Online Appendix

A Appendix: Theoretical analysis

A.1 Proof of Lemma 1

Proof. Let $\beta^t \Pr\left(s^t\right) M\left(s^t\right) \mathfrak{N}\left(s^t\right)$ be the vector of Lagrange multipliers on constraints (5) and let $\mathbb{W}_{t,s_k}\left(V_{t+1}\right)$ be the derivative of $\mathbb{W}_t$ with respect to $V_{t+1}\left(s^t, s_k\right)$ for any s_k. The optimality condition for $b_t^i\left(s^t\right)$ in any competitive equilibrium can be written as

$$\left[\left(\frac{\mathbb{W}_{0,s_1} \times \ldots \times \mathbb{W}_{t-1,s_t}}{\Pr\left(s^t\right) M_t\left(s^t\right)}\right) \frac{\partial U_t\left(s^t\right)}{\partial\left(Q_t^i b_t^i\right)} + \mathfrak{N}\left(s^t\right) \cdot \frac{\partial \varphi_t\left(s^t\right)}{\partial\left(Q_t^i b_t^i\right)} - 1\right] + \beta \sum_{s^{t+1}|s^t} \Pr\left(s^{t+1}|s^t\right) \frac{M_{t+1}\left(s^{t+1}\right)}{M_t\left(s^t\right)} R_{t+1}^i\left(s^{t+1}\right) = 0, \tag{39}$$

where $s^t = (s_1, .., s_t)$. When government securities are perfect substitutes, $\frac{\partial U_t\left(s^t\right)}{\partial\left(Q_t^i b_t^i\right)}$ and $\frac{\partial \varphi_t\left(s^t\right)}{\partial\left(Q_t^i b_t^i\right)}$ are the same for all $i \in \mathcal{G}_t$. Therefore, this equation implies that $A_t^i = A_t^0$ and

$$\mathbb{E}_t \frac{\beta M_{t+1}}{M_t} r_{t+1}^j = 0 \text{ for all } t, j \in \mathcal{G}_t \tag{40}$$

in any competitive equilibrium. □

A.2 Proofs for Section 3.2

Let $x_t(\sigma)$ be any equilibrium variable in the σ-economy. We use second order Taylor expansions of the equilibrium conditions with respect to σ around $\sigma = 0$. Let $\overline{x}_t, \partial_\sigma x_t, \partial_{\sigma\sigma} x_t$ be the zeroth-, first- and second-order terms in these expansions. In this notation,

$$x_t(\sigma) \simeq \overline{x}_t + \sigma \partial_\sigma x_t + \frac{\sigma^2}{2} \partial_{\sigma\sigma} x_t, \quad x_t(\sigma) \approx \overline{x}_t.$$

Note that the statement that $x_t \approx 0$ is equivalent to $\overline{x}_t = 0$. We first show several preliminary results that will be used throughout this section.

Lemma 2. *In the optimal equilibrium in the benchmark economy, $\overline{r}_{T+1}^j = \mathbb{E}_T \partial_\sigma r_{T+1}^j = 0$ for all $T, j \in \mathcal{G}_T$, which also implies that*

$$\overline{\mathcal{Q}}_{T+1,t} = \overline{Q}_{T+1,t}, \quad \mathbb{E}_{T+1} \partial_\sigma \mathcal{Q}_{T+1,t} = \mathbb{E}_{T+1} \partial_\sigma Q_{T+1,t} \textit{ for } T, t. \tag{41}$$

Proof. We first show that $\overline{r}^j_{T+1} = \mathbb{E}_T \partial_\sigma r^j_{T+1} = 0$ for all $T, j \in \mathcal{G}_T$ under conditions of the lemma. The zeroth order expansion of equation (12) for $t = 1$ is

$$\overline{M}_{T+1} \frac{\overline{r}^j_{T+1}}{\overline{\xi}_{T+1}} = 0 \text{ for all } T, j \in \mathcal{G}_T. \tag{42}$$

Neither $\overline{M}_{T+1}$ nor $\overline{\xi}_{T+1}$ can be zero, which implies that $\overline{r}^j_{T+1} = 0$. Using this result, the first-order approximation of equation (12) for $t = 1$ is

$$\overline{M}_{T+1} \frac{\overline{r}^j_{T+1}}{\overline{\xi}_{T+1}} \mathbb{E}_T \partial_\sigma r^j_{T+1} = 0 \, \text{for all } T, j \in \mathcal{G}_T,$$

which implies that $\mathbb{E}_T \partial_\sigma r^j_{T+1} = 0$. Applying this result to first-order expansions of $Q_{t,k}$ and $\mathcal{Q}_{t,k}$ in equation (8) gives (41). □

The previous lemma also implies the following useful corollary.

Corollary 7. *For any equilibrium variables x_t, z'_t, z''_t in the optimal equilibrium in the benchmark economy, the following relationship holds for any T, $j \in \mathcal{G}_T$.*

$$\mathbb{E}_T \left[z'_{T+1} z''_{T+1} \right] cov_T \left(x_{T+1}, r^j_{T+1} \right) \simeq \mathbb{E}_T \left[z'_{T+1} \right] \mathbb{E}_T \left[z''_{T+1} \right] cov_T \left(x_{T+1}, r^j_{T+1} \right) \simeq \overline{z}'_{T+1} \overline{z}''_{T+1} \mathbb{E}_T \partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1}.$$

Proof. Using Lemma 2, we have

$$\mathbb{E}_T \left[x_{T+1} r^j_{T+1} \right] \simeq \mathbb{E}_T \left[\partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1} + \frac{1}{2} \overline{x}_{T+1} \partial_{\sigma\sigma} r^j_{T+1} \right], \; \left(\mathbb{E}_T x_{T+1} \right) \left(\mathbb{E}_T r^j_{T+1} \right) \simeq \mathbb{E}_T \left[\frac{1}{2} \overline{x}_{T+1} \partial_{\sigma\sigma} r^j_{T+1} \right],$$

and, therefore,

$$cov_T \left(x_{T+1}, r^j_{T+1} \right) = \left[\mathbb{E}_T x_{T+1} r^j_{T+1} - \mathbb{E}_T x_{T+1} \mathbb{E}_T r^j_{T+1} \right] \simeq \mathbb{E}_T \partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1}.$$

Since $\overline{cov_T \left(x_{T+1}, r^j_{T+1} \right)} = \partial_\sigma cov_T \left(x_{T+1}, r^j_{T+1} \right) = 0$, we obtain

$$\begin{aligned} \mathbb{E}_T \left[z'_{T+1} z''_{T+1} \right] cov_T \left(x_{T+1}, r^j_{T+1} \right) &\simeq \overline{z}'_{T+1} \overline{z}''_{T+1} \mathbb{E}_T \partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1}, \\ \mathbb{E}_T \left[z'_{T+1} \right] \mathbb{E}_T \left[z''_{T+1} \right] cov_T \left(x_{T+1}, r^j_{T+1} \right) &\simeq \overline{z}'_{T+1} \overline{z}''_{T+1} \mathbb{E}_T \partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1}. \end{aligned}$$

□

Lemma 3. *Equation (14) holds in the optimal equilibrium of the benchmark economy.*

Proof. The second-order expansion of (13), invoking results of Lemma 2, gives

$$\begin{aligned} 0 &= \frac{1}{2} \left[\frac{\overline{M_{T+t}}}{Q^{pvt}_{T+1,t-1}} \right] \mathbb{E}_T \partial_{\sigma\sigma} r^j_{T+1} + \left[\frac{\overline{M_{T+t}}}{Q^{pvt}_{T+1,t-1}} \right] \mathbb{E}_T \partial_\sigma \ln M_{T+t} \partial_\sigma r^j_{T+1} - \left[\frac{\overline{M_{T+t}}}{Q^{pvt}_{T+1,t-1}} \right] \mathbb{E}_T \partial_\sigma \ln Q^{pvt}_{T+1,t-1} \partial_\sigma r^j_{T+1} \\ &= \frac{1}{2} \mathbb{E}_T \partial_{\sigma\sigma} r^j_{T+1} + \mathbb{E}_T \partial_\sigma \ln M_{T+t} \partial_\sigma r^j_{T+1} - \mathbb{E}_T \partial_\sigma \ln Q^{pvt}_{T+1,t-1} \partial_\sigma r^j_{T+1}. \end{aligned}$$

Similarly, the second-order expansion of (12) gives

$$0 = \frac{1}{2}\mathbb{E}_T \partial_{\sigma\sigma} r^j_{T+1} + \mathbb{E}_T \partial_\sigma \ln M_{T+t} \partial_\sigma r^j_{T+1} - \mathbb{E}_T \partial_\sigma \ln Q_{T+1,t-1} \partial_\sigma r^j_{T+1} - \mathbb{E}_T \partial_\sigma \ln \xi_{T+1} \partial_\sigma r^j_{T+1}.$$

Combine these two equations and use the fact that $\ln A_{T+1,t-1} = \ln Q_{T+1,t-1} - \ln Q^{pvt}_{T+1,t-1}$ to get

$$\mathbb{E}_T \partial_\sigma \ln \xi_{T+1} \partial_\sigma r^j_{T+1} = -\mathbb{E}_T \partial_\sigma \ln A_{T+1,t-1} \partial_\sigma r^j_{T+1}.$$

This equation implies (14) by Corollary 7. □

Lemma 4. *Equation (15) holds in the optimal equilibrium of the benchmark economy.*

Proof. Lemma 2 implies that the zeroth order approximation of equation (9) is

$$\sum_{t=1}^{\infty} \overline{Q}_{T+1,t} \overline{X}_{T+t-1} = \overline{B}_T \overline{R}^0_{T+1}. \tag{43}$$

Multiply equation (9) by r^j_{T+1} and take expectations at time T. The Law of Iterated Expectations implies

$$\mathbb{E}_T \sum_{t=1}^{\infty} \mathcal{Q}_{T+1,t-1} X_{T+t} r^j_{T+1} = \mathbb{E}_T B_T \left[R^0_{T+1} + \sum_{i \geq 1} \omega^i_T r^i_{T+1} \right] r^j_{T+1}. \tag{44}$$

Take the second-order expansion of equation (44), note that the terms multiplying $\partial_{\sigma\sigma} r^j_{T+1}$ cancel out due to (43) and that $\mathbb{E}_T \partial_\sigma \mathcal{Q}_{T+1,t-1} \partial_\sigma r^j_{T+1} = \mathbb{E}_T \left(\mathbb{E}_{T+1} \partial_\sigma \mathcal{Q}_{T+1,t-1} \right) \partial_\sigma r^j_{T+1} = \mathbb{E}_T \partial_\sigma Q_{T+1,t-1} \partial_\sigma r^j_{T+1}$ by Lemma 2 to obtain

$$\mathbb{E}_T \sum_{t=1}^{\infty} \overline{X}_{T+t} \partial_\sigma Q_{T+1,t-1} \partial_\sigma r^j_{T+1} + \mathbb{E}_T \sum_{t=1}^{\infty} \overline{Q}_{T+1,t-1} \partial_\sigma X_{T+t} \partial_\sigma r^j_{T+1} = \overline{B}_T \mathbb{E}_T \left[\sum_{i \geq 1} \overline{\omega}^i_T \partial_\sigma r^i_{T+1} \partial_\sigma r^j_{T+1} \right]. \tag{45}$$

Finally, note that $Q_{T+1,0} = 1$ and, therefore, $\partial_\sigma Q_{T+1,0} = 0$. Together with Corollary 7 this establishes equation (15). □

Lemma 5. *Equation (16) holds in the optimal equilibrium of the benchmark economy.*

Proof. From (7), we have $\ln Y_t = \ln \theta_t + \gamma \ln (1 - \tau_t)$, therefore $\ln Y^\perp_t = \ln \theta_t$. Thus, using Corollary 7, we have

$$cov_T \left(X^\perp_{T+t}, r^j_{T+1} \right) \simeq \overline{\tau_{T+t} Y_{T+t}} \mathbb{E}_T \partial_\sigma \ln Y^\perp_{T+t} \partial_\sigma r^j_{T+1} - \overline{G}_{T+t} \mathbb{E}_T \partial_\sigma \ln G_{T+t} \partial_\sigma r^j_{T+1}.$$

Similarly, $X_{T+t} = \tau_{T+t} Y_{T+t} - G_{T+t}$ and thus

$$cov_T\left(X_{T+t}, r^j_{T+1}\right) \simeq \overline{\tau_{T+t} Y_{T+t}}\Bigg[\mathbb{E}_T \partial_\sigma \ln Y^\perp_{T+t} \partial_\sigma r^j_{T+1} - \frac{\gamma}{1-\overline{\tau}_{T+t}} \mathbb{E}_T \partial_\sigma \tau_{T+t} \partial_\sigma r^j_{T+1} + \frac{1}{\overline{\tau}_{T+t}} \mathbb{E}_T \partial_\sigma \tau_{T+t} \partial_\sigma r^j_{T+1}\Bigg] - \overline{G}_{T+t} \mathbb{E}_T \partial_\sigma \ln G_{T+t} \partial_\sigma r^j_{T+1}$$
$$\simeq cov_T\left(X^\perp_{T+t}, r^j_{T+1}\right) + \overline{Y}_{T+t}\left(\frac{1-\overline{\tau}_{T+t} - \gamma\overline{\tau}_{T+t}}{1-\overline{\tau}_{T+t}}\right) \mathbb{E}_T \partial_\sigma \tau_{T+t} \partial_\sigma r^j_{T+1}.$$

Direct calculations show that $\partial_\sigma \ln \xi_{T+t} = -\frac{\gamma}{(1-\overline{\tau}_{T+t}-\gamma\overline{\tau}_{T+t})(1-\overline{\tau}_{T+t})} \partial_\sigma \tau_{T+t}$. Therefore, we have

$$\frac{1-\overline{\tau}_{T+t} - \gamma\overline{\tau}_{T+t}}{1-\overline{\tau}_{T+t}} \mathbb{E}_T \partial_\sigma \tau_{T+t} \partial_\sigma r^j_{T+1} = -\frac{\left(1-\overline{\tau}_{T+t} - \gamma\overline{\tau}_{T+t}\right)^2}{\gamma} \mathbb{E}_T \partial_\sigma \ln \xi_{T+t} \partial_\sigma r^j_{T+1}.$$

Combine with the previous equation, use definition of ζ_{T+t} and Corollary 7 to obtain (16). □

We need to prove the following result before proceeding to study stationary economy.

Lemma 6. *In any equilibrium $\overline{\mathbb{W}_{t,s_k}(V_{t+1})} = \Pr\left(s_k|s^t\right)$ and, therefore, in any equilibrium the benchmark economy* $\left(\overline{\frac{1}{R^{j,pvt}_{t+1}}}\right) = \frac{\overline{\delta}_{t+1}}{\overline{\delta}_t} \frac{U_c\left(\overline{c}_{t+1} - \frac{1}{\overline{\theta}^{1/\gamma}_{t+1}} \frac{\overline{y}^{1+1/\gamma}_{t+1}}{1+1/\gamma}, \left\{\overline{Q}^i_{t+1}\overline{b}^i_{t+1}\right\}_{i\in\mathcal{G}_{t+1}}, \overline{G}_{t+1}\right)}{U_c\left(\overline{c}_t - \frac{1}{\overline{\theta}^{1/\gamma}_t} \frac{\overline{y}^{1+1/\gamma}_t}{1+1/\gamma}, \left\{\overline{Q}^i_t\overline{b}^i_t\right\}_{i\in\mathcal{G}_t}, \overline{G}_t\right)}$ *for all $t \geq T$.*

Proof. Consider any random variable $\epsilon = \{\epsilon(s_l)\}_l$ with $\sum_l \Pr\left(s_l|s^t\right)\epsilon(s_l) = 0$. Let $F_t(\sigma) \equiv \mathbb{W}_t\left(\{\overline{x} + \sigma\epsilon_l\}_l\right)$. Its derivatives is $F'_t(0) = \sum_{s_k} \mathbb{W}_{t,s_k}\left(\{\overline{x}\}_l\right)\epsilon(s_k)$. Since $\mathbb{W}$ is increasing in the second-order stochastic dominance, $F'_t(0) \leq 0$. Together with $\sum_l \Pr\left(s_l|s^t\right)\epsilon(s_l) = 0$ then the first condition can be written as

$$\sum_{s_k}\left[\mathbb{W}_{t,s_k}\left(\{\overline{x}\}_l\right) - \Pr\left(s_k|s^t\right)\right]\epsilon(s_k) \leq 0.$$

Since ϵ is arbitrary, $\mathbb{W}_{t,s_k}\left(\{\overline{x}\}_l\right) = \Pr\left(s_k|s^t\right)$.

The consumption optimality condition for household is

$$\mathbb{W}_{0,s_1} \times ... \times \mathbb{W}_{t-1,s_t} \times U_{c,t}\left(s^t\right) = \Pr\left(s^t\right) M_t\left(s^t\right),$$

where $s^t = (s_1, .., s_t)$, which implies that

$$\overline{\delta}_{T+t} U_c\left(\overline{c}_{T+t} - \frac{1}{\overline{\theta}^{1/\gamma}_{T+t}} \frac{\overline{y}^{1+1/\gamma}_{T+t}}{1+1/\gamma}, \left\{\overline{Q}^i_{T+t}\overline{b}^i_{T+t}\right\}_{i\in\mathcal{G}_{T+t}}, \overline{G}_{T+t}\right) = \overline{M}_{T+t}.$$

Since $R^{j,pvt}_{T+t}$ satisfies $\left(\overline{\frac{1}{R^{j,pvt}_{T+t+1}}}\right) = \frac{\beta\overline{M}_{T+t+1}}{\overline{M}_{T+t}}$, for all j,we obtain the result of the lemma. □

We are now ready to show the properties of stationary economy.

Lemma 7. *In a stationary optimal equilibrium of the benchmark economy, for all* $t \geq 1$, $\tau_{T+t} \approx \tau_T$, $Q_{T+t} \approx \beta \Gamma^{-1/IES}$,

$$\mathbb{E}_T \frac{X_{T+t+1}}{X_{T+t}} \approx \mathbb{E}_T \frac{Y_{T+t+1}}{Y_{T+t}} \approx \mathbb{E}_T \frac{B_{T+t+1}}{B_{T+t}} \approx \Gamma,$$

and $\mathbb{E}_T \frac{X_{T+t}}{Y_{T+t}} \approx \frac{(1-\hat{\beta})}{\hat{\beta}} \frac{B_T}{Y_T}$ *where* $\hat{\beta} = \beta \Gamma^{1-1/IES}$.

Proof. Since $A_t^{j,pvt} = 1$ for all j, t,

$$1 = \left(\overline{\frac{\beta M_{T+t+1}}{M_{T+t}}} \right) \overline{R}^{j,pvt}_{T+t+1} = \left(\overline{\frac{\beta M_{T+t+1}}{M_{T+t}}} \right) R \text{ for all } j, t$$

by condition (iii) of the definition of stationarity. Therefore, equations (11) and (12) imply

$$\frac{1}{\overline{\xi}_{T+t}} = \left(\overline{\frac{\beta M_{T+t+1}}{M_{T+t}}} \right) R \frac{1}{\overline{\xi}_{T+t+1}} \text{ for all } t,$$

which, in turn, implies $\overline{\tau}_{T+t} = \overline{\tau}_T$ for all t. The optimality condition of households (7) and condition (i) then implies that $\left(\overline{\frac{Y_{T+t+1}}{Y_{T+t}}} \right) = \Gamma$ and, therefore, $\left(\overline{\frac{X_{T+t+1}}{X_{T+t}}} \right) = \Gamma$ for all t.

Let $Q \equiv 1/R$. The government budget constraint (9) is $\overline{B}_T = \sum_{t=1}^{\infty} Q^t \overline{X}_{T+t} = \frac{Q\Gamma}{1-Q\Gamma} \overline{X}_T$, which implies that $\left(\overline{\frac{B_{T+t+1}}{B_{T+t}}} \right) = \Gamma$ and

$$\left(\overline{\frac{X_{T+t}}{Y_{T+t}}} \right) = \left(\frac{1 - Q\Gamma}{Q\Gamma} \right) \left(\overline{\frac{B_{T+t}}{Y_{T+t}}} \right) = \left(\frac{1 - Q\Gamma}{Q\Gamma} \right) \left(\overline{\frac{B_T}{Y_T}} \right). \tag{46}$$

The household optimality condition for c_t in the benchmark economy is

$$\mathbb{W}_{0,s_1} \times ... \times \mathbb{W}_{t-1,s_t} \times \delta_t \left(s^t \right) U_c \left(s^t \right) = \Pr \left(s^t \right) M_t \left(s^t \right).$$

Lemma 6 implies that the zeroths order approximation of this equation is

$$\overline{\delta}_{T+t} U_c \left(\overline{c}_{T+t} - \frac{1}{\overline{\theta}_{T+t}^{1/\gamma}} \frac{\overline{y}_{T+t}^{1+1/\gamma}}{1 + 1/\gamma}, \left\{ \overline{b}^i_{T+t} \right\}_{i \in \mathcal{G}_t}, \overline{G}_{T+t} \right) = \overline{M}_{T+t}$$

and, therefore, by the properties (iii) and (iv) and the definition of IES we have

$$Q = \left(\overline{\frac{\beta M_{T+t+1}}{M_{T+t}}} \right) = \beta \Gamma^{-1/IES}.$$

Thus, $Q\Gamma = \beta \Gamma^{1-1/IES}$. This, together with definition of $\hat{\beta}$ and (46), implies

$$\left(\overline{\frac{X_{T+t}}{Y_{T+t}}} \right) = \frac{1 - \hat{\beta}}{\hat{\beta}} \left(\overline{\frac{B_T}{Y_T}} \right).$$

The statement of the lemma then follows from applying Corollary 7. □

Note that the only place where we used condition (iv) in the proof is in deriving expression Q in terms of growth rates of real variables and obtaining weights $\hat{\beta}$ in terms of growth rates of real variables on balanced growth path. Condition (iv) is necessary for a balanced growth path but not for our results. For concreteness, suppose that U is separable in the first argument and let $u_t \equiv c_t - \frac{1}{\theta_t^{1/\gamma}} \frac{y_t^{1+1/\gamma}}{1+1/\gamma}$ and $\Gamma_t^u \equiv \left(\frac{u_t}{u_{t-1}}\right)$. Conditions (i)-(iii) imply that Γ_t^u is independent of t and so that $Q = \beta\left(\Gamma^u\right)^{-1/IES}$. Thus, the only thing that changes in the analysis is that we replace $\Gamma^{1-1/IES}$ with $\Gamma \times \left(\Gamma^u\right)^{-1/IES}$ and adjust the definition of $\hat{\beta}$ accordingly.

A.2.1 Proof of Corollary 1

Proof. Equation (17) can equivalently be written as

$$\sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \left(\frac{\overline{X}_{T+t}}{\overline{Y}_{T+t}}\right) \mathbb{E}_T \partial_\sigma \ln Q_{T+1,t-1} \partial_\sigma r_{T+1}^j + \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \mathbb{E}_T \frac{\partial_\sigma X_{T+t}^{\perp}}{\overline{Y}_{T+t}} \partial_\sigma r_{T+1}^j$$
$$+ \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \overline{\zeta}_{T+t} \mathbb{E}_T \partial_\sigma r_{T+1}^j \partial_\sigma \ln A_{T+1,t-1} = \overline{Q}_T^0 \frac{\overline{B}_T}{\overline{Y}_T} \mathbb{E}_T \left(\sum_{i \geq 1} \partial_\sigma r_{T+1}^j \partial_\sigma r_{T+1}^i \overline{\omega}_T^i\right).$$

By Lemma 7, in the stationary economy $\overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) = \hat{\beta}^t$, $\overline{\zeta}_{T+t} = \overline{\zeta}_T$ and $\frac{\overline{X}_{T+t}}{\overline{Y}_{T+t}} = \frac{1-\hat{\beta}}{\hat{\beta}} \frac{\overline{B}_T}{\overline{Y}_T}$. Substitute these expressions and re-arrange to obtain

$$\left(1-\hat{\beta}\right) \sum_{t=1}^{\infty} \hat{\beta}^t \mathbb{E}_T \frac{\partial_\sigma \ln Q_{T+1,t}}{Q_T^0} \partial_\sigma r_{T+1}^j + \hat{\beta}^{-1} \Gamma \frac{\overline{Y}_T}{\overline{B}_T} \sum_{t=1}^{\infty} \hat{\beta}^t \mathbb{E}_T \frac{\partial_\sigma X_{T+t}^{\perp}}{\overline{Y}_{T+t}} \partial_\sigma r_{T+1}^j$$
$$+ \overline{\zeta}_T \Gamma \frac{\overline{Y}_T}{\overline{B}_T} \sum_{t=1}^{\infty} \hat{\beta}^t \mathbb{E}_T \partial_\sigma r_{T+1}^j \partial_\sigma \ln A_{T+1,t} = \mathbb{E}_T \left(\sum_{i \geq 1} \partial_\sigma r_{T+1}^j \partial_\sigma r_{T+1}^i \overline{\omega}_T^i\right).$$

Apply Corollary 7 and write in matrix form to get (18). □

A.2.2 Proof of Corollary 2

Corollary 2 follows from the following lemma.

Lemma 8. *Let Q_T^t, r_T^t be the period-T price and excess return of a pure discount bond that expires in period $T+t$. Then in the optimal equilibrium of the baseline economy*

$$cov_T\left(\ln Q_{T+1,t}^t, r_{T+1}^j\right) \simeq cov_T\left(\ln Q_{T+1,t}, r_{T+1}^j\right) \tag{47}$$

and

$$Q_T^0 cov_T\left(r_{T+1}^t, r_{T+1}^j\right) \simeq cov_T\left(\ln Q_{T+1,t}, r_{T+1}^j\right) \tag{48}$$

for all $j \in \mathcal{G}_T$. The latter implies $\Sigma_T \simeq \Sigma_T^Q$ when the government trades the full set of pure discount bonds.

Proof. From the definition of A_T^{t+1} and R_T^t, we have the following recursion: $Q_T^{t+1} = \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} A_T^{t+1} Q_T^t$. This implies that price Q_T^{t+1} must satisfy

$$\begin{aligned}
&= \mathbb{E}_{T+1}\left[\left(\frac{\beta^t M_{T+t}}{M_{T+1}}\right)\left(A_{T+1}^t \times A_{T+2}^{t-1} \times ... \times A_{T+t-1}^0\right)\right] \\
&= \mathbb{E}_{T+1}\left[\left(\frac{\beta^t M_{T+t}}{M_{T+1}}\right)\left(A_{T+1}^0 \times A_{T+2}^0 \times ... \times A_{T+t-1}^0\right)\right] \\
&= \mathbb{E}_{T+1}\left[\frac{\beta M_{T+2} A_{T+1}^0}{M_{T+1}} \times ... \times \frac{\beta M_{T+t} A_{T+t-1}^0}{M_{T+t-1}}\right],
\end{aligned}$$

where the second equation follows form Lemma 1. Similarly, $Q_{T+1,t}$ is given by

$$Q_{T+1,t} = \mathbb{E}_{T+1}\frac{\beta M_{T+2} A_{T+1}^0}{M_{T+1}} \times ... \times \mathbb{E}_{T+t-1}\frac{\beta M_{T+t} A_{T+t-1}^0}{M_{T+t-1}}.$$

Using the Law of Iterated Expectations, we obtain

$$\begin{aligned}
\mathbb{E}_T \partial_\sigma \ln Q_{T+1,t} \partial_\sigma r_{T+1}^j &= \mathbb{E}_T\left[\mathbb{E}_{T+1}\partial_\sigma \ln \frac{\beta M_{T+2} A_{T+1}^0}{M_{T+1}} \partial_\sigma r_{T+1}^j + ... + \mathbb{E}_{T+t-1}\partial_\sigma \ln \frac{\beta M_{T+t} A_{T+t-1}^0}{M_{T+1}} \partial_\sigma r_{T+1}^j\right] \\
&= \mathbb{E}_T\left[\partial_\sigma \ln \frac{\beta M_{T+2} A_{T+1}^0}{M_{T+1}} \partial_\sigma r_{T+1}^j + ... + \partial_\sigma \ln \frac{\beta M_{T+t} A_{T+t-1}^0}{M_{T+1}} \partial_\sigma r_{T+1}^j\right] \\
&= \mathbb{E}_T \partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j.
\end{aligned}$$

This expression is equivalent to (47) by Corollary 7.

To show (48), first observe that $r_{T+1}^t = Q_{T+1}^t / Q_T^{t+1} - 1/Q_T^0$ and, therefore,

$$\begin{aligned}
\mathbb{E}_T \partial_\sigma r_{T+1}^t \partial_\sigma r_{T+1}^j &= \left(\overline{\frac{Q_{T+1}^t}{Q_T^{t+1}}}\right)\mathbb{E}_T\left[\partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j - \partial_\sigma \ln Q_T^{t+1} \partial_\sigma r_{T+1}^j\right] - \left(\overline{\frac{1}{Q_T^0}}\right)\mathbb{E}_T \partial_\sigma \ln Q_T^0 \partial_\sigma r_{T+1}^j \\
&= \left(\overline{\frac{Q_{T+1}^t}{Q_T^{t+1}}}\right)\mathbb{E}_T\left[\partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j\right] = \left(\overline{\frac{1}{\beta M_{T+1} A_T^0 / M_t}}\right)\mathbb{E}_T\left[\partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j\right] \\
&= \left(\overline{\frac{1}{Q_T^0}}\right)\mathbb{E}_T\left[\partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j\right],
\end{aligned}$$

where the second equation follows from the fact that $\partial_\sigma \ln Q_T^{t+1}$ and $\partial_\sigma \ln Q_T^0$ are measurable with respect to T and $\mathbb{E}_T \partial_\sigma r_{T+1}^j = 0$ by Lemma 2. This equation is equivalent to (48) by Corollary 7. □

A.3 Nominal economy

We now describe a nominal version of the economy. Let P_t be the price level and suppose all securities are nominal. Security 0 now refers to a nominal one-period bond that pays one dollar next period. The household and government budget constraint in the nominal economy are

$$P_t c_t + \sum_i Q_t^i b_t^i = P_t y_t - \tau_t P_t y_t + \sum_i \left(Q_t^i + D_t^i\right) b_{t-1}^i$$

and

$$P_t X_t + \sum_{i \in \mathcal{G}_t} Q_t^i B_t^i = \sum_{i \in \mathcal{G}_{t-1}} \left(Q_t^i + D_t^i\right) B_{t-1}^i,$$

respectively. All returns and liquidity premia are now in nominal terms. It is irrelevant in the benchmark economy where U_t, φ_t, $\left\{\mathcal{B}_t^i\right\}_i$ are functions of nominal or real value of security holdings. The definition of competitive equilibrium and optimum competitive equilibrium remain unchanged except in nominal economy they are defined for $(\mathbf{G}, \mathbf{P})$ rather than $\mathbf{G}$.

Our analysis of the benchmark economy extends with minimal changes to nominal economy. For any real variable x_t we use notation $x_t^\$$ to denote its nominal value, $x_t^\$ \equiv P_t x_t$. All variables are defined in the same way as in Section 3 and it is easy to see that Lemma 1 continues to hold in this settings.

It is easy to see that the perturbation we considered in Section 3 requires tax adjustments $r_{T+1}^i / \left(Q_{T+1,t-1} \xi_{T+t} Y_{T+t}^\$\right)$ so that equation (12) remain unchanged. The budget constraint now holds in nominal terms so that equation (15) in the nominal economy becomes

$$\sum_{t=2}^{\infty} \mathbb{E}_T X_{T+t}^\$ cov_T \left(Q_{T+1,t-1}, r_{T+1}^j\right) + \sum_{t=1}^{\infty} \mathbb{E}_T Q_{T+1,t-1} cov_T \left(X_{T+t}^\$, r_{T+1}^j\right) \simeq B_T \sum_{i \geq 1} \omega_T^i cov_T \left(r_{T+1}^i, r_{T+1}^j\right). \tag{49}$$

If we define $X_{T+t}^{\perp,\$}$ as

$$X_{T+t}^{\perp,\$} \equiv \mathbb{E}_T \mathcal{T}_{T+t}^\$ \times \left(\ln Y_t^\$ - \gamma \ln \left(1 - \tau_t\right)\right) - \mathbb{E}_T G_{T+t}^\$ \times \ln G_{T+t}^\$$$

and follow the steps of Lemma 5, we obtain

$$cov_T \left(X_{T+t}^\$, r_{T+1}^j\right) \simeq cov_T \left(X_{T+t}^{\perp,\$}, r_{T+1}^j\right) - \mathbb{E}_T \zeta_{T+t} \mathbb{E}_T Y_{T+t}^\$ cov_T \left(\ln \xi_{T+t}, r_{T+1}^j\right). \tag{50}$$

Use equations (49) and (50) and follow the steps of proof of equation (17) to obtain

$$\begin{aligned}
&\mathbb{E}_T \left(\sum_{i \geq 1} \partial_\sigma r_{T+1}^j \partial_\sigma r_{T+1}^i \overline{\omega}_T^i\right) \overline{Q}_T^0 \frac{\overline{B}_T}{\overline{P}_T \overline{Y}_T} = \\
&- \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{P}_{T+t}}{\overline{P}_T} \frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \overline{\zeta}_{T+t} \left(\mathbb{E}_t \partial_\sigma r_{T+1}^j \partial_\sigma \ln A_{T+1,t-1}\right) \\
&+ \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{P}_{T+t}}{\overline{P}_T} \frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \mathbb{E}_T \frac{\partial_\sigma X_{T+t}^{\perp,\$}}{\overline{P}_{T+t} \overline{Y}_{T+t}} \partial_\sigma r_{T+1}^j \\
&+ \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{P}_{T+t}}{\overline{P}_T} \frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \left(\frac{\overline{X}_{T+t}}{\overline{Y}_{T+t}}\right) \mathbb{E}_T \partial_\sigma \ln Q_{T+1,t-1} \partial_\sigma r_{T+1}^j.
\end{aligned} \tag{51}$$

In the stationary nominal economy, $\frac{\overline{P}_{T+t}}{\overline{P}_T}\frac{\overline{Y}_{T+t}}{\overline{Y}_T} = (\Pi_T\Gamma)^t$, the price of a nominal one-period government bond to the zeroth order satisfies

$$\overline{Q}^0_{T+t} = \beta\frac{U_{c,T+t+1}}{U_{c,T+t}}\frac{\overline{P}_T}{\overline{P}_{T+1}} = \frac{\beta\Gamma^{-1/IES}}{\Pi_T},$$

the nominal discount rate $Q_{T+1,t-1}$ satisfies $\overline{Q}_{T+1,t-1} = \left(\frac{\beta}{\Pi_T}\Gamma^{-1/IES}\right)^{t-1}$, and $\frac{\overline{B}_T}{\overline{P}_T\overline{X}_T} = \frac{\sum_{t=1}^{\infty}\left(\frac{\beta}{\Pi_T}\Gamma^{-1/IES}\Gamma\Pi_T\right)^t}{\frac{\beta}{\Pi_T}\Gamma^{-1/IES}}$, $\overrightarrow{\hat{\beta}}_T[t] = \overline{Q}^0_T\overline{Q}_{T+1,t-1}\left(\frac{\overline{P}_{T+t}}{\overline{P}_T}\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) = \left(\beta\Gamma^{1-1/IES}\right)^t = \hat{\beta}^t$. Use these expressions in equation (51) and apply Corollary 7 to get

$$\mathbb{E}_T\left(\sum_{i\geq1}\partial_\sigma r^j_{T+1}\partial_\sigma r^i_{T+1}\overline{\omega}^i_T\right) = \left(1-\hat{\beta}\right)\sum_{t=1}^{\infty}\hat{\beta}^t cov_T\left(\frac{\ln Q_{T+1,t}}{Q^0_T}, r^j_{T+1}\right)$$
$$+\left(\frac{\Gamma\Pi_T}{\hat{\beta}}\right)\left(\frac{Y^\$_T}{B_T}\right)\sum_{t=1}^{\infty}\hat{\beta}^t cov_T\left(\frac{X^{\perp,\$}_{T+t}}{\mathbb{E}_T Y^\$_{T+t}}, r^j_{T+1}\right) + (\Gamma\Pi_T\zeta_T)\left(\frac{Y^\$_T}{B_T}\right)\sum_{t=1}^{\infty}\hat{\beta}^t cov_T\left(\ln A_{T+1,t}, r^j_{T+1}\right)$$

In matrix form this equation becomes

$$\Sigma_T\overrightarrow{\boldsymbol{\omega}}_T \simeq \left[\pi^Q\Sigma^Q_T + \Pi_T\pi^X_T\Sigma^X_T + \Pi_T\pi^A_T\Sigma^A_T\right]\overrightarrow{\hat{\beta}},$$

which proves the nominal version of equation (18) given in Corollary 3. The returns with the full set of nominal bonds satisfy $\Sigma_T \simeq \Sigma^Q_T$, which implies the nominal version of (20) given in Corollary 3.

A.4 Household Heterogeneity

Suppose household h has household specific productivity $\theta_{h,t}$ and we partition the households into two groups: $\mathbb{T}$ represent the set of households who can trade bonds and $\mathbb{N}$ represent the set of households who cannot trade bonds. Other than that, we focus on the baseline economy, in which all government assets are perfect substitutes and economy is small and open. Individual optimality implies

$$y_{h,t} = \theta^{1+\gamma}_{h,t}(1-\tau_t)^\gamma$$

and we can define total output as $Y_t = \sum_h y_{h,t}$. Assuming a linear tax function, we have

$$\frac{\partial\Upsilon_t}{\partial\tau_t} = Y_t + \tau_t\sum_h\frac{\partial y_{h,t}}{\partial\tau_t} = Y_t - \gamma\frac{\tau_t}{1-\tau_t}\sum_h y_{h,t} = Y_t\left(1-\gamma\frac{\tau_t}{1-\tau_t}\right),$$

so tax revenue elasticity is the same as before.

$$\xi_t = 1-\gamma\frac{\tau_t}{1-\tau_t}.$$

Consider a perturbation in which the government swaps ϵ of security $j \in \mathcal{G}_T$ for a risk-free bond in period T, undoes this perturbation in period $T+1$ and realized excess return r^j_{T+1} and then rolls it over for t periods using one-period bond before returning it back to the household. The same analysis as with the representative agent yields a welfare gain for agent h as

$$\partial_{j,T,\epsilon} V_{h,0} \propto \mathbb{E}_T M_{h,T+t} \left(Q_{T+1,t-1}\right)^{-1} r^j_{T+1} y_{h,T+t} \frac{\partial \tau_{T+t}}{\partial \Upsilon_{T+t}} = \mathbb{E}_T \beta^t M_{h,T+t} \left(Q_{T+1,t-1}\right)^{-1} r^j_{T+1} \frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}.$$

So an optimality condition will be

$$\mathbb{E}_T \sum_h \varpi_h M_{h,T+t} \left(Q_{T+1,t-1}\right)^{-1} r^j_{T+1} \frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}} = 0,$$

where ϖ_h are Pareto weights. A second-order expansion of this equation then yields

$$\begin{aligned}
0 = \mathbb{E}_T \Bigg\{ & \frac{1}{2} \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_{\sigma\sigma} r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \\
& + \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_\sigma \ln\left(M_{h,T+t}\right) \partial_\sigma r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \\
& + \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_\sigma \ln\left(\frac{y_{h,T+t}}{Y_{T+t}}\right) \partial_\sigma r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \\
& - \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_\sigma \ln\left(\xi_{T+t}\right) \partial_\sigma r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \\
& - \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_\sigma \ln\left(Q^{rf}_{T+1,t-1}\right) \partial_\sigma r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \Bigg\}.
\end{aligned}$$

Canceling out the terms that do not depend on h and dividing out by the coefficient on $\mathbb{E}_T \partial_{\sigma\sigma} r^j_{T+1}$ yields the optimality condition

$$\begin{aligned}
0 = & \mathbb{E}_T \Bigg[\frac{1}{2} \partial_{\sigma\sigma} r^j_{T+1} + \sum_h \mu_{h,T+t} \partial_\sigma \ln\left(M_{h,T+t}\right) \partial_\sigma r^j_{T+1} + \partial_\sigma \ln\left(\xi_{T+t}\right) \partial_\sigma r^j_{T+1} + \partial_\sigma \ln\left(Q_{T+1,t-1}\right) \partial_\sigma r^j_{T+1} \\
& \quad + \sum_h \mu_{h,T+t} \partial_\sigma \ln\left(\frac{y_{h,T+t}}{Y_{T+t}}\right) \partial_\sigma r^j_{T+1} \Bigg] \qquad (52)
\end{aligned}$$

where $\mu_{h,T+t} \equiv \varpi_h \left[\overline{M_{h,T+t}}\right] \overline{s_{h,T+t}} \Big/ \left(\sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \overline{s_{h,T+t}}\right)$ are a deterministic sequence of weights that sum to one with $s_{h,T+t} \equiv \frac{y_{h,T+t}}{Y_{T+t}}$.

As government bonds are perfect substitutes, for all $h \in \mathbb{T}$ we must have

$$\mathbb{E}_T M_{h,T+t} \left(Q^{pvt}_{T+1,t-1}\right)^{-1} r^j_{T+1} = 0.$$

Expanding this equation yields

$$0=\frac{1}{2}\left[\overline{M_{h,T+t}}\right]\left(\overline{Q^{pvt}_{T+1,t-1}}\right)^{-1}\partial_{\sigma\sigma}r^{j}_{T+1}-\left[\overline{M_{h,T+t}}\right]\left(\overline{Q^{pvt}_{T+1,t-1}}\right)^{-1}\partial_{\sigma}\ln\left(Q^{pvt}_{T+1,t-1}\right)\partial_{\sigma}r^{j}_{T+1}$$
$$+\left[\overline{M_{h,T+t}}\right]\left(\overline{Q^{pvt}_{T+1,t-1}}\right)^{-1}\partial_{\sigma}\ln\left(M_{h,T+t}\right)\partial_{\sigma}r^{j}_{T+1}$$

for all $h\in\mathbb{T}$. This simplifies to

$$0=\mathbb{E}_T\left[\frac{1}{2}\partial_{\sigma\sigma}r^{j}_{T+1}-\partial_{\sigma}\ln\left(Q^{pvt}_{T+1,t-1}\right)\partial_{\sigma}r^{j}_{T+1}+\partial_{\sigma}\ln\left(M_{h,T+t}\right)\partial_{\sigma}r^{j}_{T+1}\right].$$

Since $\ln Q^{pvt}_{T+1,t-1}=\ln Q_{T+1,t-1}-\ln A_{T+1,t-1}$, we have.

$$\frac{1}{2}\mathbb{E}_T\partial_{\sigma\sigma}r^{j}_{T+1}=\mathbb{E}_T\left[-\partial_{\sigma}\ln A_{T+1,t-1}\partial_{\sigma}r^{j}_{T+1}+\partial_{\sigma}\ln\left(Q_{T+1,t-1}\right)\partial_{\sigma}r^{j}_{T+1}-\partial_{\sigma}\ln\left(M_{h,T+t}\right)\partial_{\sigma}r^{j}_{T+1}\right] \tag{53}$$

As this holds for all $h\in\mathbb{T}$ we can average over all traders, using weights $\mu_{h,T+t}$, to obtain

$$\frac{1}{2}\mathbb{E}_T\partial_{\sigma\sigma}r^{j}_{T+1}=\mathbb{E}_T\left[-\partial_{\sigma}\ln A_{T+1,t-1}\partial_{\sigma}r^{j}_{T+1}+\partial_{\sigma}\ln\left(Q_{T+1,t-1}\right)\partial_{\sigma}r^{j}_{T+1}-\partial_{\sigma}\ln\left(M_{\mathcal{T},T+t}\right)\partial_{\sigma}r^{j}_{T+1}\right] \tag{54}$$

where $\ln\left(M_{\mathbb{T},T+t}\right)$ is the average SDF of all traders:

$$\ln\left(M_{\mathbb{T},T+t}\right)\equiv\sum_{h\in\mathbb{T}}\mu_{h,T+t}\ln\left(M_{h,T+t}\right)\Big/\sum_{h\in\mathbb{T}}\mu_{h,T+t}.$$

The same equation does not hold for the non-traders but we do have that for all $h\in\mathbb{N}$

$$\begin{aligned}\frac{1}{2}\mathbb{E}_T\partial_{\sigma\sigma}r^{j}_{T+1}=&\mathbb{E}_T\Big[-\partial_{\sigma}\ln A_{T+1,t-1}\partial_{\sigma}r^{j}_{T+1}+\partial_{\sigma}\ln\left(Q_{T+1,t-1}\right)\partial_{\sigma}r^{j}_{T+1}-\partial_{\sigma}\ln\left(M_{h,T+t}\right)\partial_{\sigma}r^{j}_{T+1}\\&+\left(\partial_{\sigma}\ln\left(M_{h,T+t}\right)-\partial_{\sigma}\ln\left(M_{\mathcal{T},T+t}\right)\right)\partial_{\sigma}r^{j}_{T+1}\Big].\end{aligned} \tag{55}$$

We can now use equations (53) and (55) substitute for $\frac{1}{2}\partial_{\sigma\sigma}r^{j}_{T+1}$ in (52) to get

$$\begin{aligned}-\mathbb{E}_T\partial_{\sigma}\ln\xi_{T+t}\partial_{\sigma}r^{j}_{T+1}=&\mathbb{E}_T\Bigg[\partial_{\sigma}\ln A_{T+1,t-1}\partial_{\sigma}r^{j}_{T+1}+\partial_{\sigma}\left\{\sum_{h}\mu_{h,T+t}\ln\left(\frac{1}{s_{h,T+t}}\right)\right\}\partial_{\sigma}r^{j}_{T+1}\\&+\partial_{\sigma}\left\{\sum_{h\in\mathbb{N}}\mu_{h,T+t}\left(\ln\left(M_{\mathbb{T},T+t}\right)-\ln\left(M_{h,T+t}\right)\right)\right\}\partial_{\sigma}r^{j}_{T+1}\Bigg].\end{aligned}$$

We can further simplify this expression by defining

$$\ln\left(M_{\mathbb{N},T+t}\right)\equiv\sum_{h\in\mathbb{N}}\mu_{h,T+t}\ln\left(M_{h,T+t}\right)\Big/\left(\sum_{h\in\mathbb{N}}\mu_{h,T+t}\right)$$

as the "average" SDF of the non-traders, then

$$-\text{cov}_T\left(\ln \xi_{T+t}, r^j_{T+1}\right) \simeq \text{cov}_T\left(\partial_\sigma \ln A_{T+1,t-1}\partial_\sigma r^j_{T+1}\right) + \text{cov}_T\left(\sum_h \mu_{h,T+t}\ln\left(\frac{1}{s_{h,T+t}}\right), r^j_{T+1}\right) + \mu_{\mathbb{N},T+t}\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right), \partial_\sigma r^j_{T+1}\right) \tag{56}$$

where $\mu_{\mathbb{N},T+t} \equiv \left(\sum_{h\in\mathbb{N}} \mu_{h,T+t}\right)$ is the "share" of non-traders. Equation (56) adds two additional terms to equation (14) in main text that capture the effect of heterogeneity on the planners desire to smooth taxes. The first term, $\text{cov}_T\left(\sum_h \mu_{h,T+t}\ln\left(\frac{1}{s_{h,T+t}}\right), r^j_{T+1}\right)$, captures the planners desire to raise taxes in states of the world where inequality is high. The second term, $\mu_{\mathbb{N},T+t}\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right), \partial_\sigma r^j_{T+1}\right)$, captures the fact that the planner is trading on behalf of agents without access to asset markets and therefore will want to raise taxes in states of which the non-traders place less weight on relative to those agents with access to asset markets. This effect is scaled by the relative size of the non-traders. Following the steps of Theorem 1 and Corollary 1 we get

$$\Sigma_T\boldsymbol{\omega}_T \simeq \left[\pi^Q\Sigma^Q_T + \pi^X_T\Sigma^X_T + \pi^A_T\Sigma^A_T + \pi^A_T\Sigma^{ineq}_T + \pi^A_T\Sigma^M_T\right]\overrightarrow{\hat{\beta}}$$

where $\Sigma^{ineq}_T[t,j] = cov_T\left(\sum_h \mu_{h,T+t}\ln\left(\frac{1}{s_{h,T+t}}\right), r^j_{T+1}\right)$ is covariance matrix of returns with inequality and

$$\Sigma^M_T[j,t] = \mu_{\mathbb{N},T+t}\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right), r^j_{T+1}\right),$$

is the covariance of returns with the relative stochastic discount factors of traders and non-traders.

We can get a feel for how trading frictions affect the optimal portfolio by studying a special case. We further specialize to a simpler market structure in which the government trades only a risk-free security and a growth-adjusted consol. Let excess return on the consol be denoted by r^∞_t. Finally, we impose that the stochastic discount factor of the non-traders is scaled version of the stochastic discount factor of the traders: $\ln\left(M_{\mathbb{N},T+t}\right) = (1+\psi)\ln\left(M_{\mathbb{T},T+t}\right)$. This introduces a new parameter, ψ, that captures the severity of trading frictions as $\psi > 0$ implies that the SDF of the non-traders is more volatile of those of the traders.

Under this last assumption the covariance of the relative stochastic discount factors simplifies to

$$\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right), r^j_{T+1}\right) = -\psi\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right), r^j_{T+1}\right).$$

As the traders trade the consol, we can use the traders' Euler equation, equation (54), to substitute out for this covariance and obtain

$$-\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right), r^j_{T+1}\right) \simeq \mathbb{E}_T r^j_{T+1} - \text{cov}_T\left(\ln Q_{T+1,t-1}, r^j_{T+1}\right) + \text{cov}_T\left(\ln A_{T+1,t-1}, r^j_{T+1}\right).$$

Under our stationarity assumptions we have $\mu_{\mathcal{N},T+t} = \mu_{\mathcal{N},T}$ and can therefore express Σ_T^M as the sum of three terms

$$\Sigma_T^M = \mu_{\mathbb{N},T}\psi\left(\mathcal{R}_T - \overline{Q}_T^0\Sigma_T^Q + \Sigma_T^A\right)$$

where $\mathcal{R}_T[j,t] = \hat{\beta}^{-1}\mathbb{E}_T r_{T+1}^j$.

The effect of non-traders on the optimal portfolio is given by $\pi_T^A\Sigma_T^{-1}\Sigma_T^M\overrightarrow{\hat{\beta}}$. This simplifies under this market structure of a growth adjusted consol and a risk free bond as Σ_T is now a single number representing the conditional covariance of the growth adjusted consol. We can also make progress on the components of $\Sigma_T^M\overrightarrow{\hat{\beta}}$, starting with $\mathcal{R}_T\overrightarrow{\hat{\beta}} = \frac{\mathbb{E}_T r_{T+1}^j}{1-\hat{\beta}}$. Next we note that

$$\begin{aligned}\Sigma_T^Q\overrightarrow{\hat{\beta}} = \sum_{t=1}^{\infty}\hat{\beta}^t\mathrm{cov}_T\left(\frac{1}{Q_T^0}\ln Q_{T+1,t}, r_{T+1}^{\infty}\right) &\simeq \frac{\Gamma}{\hat{\beta}}\sum_{t=1}^{\infty}\hat{\beta}^t\mathbb{E}_T\partial_\sigma\ln Q_{T+1,t}\partial_\sigma r_{T+1}^{\infty}\\ &\simeq \frac{\Gamma}{\hat{\beta}}\mathbb{E}_T\sum_{t=1}^{\infty}\Gamma^t\partial_\sigma Q_{T+1,t}\partial_\sigma r_{T+1}^{\infty}\\ &\simeq \frac{\Gamma}{\hat{\beta}}\mathbb{E}_T\partial_\sigma Q_{T+1,t}^{\infty}\partial_\sigma r_{T+1}^{\infty}\\ &\simeq \frac{\Gamma}{1-\hat{\beta}}\mathrm{cov}_T(r_{T+1}^{\infty}, r_{T+1}^{\infty}) = \frac{\Gamma}{1-\hat{\beta}}\Sigma_T.\end{aligned}$$

Finally, we have that $\Sigma_T^A\overrightarrow{\hat{\beta}} = \sum_{t\geq 1}\hat{\beta}^t cov_T\left(\ln A_{T+1,t}^{rf}, r_{T+1}^{\infty}\right)$. All put together we have that

$$\Sigma_T^{-1}\Sigma_T^M\overrightarrow{\hat{\beta}} \approx \frac{\mu_{\mathbb{N},T}\psi}{1-\hat{\beta}}\left(\frac{\mathbb{E}_T r_{T+1}^{\infty}}{\mathrm{var}_T(r_{T+1}^{\infty})} - \hat{\beta} + (1-\hat{\beta})\frac{\sum_{t\geq 1}\hat{\beta}^t\mathrm{cov}_T\left(\ln A_{T+1,t}^{rf}, r_{T+1}^{\infty}\right)}{\mathrm{var}_T(r_{T+1}^{\infty})}\right).$$

Our empirical estimates have found that holding period returns on government debts of all maturities co-vary positively with the liquidity premium so we can assume that $\sum_{t\geq 1}\hat{\beta}^t cov_T\left(\ln A_{T+1,t}^{rf}, r_{T+1}^{\infty}\right)$. This implies that

$$\Sigma_T^{-1}\Sigma_T^M\overrightarrow{\hat{\beta}} > \frac{\mu_{\mathbb{N},T}\psi}{1-\hat{\beta}}\left(\frac{\mathbb{E}_T r_{T+1}^{\infty}}{\mathrm{var}_T(r_{T+1}^{\infty})} - \hat{\beta}\right). \tag{57}$$

So the presence of non-traders will lengthen the maturity as long as $\frac{\mathbb{E}_T r_{T+1}^{\infty}}{\mathrm{var}_T(r_{T+1}^{\infty})} > \hat{\beta}$. We can construct estimates for both $\mathbb{E}_T r_{T+1}^{\infty}$ and $\mathrm{var}_T(r_{T+1}^{\infty})$ using the fact that the growth adjusted consol is the infinite sum of zero coupon bonds of all maturities weighted by $\hat{\beta}^j$. To check equation (57), we use the estimates of the factor model and find that the left hand side is 137 which is significantly larger that $\hat{\beta} < 1$.

A.5 Price Effects

In this section we fill in the steps to derive formula (35) and the proof of Corollary 6.

Lemma 9. *In the preferred habitat benchmark economy with $\phi_T B_{i,T} \approx b_{i,T}$, the government optimality is given by*

$$cov_T\left(\ln \xi_{T+1+k}, r^j_{T+1}\right) + cov_T\left(\ln A^0_{T+1,k}, r^j_{T+1}\right) \simeq \overline{\xi_T} Q^{-1} \vartheta^j_T. \tag{58}$$

and equation (35) is satisfied.

Proof. Assumption $\phi_T B_{i,T} \approx b_{i,T}$ implies

$$-\sum_i \left[\partial_{j,T,t,\epsilon} Q^i_T\left(s^T\right)\left(b^i_T\left(s^T\right) - b^i_{T-1}\left(s^{T-1}\right)\right)\right] \approx \phi_T \sum_i \left[\partial_{j,T,t,\epsilon} Q^i_T\left(s^T\right)\left(B^i_T\left(s^T\right) - B^i_{T-1}\left(s^{T-1}\right)\right)\right].$$

From assumption $\overline{\Lambda} = \partial_\sigma \Lambda = \mathbf{0}$ and equation (34) it follows that $\overline{r}^j_{T+1} = \mathbb{E}_T \partial_\sigma r^j_{T+1} = 0$ and as before, stationarity implies $A^i_T \approx 1$. Now, take second-order expansion of equation (34) to get

$$\begin{aligned} 0 = (1 - \phi_T \xi_T) &\left(\frac{\sum_{i\geq 1} \partial_{\sigma\sigma} \partial_{j,T,t,\epsilon} Q\left(\overline{B}^i_T - \overline{B}^{i+1}_{T-1}\right)}{\xi_T}\right) \\ &+ \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} \frac{r^j_{T+1}}{\xi_{T+1+k}} \left[\left(\frac{\beta M_{T+2}}{M_{T+1}} R^0_{T+2}\right) \times ... \left(\frac{\beta M_{T+1+k}}{M_{T+k}} R^0_{T+1+k}\right)\right] \end{aligned} \tag{59}$$

Under our perturbation and exploiting the form (33), we get

$$\partial_{j,T,t,\epsilon} \ln Q^i_T = \frac{\Lambda_T[i,j]}{Q^j_T} - \frac{\Lambda_T[0,j]}{Q^0_T}$$

which combined with the GV assumption $\Lambda_T[0,j] = 0$ and $\overline{\Lambda} = \partial_\sigma \Lambda = \mathbf{0}$ imply that $\partial_{\sigma\sigma} \partial_{j,T,t,\epsilon} Q^i_T = \frac{\overline{Q}^i_T \Lambda_T[i,j]}{\overline{Q}^j_T}$. Using this, it is easy to see that

$$\begin{aligned} \sum_{i\geq 1} \partial_{\sigma\sigma} \partial^j_\epsilon Q^i_T \left(B^i_T - B^{i+1}_{T-1}\right) &= \left(\frac{B_T}{Y_T}\right) \sum_{i\geq 1} \left(\frac{Y_T \Lambda_T[i,j]}{\overline{Q}^j_T}\right) \left(\overline{\omega}_T[i] - \overline{\omega}_{T-1}[i+1] \frac{\overline{Q}^i_T}{\overline{Q}^{i+1}_{T-1}} \frac{\overline{B}_{T-1}}{\overline{B}_T}\right) \\ &= \left(\frac{B_T}{Y_T}\right) \sum_{i\geq 1} \Lambda^{QE}_T[i,j] \left(\overline{\omega}_T[i] - \overline{\omega}^+_{T-1}[i]\right) \end{aligned} \tag{60}$$

Following the same steps as in Lemma (3), we get

$$\begin{aligned} &\mathbb{E}_T \frac{\beta M_{T+1}}{M_T} \frac{r^j_{T+1}}{\xi_{T+1+k}} \left[\left(\frac{\beta M_{T+2}}{M_{T+1}} R^0_{T+2}\right) \times ... \left(\frac{\beta M_{T+1+k}}{M_{T+k}} R^0_{T+1+k}\right)\right] \\ &= -\overline{\xi^{-1}_T Q} \mathbb{E}_T \left[\mathbb{E}_T \partial_\sigma r^j_{t+1} \partial_\sigma \ln A^0_{T+1,k}\right] - \overline{\xi^{-1}_T Q} \mathbb{E}_T \partial_\sigma \log \xi_{T+1+k} \partial_\sigma r^j_{t+1}. \end{aligned} \tag{61}$$

Substitute (60) and (61) in (59) and applying Lemma (7) to get equation (58), where the extra term ϑ^j_T simplifies to

$$\vartheta^j_T = (1 - \phi_T \xi_T) \left(\frac{B_T}{Y_T}\right) \left(\frac{1}{\overline{\xi}_T}\right) \sum_{i\geq 1} \Lambda^{QE}_T[i,j] \left(\omega_T[i] - \omega^+_{T-1}[i]\right). \tag{62}$$

where $\Lambda_T^{QE}[i,j] \equiv Y_T \Lambda[i,j] \left(\frac{\Gamma}{\hat{\beta}}\right)^j$ and $\omega_{T-1}^{+}[i] \equiv \hat{\beta}^{-1} \omega_{T-1}[i+1]$. [31]

The final expression (35) follows from substituting the budget constraint (50), and equations (58) and (62) for all j. □

Lemma 10. *Corollary 6 holds*

In a stationary economy

$$Q_T^i \approx Q = \beta \Gamma^{1-1/IES} \quad \frac{B_{T-1}^{i+1}}{B_T} \approx \Gamma^{-1} \omega_{T-1}[i+1].$$

From the definition of $\omega_T^{+}[i] = \frac{Q_T^i B_{T-1}^{i+1}}{B_T}$, we get that

$$\omega_T^{+}[i] = \beta \Gamma^{-1/IES} \omega_{T-1}[i+1]. \tag{63}$$

When $\Sigma^X = \Sigma^A = 0$, from Lemma 2 we get $\omega_T^{*} = \left(1 - \hat{\beta}\right) \overrightarrow{\hat{\beta}}$ and substituting for $\hat{\beta} = \beta \Gamma^{1-1/IES}$, we obtain

$$\omega_{T-1}^{*}[i+1] = \left(1 - \beta \Gamma^{1-1/IES}\right) \left(\beta \Gamma^{1-1/IES}\right)^{i+1}. \tag{64}$$

Substitute in (64) in (63) to get

$$\omega_T^{+}[i] = \left(1 - \beta \Gamma^{1-1/IES}\right) \left(\beta \Gamma^{1-1/IES}\right)^{i} = \omega_T^{*}[i].$$

The statement of the corollary follows from noticing that the second term in equation (35) drops out when $\omega_T^{+} = \omega_T^{*}$.

B Appendix: Empirical analysis

B.1 Results reported in Section 4

B.1.1 Data

Output, expenditures, tax revenues

We use the U.S. national income and product accounts to measure output, tax revenues. For our measure of output $Y_t^{\$}$ we use U.S. GDP. We measure nominal tax revenues $\mathcal{T}_t^{\$}$ as Federal Total Current Tax Receipts + Federal Contribution To Social Insurance and public expenditures $G_t^{\$}$ as Federal Consumption Expenditures + Federal Transfer Payments To Persons from BEA. All series are nominal and de-trended with constant time trends.

[31] Lemma 7 continues to hold in the stationary preferred habitat economy because $\bar{r}_{T+1}^{j} = \mathbb{E}_T \partial_\sigma r_{T+1}^{j} = 0$.

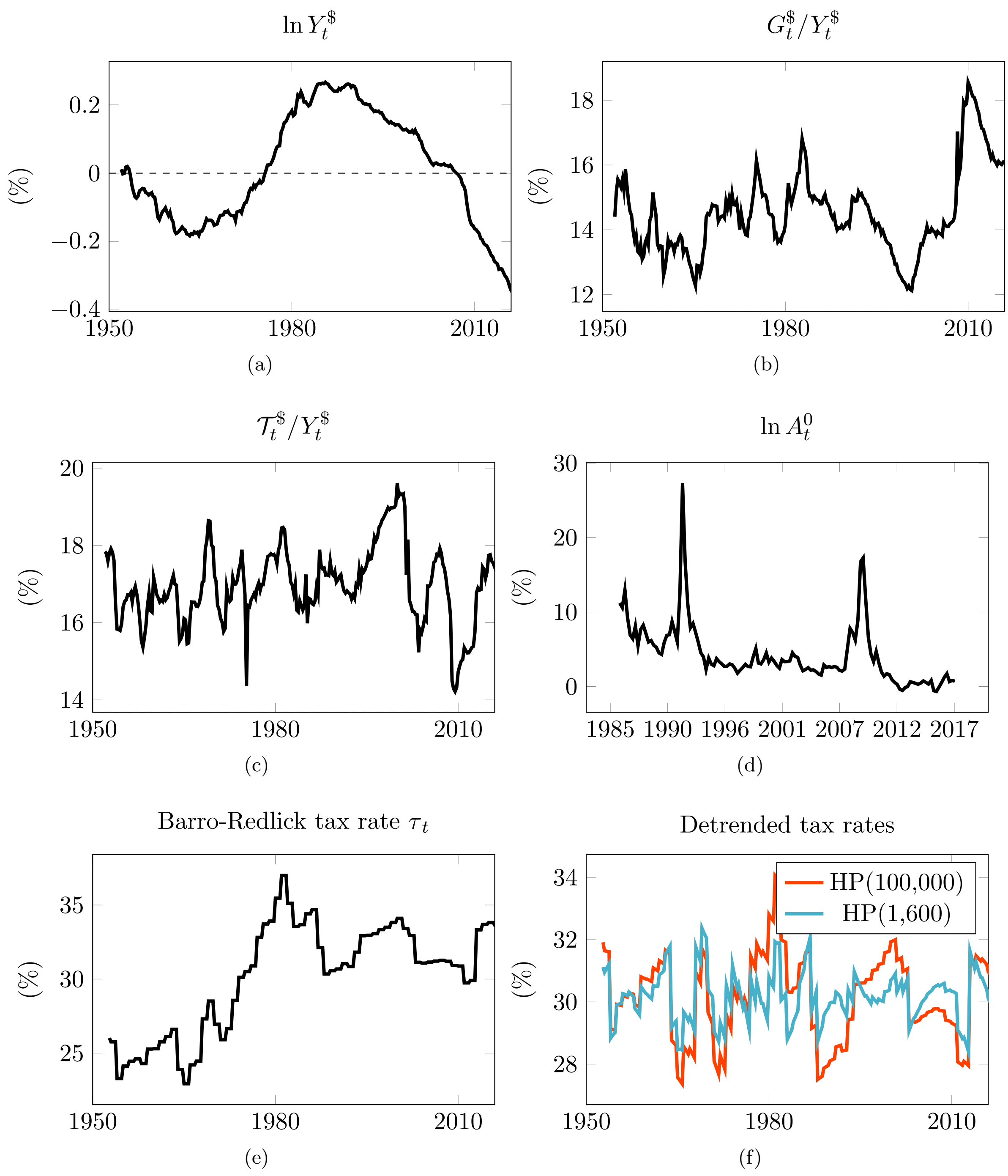


Figure 5: Summary of macroeconomic time series. Panel (a) plots detrended log nominal GDP, panel (b) plots the nominal government expenditure measured as Federal Consumption Expenditures + Federal Transfer Payments To Persons divided by nominal GDP, panel (c) plots nominal revenues divided by nominal GDP, panel (d) plots the imputed 3 month return on privately-issued debt, panel (e) plots the average marginal tax rate on income, panel (f) plots two ways of detrending the series in panel (e).

Tax rates

As a measure of tax rates τ_t we use the measure of the average marginal federal tax rate from Barro and Redlick (2011). Their series end in 2012 but we follow their steps and extrapolate this series for the years 2013-2017 using the Statistics of Income publicly available data from the Taxstats website. The series for the raw tax rates are plotted in Figure 5(e). It is clear from the series that there is a structural break in taxes around 1975. In our analysis we use want to focus on movements in taxes around business cycle frequencies and therefore we want to remove this break. We pursue two ways of doing that. First, we follow the business cycle literature and apply a Hodrick-Prescott (HP) filter with the penalty parameter set to 1,600. The resulting series is shown as the teal-blue line in the right panel Figure 5(f). While this procedure eliminates the low frequency movements in taxes, it also makes the resulting series "too smooth" post 1975. As an alternative, we adjust the penalty parameter until we achieve both goals: remove low frequency movements around 1975 and preserve the volatility of tax rates after and before 1975. The resulting series is show in the red line (at a penalty parameter of 100,000) in the right panel. We use the red line as a baseline measure of tax rates, but all our results are virtually unchanged if we use the teal line instead (see sub-section B.1.4).

Asset returns and government portfolio of bonds

We use the Fama Maturity Portfolios published by CRSP. There are 11 such portfolios, out of which ten portfolios correspond to maturities of 2 to 20 quarter in 2 quarter intervals, and a final portfolio for maturities between 30 and 40 quarters. We use the convention that the upper cut-off for each maturity corresponds to j in the mapping of data to the theory. That is, we use returns on portfolio of bonds of maturities between 2 to 4 quarter to measure r_t^j to $j = 4$, between 4 to 6 quarters to measure r_t^j for $j = 6$, etc. With this convention $j = 40$ is the largest maturity. We aggregate monthly log-returns by summing them across months within each quarter.

To measure returns on private bonds we use the yield curve of High Quality Market (HQM) Corporate Bonds computed by the U.S. Treasury.[32] The yields are available for select maturities with the shortest one being one year, while our quarterly model requires imputing returns on 3-months private bonds. For our baseline dataset, we followed McCulloch (1975) and interpolated the nominal bond yields using cubic splines and then used that interpolation to obtain the 3-month returns. We experimented with alternative extrapolation procedures, such

[32]The data can be accessed at https://www.treasury.gov/resource-center/economic-policy/corp-bond-yield/pages/corp-yield-bond-curve-papers.aspx

as using quadratic splines, and did not find any meaningful effect on our results. We use these returns to construct the liquidity premium $\ln A^0_t$.

Maturity structure of the U.S. government debt

We use the CRSP Treasuries Monthly Series to get the amount outstanding B^i_t for all (including TIPS and other inflation-protected bonds) federally issued (marketable) debt between 1952 and 2017, normalized by its face value. Each bond is uniquely identified by its cusips number n. CRSP also supplies us the Macaulay duration i for the outstanding amount, and the nominal market price $Q^{n,i}_t$ of each bond outstanding. For a few bonds where duration is absent, we set the duration equal to maturity date $-$ current date.

We follow Jiang et al. (2019), and construct at each date t, the market value $Q^i_t B^i_t$ held by the US government in bonds of Macaulay duration i, by summing across cusips n, such that $Q^i_t B^i_t = \sum_n Q^{n,i}_t B^{n,i}_t$. We then sum across all Macaulay duration i to get the market value of the government debt portfolio $B_t \equiv \sum_{i \in \mathcal{G}_t} Q^i_t B^i_t$ at each date t. We finally compute the portfolio weight in the US government debt portfolio for each maturity i using that $\omega^i_t = \frac{Q^i_t B^i_t}{B_t}$.

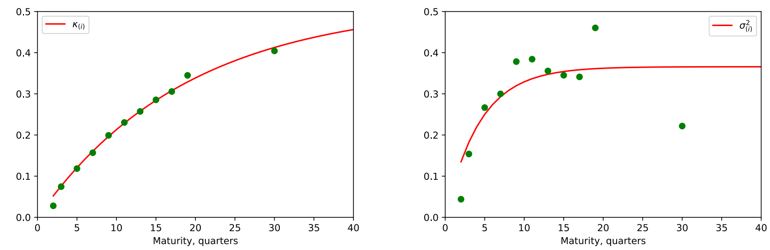
B.1.2 Estimations and extrapolations

We estimate model our factor model (21) using OLS. In the main text (Table 2), we report the estimates for the baseline specification in which we restricted $\rho_G = \rho_Y = 1$ and $\rho_f = 0$. This estimation procedure produces estimates of $\left(\alpha_j, \rho_j, \kappa_j, \sigma^2_j\right)$ for eleven j, with the highest being $j = 40$. For constructing our target portfolios, we need to extrapolate $\left(\rho_j, \kappa_j\right)$ for all $j > 1$. In the baseline extrapolation, we estimate δ_j and σ^2_j by fitting the closest exponential function: $f(j) = e^0 - e^0 \exp(-e^1 \times j)$ for $f(j) \in \left\{\delta_j, \sigma^2_j\right\}$. We fit the parameters e^0 and e^1 to minimize sum of squares between fitted and actual values of δ_j and σ^2_j. Alternatively, we also experimented to linearly extrapolate between any two adjacent j, and assume that $\left(\kappa_j, \sigma^2_j\right) = \left(\kappa_{40}, \sigma^2_{40}\right)$ for $j > 40$. The point estimates and this extrapolation is reported in Figure 6(a). We also experimented with alternative extrapolation, presented in Figure 6(b) that we report in Section B.1.4.

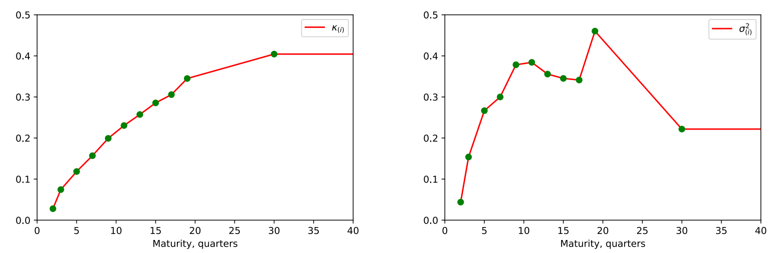
B.1.3 Deriving equations (22) and (23)

We start from the factor structure (21), which we rewrite in matrix form for each subscripts $k \in \{Y, G, A\}$ as:

$$\mathbb{A}^k \begin{bmatrix} z^k_{t+T} \\ f_{T+t} \end{bmatrix} = \boldsymbol{\alpha}^k + \mathbb{B}^k \begin{bmatrix} z^k_{t+T-1} \\ f_{T+t-1} \end{bmatrix} + \boldsymbol{\varepsilon}^k_{T+t},$$



(a) Fit for extrapolation of the factor model estimates of $\left(\kappa_j, \sigma_j^2\right)$ using $f(j) = e^0 - e^0 \exp(-e^1 \times j)$(21)



(b) Fit for extrapolation of the factor model estimates of $\left(\kappa_j, \sigma_j^2\right)$ using linear splines(21)

Figure 6

where we have stacked the coefficients as follows:

$$\boldsymbol{\alpha}^k \equiv \begin{bmatrix} \alpha_y \\ \alpha_f \end{bmatrix}, \mathbb{A}^k \equiv \begin{bmatrix} 1 & -\kappa_k \\ 0 & 1 \end{bmatrix}, \mathbb{B}^k = \begin{bmatrix} \rho_k & 0 \\ 0 & \rho_f \end{bmatrix}, \boldsymbol{\varepsilon}_{T+t}^k = \begin{bmatrix} \varepsilon_{T+t}^k \\ \varepsilon_{f,T+t} \end{bmatrix}.$$

We then invert this VAR(1) representation to get to the vector MA(t):

$$\begin{aligned}
\begin{bmatrix} z_{T+t}^k \\ f_{T+t} \end{bmatrix} &= \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\alpha}^k + \left(\mathbb{A}^k\right)^{-1} \mathbb{B}^k \begin{bmatrix} z_{T+t-1}^k \\ f_{T+t-1} \end{bmatrix} + \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\varepsilon}_{T+t}^k \\
&= \left(\mathbb{A}^k\right)^{-1} \left[\boldsymbol{\alpha}^k + \mathbb{B}^k \begin{bmatrix} z_{T+t-2}^k \\ f_{T+t-2} \end{bmatrix}\right] + \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\varepsilon}_{T+t}^k + \left(\mathbb{A}^k\right)^{-1} \mathbb{B}^k \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\varepsilon}_{T+t-1}^k \\
&\vdots \\
&= \mathbb{E}_T\left(\begin{bmatrix} z_{T+t}^k \\ f_{T+t} \end{bmatrix}\right) + \sum_{\tau=0}^{t-1} \left(\left(\mathbb{A}^k\right)^{-1} \mathbb{B}^k\right)^\tau \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\varepsilon}_{T+t-\tau}^k.
\end{aligned}$$

From the first row of this vector MA(t)representation, we can read MA(t) representation for each component z_{T+t}^k:

$$z_{T+t}^k = \mathbb{E}_T\left[z_{T+t}^k\right] + \sum_{\tau=0}^{t-1} \left[\rho_k^\tau \varepsilon_{T+t-\tau}^k + \frac{\kappa_k\left(\rho_f^{\tau+1} - \rho_k^{\tau+1}\right)}{\rho_f - \rho_k} \varepsilon_{f,T+t-\tau}\right].$$

We use the MA(t) representation to obtain formula for the matrices Σ_T^{-1}, Σ_T^Q, Σ_T^A, Σ_T^X. First, note that for any k and any $t \geq 1$:

$$cov_T\left(z_{T+t}^k, r_{T+1}^j\right) = cov_T\left(\rho_k^{t-1}\varepsilon_{T+1}^k + \frac{\kappa_k\left(\rho_f^t - \rho_k^t\right)}{\rho_f - \rho_k}\varepsilon_{f,T+1}, \varepsilon_{T+1}^j + \kappa_j\varepsilon_{f,T+1}\right)$$
$$= \frac{\left(\rho_f^t - \rho_k^t\right)}{\rho_f - \rho_k}\kappa_k\kappa_j\sigma_f^2 + \iota_{\{k=j\}}\sigma_j^2.$$

Applying that formula for $t = 1$ and $k = j$, we get

$$\Sigma_T[j,t] = cov_T\left(r_{T+1}^j, r_{T+1}^t\right) = \kappa_j\kappa_t\sigma_f^2 + \iota_{\{t=j\}}\sigma_j^2.$$

Furthermore we can easily check that, using that $\chi^{-2} = \sigma_f^{-2} + \sum_{t\in\mathcal{G}}\kappa_t^2\sigma_t^{-2}$,

$$\Sigma^{-1}[i,j] = \iota_{\{i=j\}}\sigma_i^{-2} - \chi^2\kappa_i\kappa_j\sigma_i^{-2}\sigma_j^{-2}.$$

Using Corollary 2, when the set of government securities consists of the full set of pure discount bonds, we have $\Sigma_T^Q[j,t] \simeq \Sigma_T[j,t]$, and hence

$$\Sigma_T^Q[j,t] \simeq \kappa_j\kappa_t\sigma_f^2 + \iota_{\{t=j\}}\sigma_j^2.$$

We then use stationarity and the definition of $X_{T+t}^{\perp,\$} \equiv \Gamma\left[\mathcal{T}_T^\$ \times \ln Y_{T+t}^{\perp,\$} - G_T^\$ \times \ln G_{T+t}^\$\right]$ to get that

$$\Sigma_T^X[j,t] = cov_T\left(\frac{X_{T+t}^{\perp,\$}}{\mathbb{E}_T Y_{T+t}^\$}, r_{T+1}^j\right)$$
$$= \frac{\mathcal{T}_T^\$}{Y_T^\$}cov_T\left(\ln Y_{T+t}^{\perp,\$}, r_{T+1}^j\right) - \frac{G_T^\$}{Y_T^\$}cov_T\left(\ln G_{T+t}^\$, r_{T+1}^j\right)$$
$$= \kappa_j\left(\frac{\mathcal{T}_T^\$}{Y_T^\$}\kappa_Y\frac{\rho_f^t - \rho_Y^t}{\rho_f - \rho_Y} - \frac{G_T^\$}{Y_T^\$}\kappa_G\frac{\rho_f^t - \rho_G^t}{\rho_f - \rho_G}\right)\sigma_f^2. \tag{65}$$

Finally, we use the definition of $A_{t,k} \equiv A_t^0 \times ... \times A_{t+k-1}^0$ and that $\overline{A_t^0} = 1$ to compute

$$\Sigma_T^A[j,t] = cov_T\left(A_{T+1,t}, r_{T+1}^j\right)$$
$$= cov_T\left(A_{T+1}^0 \times ... \times A_{t+T}^0, r_{T+1}^j\right)$$
$$\simeq \sum_{\ell=0}^{t-1} cov_T\left(A_{T+1+\ell}, r_{T+1}^j\right)$$
$$\simeq \sum_{\ell=0}^{t-1}\frac{\left(\rho_f^{\ell+1} - \rho_A^{\ell+1}\right)}{\rho_f - \rho_A}\kappa_A\kappa_j\sigma_f^2$$
$$\simeq \frac{\kappa_A\kappa_j\sigma_f^2}{\rho_f - \rho_A}\left[\rho_f\frac{1-\rho_f^t}{1-\rho_f} - \rho_A\frac{1-\rho_A^t}{1-\rho_A}\right]. \tag{66}$$

In the third line, we use the $\simeq$ sign because we take a first-order approximation of the product of A^0. Note that in our baseline case with $\rho_G = \rho_Y = 1$ and $\rho_f = 0$, those formula simplify to

$$\Sigma_T^X [j,t] = \kappa_j \left(\frac{\mathcal{T}_T^{\$}}{Y_T^{\$}} \kappa_Y - \frac{G_T^{\$}}{Y_T^{\$}} \kappa_G \right) \sigma_f^2, \Sigma_T^A [j,t] = \kappa_A \kappa_j \sigma_f^2 \frac{1-\rho_A^t}{1-\rho_A}.$$

We can now compute the three components determining the portfolio allocation. First note that:

$$\begin{aligned}
\sum_{\ell \in \mathcal{G}} \Sigma_T^{-1} [j,\ell] \kappa_\ell &= \sum_{\ell \in \mathcal{G}} \left(\iota_{\{j=\ell\}} \sigma_j^{-2} - \chi^2 \kappa_j \kappa_\ell \sigma_j^{-2} \sigma_\ell^{-2} \right) \kappa_\ell \\
&= \kappa_j \sigma_j^{-2} \left[1 - \frac{\sum_{\ell \in \mathcal{G}} \kappa_\ell^2 \sigma_\ell^{-2}}{\sigma_f^{-2} + \sum_{t \in \mathcal{G}} \kappa_t^2 \sigma_t^{-2}} \right] \\
&= \frac{\kappa_j \sigma_j^{-2}}{1 + \sum_{t \in \mathcal{G}} \kappa_t^2 \sigma_f^2 \sigma_t^{-2}}.
\end{aligned}$$

The primary surplus component $\pi_T^X \Sigma_T^{-1} \Sigma_T^X \overrightarrow{\beta}$ is given by

$$\begin{aligned}
\pi_T^X \Sigma_T^{-1} \Sigma_T^X \overrightarrow{\beta} [j] &= \pi_T^X \left(\sum_{\ell \geq 1} \Sigma_T^{-1} [j,\ell] \kappa_\ell \right) \left(\sum_{t \geq 1} \left(\frac{\mathcal{T}_T^{\$}}{Y_T^{\$}} \kappa_Y \frac{\rho_f^t - \rho_Y^t}{\rho_f - \rho_Y} - \frac{G_T^{\$}}{Y_T^{\$}} \kappa_G \frac{\rho_f^t - \rho_G^t}{\rho_f - \rho_G} \right) \sigma_f^2 \hat{\beta}^t \right) \\
&= K_{X,T} \frac{\kappa_j}{\sigma_j^2} \chi^2,
\end{aligned}$$

with $K_{X,T} = \pi_T^X \hat{\beta} \left(\frac{\mathcal{T}_T^{\$}}{Y_T^{\$}} \frac{\kappa_Y}{\rho_f - \rho_Y} \left[\frac{\rho_f}{1-\hat{\beta}\rho_f} - \frac{\rho_Y}{1-\hat{\beta}\rho_Y} \right] - \frac{G_T^{\$}}{Y_T^{\$}} \frac{\kappa_G}{\rho_f - \rho_G} \left[\frac{\rho_f}{1-\hat{\beta}\rho_f} - \frac{\rho_G}{1-\hat{\beta}\rho_G} \right] \right)$. Similarly, the liquidity component $\pi_T^A \Sigma_T^{-1} \Sigma_T^A \overrightarrow{\beta}$ is

$$\begin{aligned}
\pi_T^A \Sigma_T^{-1} \Sigma_T^A \overrightarrow{\beta} [j] &\simeq \pi_T^A \left(\sum_{\ell \geq 1} \Sigma_T^{-1} [j,\ell] \kappa_\ell \right) \left(\sum_{t \geq 1} \frac{\kappa_A \sigma_f^2}{\rho_f - \rho_A} \left[\rho_f \frac{1-\rho_f^t}{1-\rho_f} - \rho_A \frac{1-\rho_A^t}{1-\rho_A} \right] \hat{\beta}^t \right) \sigma_f^2 \\
&= K_{A,T} \frac{\kappa_j}{\sigma_j^2} \chi^2,
\end{aligned}$$

with $K_{A,T} = \pi_T^A \hat{\beta} \left(\frac{\kappa_A}{\rho_f - \rho_A} \left[\frac{\rho_f}{1-\rho_f} \left[\frac{1}{1-\hat{\beta}} - \frac{\rho_f}{(1-\hat{\beta}\rho_f)} \right] - \frac{\rho_A}{1-\rho_A} \left[\frac{1}{1-\hat{\beta}} - \frac{\rho_A}{(1-\hat{\beta}\rho_A)} \right] \right] \right)$. Finally, the interest rate risk component $\pi^Q \Sigma_T^{-1} \Sigma_T^Q \overrightarrow{\beta}$ is given by

$$\begin{aligned}
\pi^Q \Sigma_T^{-1} \Sigma_T^Q \overrightarrow{\beta}\,[j] &= \pi^Q \left(\sum_{\ell \in \mathcal{G}} \Sigma_T^{-1}[j,\ell]\, \kappa_\ell \right) \left(\sum_{t \geq 1} \left(\kappa_t \sigma_f^2 + \frac{\iota_{\{t=\ell\}} \sigma_\ell^2}{\kappa_\ell} \right) \hat{\beta}^t \right) \\
&= \pi^Q \kappa_j \sigma_j^{-2} \left[\frac{\sum_{t \geq 1} \kappa_t \sigma_f^2 \hat{\beta}^t}{\sigma_f^{-2} + \sum_{t \in \mathcal{G}} \kappa_t^2 \sigma_t^{-2}} \right] \\
&\quad \cdots + \pi^Q \left(\sum_{\ell \in \mathcal{G}} \left[\iota_{\{j=\ell\}} \sigma_j^{-2} - \frac{\kappa_j \kappa_\ell \sigma_j^{-2} \sigma_\ell^{-2}}{\sigma_f^{-2} + \sum_{t \in \mathcal{G}} \kappa_t^2 \sigma_t^{-2}} \right] \sigma_\ell^2 \hat{\beta}^\ell \right) \\
&= \hat{\beta}^j + K_{Q,T} \frac{\kappa_j}{\sigma_j^2} \chi^2,
\end{aligned}$$

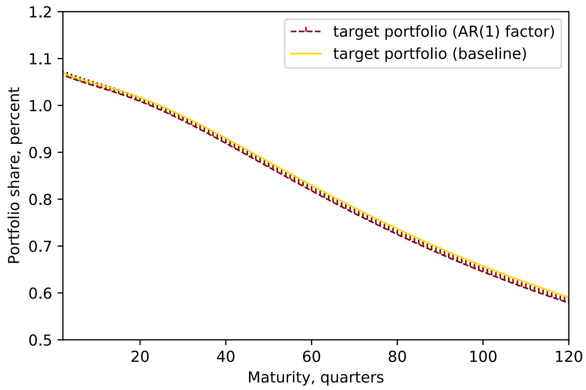
where $K_{Q,T} = \pi^Q \left[\sum_{t \geq 1} \kappa_t \hat{\beta}^t - \sum_{t \in \mathcal{G}} \kappa_t \hat{\beta}^t \right]$. From here it is easy to see that as the number of maturities go to infinity, $K_{Q,T} \to 0$.

In our baseline case with $\rho_G = \rho_Y = 1$ and $\rho_f = 0$, these formula simplify to expressions (22) and (23).

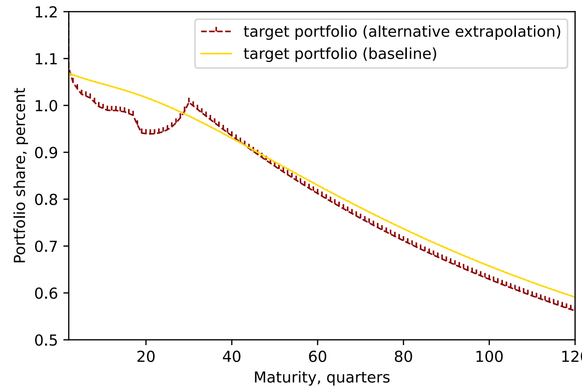
We quantify the coefficients K_X, K_A, K_Q for the capped portfolio using the estimates from our factor model. The constants $\chi^2 = \underbrace{\frac{1}{\underbrace{\sigma_f^{-2}}_{0.015} + \underbrace{\sum_{i \in \mathcal{G}} \kappa_i^2 \sigma_i^{-2}}_{74.76}}}_{0.013}$, $\pi_T^X = \underbrace{\underbrace{\hat{\beta}^{-1}}_{1.015} \Gamma \underbrace{Y_T / B_T}_{0.25}}_{0.2537}$, $\pi_T^A = \underbrace{\underbrace{\Gamma}_{1.005} \underbrace{\zeta_T}_{0.605} \underbrace{Y_T / B_T}_{0.25}}_{0.152}$, $\pi^Q = \underbrace{1 - \hat{\beta}}_{0.01}$ and

$$K_{X,T} = \underbrace{\underbrace{\left(\frac{\hat{\beta}}{1 - \hat{\beta}} \right)}_{98.99} \underbrace{\left(\underbrace{\kappa_Y}_{-0.47} \underbrace{\frac{\mathcal{T}_T^\$}{Y_T^\$}}_{0.17} - \underbrace{\kappa_G}_{-0.0317} \underbrace{\frac{G_T^\$}{Y_T^\$}}_{0.15} \right)}_{-0.00321}}_{-0.320},$$

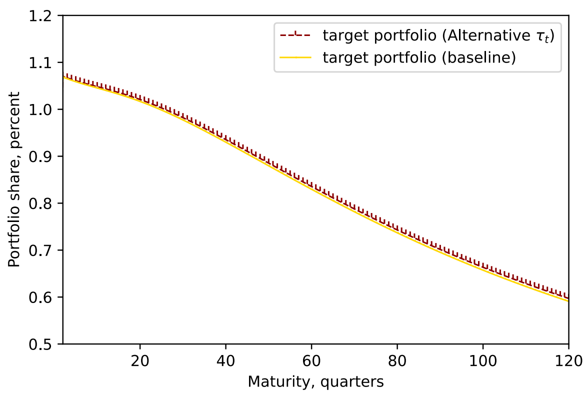
$$K_{A,T} = \underbrace{\underbrace{\left(\frac{\underbrace{\kappa_A}_{0.00098} \underbrace{\hat{\beta}}_{0.99}}{1 - \underbrace{\rho_A}_{0.827}} \right)}_{0.0056} \underbrace{\left[\underbrace{\frac{1}{1 - \hat{\beta}}}_{99.99} - \underbrace{\frac{\overbrace{\rho_A}^{0.827}}{1 - \underbrace{\hat{\beta} \rho_A}_{0.819}}}_{4.59} \right]}_{95.4}}_{0.534},$$



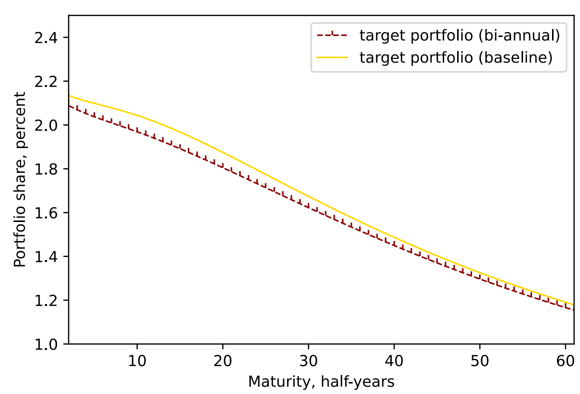
(a) Autocorrelated factor f_t



(b) Alternative extrapolation of the factor model (21)



(c) Alternative tax series



(d) Bi-annual frequency

Figure 7: The capped target portfolio $\overrightarrow{\omega}_T^*$ for various alternatives (in yellow) versus the baseline capped target portfolio (in red)

$$K_{Q,T} = \underbrace{\left(\sum_{t \notin G}^{\infty} \hat{\beta}^t \kappa_t\right)}_{0.153}.$$

We see that the magnitude of $\pi_T^X K_{X,T}$ is roughly equal to $\pi_T^A K_{A,T}$ but they have opposite signs and is much smaller than $\left(1 - \hat{\beta}\right)$. This explains why the portfolio that hedges primary surplus offsets the portfolio that hedges liquidity risks but both are less important than hedging interest rate risks.

Table 3: FACTOR MODEL ESTIMATION RESULTS (AR(1) FACTOR STRUCTURE)

	Excess returns r_t^j for various maturities j														
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	$\ln G_t^{\$}$	$\ln Y_t^{\perp,\$}$	$\ln A_t^0$	f_t
α_k	0.086 (0.014)	0.155 (0.025)	0.220 (0.033)	0.245 (0.035)	0.284 (0.039)	0.315 (0.039)	0.346 (0.038)	0.344 (0.037)	0.372 (0.037)	0.304 (0.043)	0.444 (0.030)	-0.177 (0.016)	-0.319 (0.008)	0.006 (0.003)	0.026 (0.502)
ρ_k	-0.107 (0.043)	-0.057 (0.035)	-0.041 (0.030)	-0.043 (0.025)	-0.042 (0.023)	-0.025 (0.020)	-0.022 (0.018)	-0.008 (0.016)	-0.022 (0.015)	-0.027 (0.015)	0.003 (0.009)	1.001 (0.008)	1.009 (0.004)	0.828 (0.047)	-0.035 (0.063)
κ_k	0.028 (0.002)	0.074 (0.003)	0.118 (0.004)	0.157 (0.004)	0.199 (0.005)	0.230 (0.005)	0.257 (0.005)	0.285 (0.005)	0.306 (0.005)	0.345 (0.005)	0.404 (0.004)	-0.032 (0.016)	-0.048 (0.008)	0.001 (0.000)	0.000 (nan)
σ_k^2	0.044 (0.004)	0.154 (0.014)	0.267 (0.024)	0.300 (0.027)	0.378 (0.034)	0.384 (0.034)	0.356 (0.031)	0.345 (0.031)	0.341 (0.030)	0.460 (0.041)	0.222 (0.020)	4.231 (0.375)	1.125 (0.100)	0.000 (0.000)	63.676 (5.65)
R2	0.536	0.698	0.771	0.840	0.870	0.898	0.922	0.938	0.946	0.943	0.979	0.985	0.996	0.727	0.001

Notes: This table records the OLS estimates of the factor model (21) without imposing $\rho_f = 0, \rho_Y = \rho_G = 1$. Standards errors are in parenthesis. The sample for excess returns and primary surpluses normalized by outputs is 1952-2017, and the sample for the one-period liquidity premium is 1984-2017. The time period is a quarter.

B.1.4 Robustness

AR(1) factor structure In this section, we consider the general estimation of (21) without any a-priori restrictions on parameters. Table 3 presents estimation results.

The construction of matrices Σ_T^{-1} and Σ_T^Q remain unchanged while matrices Σ_T^A and Σ_T^X are now constructed using expressions (65) and (66). We construct the capped target portfolio implied by the general AR(1) structure and compare to our baseline portfolio in panel (a) of Figure 7.

Alternative extrapolation method, tax series, time aggregation, calculation of returns In Section B.1.2 we discussed an alternative extrapolation procedure for coefficients $\left(\kappa_j, \sigma_j^2\right)$, while in Section B.1.1 we presented an alternative procedure to de-trend tax series. None of these alternative approached affect our conclusions in any meaningful way. We reported the implied unrestricted target portfolios under this alternative procedures respectively in panel (b) and (c) of Figure 7.

We also experimented with alternative ways to calculate returns with different time frequencies. In the baseline, we used quarterly measures of returns, surpluses and taxes to ensure the largest sample such that we could measure asset prices and macro data in a consistent way. To verify if our results are driven by our choice of the frequency, we use returns and other macro variables at biannual frequencies. The shortest maturity available is now of 6 months, which we take as our measure of the one-period government bond R_t^0. As before, we construct the biannual holding period return by summing monthly returns for each portfolio which are separated by 6 month intervals. For other macro variables, we aggregate two consecutive quarters to obtain the biannual series. Using this data, we apply the same procedure as the baseline (extracting the factor, estimating the factor model, constructing the conditional covariances) and obtain the optimal portfolio. We show the implied unrestricted target portfolios in the panel (d) of Figure 7. In order to compare it to our baseline results which have portfolios by quarterly bins, we aggregate the baseline portfolio weights to biannual weights using $\omega_{biannual}[i] \equiv \omega[2i-1] + \omega[2i]$,where i indexes the 6 month intervals and the right hand size is the baseline target portfolio. We find that that the two biannual portfolios are very similar.

B.2 Results reported in Section 5

B.2.1 Additional discussion for Section 5.1

To simulate the neoclassical model, we solve a complete markets Ramsey allocation as in Lucas and Stokey (1983) by posing the following maximization problem. Given some $t = 0$ state $s_0 \in \mathcal{S}$ and household savings $b_0\left(s^0\right)$, the Ramsey problem can be expressed as

$$\max_{c_t(s^t), y_t(s^t)} \mathbb{E}_0 \sum_{t=0}^{\infty} U\left(c_t, \frac{y_t}{\theta_t}\right) \tag{67}$$

subject to

$$y_t\left(s^t\right) = c_t\left(s^t\right) + G\left(s_t\right), \tag{68}$$

$$b_0\left(s^0\right) u_c\left(s^0\right) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right)\left[u_c\left(s^t\right) c_t\left(s^t\right) + u_y\left(s^t\right) y_t\left(s^t\right)\right], \tag{69}$$

where the *implementability constraints,* equation (69) is derived by taking the time-0 budget constraint and replacing after-tax wages as well as bond prices.

We assume that the state space $\mathcal{S}$ is discrete (described below) and non-linearly solve the optimal allocation using the first-order conditions of the Ramsey planning problem. The resulting optimal allocation is represented using two sets of vectors of dimension $2|\mathcal{S}|$, one set for consumption and labor choices at $t = 0$ and another set for all $s_t \in \mathcal{S}$ for $t \geq 1$. Using the Ramsey allocation $\{c_t, y_t\}$, we can back out other related objects such tax rates $\tau_t = 1 - \frac{\left(\frac{y_t}{\theta_t}\right)^{\gamma}}{c_t^{-\sigma}}$; primary surplus $X_t = \tau_t y_t - G_t$; and zero-coupon bond prices $Q_t^n = \mathbb{E}_t \frac{c_{t+n}^{-\sigma}}{c_t^{-\sigma}}$.

We follow Buera and Nicolini (2004) and assume that the preferences of households are isoelastic $U\left(c_t, \frac{y_t}{\theta_t}\right) = \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{\left(\frac{y_t}{\theta_t}\right)^{1+\gamma}}{1+\gamma}$ with parameters $\sigma = 2$ and $\gamma = 1$. The economy is closed, so the demand of assets from foreign investors is zero and there are liquidity services provided by government bonds. The only source of uncertainty comes from the exogenous stochastic process of government expenditures G_t, which follows an AR(1) process

$$\ln G_t = \alpha_G + \rho_G \ln G_{t-1} + \sigma_G \epsilon_t$$

We set $(\alpha_G, \rho_G, \sigma_G)$ to obtain a mean G/Y of 15%, auto correlation of 0.95 and a standard deviation $\frac{1.2}{15}$ which are in line with the U.S. data that we use in Section 4.1. We discretize the $\ln G_t$ process with $|\mathcal{S}| = 50$ grid points. For our calculations, we set the level of initial debt B_0 so that the annualized initial level of government liabilities to GDP is 100%.

We use this parameterization to construct several versions of the optimal portfolio. First, for a given $s \in \mathcal{S}$, we apply Corollary to Theorem 1 in Angeletos (2002) and obtain the optimal portfolio $\omega_T^{CM,n}\left(s^T\right) = \omega^{CM,n}\left(s_T = s\right)$ for $n = 1, \ldots 50$ maturities that implements the complete markets allocation. We use the bond prices and present value of primary surpluses all of which can be backed out given the objects from the Ramsey allocation. In Figure 2, red color line, we plot $\left\{\omega^{CM,n}\right\}_{n=1}^{50}$ for $s_T = s_{24}$ which is the modal state.

Details for Figure 2 To obtain the target portfolio $\overrightarrow{\omega}_T$ given some history s^T, we need to solve for a vector of portfolio shares that satisfies

$$\Sigma_T \overrightarrow{\omega}_T = \left[\pi_T^Q \Sigma_T^Q + \pi_T^X \Sigma_T^X\right] \overrightarrow{\hat{\beta}}.$$

Before explaining how we get $\overrightarrow{\omega}_T$, we make two observations. First, given the properties of the Ramsey allocations, $\Sigma_T, \pi_T^Q, \pi_T^X, \Sigma_T^X$ only depend on state s_T, which we set to s_{24} and as before can be computed in closed form using the complete market allocation that we have already solved. Second, as mentioned in the main text the returns of different bonds are highly correlated in the neoclassical economy, which makes the matrix of returns Σ_T to be effectively non-invertible and there are a range of portfolios that satisfy inequality (26) for a given ϵ. To obtain the target portfolio that is plotted in Figure 2 blue color, we set $\epsilon = 1e-3$ and pose the following minimization problem

$$\min_{\tilde{\boldsymbol{\omega}}} \left\|\tilde{\boldsymbol{\omega}} - \overrightarrow{\omega}_T^{ABN}\right\|$$

such that

$$\left\|\Sigma_T \tilde{\boldsymbol{\omega}}^n - \left[\pi^Q \Sigma_T^Q + \pi_T^X \Sigma_T^X + \pi_T^A \Sigma_T^A\right] \overrightarrow{\hat{\beta}}\right\| \leq \mathbf{1}^\intercal \epsilon.$$

where $||\cdot||$ we mean the sup norm . This formulation conveniently delivers an objective that is quadratic while the constraint set is linear and convex; and we use a standard methods (OSQP library) to solve the minimization problem.

From the outcomes in Figure 2, there are two clear observations. First, even though Corollary 1 abstracted from income and price effects, the resulting target portfolio provides an excellent fit to the optimal portfolio that replicated the complete market allocation in this neoclassical economy. Second, the portfolio in the neoclassical economy are very different than our target portfolio computed using U.S. data. It has large negative (695 times annual GDP in risk-free assets) positions in the risk-free bond and large and offsetting positions in risky bonds with flipping signs. As a point of reference, in the target portfolio that is calibrated to US data, positive debt is issued in all maturities with the maximum being around 1 percent, and the share in the risk-free debt is quite small 0.7 percent.

Risk-free bond and consol We chose the first 50 maturity zero coupon bonds to keep the discussion closer to the market structure analyzed in Section 4. The formula Theorem 1 of Angeletos (2002) can be applied to any $|\mathcal{S}|$ securities. Furthermore, if we replace inverse in equation (12) of Angeletos (2002) of with pseudo-inverse[33], we can also obtain the formula for a set of securities smaller than $|\mathcal{S}|$. Our online codes are flexible to implement any market structure but since the general patterns are not that different we discuss only one special case, in which the market structure has a risk-free bond and consol. Besides being a case analyzed in detail in Angeletos (2002), an attractive feature of risk-free bond and consol market structure is that Σ_T is a scalar, and the target portfolio (also a scalar) is uniquely pinned down. In this case, using the modified Angeletos (2002) formula, we find that the holdings in the consol, ω^{consol} is 746 percent of annual GDP and an offsetting position of in the risk free bond ω^0 of -646 percent of annual GDP. The counterpart $\overrightarrow{\omega}_T = \left[\frac{\pi_T^Q \Sigma_T^Q}{\Sigma_T} + \frac{\pi_T^X \Sigma_T^X}{\Sigma_T}\right] \overrightarrow{\hat{\beta}}$ for our target portfolio, we get 677 percent in the consol and an offsetting -577 percent in the risk-free bond. This fact both the portfolios are so close to each other reasserts the validity of our formula for the target portfolio.

Next comparing the share of holdings in the consol (741 percent of total debt) to the sum of shares of the portfolio that hedges primary surplus risk $\mathbf{1}^\intercal \Sigma_T^{-1} \pi_T^X \Sigma_T^X$ (-15% in the baseline with 120 maturities and -17% in the theoretical unrestricted target portfolio which we estimate from the US data in Section 4), we find that it is about 40-50 times larger with an opposite sign.

Understanding the sources of differences between the optimal portfolios We want to understand the sources that drive the differences between the optimal portfolios computed

[33] A pseudoinverse is minimum (Euclidean) norm solution to a system of linear equations with multiple solutions. See also discussion in Section V.B in Angeletos (2002).

using U.S. data and the neoclassical model. First, we construct the counterpart of Table 1 using data simulated for 265 quarters from neoclassical model economy. The results are reported in Table 4 below. Compared to Table 1 in the main text, we see that returns in the simulated economy are much less volatile. For instance, for long maturities the variance of returns is between 0.025 and 0.035 which is 300 times smaller than what we get for the U.S. counterparts. The covariances of returns with primary surplus are only 10-20 times smaller signaling a much higher correlation. Furthermore, the sign of the covariance with primary surplus is positive for long maturities while it is negative in for the U.S. data.

Table 4: COVARIANCE MATRIX FOR NEOCLASSICAL MODEL

	Excess returns r_t^j for various maturities j											Surplus to GDP	Tax rate	Liquidity premium
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	X_t/Y_t	τ_t	$\ln A_t^0$
6m	0.00015	0.00042	0.00067	0.00089	0.0011	0.0013	0.0014	0.0016	0.0017	0.0019	0.0023	0.0029	0	0
12m		0.0012	0.0019	0.0026	0.0031	0.0037	0.0041	0.0046	0.0049	0.0053	0.0066	0.0083	0	0
18m			0.003	0.0041	0.005	0.0058	0.0066	0.0072	0.0079	0.0084	0.01	0.013	0	0
24m				0.0054	0.0066	0.0078	0.0088	0.0097	0.01	0.011	0.014	0.018	0	0
30m					0.0082	0.0095	0.011	0.012	0.013	0.014	0.017	0.022	0	0
36m						0.011	0.013	0.014	0.015	0.016	0.02	0.025	0	0
42m							0.014	0.016	0.017	0.018	0.023	0.029	0	0
48m								0.017	0.019	0.02	0.025	0.032	0	0
54m									0.02	0.022	0.027	0.034	0	0
60m										0.023	0.029	0.037	0	0
120m											0.036	0.046	0	0
X_t/Y_t												0.58	0	0
τ_t													0	0
$\ln A_t^0$														0
Mean	0.0008	0.0023	0.0036	0.0047	0.0058	0.0067	0.0076	0.0083	0.009	0.0096	0.012	4.1	18	0
Autocorr	0.11	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.92	0.84	0

We simulate the neoclassical model for 265 quarters that correspond to the sample period 1952-2017. Excess returns 6m, 12m, ... are the nominal excess returns in Fama maturity portfolios corresponding to 6-12 months, 12-18 months, ... maturity bins, respectively. All data is quarterly and in percentage points.

Our factor structure suggests a parsimonious way to understand why the neoclassical portfolio has the features we highlighted, that is, large savings in risk-free bonds and offsetting positions in risky assets. To see that, consider a limiting case when the market structure has bonds of all maturities. In the main text equations (22) provides closed-form expressions for the share in the risk-free security and the risky portfolio as a function of of the factor loadings and highlights the role of the ratio $\frac{K^X}{\kappa_\infty}$ in driving the differences.

	Excess returns r_t^j for various maturities j														
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	$\ln G_t^{\$}$	$\ln Y_t^{\perp,\$}$	$\ln A_t^0$	f_t
α_k	0.001	0.003	0.004	0.006	0.007	0.008	0.009	0.010	0.010	0.011	0.014	-0.098	5.503	0.000	-0.003
	(0.001)	(0.002)	(0.003)	(0.004)	(0.005)	(0.006)	(0.007)	(0.008)	(0.008)	(0.009)	(0.011)	(0.170)	(0.017)	(0.000)	(0.025)
ρ_k	-0.034	-0.035	-0.036	-0.037	-0.037	-0.038	-0.038	-0.039	-0.039	-0.040	-0.041	0.455	0.452	0.000	0.861
	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.060)	(0.025)	(0.025)	(0.000)	(0.032)
κ_k	-0.006	-0.019	-0.030	-0.039	-0.049	-0.057	-0.064	-0.071	-0.077	-0.082	-0.103	3.869	0.387	0.000	0.000
	(0.001)	(0.003)	(0.004)	(0.006)	(0.007)	(0.008)	(0.009)	(0.010)	(0.011)	(0.012)	(0.014)	(0.170)	(0.017)	(0.000)	(nan)
σ_k^2	0.000	0.001	0.003	0.004	0.007	0.009	0.012	0.014	0.017	0.019	0.030	1.908	0.019	0.000	0.163
	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.002)	(0.003)	(0.169)	(0.002)	(0.000)	(0.014)
R2	0.173	0.174	0.174	0.175	0.175	0.175	0.176	0.176	0.176	0.176	0.177	0.936	0.936	nan	0.745

Table 5: Factor model estimation results Neoclassical model

Notes: This table records the OLS estimates of the factor model (21) without imposing $\rho_f = 0, \rho_Y = \rho_G = 1$. Standards errors are in parenthesis. All series are for a 265 quarters that correspond to the sample period 1952-2017.

We follow the same steps as in Section 4.2 and estimate the factor model but now using data simulated from the neoclassical economy. We make two changes relative to the baseline. We allow for ρ_G and ρ_Y to be smaller than 1 as the neoclassical model had AR(1) as the data generating process. Second, to estimate κ_∞ we extrapolate using $\kappa_j = a^0 - a^1 \exp\left(-a^2 \times j\right)$.[34] Plugging in for the values of K^X, κ_∞, we obtain $\frac{K^X}{\kappa_\infty}$ equals 9.72 in the neoclassical economy as compared to -0.6 using the U.S. data. Thus, from the lens of our expression (25), a big part of the differences between the optimal portfolios can be understood from the fact that $\frac{K^X}{\kappa_\infty}$ in the neoclassical model is much larger and of opposite sign from what we obtained in Section 4.2 using US data. Qualitatively it implies a larger and position with opposite opposite in the portfolio that hedges primary surplus risk. Quantitatively, the finding that $\frac{K^X}{\kappa_\infty}$ is about 20 times larger than in the neoclassical economy explains well the gaps in the portfolio that hedges the primary surplus risk. The reason for this can be traced back to observations we made about covariance of returns with each other and primary deficits in Table 4.

B.2.2 Additional discussion for Section 5.2

Heteroskedastic shocks In the main text, we assumed that the shocks ε_t were homoskedastic, that is, we imposed that $\{\sigma_k\}$ for $k \in \{j, Y, G, A, f\}$ are constant through time. We relax that assumption and augment the baseline factor model 21 with the following univariate GARH

[34]This is a slightly general version so that its fits the neoclassical data well. Our results are virtually the same if we use the linear splines extrapolation as in the Online Appendix B.1.2.

processes $\{\sigma_k\}$

$$\sigma_{k,t}^2 = \overline{\sigma}_k^2 + \sum_{j=1}^{p} \rho_{kp}^{GARCH} \varepsilon_{zt-p}^2 + \sum_{j=1}^{q} \varrho_{kq}^{GARCH} \sigma_{\varepsilon z,t-q}^2$$

and impose that all ε are standard Gaussian and independent of each other. We now estimate the system using maximum likelihood and assuming $p = 2$ and $q = 1$.

The consequence of heteroskedastic shocks is that structure of the expressions for Σ_T and Σ_T^{-1} as well as $\Sigma_T^k$for $k \in \{X, A, Q\}$ remains the same but they have time-varying parameters $\sigma_{f,t}$ and $\sigma_{j,t}$ for each return maturities j.[35] We use the same extrapolation scheme as the baseline to obtain (σ_j, κ_j) for other maturities. And finally, as an implication, the optimal target portfolio and its components also inherit that time-variation.[36]

Results In Figure 8, we plot the time-series for elements in $\{\sigma_{j,t}\}$ and $\sigma_{f,t}$. The volatilities for returns (including the factor) and macroaggregates are high in the early 80s and the great recession of 2008-2010 and quite stable in the intervening periods.

Keeping everything else the same, periods when the factor is more volatile increases the covariance of returns with each other as well as the covariance of returns with surpluses and liquidity risk. Thus, a priori the effect on the optimal portfolio is ambiguous. To gauge how much the portfolio moves overtime, we start by plotting in Figure 9), the 90-10 interval by maturity, that is, for each maturity we construct the 90th and 10th percentile across dates. We see that for lower maturities the portfolio shares varies by as much as 20-25 basis points and the fluctuations are much smaller for larger maturities.

To understand the sources of this variation, we separate out the primary surplus risk portfolio and the liquidity risk portfolios using expressions (22) and report the sum of the portfolio shares across maturities for every period. In Figure 10, we see that both these shares are quite stable through time, and more or less offset each other.

B.3 Additional details for Section 5.3

In this section we estimate excess liquidity premia a^i and statistically test $\mathbb{E}a_t^i = 0$. We first describe the estimation framework and then our findings.

[35]The time-variation in $\{\sigma_G^2, \sigma_Y^2, \sigma_A^2\}$ drops out because the covariances of hedging terms are driven by the common component captured in the factor $\{\sigma_{f,t}^2\}$.

[36]In principle, the fiscal risk and liquidity risk portfolio could vary because quasi-weights π_T^X and π_T^A or $\vec{\beta}$ vary with time. To focus on the impact of heteroskedastic shocks, we keep them constant and equal to the values that we used in the main text and only allow the target portfolio to vary due to time-varying covariances.

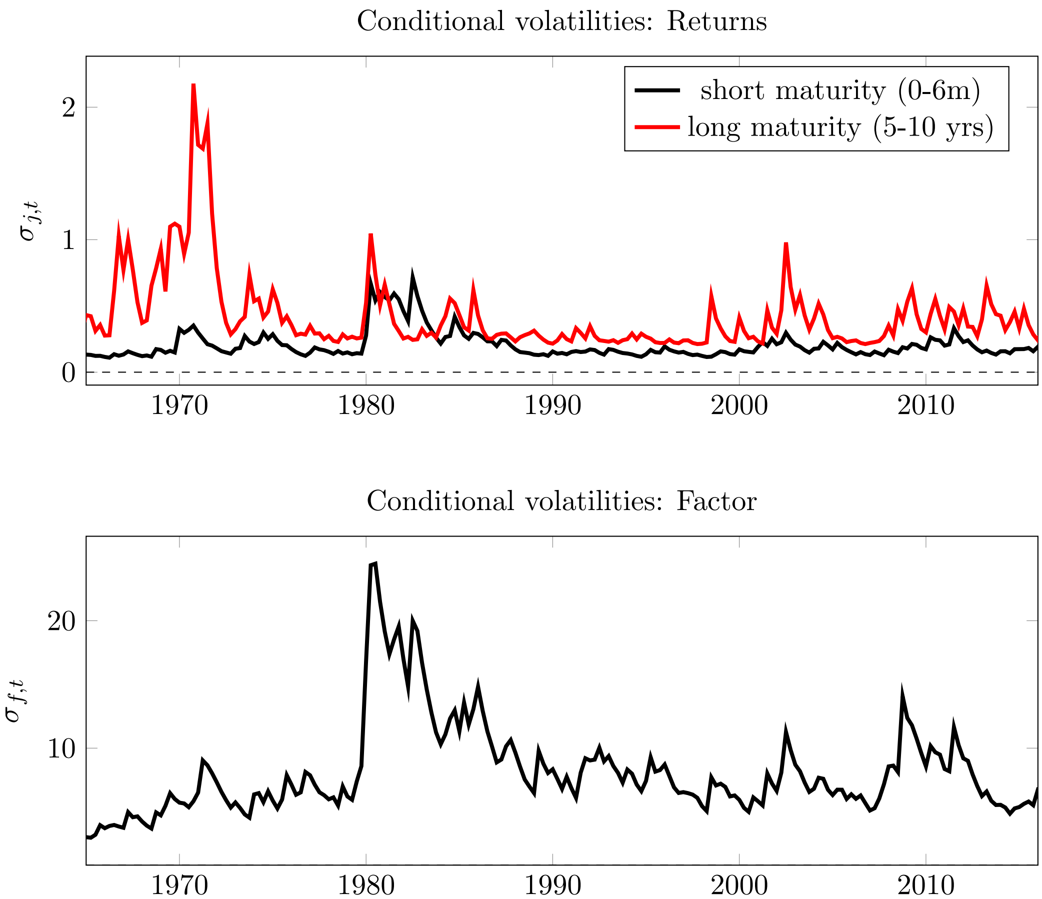


Figure 8: Conditional volatilities of returns, factor, using the estimated GARCH model

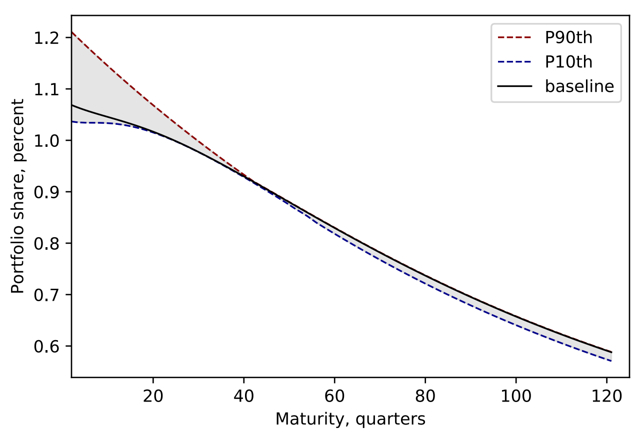


Figure 9: 90-10 interval of portfolio shares (maturities from 2 quarters to 120 quarters) with heteroskedastic shocks.

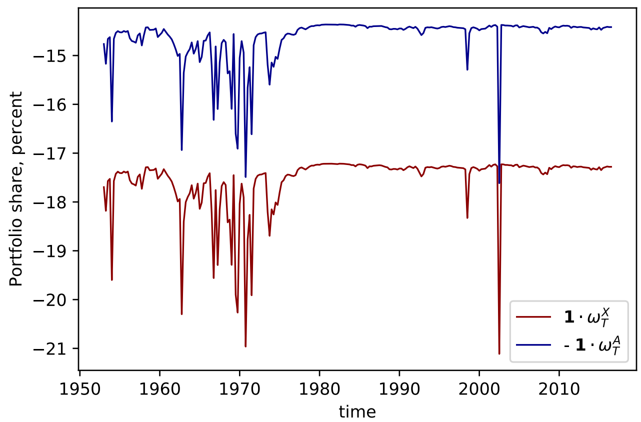


Figure 10: Components of target portfolio with heteroskedastic shocks. The blue line plots the sum of the shares of the portfolio that hedges the primary surplus risk, that is, $\mathbf{1} \cdot \pi_T^X \Sigma_t^{-1} \Sigma_t^X$ and the red line plots the negative of sum of the shares of the portfolio that hedges the liquidity risk, that is, $-\mathbf{1} \cdot \pi_T^A \Sigma_t^{-1} \Sigma_t^A$.

Framework From equation (10), we know that

$$a_t^i = -\mathbb{E}_t \frac{\beta M_{t+1}}{M_t} \left(R_{t+1}^i - R_t^0\right) = -\mathbb{E}_t \frac{\beta M_{t+1}}{M_t} r_t^i.$$

To back out a, we need to estimate $\frac{\beta M_{t+1}}{M_t}$. We start by assuming that the SDF is affine in a vector of some demeaned factors $\mathbf{f}_t^{pvt}$:

$$\frac{\beta M_{t+1}}{M_t} = -c_0 - \mathbf{c_1} \cdot \mathbf{f}_t^{pvt},$$

and then use the fact that the there is no liquidity wedge for privately traded bonds. Thus we are looking for $(c_0, \mathbf{c_1})$ that minimize the error in $\mathbb{E}_t \frac{\beta M_{t+1}}{M_t} \boldsymbol{R}_t^{pvt} = \mathbf{1}$ for a given set of returns on private bonds $\boldsymbol{R}_t^{pvt}$. This yields a familiar expression for estimates of $(c_0, \mathbf{c_1})$

$$c_0 = -\left(\mathbb{E} R_t^{0,pvt}\right)^{-1},$$

$$\boldsymbol{c}_1^{'} = -c_0 \mathbb{E}\left(\boldsymbol{r}_{t+1}^{pvt}\right)^{\intercal} \left(\mathbb{E}\left[\boldsymbol{r}_{t+1}^{pvt}\right]\left(\boldsymbol{f}_{t+1}^{pvt}\right)^{\intercal}\right)\left(\left[\mathbb{E}\mathbf{f}_{t+1}^{pvt}\left(\left[\boldsymbol{r}_{t+1}^{pvt}\right]\right)^{\intercal}\right]\left[\mathbb{E}\left[\boldsymbol{r}_{t+1}^{pvt}\right]\left(\boldsymbol{f}_{t+1}^{pvt}\right)^{\intercal}\right]\right)^{-1}.$$

Fama and MacBeth (1973) show that $(c_0, \mathbf{c_1})$ can be estimated using a two step process in which we first run a return by return time-series regression to estimate security specific "betas" and then we run a cross section regression for each date to back out "lambdas" or factor risk premia, $\boldsymbol{\lambda}_t^{pvt}$

$$r_t^{pvt,j} = \gamma^j + \boldsymbol{\beta}^j \cdot \mathbf{f}_t^{pvt} + \epsilon_t^j,$$
$$\mathbf{r}_t^{pvt} = \boldsymbol{\alpha}_t + \boldsymbol{\beta}' \cdot \boldsymbol{\lambda}_t^{pvt},$$

and we can recover $(c_0, \mathbf{c_1})$ using

$$c_0 = -\mathbb{E}\left[R_t^{0,pvt}\right]^{-1},$$
$$c_1 = -c_0\left(\Sigma_t^{\mathbf{f}}\right)^{-1}\mathbb{E}\left[\boldsymbol{\lambda}_t^{pvt}\right].$$

Using Fama McBeth procedure is useful because it immediately lends to an application for Delta method for computing the standard errors on the a. Let $\widehat{}$ be the estimated counterparts of the theoretical objects. We can express $\widehat{\mathbb{E}\left[a_t^i\right]}$ as some function $s(\cdot)$ such that

$$\widehat{\mathbb{E}\left[a_t^i\right]} \equiv s\left(\left\{r_{t+1}^i\right\}_{t\geq 0}, \left\{R_t^{0,pvt}\right\}_{t\geq 1}, \left\{\mathbf{f}_t^{pvt}\right\}_{t\geq 1}, \left\{\boldsymbol{\lambda}_t^{pvt}\right\}_{t\geq 1}\right).$$

Applying the Delta method, we get that:

$$\sigma^2(\mathbb{E}\left[a_t^i\right]) = T \times \nabla s' \Sigma^a \nabla s,$$

where

$$\nabla s = -\frac{1}{T}\begin{bmatrix} \widehat{\mathbb{E}}\left[R_t^{0,pvt}\right]^{-1}\mathbf{1}_{T\times 1} \\ \left[\widehat{\mathbb{E}}\left[r_{t+1}^i\right] + \left(\widehat{\Sigma_t^{\mathbf{f}}}\right)^{-1}\widehat{\mathbb{E}}\left[\boldsymbol{\lambda}_t^{pvt}\right]\widehat{cov}_t(\mathbf{f}_t^{pvt}, r_{t+1}^i)\right]\widehat{\mathbb{E}}\left[R_t^{0,pvt}\right]^{-2}\mathbf{1}_{T\times 1} \\ 0 \\ \widehat{\mathbb{E}}\left[R_t^{0,pvt}\right]^{-1}\left(\widehat{\Sigma_t^{\mathbf{f}}}\right)^{-1}\widehat{cov}_t(\mathbf{f}_t^{pvt}, r_{t+1}^i)\mathbf{1}_{T\times 1} \end{bmatrix} \quad \Sigma^a = cov\begin{bmatrix} \left\{r_{t+1}^i\right\}_{t\geq 0} \\ \left\{R_t^{0,pvt}\right\}_{t\geq 1} \\ \left\{\mathbf{f}_t^{pvt}\right\}_{t\geq 1} \\ \left\{\boldsymbol{\lambda}_t^{pvt}\right\}_{t\geq 1} \end{bmatrix}.$$

Estimation To estimate the excess liquidity premia and its standard errors, we need three things: a set of factors Factors $\mathbf{f}_t^{pvt}$, a measure of private risk-free rate $R^{0,pvt}$, and a set of excess returns $\boldsymbol{r}_t^{pvt}$. We describe those choices and then our results.

For the SDF $\frac{\beta M_{t+1}}{M_t}$ estimation, we impose a 3 factors structure to the SDF as in Koijen et al. (2017). The first factor is Cochrane and Piazzesi (2005)'s "CP factor". The second factor is the level (LVL) factor, which is constructed as the first component of the forward rate covariance matrix, following Cochrane and Piazzesi (2008). The third factor is the value-weighted stock market excess return from CRSP. We then estimate the SDF with the returns on 5 portfolios of corporate bonds of credit ratings AAA, AA, and A, constructed from Bloomberg (formerly Barclays) indices,[37] and available from 1989 to 2015. These indices measure the investment grade, fixed-rate, taxable corporate bond market. They include USD-denominated securities publicly issued by US and non-US industrial, utility and financial issuers.[38] We use as a private

[37]We thank Alexandros Kontonikas for sharing with us the data used in and used in Guo, Kontonikas and Maio (2020)

[38]For more details, see https://www.bloomberg.com/professional/product/indices/bloomberg-fixed-income-indices-fact-sheets-publications/

risk free rate $R_t^{0,pvt}$ our previous estimates of A_t^0 such that $R_t^{0,pvt} = \frac{R_t^0}{1-A_t^0}$.[39]

We then apply our estimation framework. Our findings are reported in Table 6. We see that although the point estimates are negative relfecting the larger share of risk-adjustment, the main takeaway is that that all maturities the estimates are statistically not different from zero. Thus we cannot reject $\mathbb{E}a_t^i = 0$.

Table 6: Estimates of the time-averaged excess liquidity premium $\widehat{\mathbb{E}\left[a_t^i\right]}$

maturity i	$\hat{\mathbb{E}}\left[a_t^i\right]$	s.e	t-stat	p-value
6 months	-0.03	(0.12)	-0.24	0.81
12 months	-0.05	(0.33)	-0.16	0.87
18 months	-0.05	(0.63)	-0.07	0.94
24 months	-0.06	(0.83)	-0.07	0.94
30 months	-0.08	(1.04)	-0.07	0.94
36 months	-0.09	(1.23)	-0.07	0.94
42 months	-0.10	(1.47)	-0.07	0.95
48 months	-0.10	(1.65)	-0.06	0.95
54 months	-0.10	(1.85)	-0.05	0.96
60 months	-0.10	(1.92)	-0.05	0.96
120 months	-0.11	(2.28)	-0.05	0.96

Notes: This table records the estimates of the average excess liquidity premium, its standard errors, and the associated t-statistics and p-value for the 11 Fama Maturity Portfolios. We take an average liquidity premium of 0 as our null hypothesis. The sample is 1989-2015. The units of the average excess liquidity premium is quarterly and in percentage points. We follow Fama and MacBeth (1973) and control for cross-sectional correlations but we assume that there is no serial correlations in the estimation of the SDF. We compute the standard errors using the Delta method.

B.4 Details for Section 5.8

Next we describe the how we build Λ_T^{QE} from the Greenwood and Vayanos (2014) estimates. We use the GV point estimates of b^i for their reported maturities and extrapolate for other maturities by fitting the same functional form that we used in the baseline for factor loadings.[40] The fit is reported in the left panel of Figure 11. In the right panel of Figure 11, we show the heatmap of Λ^{QE} (all normalized by its mean value) computed using expression (38) and the extrapolated $\{b^n\}$. The price impacts are larger around the south east region around the

[39] Alternatively, we also use the return on the portfolio of AAA corporate bonds of intermediate maturities (1 to 10 years) from Barclays as our (private) risk-free rate $R_t^{0,pvt}$. That doesn't affect much the results (the point estimates are modified but still very non-significant).

[40] In particular, Table 2 of GV reports estimates for bonds of maturities $2, 3, 4, 5, 10$ years. We assume that $b^n = \bar{b}_0 + \bar{b}_0 \exp\left(-\bar{b}_1 \times n\right)$ and find coefficients $\left\{\bar{b}_0, \bar{b}_1\right\}$ that minimize the least square errors. The results are robust to other extrapolation schemes.

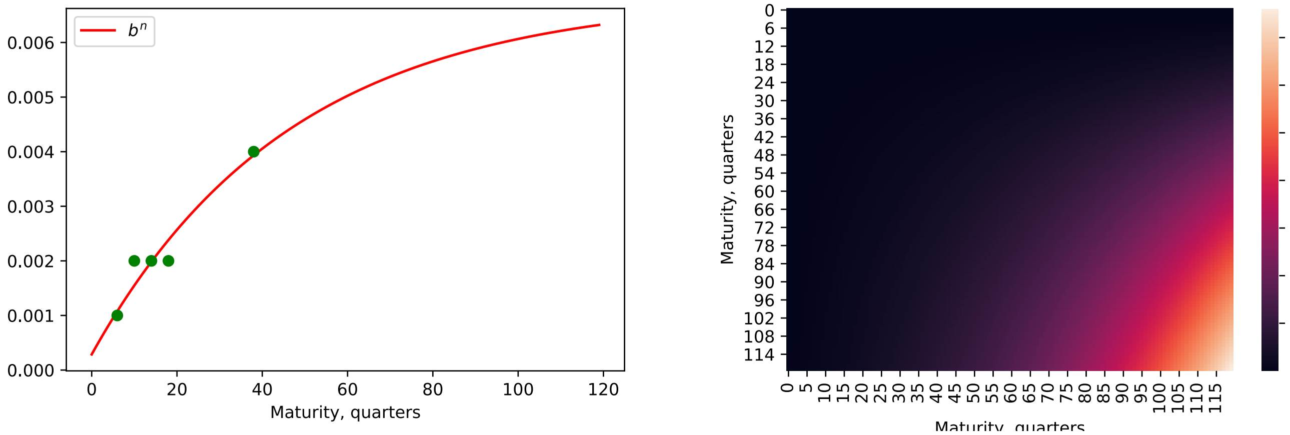


Figure 11: The left panel plots the fit for coefficients b^n. The right panel shows the normalized heatmap for the price impact matrix: Λ^{QE}

diagonal. Thus price impacts are large when securities involved are both of longer maturities.

C Closed Economy

In this appendix, we study a closed neoclassical version of our *benchmark economy*. Unlike the benchmark open economy specification in Section 3, a change in the governments portfolio will necessarily change the price of assets in economy; and, compared to the segmented markets version of the benchmark economy presented in Section 5.8, a change in the portfolio composition at date T will also affect the price of securities in all other periods.

In what follows, we show how to to adjust our variational approach to incorporate such effects on prices. Our main result is to characterize the price effects and using that we show that the closed economy neoclassical setting implies price responses that are counterfactual relative to the evidence reviewed in Section 5.8. Besides the different structure on price effects, the rest of the analysis of a closed economy including the steps to obtain the expression for the optimal portfolio are identical to Section 3.2. In Section C.1, we formally describe the neoclassical closed economy environment that we study, then introduce the perturbation and analyze the welfare effects and optimality of the government. The proofs of the main results are in Section C.2.

C.1 Analysis

In addition to the assumptions of the benchmark economy we assume that:

1. Household preferences are time separable

$$V_t = U_t\left(c_t - \frac{(y_t/\theta_t)^{1+1/\gamma}}{1+1/\gamma}\right) + \beta \mathbb{E}_t V_{t+1}.$$

   Following the neoclassical tradition, we abstract from trading frictions and non-pecuniary benefits of government debt by assuming that $\{Q_t^i b_t^i\}_{i\in\mathcal{G}_t}$ does not enter into the utility function and $\varphi_t\left(\{Q_t^i b_t^i\}\right) \equiv 0$.

2. Government expenditures $\mathbf{G}$ are exogenous.

3. Foreign investors are absent, $\boldsymbol{\mathcal{B}}^i = \mathbf{0}$, for all i and all assets are in zero net supply.[41]

4. The set of available securities can replicate a consol. We will let Q_t^∞ denote the price of the consol at date t.

Under these assumptions asset market clearing implies that

$$b_t^i = B_t^i$$

and

$$c_t + G_t = Y_t.$$

Absence of trading frictions and non-pecuniary benefits of government securities the household optimality conditions imply

$$\mathbb{E}_t M_{t+1} R_{t+1}^i = M_t \text{ or } M_t Q_t^i = \mathbb{E}_t\left[M_{t+1}\left(d_{t+1}^i + Q_{t+1}^i\right)\right] \tag{70}$$

Perturbation Following Section 3.2, we use a variational approach to isolate the optimal public portfolio. We consider any competitive equilibrium and introduce a perturbation at a particular history s^T by assuming that the government purchases $\frac{\epsilon}{Q_T^j(s^T)}$ units of security j which is funded by selling $\frac{\epsilon}{Q_T^{rf}(s^T)}$ of the risk free bond. This asset swap produces an additional $r_{T+1}^j(s^{T+1})\epsilon$ of excess returns at all histories s^{T+1} following s^T. We assume that the government uses those resources to purchase an additional $\frac{r_{T+1}^j(s^T)\epsilon}{1+Q_{T+1}^\infty(s^{T+1})}$ of the consol while keeping its holdings of all other assets constant. Due to its nature of swapping a longer security for a risk-free bond we will refer to this as a Quantitative Easing (or QE) perturbation and

[41] That all assets are in zero net supply is for notational simplicity. Assuming positive net supply simply adds another term to the resource constraint equivalent to changing exogenous government expenditures.

formally define it by

$$\partial_{j,T,\epsilon}B_t^i(s^t) = \begin{cases} \frac{\epsilon}{Q_T^0(s^T)} & \text{if } i = rf \text{ and } s^t = s^T, \\ -\frac{\epsilon}{Q_T^j(s^T)} & \text{if } i = j \text{ and } s^t = s^T, \\ -\frac{1}{1+Q_{T+1}^{\infty}(s^T)}\left(r_{T+1}^j\left(s^{T+1}\right)\right)\epsilon & \text{if } i = \infty \text{ and } s^t \succ s^T, t > T, \\ 0 & \text{otherwise.} \end{cases}$$

The change in portfolio composition necessarily requires a change in taxes to balance the governments budget constraint,

$$G_t + \sum_{i \geq 0}(Q_t^i + d_t^i)B_{t-1}^i = \tau_t Y_t + \sum_{i \geq 0} Q_t^i B_t^i.$$

Differentiating with respect to ϵ in the direction of the QE perturbation yields the following response of tax revenues

$$-\partial_{j,T,\epsilon}\left(\tau_t Y_t\right) = \frac{r_{T+1}^j\left(s^{T+1}\right)}{1 + Q_{T+1}^{\infty}\left(s^{T+1}\right)}\left(I_{\{s^t \succ s^T\}}\right) + \sum_{i \geq 0} \partial_{j,T,\epsilon}Q_t^i(s^t)\left(B_t^i(s^{t-1}) - B_{t-t}^i(s^t)\right) \tag{71}$$

where $I_{\{s^t \succ s^T\}}$ is an indicator returning 1 if history s^t follows from s^T and zero otherwise. Intuitively the effect of the perturbation on tax revenues is a combination of two effects. The first, $\frac{r_{T+1}^j\left(s^{T+1}\right)}{1+Q_{T+1}^{\infty}(s^{T+1})}\left(I_{\{s^t \succ s^T\}}\right)$, are the direct effects that are a result of the excess returns generated by the asset swap. The second, $\sum_{i \geq 0} \partial_{j,T,\epsilon}Q_t^i\left(B_t^i(s^t) - B_{t-1}^i(s^{t-1})\right)$, is the indirect effect that arises because the asset swap in period T changes prices not only in all future periods but also in all past periods starting from the initial date 0.

Assuming that the equilibrium manifold is sufficiently smooth, we can apply the envelope theorem to the household's maximization problem to obtain the welfare impact of this perturbation as $\epsilon \to 0$. The welfare effect of this perturbation comes from its effect on both tax rates and security prices and is given by

$$\begin{aligned}\partial_{j,T,\epsilon} V_0 &= \mathbb{E}_0 \sum_{t\geq 0} M_t \left(-\frac{\partial_{j,T,\epsilon}\left(\tau_t Y_t\right)}{\xi_t} + \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(b_{t-1}^i - b_t^i\right)\right) \\ &= \mathbb{E}_0 \sum_{t\geq 0} M_t \left(-\frac{\partial_{j,T,\epsilon}\left(\tau_t Y_t\right)}{\xi_t} + \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(B_{t-1}^i - B_t^i\right)\right) \\ &= \mathbb{E}_0 \left[\sum_{t\geq 0} M_t \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(\frac{\xi_t B_{t-1}^i - B_{t-1}^i}{\xi_t} - \frac{\xi_t B_t^i - B_t^i}{\xi_t}\right) + \sum_{t\geq T+1} \left(\frac{M_t}{\xi_t}\right)\left(I_{\{\tilde{s}^t \succ s^T\}}\right)\left(\frac{r_{T+1}^j}{1+Q_{T+1}^{\infty}}\right)\right] \\ &= \mathbb{E}_0 \left[\sum_{t\geq 0} M_t \left(\frac{\xi_t - 1}{\xi_t}\right) \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(B_{t-1}^i - B_t^i\right) + \sum_{t\geq T+1} \left(\frac{M_t}{\xi_t}\right)\left(I_{\{\tilde{s}^t \succ s^T\}}\right)\left(\frac{r_{T+1}^j}{1+Q_{T+1}^{\infty}}\right)\right] \\ &= \Pr_0\left(s^T\right) M_T\left(s^T\right) \left[PE_{j,T,\epsilon} + \mathbb{E}_T \sum_{k\geq 1} \left(\frac{M_{T+k}}{M_T}\right)\left(\frac{r_{T+1}^j}{1+Q_{T+1}^{\infty}}\right)\frac{1}{\xi_{T+k}}\right] \qquad (72)\end{aligned}$$

with

$$PE_{j,T,\epsilon} = \frac{1}{\Pr_0\left(s^T\right) M_T\left(s^T\right)} \mathbb{E}_0 \left[\sum_{t\geq 0} M_t \left(\frac{\xi_t - 1}{\xi_t}\right) \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(B_{t-1}^i - B_t^i\right)\right].$$

The term $\mathbb{E}_T \sum_{k\geq 1} \left(\frac{M_{T+k}}{M_T}\right)\left(\frac{r_{T+1}^j}{1+Q_{T+1}^{\infty}}\right)\frac{1}{\xi_{T+k}}$ parallels the effect of the same perturbation in the open economy benchmark model, and can be analyzed in a similar manner. Now, in addition to that term, we also have $PE_{j,T,\epsilon}$ that captures the effect on asset prices for all histories starting from time 0 onward. In the next section we will show how our second order expansions can allow us express that term using covariances that can be measured in the data.

Characterizing the Price Effects The perturbation affects asset prices through its effect on the stochastic discount factor of the household. This can be seen by differentiating the household Euler equation (70) with respect to ϵ in the direction of the perturbation to get

$$\left(\partial_{j,T,\epsilon} M_t\right) Q_t^i + M_t \left(\partial_{j,T,\epsilon} Q_t^i\right) = \mathbb{E}_t \left[\partial_{j,T,\epsilon} M_{t+1}\left(d_{t+1}^i + Q_{t+1}^i\right) + M_{t+1}\left(\partial_{j,T,\epsilon} Q_{t+1}^i\right)\right].$$

As the perturbation affects the stochastic discount factor through changes in tax rates we define $\xi_{M,t} \equiv \frac{\partial \log M_t}{\partial(\tau_t y_t)}$ as the semi-elasticity of $\log M_t$ with respect to the tax revenues which implies $\partial_{j,T,\epsilon} M_t = M_t \xi_{M,t} \partial_{j,T,\epsilon}\left(\tau_t y_t\right)$. Under our assumptions, this semi-elasticity is given by

$$\xi_{M,t} = -\psi_t \times \frac{1}{Y_t - G_t - \theta_t v\left(Y_t\right)} \times \left(\frac{\xi_t - 1}{\xi_t}\right)$$

where $\psi_t \equiv \frac{-[c_t - v_t(Y_t)]U''(c_t - v_t(Y_t))}{U'(c_t - v_t(Y_t))}$ is the coefficient of relative risk aversion.

To get a better understanding of how these terms contribute the price effects in the closed economy we'll focus on a stationary version of the economy

Definition 5. An optimal competitive equilibrium is *stationary from time T* if there exists a constant R_T such that for all $t > T$ (i) $\mathbb{E}_T G_t \approx G_T$ (ii) $\mathbb{E}_T \delta_t \approx \delta_T$ (iii) $\mathbb{E}_T R_t^i \approx R_T$ for all i and (iv) $\mathbb{E}_T c_t \approx c_T$.

This definition of stationary differs from the stationarity of the main text in that we assume a growth rate of $\Gamma = 1$. All of our results extend to a positive growth rate assuming that the utility function is CRRA.[42] Our first set of results concern the asset pricing implications of the QE perturbation. We will leave the proof of both propositions to the end of the section.

Proposition 1. *For a neoclassical model which is stationary from time T*

1. The QE perturbation keeps asset prices zero to the first-order

$$\partial_\sigma \partial_{j,T,\epsilon} Q_t^i = 0 \quad \forall \quad i, t \geq 0$$

2. The QE perturbation only affects risk-premia at T

$$\mathbb{E}_t \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{t+1}^i = 0 \quad \forall \quad t \neq T$$

and at date T

$$\mathbb{E}_T \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{T+1}^i = \frac{2\psi_T}{Y_T - G_T - \theta_T v\left(Y_T\right)} \times \left(\frac{1-\xi_T}{\xi_T}\right) \left(\frac{1}{1+\overline{Q}_{T+1}^{\infty}}\right) \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma r_{t+1}^i > 0,$$

where ψ_T is coefficient of relative risk aversion.

This proposition states that the QE perturbation does not effect prices to zeroth or first order. This is inline with our modeling of price effects in Section 5.8 where we assume that the effect prices is at second order. Intuitively, to zeroth and first-order all assets have the same expected return so the QE perturbation only changes the risk profile of the household's stochastic discount factor which, in turn, will only effect prices to second order. Moreover, the proposition states that the effect on asset prices in the closed economy are counterfactual to what has been documented in the data. Estimates by Greenwood and Vayanos (2014) and others find that find that $\Lambda^{QE}[rf, j] \approx 0$ and $\Lambda^{QE}[i, j] > 0$ for $i > rf$ which implies that expected excess returns should decrease with the QE perturbation rather than increase:

$$\mathbb{E}_T \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{T+1}^i = -\frac{\overline{Q}_{T+1}^i}{\overline{Q}_T^i} \frac{\partial_{\sigma\sigma} \partial_{j,T,\epsilon} Q_T^i}{\overline{Q}_T^i} < 0.$$

[42] The main difference is that we will require that the government smooth excess returns using a growth-adjusted consol rather than a pure consol.

When governments buy back long term debt by issuing short term debt, short term rates appear to be unchanged so expected excess returns are driven by the fall in the term premia as the increased demand drives up prices.

In contrast, in the closed economy, the government returns the excess returns from the QE swap via taxes which results in making states of the world where excess returns are high (low) better (worse) for the household by lowering (raising) tax rates in those states. As a result, the value of the asset decreases which raises the risk-premia. As noted, this is in inconsistent with the segmented market literature which finds that the excess returns on long maturity debt are lower after QE.

Finally, we are able to use our expansions to characterize the price effects

Proposition 2. *For a neoclassical economy which is stationary from time 0, if all initial debt* $\left\{B^i_{-1}\right\}_i$ *was risk-free then* $PE_{j,T,\epsilon} \simeq \left(\frac{\bar{\xi}}{\bar{\xi}-1}\right)^{-1} \Psi_T\left(s^T\right)$ *where*

$$\begin{aligned}
\Psi_T\left(s^T\right) =& \frac{-2\overline{B}\bar{\xi}_M(\overline{Q}^{rf}-1)}{\left(1-\overline{B}(\overline{Q}^{rf}-1)\bar{\xi}_M\right)} \sum_{t=T+1}^{\infty} \left(\frac{\left(\overline{Q}^{rf}_t\right)^{t-T}}{1+\overline{Q}^{\infty}_{T+1}} \right) cov_T\left(\partial_\sigma \ln M_t, \partial_\sigma r^j_{T+1}\right) \\
&- \frac{2\bar{\xi}_M\overline{B}}{\left(1-\overline{B}(\overline{Q}^{rf}-1)\bar{\xi}_M\right)} \sum_{t=T+1} \left(\frac{\left(\overline{Q}^{rf}_t\right)^{-T}}{1+\overline{Q}^{\infty}_{T+1}} \right) cov_T\left(\partial_\sigma r^j_{T+1}, \partial_\sigma \ln Q^{rf}_t\right) \\
&- \frac{2\bar{\xi}_M}{\left(1-\overline{B}(\overline{Q}^{rf}-1)\bar{\xi}_M\right)} \sum_{j\geq 1} \frac{\overline{Q}^{rf}_t}{1+\overline{Q}^{\infty}_{T+1}\left(s^{T+1}\right)} cov_T\left(\partial_\sigma r^j_{T+1}, \partial_\sigma r^j_{T+1}\right) \\
&- \frac{2\overline{B}}{\left(1-\overline{B}(\overline{Q}^{rf}-1)\bar{\xi}_M\right)} \sum_{t=T}^{\infty} \left(\frac{\left(\overline{Q}^{rf}_t\right)^{t-T}}{1+\overline{Q}^{\infty}_{T+1}} cov_T\left(\partial_\sigma \xi_{M,t} - \partial_\sigma \xi_{M,t+1}, \partial_\sigma r^j_{T+1}\right) \right)
\end{aligned}$$

As we have noted without any assumptions price effects are given by

$$PE_{j,T,\epsilon} = \frac{1}{\Pr_0\left(s^T\right) M_T\left(s^T\right)} \mathbb{E}_0 \left[\sum_{t\geq 0} M_t \left(\frac{\xi_t - 1}{\xi_t}\right) \sum_{i\geq 0} \partial_{j,T,\epsilon} Q^i_t \left(B^i_{t-1} - B^i_t\right) \right]$$

where a swap of securities at a particular history can affect asset prices at all other histories—past and future—due to general equilibrium effects on the stochastic discount factor that now directly depends on the tax rates. Proposition 2 allows us to characterize these price effects with a closed form expression using entirely time T covariances that are measurable in the data.

C.2 Proofs for Propositions 1 and 2

C.2.1 Proof of Proposition 1

We begin by noting that at the zeroth order, we get $\overline{\xi}^{j}_{M,t} = -\frac{\overline{\psi_t}}{\overline{Y}_t - \overline{G}_t - \overline{\theta}_t v(\overline{Y}_t)} \times \left(\frac{\overline{\xi}_t - 1}{\overline{\xi}_t}\right) = \overline{\xi}_{M,T}$, is independent of time and the details of the perturbation. We proceed by proving a series of lemmas documenting the results of Proposition 1

Lemma 11. *Expected excess returns are zero to the zeroth and the first order*

Proof. The zeroth of (70) gives us

$$\overline{r}^{i}_{t+1} = 0$$

Take first-order expansion to get

$$\mathbb{E}_t \partial_\sigma r^{i}_{t+1} \overline{M_{t+1}} + \mathbb{E}_t \overline{r^{i}_{t+1}} \partial_\sigma M_{t+1} = 0$$

and thus

$$\mathbb{E}_t \partial_\sigma r^{i}_{t+1} = 0.$$

□

Lemma 12. *To the first-order, price effects are zero, that is, for all i, t: $\partial_\sigma \partial_{j,T,\epsilon} Q^{i}_{t} = 0$*

Proof. Start from the definition of Q^{i}_{t}

$$Q^{i}_{t}\left(s^{t}\right) = \mathbb{E}_{s^t} \sum_{k \geq 1} \frac{M_{t+k}}{M_t} D^{i}_{t+k}.$$

$$\partial_\sigma \partial_{j,T,\epsilon} Q^{i}_{t} = \mathbb{E}_t \sum_{k \geq 1} \left(\partial_{j,T,\epsilon} \partial_\sigma \log M_{t+k} - \partial_{j,T,\epsilon} \partial_\sigma \log M_t\right) \left(\frac{\overline{M_{t+k}}}{\overline{M}_t}\right) D^{i}_{T+k}.$$

A necessary and sufficient condition for price effects to be zero at the firs-order is that $k \geq 1$

$$\mathbb{E}_t \left(\partial_{j,T,\epsilon} \partial_\sigma \log M_{t+k} - \partial_{j,T,\epsilon} \partial_\sigma \log M_t\right) = 0 \tag{73}$$

Use the definition of $\xi_M\left(s^t\right)$ to get $\partial_{j,T,\epsilon} \log M_t\left(s^t\right) = \partial_{j,T,\epsilon}\left(\tau_t\left(s^t\right) Y_t\left(s^t\right)\right) \times \xi_M\left(s^t\right)$. To first-order

$$\partial_\sigma \partial_{j,T,\epsilon} \log M_t = \partial_\sigma \partial_{j,T,\epsilon}\left(\tau_t Y_t\right) \times \overline{\xi}_{M,t}$$

Then (73) is equivalently expressed as

$$\mathbb{E}_t \left(\partial_\sigma \partial_{j,T,\epsilon} \log M_{t+k} - \partial_\sigma \partial_{j,T,\epsilon} \log M_t\right) = \overline{\xi}_{M,t} \left(\mathbb{E}_t \partial_\sigma \partial_{j,T,\epsilon}\left(\tau_{t+k} Y_{t+k}\right) - \partial_\sigma \partial_{j,T,\epsilon}\left(\tau_t Y_t\right)\right)$$

We check condition (73) by guess and verify.

Suppose $\partial_\sigma \partial_{j,T,\epsilon} Q_t^i = 0$ for $t \geq 0$, then for all $t \geq 0$ and from equations (71)

$$-\partial_\sigma \partial_{j,T,\epsilon} (\tau_t Y_t) = \partial_\sigma \left(\frac{r_{T+1}^j (s^{T+1})}{1 + Q_{T+1}^\infty (s^{T+1})} \right) I_{\{s^t \succ s^{T+1}\}} = \frac{\partial_\sigma r_{T+1}^j (s^{T+1})}{1 + \overline{Q}_{T+1}^\infty} I_{\{s^t \succ s^{T+1}\}}$$

When $t \geq T+1$

$$\mathbb{E}_t (\partial_\sigma \partial_{j,T,\epsilon} \log M_{T+1+k} - \partial_\sigma \partial_{j,T,\epsilon} \log M_{T+1}) = \overline{\xi}_{M,T+1} \left(\frac{\partial_\sigma r_{T+1}^j (s^{T+1})}{1 + \overline{Q}_{T+1}^\infty} - \frac{\partial_\sigma r_{T+1}^j (s^{T+1})}{1 + \overline{Q}_{T+1}^\infty} \right) I_{\{s^t \succ s^{T+1}\}} = 0$$

When $t \leq T$, we can use the fact that to the first order, expected excess returns are zero from Lemma (12) to establish that (73) holds. □

Lemma 13. *In the closed economy the effect of the perturbation on expected excess returns is*

$$\mathbb{E}_t \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{t+1}^i = 0 \quad \forall \quad t \neq T$$

and at date T

$$\mathbb{E}_T \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{T+1}^i = \frac{2\psi_T}{Y_T - G_T - \theta_T v(Y_T)} \times \left(\frac{1 - \xi_T}{\xi_T} \right) \left(\frac{1}{1 + \overline{Q}_{T+1}^\infty} \right) \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma r_{t+1}^i > 0$$

Proof. The first-order expansion $\partial_{j,T,\epsilon} M_t$ after using Lemma 12 gives us

$$\partial_\sigma \partial_{j,T,\epsilon} M_{t+1} = -\overline{\xi}_{M,t} \overline{M}_{t+1} \left\{ \partial_\sigma \left(\frac{r_{T+1}^j}{1 + Q_{T+1}^\infty} \right) I_{\{\boldsymbol{s}^t \succ \boldsymbol{s}^{T+1}\}} \right\}$$

Use this along with the second order expansion of households optimality condition (70) to obtain

$$0 = \mathbb{E}_t \partial_\sigma r_{t+1}^i \left(-\overline{\xi}_{M,t} \overline{M}_{t+1} \left\{ \partial_\sigma \left(\frac{r_{T+1}^j}{1 + q_{T+1}^\infty} \right) I_{\{\boldsymbol{s}^t \succ \boldsymbol{s}^{T+1}\}} \right\} \right) + \mathbb{E}_t \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{t+1}^i \overline{M}_{t+1}$$

For $t < T$, $I_{\{\boldsymbol{s}^t \succ \boldsymbol{s}^{T+1}\}} = 0$ and thus $\mathbb{E}_t \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{t+1}^i = 0$.

For $\boldsymbol{s}^t \succ \boldsymbol{s}^{T+1}$, use Law of iterated expectations to get

$$0 = \mathbb{E}_{T+1+k} \partial_\sigma r_{T+1+k}^i \left(-\overline{\xi}_{M,T+1+k} \overline{M}_{T+1+k} \underbrace{\mathbb{E}_{T+1} \left\{ \partial_\sigma \left(\frac{r_{T+1}^j}{1 + Q_{T+1}^\infty} \right) \right\}}_{=0} \right) + \mathbb{E}_{T+1+k} \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{T+k+2}^i \overline{M}_{T+2+k}$$

and use Lemma (12) to get $\mathbb{E}_t \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{t+1}^i = 0$ for $\boldsymbol{s}^t \succ \boldsymbol{s}^{T+1}$.

Finally for $t = T$

$$0 = \mathbb{E}_T \partial_\sigma r_{T+1}^i \left(-\overline{\xi}_{M,T} \overline{M}_{T+1} \left\{ \partial_\sigma \left(\frac{r_{T+1}^j}{1 + q_{T+1}^\infty} \right) \right\} \right) + \mathbb{E}_t \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{t+1}^i \overline{M}_{T+1}.$$

Substitute for $\overline{\xi}_{M,T}$ and simplify to to get

$$\mathbb{E}_T \partial_{\sigma\sigma}\partial_{j,T,\epsilon} r^i_{T+1} \simeq \frac{2\psi_T}{Y_T - G_T - \theta_T v\left(Y_T\right)} \times \left(\frac{1-\xi_T}{\xi_T}\right)\left(\frac{1}{1+\overline{Q}^{\infty}_{T+1}}\right)\mathbb{E}_t \partial_\sigma r^j_{T+1}\partial_\sigma r^i_{t+1}.$$

Since $\xi_T = 1 - \gamma\frac{\tau_T}{1-\tau_T} < 1$, $Y_T - G_T - \theta_T v\left(Y_T\right) > 0$ from Inada conditions, and $\psi_T > 0$, we get that $\mathbb{E}_T \partial_{\sigma\sigma}\partial_{j,T,\epsilon} r^j_{T+1} > 0$. □

C.2.2 Proof of Proposition 2

The second order expansion of the price effects

$$\partial_{\sigma\sigma}\left(\Pr_0\left(s^T\right)M_T\left(s^T\right)PE_{j,T,\epsilon}\right) = \mathbb{E}_0\left[\sum_{t\geq 0}\left(\frac{\bar{\xi}_t - 1}{\bar{\xi}_t}\right)M_t\sum_{i\geq 0}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t\left(\overline{B}^i_{t-1} - \overline{B}^i_t\right)\right] \tag{74}$$

which equals

$$\left(\frac{\bar{\xi}_0 - 1}{\bar{\xi}_0}\right)\overline{M}_0\sum_{i\geq 0}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_0\overline{B}^i_{-1} + \left(\frac{\bar{\xi}_0 - 1}{\bar{\xi}_0}\right)\mathbb{E}_0\left[\sum_{t\geq 0}\sum_{i\geq 0}\overline{B}^i_t\left(\overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_{t+1} - \overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t\right)\right]. \tag{75}$$

Its easy to see that $\left(\frac{\bar{\xi}_0-1}{\bar{\xi}_0}\right)\overline{M}_0\sum_{i\geq 0}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_0\overline{B}^i_{-1} = \left(\frac{\bar{\xi}_0-1}{\bar{\xi}_0}\right)\overline{M}_0\sum_{i\neq rf}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_0\overline{B}^i_{-1} = 0$ under the assumption that initial debt was risk-free.

The household pricing equation implies

$$M_t Q^i_t = \mathbb{E}_t\left[M_{t+1}\left(Q^i_{t+1} + D^i_{t+1}\right)\right] \tag{76}$$

Differentiating by $\partial_{j,T,\epsilon}$ gives

$$\left(\partial_{j,T,\epsilon}M_t\right)Q^i_t + M_t\partial_{j,T,\epsilon}Q^i_t = \mathbb{E}_t\left[\left(\partial_{j,T,\epsilon}M_{t+1}\right)\left(Q^i_{t+1} + D^i_{t+1}\right) + M_{t+1}\partial_{j,T,\epsilon}Q^i_{t+1}\right]$$

Let's start by looking at $t < T$, We know that $\partial_\sigma\partial_{j,T,\epsilon}M_{t+1} = 0$ so taking the second derivative with respect to σ yields

$$\mathbb{E}_t\left[\overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_{t+1} - \overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t\right] = \overline{Q}^i_t\mathbb{E}_t\left[\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_t\right) - \left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_{t+1}\right)\overline{R}^{rf}_{t+1}\right].$$

For $t > T$ and $s^t \succ s^T$ we have $\frac{\partial_{j,T,\epsilon}M_t}{M_t} = \xi^j_{M,t}\partial_{j,T,\epsilon}\left(\tau_t Y_t\right)$ and hence $\partial_\sigma\partial_{j,T,\epsilon}M_t = \overline{M}_t\bar{\xi}_{M,t}\frac{\partial_\sigma r^j_{T+1}}{1+\overline{Q}^{\infty}_{T+1}}$. The second-order expansion of equation (76) is

$$2\partial_\sigma\partial_{j,T,\epsilon}M_t\partial_\sigma Q^i_t + \left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_t\right)\overline{Q}^i_t + \overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t = \mathbb{E}_t\left[2\partial_\sigma\partial_{j,T,\epsilon}M_{t+1}\partial_\sigma\left(Q^i_{t+1} + D^i_{t+1}\right)\right]$$
$$+ \left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_{t+1}\right)\left(\overline{Q}^i_{t+1} + D^i_{t+1}\right) + \overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_{t+1}$$

We know that

$$\mathbb{E}_t\left[\partial_\sigma\partial_{j,T,\epsilon}M_{t+1}\partial_\sigma\left(Q^i_{t+1}+D^i_{t+1}\right)\right]=\overline{\xi}_{M,T+1}\frac{\partial_\sigma r^j_{T+1}}{1+\overline{Q}^\infty_{T+1}}\mathbb{E}_t\left[\overline{M}_{t+1}\partial_\sigma\left(Q^i_{t+1}+D^i_{t+1}\right)\right]$$

so we get

$$\begin{aligned}&\mathbb{E}_t\left[\partial_\sigma\partial_{j,T,\epsilon}M_{t+1}\partial_\sigma\left(Q^i_{t+1}+D^i_{t+1}\right)\right]-\partial_\sigma\partial_{j,T,\epsilon}M_t\partial_\sigma Q^i_t\\&=\frac{\partial_\sigma r^j_{T+1}}{1+\overline{Q}^\infty_{T+1}}\overline{\xi}_{M,T+1}\left(\mathbb{E}_t\left[\overline{M}_{t+1}\partial_\sigma\left(Q^i_{t+1}+D^i_{t+1}\right)\right]-\overline{M}_t\partial_\sigma Q^i_t\right)\\&=\frac{\partial_\sigma r^j_{T+1}}{1+\overline{Q}^\infty_{T+1}}\overline{\xi}_{M,T+1}\overline{Q}^i_t\left(\partial_\sigma M_t-\partial_\sigma M_{t+1}\overline{R}^{rf}_{t+1}\right)\end{aligned}$$

with the last equality coming from

$$\partial_\sigma M_t\overline{Q}^i_t+\overline{M}_t\partial_\sigma Q^i_t=\mathbb{E}_t\left[\partial_\sigma M_{t+1}\left(\overline{Q}^i_{t+1}+\overline{D}^i_{t+1}\right)+\overline{M}_{t+1}\partial_\sigma\left(Q^i_{t+1}+D^i_{t+1}\right)\right].$$

Note that this only depends on i through $\overline{Q}^i$ thus for $t>T$

$$\begin{aligned}&\mathbb{E}_t\left[\overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_{t+1}-\overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t\right]\\&=\overline{Q}^i_t\mathbb{E}_t\left[\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_t\right)-\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_{t+1}\right)\overline{R}^{rf}_{t+1}-\overline{\xi}_{M,T+1}\overline{M}_t\frac{\partial_\sigma r^j_{T+1}}{1+\overline{Q}^\infty_{T+1}}\frac{\partial_\sigma Q^{rf}_t}{\overline{Q}^{rf}_t}\right]\end{aligned}$$

where the last term is simplified by noting that $\overline{M}_t\frac{\partial_\sigma Q^{rf}_t}{\overline{Q}^{rf}_t}=\mathbb{E}_t\left[\frac{1}{\overline{Q}^{rf}_t}\partial_\sigma M_{t+1}-\partial_\sigma M_t\right]$.

Finally, we have the $t=T$ and $s^t=s^T$ term which gives

$$\begin{aligned}&\mathbb{E}_T\left[\overline{M}_{T+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_{T+1}-\overline{M}_T\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_T\right]\\&=\overline{Q}^i_T\mathbb{E}_t\left[\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_T\right)-\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_{T+1}\right)\overline{R}^{rf}_{T+1}-\frac{\overline{\xi}_{M,T+1}\overline{M}_{T+1}}{1+\overline{Q}^\infty_{T+1}\left(s^{T+1}\right)}\partial_\sigma r^j_{T+1}\partial_\sigma r^i_{T+1}\right].\end{aligned}$$

Now we note that all the terms $\overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_{t+1}-\overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t$ in the price effect sum have a component $\overline{Q}^i_t\left(\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_t\right)-\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_{t+1}\right)\overline{R}^{rf}_{t+1}\right)$ in them. We gain some tractability by substituting $\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_t=\overline{M}_t\overline{\xi}_M\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)+2\overline{M}_t\partial_\sigma\xi_{M,t}\partial_\sigma\partial_{j,T,\epsilon}\left(\tau_tY_t\right)$ and doing so makes

$$
\begin{aligned}
&\mathbb{E}_0\left[\sum_{t\geq 0}\sum_{i\geq 0}\overline{B}_t^i\left(\overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q_{t+1}^i-\overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q_t^i\right)\right]\\
&=\mathbb{E}_0\left[\sum_{t=0}^{\infty}\overline{B}_t\overline{M}_t\overline{\xi}_{M,t}\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right]\\
&-2\Pr(s^T)\overline{\xi}_{M,T+1}\mathbb{E}_T\left[\sum_{t=T+1}\overline{B}_t\overline{M}_t\frac{\partial_\sigma r_{T+1}^j}{1+\overline{Q}_{T+1}^{\infty}}\frac{\partial_\sigma Q_t^{rf}}{\overline{Q}_t^{rf}}\right] &(77)\\
&-2\Pr(s^T)\overline{\xi}_{M,T+1}\mathbb{E}_T\left[\sum_{j\geq 1}\frac{\overline{M}_{T+1}}{1+\overline{Q}_{T+1}^{\infty}\left(s^{T+1}\right)}\partial_\sigma r_{T+1}^j\partial_\sigma r_{T+1}^i\right]\\
&+2\Pr(s^T)\mathbb{E}_T\left[\sum_{t=T}^{\infty}\overline{B}_t\overline{M_t}\left(\partial_\sigma\xi_{M,t}\partial_\sigma\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_\sigma\xi_{M,t+1}\partial_\sigma\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right] &(78)
\end{aligned}
$$

Most of these objects we can easily put some structure on except for

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\overline{B}_t\overline{M}_t\overline{\xi}_{M,t}\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right],$$

there we have note that $\overline{B}_t=\overline{B}_0=\overline{B}$, $\overline{M}_t=\left(Q^{rf}\right)^t\overline{M_0}$ and $\overline{\xi}_{M,t}=\overline{\xi}_{M,0}=\overline{\xi}_M$. Put together we have

$$
\begin{aligned}
&\mathbb{E}_0\left[\sum_{t=0}^{\infty}\overline{B}_t\overline{M}_t\overline{\xi}_{M,t}\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right]\\
&=\overline{B\xi}_{M,}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(Q^{rf}\right)^t\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right]\overline{M}_0\\
&=\overline{B\xi}_{M,}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(Q^{rf}\right)^t\left(Q^{rf}-1\right)\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)\right]\overline{M}_0
\end{aligned}
$$

we can then plug into $\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_t Y_t\right)$ to get

$$\begin{aligned}
&\mathbb{E}_0\left[\sum_{t=0}^{\infty}\overline{B}_t\overline{M}_t\overline{\xi}_{M,t}\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_t Y_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right]\\
&=\overline{B}(Q^{rf}-1)\overline{\xi}_M\overline{M}_0\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(Q^{rf}\right)^t\sum_{i\geq 0}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q_t^i\left(\overline{B}_{t-1}^i-\overline{B}_t^i\right)\right]\\
&+\overline{B\xi}_M\overline{M}_0(Q^{rf}-1)\Pr(s^T)\mathbb{E}_T\left[\sum_{t=t+1}^{\infty}\left(Q^{rf}\right)^t\partial_{\sigma\sigma}\left(\frac{r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right)\right]\\
=&\overline{B}(Q^{rf}-1)\overline{\xi}_M\frac{\overline{\xi}}{\overline{\xi}-1}\partial_{\sigma\sigma}\left(\Pr_0\left(s^T\right)M_T\left(s^T\right)PE_0^j\left(s^T\right)\right) \qquad (79)\\
&+\overline{B\xi}_M\overline{M}_0(Q^{rf}-1)\Pr(s^T)\mathbb{E}_T\left[\sum_{t=t+1}^{\infty}\left(Q^{rf}\right)^t\partial_{\sigma\sigma}\left(\frac{r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right)\right]
\end{aligned}$$

Going back to the HH version of this perturbation we get

$$\mathbb{E}_T\left[\sum_{t=T+1}^{\infty}M_t\frac{r_{T+1}^j}{1+\overline{Q}_{T+1}^{\infty}}\right]=0$$

As second order expansion of this gives

$$\overline{M}_0\mathbb{E}_T\left[\sum_{t=T+1}^{\infty}\left(Q^{rf}\right)^t\partial_{\sigma\sigma}\left(\frac{r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right)\right]=-2\mathbb{E}_T\left[\sum_{t=T+1}^{\infty}\partial_\sigma M_t\frac{\partial_\sigma r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right] \qquad (80)$$

Putting all together we get (combining equations (74),(77),(79), and (80))

$$\begin{aligned}
\left(\frac{\overline{\xi}}{\overline{\xi}-1}\right)\partial_{\sigma\sigma}PE_{j,T,\epsilon}=&\frac{-2\overline{\xi}_M}{\left(1-\overline{B}_0(Q^{rf}-1)\right)\left(Q^{rf}\right)^T}\mathbb{E}_T\left[\sum_{t=T+1}\overline{B}_t\frac{\partial_\sigma r_{T+1}^j}{1+\overline{Q}_{T+1}^{\infty}}\frac{\partial_\sigma Q_t^{rf}}{\overline{Q}_t^{rf}}\right]\\
&-\frac{2\overline{B\xi}_M(Q^{rf}-1)}{\left(1-\overline{B}(Q^{rf}-1)\right)}\mathbb{E}_T\left[\sum_{t=T+1}^{\infty}\left(Q^{rf}\right)^{t-T}\partial_\sigma\ln M_t\frac{\partial_\sigma r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right]\\
&\frac{-2\overline{\xi}_M}{\left(1-\overline{B}(Q^{rf}-1)\right)\left(Q^{rf}\right)^T}\mathbb{E}_T\left[\sum_{j\geq 1}\frac{Q^{rf}}{1+\overline{Q}_{T+1}^{\infty}\left(s^{T+1}\right)}\partial_\sigma r_{T+1}^j\partial_\sigma r_{T+1}^i\right]\\
&-\frac{2\overline{B}}{\left(1-\overline{B}(Q^{rf}-1)\right)}\mathbb{E}_T\left[\sum_{t=T}^{\infty}\left(Q^{rf}\right)^{t-T}\frac{\partial_\sigma r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\left(\partial_\sigma\xi_{M,t}-\partial_\sigma\xi_{M,t+1}\right)\right]
\end{aligned}$$

as desired.