tau} < y_R - \bar{x}\frac{\xi(Y-\bar{x})^\theta}{Y^\theta}$.

- Case 3: $\psi > 0$ and $\mathcal{C}_{i,P,2}(\mathbf{s}), \mathcal{C}_{j,P,2}(\mathbf{s}), \mathcal{G}_{P,2}(\mathbf{s}) > \bar{x}$. By eqs. (24)-(27), we have that $\mathcal{G}_{P,2}(\mathbf{s}) = \frac{\theta Y}{1+\theta} = g^*$. By eq. (28), we have that $\mathcal{C}_{R,P,2}(\mathbf{s}) = \overline{c_R}\frac{1}{\xi}$ and from eq. (27) we have that $\mathcal{C}_{P,P,2}(\mathbf{s}) = \frac{Y}{1+\theta} - \overline{c_R}\frac{1}{\xi}$. This case holds if and only if $\mathcal{C}_{P,P,2}(\mathbf{s}) > \bar{x}$, which implies $\overline{c_R} < \omega\xi$, where $\omega = \frac{Y}{1+\theta} - \bar{x}$. From the rich's budget constraint, we have that $\Psi_{P,2}(\mathbf{s}) = y_R - \overline{c_R}\xi$. The expression for $\mathcal{E}_{P,2}(\mathbf{s})$ follows by using P's budget constraint. In terms of policies, this case holds if and only if $\overline{\tau} > y_R - \omega\xi$.

- Case 4: $\psi > 0$ and $\mathcal{C}_{i,P,2}(\mathbf{s}) = \bar{x}$. By eq. (27) and eq. (28), we have that g_2 is the solution for

$$\ln(Y - \bar{x} - g_2) + \theta \ln(g_2) - \ln(c_{j,1}) - \theta \ln(\bar{x}) = 0 \tag{29}$$

By P's budget constraint we have that $\mathcal{E}_{P,2}(\mathbf{s}) = \bar{x} - y_P = e_R^D$. $\Psi_{P,2}(\mathbf{s}) = \tau_P^D$ can be obtained from the budget constraint of rich agents. This case holds if and only if $\bar{\tau} \leq y_R - \omega \xi$.

Therefore, we have

$$\mathcal{E}_{P,2}(\mathbf{s}) = \begin{cases} e_R^D, & \text{if} \quad \bar{\tau} < y_R - \omega\xi \\ e_P^D + \frac{\bar{x}}{1+\theta} - (y_R - \bar{\tau})\xi, & \text{if} \quad \bar{\tau} \in [y_R - \omega\xi, y_R - \bar{x}\xi) \\ y_R - \bar{x}\left(1 + \left(\frac{y_R - \bar{\tau}}{\bar{x}}\right)^{\frac{1}{\theta}}\right), & \text{if} \quad \bar{\tau} \in \left[y_R - \bar{x}\xi, y_R - \bar{x}\xi\frac{(Y-\bar{x})^\theta}{Y}\right) \\ e_P^D, & \text{if} \quad \bar{\tau} \geq y_R - \bar{x}\xi\frac{(Y-\bar{x})^\theta}{Y} \end{cases}$$

$$\Psi_{P,2}(\mathbf{s}) = \begin{cases} \bar{x} + G_2(\bar{\tau}) - y_P, & \text{if} \quad \bar{\tau} < y_R - \omega\xi \\ y_R - (y_R - \bar{\tau})\frac{\bar{x}}{\xi}, & \text{if} \quad \bar{\tau} \in [y_R - \omega\xi, y_R - \bar{x}\xi) \\ \tau_P^D, & \text{if} \quad \bar{\tau} \in \left[y_R - \bar{x}\xi, y_R - \bar{x}\xi\frac{(Y-\bar{x})^\theta}{Y}\right) \\ \tau_P^D, & \text{if} \quad \bar{\tau} \geq y_R - \bar{x}\xi\frac{(Y-\bar{x})^\theta}{Y} \end{cases}$$

and

$$\mathcal{G}_{P,2}(\mathbf{s}) = \begin{cases} G_2(\bar{\tau}), & \text{if} \quad \bar{\tau} < y_R - \omega\xi \\ g^*, & \text{if} \quad \bar{\tau} \in [y_R - \omega\xi, y_R - \bar{x}\xi) \\ \bar{x}\left(\frac{y_R - \bar{\tau}}{\bar{x}}\right)^{\frac{1}{\theta}}, & \text{if} \quad \bar{\tau} \in \left[y_R - \bar{x}\xi, y_R - \bar{x}\xi\frac{(Y-\bar{x})^\theta}{Y}\right) \\ g^D, & \text{if} \quad \bar{\tau} \geq y_R - \bar{x}\xi\frac{(Y-\bar{x})^\theta}{Y} \end{cases}$$

with $G_2(s)$ is the unique solution to eq. (29).
The associated private consumption allocations are

$$\mathcal{C}_{P,P,2}(s) = y_P + \mathcal{E}_{P,2}(s) \quad \text{and} \quad \mathcal{C}_{R,P,2}(s) = y_R - \Psi_{P,2}(s).$$

In the first-period, assuming the case in which the individual rationality constraint is not binding, party's P Lagrangian for this problem at $t = 1$ is given by:

$$\mathcal{L} = \ln(c_{P,1}) + \ln(g_1) + \beta\left[q V_P(c_{R,1}) + (1-q) W_P(c_{P,1})\right] + \lambda\left[Y - c_{P,1} - c_{R,1} - g_1\right]$$

The first-order and Kuhn-Tucker conditions party P are $c_{P,1}, c_{R,1}, g_1 \geq \bar{x}, \lambda, \psi \geq 0$ and

$$[c_{P,1}] \quad \frac{1}{c_{P,1}} + \beta(1-q)\frac{dW^P(c_{P,1})}{dc_{P,1}} - \lambda \leq 0.$$

$$(c_{P,1} - \bar{x})\left[\frac{1}{c_{P,1}} + \beta(1-q)\frac{dW^P(c_{P,1})}{dc_{P,1}} - \lambda\right] = 0. \tag{30}$$

$$[c_{R,1}] \quad \beta q \frac{dV^P(c_{R,1})}{dc_{R,1}} - \lambda \leq 0.$$

$$(c_{R,1} - \bar{x})\left[\beta q \frac{dV^P(c_{R,1})}{dc_{R,1}} - \lambda\right] = 0. \tag{31}$$

$$[g_1] \quad \frac{\theta}{g_1} - \lambda \leq 0.$$

$$(g_1 - \bar{x})\left[\frac{\theta}{g_1} - \lambda\right] = 0. \tag{32}$$

$$[RC] \quad [Y - c_{P,1} - c_{R,1} - g_1] \geq 0.$$

$$\lambda\left[Y - c_{P,1} - c_{R,1} - g_1\right] = 0. \tag{33}$$

where

$$W_P(c_{P,1}) = \begin{cases} \ln(\bar{x}) + \theta \ln\left(\frac{\theta(Y-\bar{x})}{1+\theta}\right), & \text{if} \quad c_{P,1} < \frac{\bar{x}\xi(Y-\bar{x})^\theta}{Y^\theta}. \\ \ln(c_{P,1}) + \theta \ln(\bar{x}), & \text{if} \quad c_{P,1} \in \left[\frac{\bar{x}\xi(Y-\bar{x})^\theta}{Y^\theta}, \omega\xi\right) \\ \ln(Y - \bar{x} - G_2(c_{P,1})) + \theta \ln(G_2(c_{P,1})), & \text{if} \quad c_{P,1} \geq \omega\xi \end{cases}$$

Note that the only relevant state for party P when in power is $c_{R,1}$ and for when out of power is $c_{P,1}$. By eq. (31), we have that $c_{R,1} = \bar{x}$.

Since $u(.)$ is increasing in its arguments, $\lambda > 0$. By eq. (33), we have that $c_{P,1} = Y - \bar{x} - g_1$. Also, since $W_P'(c_{P,1}) = 0$ if $c_{P,1} < \bar{x}\frac{\xi(Y-\bar{x})^\theta}{Y^\theta}$ and $W_P'(c_{P,1}) = \frac{1}{c_{P,1}}$ otherwise. Therefore, we have two cases to consider:

- Case 1: By eq. (30) and eq. (9) we have that $g = \frac{\theta(Y-\bar{x})}{1+\theta} = g^D$ and $c_{P,1} = \frac{Y-\bar{x}}{1+\theta} = c_{i,i}^D \geq \bar{x}\frac{\xi(Y-\bar{x})^\theta}{Y^\theta}$, a contradiction. Therefore, this case never holds.

- Case 2: By eq. (30) and eq. (32) we have that $g = \frac{\theta(Y-\bar{x})}{1+\theta+\beta(1-q)} = \frac{1+\theta}{1+\theta+\beta(1-q)}g^D \leq g^D$ and $c_{P,1} = \frac{(1+\beta(1-q))(Y-\bar{x})}{1+\theta+\beta(1-p)} = \frac{(1+\beta(1-q))(1+\theta)}{1+\theta+\beta(1-q)}c_{i,i}^D \geq c_{i,i}^D$. By P's budget constraint, we have that $\mathcal{E}_{P,1} = \frac{\Delta_P - \bar{x} + \beta(1-q)(y_P - \bar{x})}{1+\theta+\beta(1-q)} = \frac{\beta}{1+\theta+\beta(1-q)}\left(\beta\tau_P^D + (1+\theta)\mathcal{E}_P^D\right)$. Using the rich's budget constraint, we have that $\tau_{P,1} = y_R - \bar{x} = \tau_P^D$.

  It is worth noting that, since the problem for the first-period considers a case in which the individual rationality constraint is not binding, the respondent R will always receive the constitutional minimum $\bar{x}$, which is the value R would receive if P is in power with all spending discretionary. This implies a tax policy that also follows the case in which all fiscal instruments are discretionary. The political wedge created by the mandatory spending on private goods is manifested in the distortion of spending between public goods and entitlements, as we can see above. Even when the individual rationality constraint is not binding, P will always provide a public good level such that $g_{P,1} \leq g^D$ and $c_{P,1} \geq c_{i,i}^D$.