Online Appendix

B Road Map

The following Sections C and D provide a precise language to analyze contracts in continuous time with finite state stochastic processes and establish key results of the optimal contract. They provide the basis for proving the specific lemmas and propositions in the paper, starting with Section E, but are more general than that. As such, Sections C and D are useful considerably beyond the particular model studied in the paper.

Section C describes the mathematical framework for finite state continuous time Markov processes and history dependence of allocations. Such a framework is necessary for the mathematically precise description of contracts in Section D . Histories are encoded by a beginning and end date, the number and dates of Markov switches in between and the value of the Markov process for all episodes (see equation (57)). Notation is introduced to describe smaller segments of a history or for how to concatenate two adjacent histories. With that, the Markov transition probability law can be stated as a probability measure on the set of all histories (see equation (59)). We proceed, using this as the appropriate probability space. Equation (61) states how to calculate conditional expectations of functions of future histories, using this probability law.

Section D describes and analyzes contracts for finite-state continuous time Markov processes in five subsections. The key properties of the contracts are established in Lemma 7 of subsection D.3. That lemma is foundational for the description of the contract properties in the main body of the text as well as for proving the results in Section E. Subsections D.1, D.2 and D.4 are the necessary preliminaries to establish this lemma, but useful in their own right. In particular, subsection D.2 extends Marimon-Marcet (2019) to continuous time. Subsection D.5 provides the dual perspective of utility maximization as an "add-on."

1. Subsection D.1 introduces contracts as mappings from histories, the current state and a promised utility level U in definition 3 and defines optimal contracts as minimizing the appropriate cost function of the principal in definition 4. Three lemmas establish that the cost function is increasing, convex and differentiable in U.

2. Subsection D.2 describes a Lagrangian approach to the cost minimization problem of subsection D.1. Starting from the somewhat heuristic formulation in equation (67), it provides for a precise definition in equation (71), using recursive Lagrange

multipliers. This extends the Marimon-Marcet (2019) approach to the continuous-time finite-state Markov case. It leads to the key first-order condition (74) and the contract property in Lemma 6 that consumption is either constant or declining when $\rho \geq r$.

3. The key Lemma 7 is established in subsection D.3. It establishes several differentiability properties, necessary for equation (80) following that lemma. That equation, together with the properties of the cumulative Lagrange multiplier and differentiability results provided by Lemma 7 in turn, is foundational for the derivatives-based analysis in the main body of the paper.

   Part 3 of the lemma establishes that consumption remains constant if the limited commitment constraint binds. That property and its proof is a central piece of the analysis and not trivial. It frequently resorts to a technique of splitting the future life of the contract into a short and immediate future of length Δ and the subsequent future history as in equation (78). That technique is more formally established and studied in subsection D.4. The proof of the lemma here builds on results established there, in particular the crucial principle of optimality in Lemma 9. From the perspective of mathematical logic, subsection D.4 precedes subsection D.3, but is only necessary for understanding the proof of Lemma 7 here. In the interest of readability, we therefore chose the current ordering.

4. Subsection D.4 proceeds to establish an equivalent recursive formulation (see definition 5). It proceeds by splitting the future into three parts as in equation (82). First, there is a short time interval $\Delta > 0$ without a state change. Second, there is the future beyond Δ and no state change until Δ. Finally, there are all of the first state changes before Δ and their continuation values. Lemma 9 establishes the principle of optimality, i.e., a key monotonicity result of the optimal contract: if the promised utility is higher, then consumption during the no-state-change epoch Δ as well as promised utility upon the first state change as well as the continuation utility beyond Δ will be higher. The proof is not entirely straightforward and requires a careful examination of inequalities and the Lagrange multipliers provided in subsection D.2. Almost as a by-product of the recursive formulation, we establish the Hamilton-Jacobi-Bellman, or HJB, equation in Proposition 14.

5. Subsection D.5 considers the dual perspective of maximizing utility, given costs.

Much of the properties here parallel the developments before, allowing us to be brief. Proposition 15 establishes equivalence. Proposition 16 provides the HJB equation.

C Mathematical Preliminaries

The purpose of this section is to provide a precise mathematical framework to describe the stochastic nature of consumption contracts in the next sections. It will turn out that we need to allow the contracts to depend on a bit more than just the history of productivities, see in particular the proof of Lemma 3 below. Furthermore, we provide these mathematical preliminaries for more than just two productivity states, in order to allow building on these in future work. The material here is not easily available elsewhere in concise form. The approach taken here and some of the material are in chapter 11 of Puterman (2005), though we need a bit more for the analysis in subsequent sections.

We assume throughout that there is a k-state Markov process for an underlying state $x(t) \in X = \{0, \ldots, k-1\}$ for each agent. The state for one agent evolves independently from that of any other agent and with constant Markov transition rates $\alpha_{i,j}$ from state $x = i$ to state $x = j$. We impose that $\alpha_{i,i} = -\sum_{j \neq i} \alpha_{i,j}$, so that α is an intensity matrix or infinitesimal generator matrix. We assume that there is a mapping $\mathbf{z} : X \to Z$ determining individual labor productivity $z = \mathbf{z}(x)$ if the individual state is s. Note that k may be larger than the cardinality of Z. The beginning of Section E provides the specific details for the case with two labor productivities used in the main text and an underlying state x that can take three values. This construction will be used in the proof of Lemma 1 (see subsection E.1).

Given dates $t < \tau < \infty$, let $x_0 = x(t)$ be the state at date $t_0 = t$. Suppose there are $n \geq 0$ switches between t and τ at switch dates $t_0 < t_1 \ldots < t_n \leq \tau$. Let $x(t_j) = x_j$ for $j > 0$ denote the new values of the state at these switching dates. The history of the state between time t and time $\tau > t$, denoted compactly as $h_{t,\tau}$, and explicitly given by

$$h_{t,\tau} = (\tau, n, t_0 = t, t_1, \ldots, t_n, x_0 = x, x_1, \ldots, x_n) \tag{57}$$

keeps track of all this information. The starting history at time t is $h_{t,t} = (t, 0, t, x)$, when the state is at $x(t) = x$ and by construction no state change has occurred yet. Generally, when $n = 0$, no switch occurs and the state remains at the initial state x_0 from t to τ. We impose[23] the condition that $n < \infty$; i.e., we only examine histories between the two dates t and τ with finitely many switching dates. This is true with probability 1.

Given some history $h_{t,\tau}$ as in (57) and some $\Delta > 0$, define the time-shifted history

$$h_{t,\tau}^{\Delta} = (\tau + \Delta, n, t_0 = t + \Delta, t_1 + \Delta, \ldots, t_n + \Delta, x_0 = x, x_1, \ldots, x_n) \quad (58)$$

This construction will be of help in the proof of Lemma 7. Given some history $h_{t,\tau}$ between dates t and τ and any two intermittent dates s, s' with $t \leq s \leq s' \leq \tau$, it will be useful to construct the **history between** s **and** s' and denote it by $h_{t,\tau}(s, s')$. To do so, starting from (57), let $m = \text{argmax}\{j \mid t_j \leq s\}$ be the index of the last switching date before the date s. Likewise, let $m' = \text{argmax}\{j \mid t_j \leq s'\}$ be the index of the last switching date before the date s'. Therefore there are $m' - m$ state transitions between dates s and s'. Starting from the new initial date $\tilde{t}_0 = s$ rewrite the dates of these state transitions as $\tilde{t}_1 = t_{m+1}, \ldots, \tilde{t}_{m'-m} = t_{m'}$. Likewise, denote the states at these redefined transition dates as $\tilde{x}_0 = x_m, \ldots, \tilde{x}_{m'-m} = x_{m'}$. Using this construction we can now define the history $h_{t,\tau}(s, s')$ between s and s' implied by the history $h_{t,\tau}$ between t and τ as

$$h_{t,\tau}(s, s') = (s', m' - m, \tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_{m'-m}, \tilde{x}_0, \ldots, \tilde{x}_{m'-m})$$

The most relevant purpose of this construction is to split the history $h_{t,\tau}$ into two non-overlapping parts $h_{t,\tau}(t, s)$ and $h_{t,\tau}(s, \tau)$, where $t \leq s \leq \tau$, or conversely, define a **concatenated history** $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ by gluing two histories $h_{t,s}$ and $h_{s,\tau}$

$$\begin{aligned} h_{t,s} &= (s, m, t_0^a = t, t_1^a, \ldots, t_m^a, x_0^a, x_1^a, \ldots, x_m^a) \\ h_{s,\tau} &= (\tau, n, t_0^b = s, t_1^b, \ldots, t_n^b, x_0^b, x_1^b, \ldots, x_n^b) \end{aligned}$$

together. This construction requires that the state x_m^a at the end of history $h_{t,s}$ equals the state $x_0^b = x_m^a$ at the beginning of history $h_{s,\tau}$.[24] Note that the last switch date before or including s is the date t_m^a, while the first subsequent switch date is t_1^b. The date s itself drops out from the switching date history, and is contained in the interval $t_m^a \leq s \leq t_1^b$. The

[23]Or better: we work with a subset of the probability space, so that this is true.

[24]We stipulate that this condition must be satisfied whenever the notation for concatenation is utilized. A more explicit notation would be cumbersome.

concatenated history is then given explicitly as

$$\begin{aligned} h_{t,\tau} &= [h_{t,s}, h_{s,\tau}] \\ &= (\tau, m+n, t_0 = t, t_1 = t_1^a, \ldots, t_m = t_m^a, t_{m+1} = t_1^b, \ldots, t_{m+n} = t_n^b, \\ &\qquad x_0^a, x_1^a, \ldots, x_m^a, x_1^b, \ldots, x_n^b) \end{aligned}$$

In particular, note that

$$h_{t,\tau} = [h_{t,\tau}(t,s), h_{t,\tau}(s,\tau)]$$

for $t \leq s \leq \tau$.

Let $\mathcal{H}_{t,\tau}(x)$ be the set of all possible histories $h_{t,\tau}$ between two given dates $\tau \geq t$, starting at $x_0 = x$. Let $\mathcal{H}_t(x)$ be their union across τ, i.e. the set of all histories $h_{t,\tau}$ for any date $\tau \geq t$, given t, and starting at $x_0 = x$. Let $\mathcal{H}_t$ be the unions of all $\mathcal{H}_t(x)$ across all $x \in X$. The transition rates $\alpha_{i,j}$ deliver a probability measure $P_{t,\tau}$ on $\mathcal{H}_{t,\tau}(x)$ for histories $h_{t,\tau}$ between two dates t and τ, given by

$$dP_{t,\tau}(h_{t,\tau}) = \exp\left((\tau - t_n)\alpha_{x_n,x_n}\right) \prod_{j=1}^{n} \exp\left((t_j - t_{j-1})\alpha_{x_{j-1},x_{j-1}}\right) \alpha_{x_{j-1},x_j} dt_j \tag{59}$$

where the dt_j are to be arranged in the sequence $dt_n\, dt_{n-1} \ldots dt_1$, when writing this out explicitly. This is important for appropriately writing the integral in equation (61). Note that a history $h_{t,\tau} = (\tau, 0, t, x)$ with $n = 0$ and thus without transitions has the point mass

$$dP_{t,\tau}(\tau, 0, t, x) = \exp((\tau - t)\alpha_{x,x})$$

More generally, and as an arbitrary example, consider a history $h_{0,3}$ between the two dates $t = 0$ and $\tau = 3$ and two switching dates, given by

$$h_{0,3} = (\tau = 3, n = 2, t_0 = 0, t_1 = 1.3, t_2 = 2.3, x_0 = 0, x_1 = 1, x_2 = 0)$$

The probability for this history is

$$\begin{aligned} dP_{t,\tau}(h_{0,3}) &= \exp((3 - 2.3)\alpha_{0,0} + (2.3 - 1.3)\alpha_{1,1} + (1.3 - 0)\alpha_{0,0})\alpha_{1,0} dt_2 \alpha_{0,1} dt_1 \\ &= \alpha_{0,1}\alpha_{1,0} \exp(2\alpha_{0,0} + \alpha_{1,1}) dt_2 dt_1 \end{aligned}$$

Note that[25]

$$P_{t,\tau}(h_{t,\tau}) = P_{t,s}(h_{t,\tau}(t,s))P_{s,\tau}(h_{t,\tau}(s,\tau)) \tag{60}$$

Write P_t for the overall probability measure $P_t(h_{t,\tau}) = P_{t,\tau}(h_{t,\tau})$. While not essential, this also allows a precise construction of a suitable probability space. Formally, let $\mathcal{H}_t(x)$ be the set of underlying events. Note that $\mathcal{H}_t(x)$ can be written as the countable union of the sets

$$\begin{aligned} H_{t,n}(x) &= \{(\tau, n, t, t_1, \ldots, t_n, x, x_1, \ldots, x_n) \mid \\ &\qquad (\tau, t_1 \ldots, t_n, x_1, \ldots, x_n) \in \mathbf{R}^{n+1} \times X^n, t < t_1 \ldots < t_n \leq \tau\} \end{aligned}$$

of $\mathbf{R}^{n+1} \times X^n$, $n \geq 0$. The sets $H_{t,n}(x)$ have the usual Borel-σ-algebra[26] of subsets, which we shall denote with $\mathcal{B}_{t,n}(x)$. Their union $\mathcal{B}_t(x)$ is the Borel-σ-algebra of the measurable subsets of $\mathcal{H}_t(x)$. With the probability measure P_t defined above, $(\mathcal{H}_t(x), \mathcal{B}_t(x), P_t)$, becomes a probability space.

The set $\mathcal{S}(x)$ of stochastic processes on $\mathcal{H}_t(x)$ is the set of measurable functions from $\mathcal{H}_t(x)$ to the real line,

$$\mathcal{S}(x) = \{\mathbf{f} : \mathcal{H}_t(x) \rightarrow \mathbf{R} \mid \mathbf{f}^{-1}([a,b]) \in \mathcal{B}_t(x) \text{ for } a, b \in \mathbf{R}, a \leq b\}$$

The value $f(\tau) = \mathbf{f}(h_{t,\tau})$ is the value of the stochastic process $\mathbf{f}$ at date τ, given the history up to and including τ. (Re-)define the stochastic process $\mathbf{x} \in \mathcal{S}(x)$ as the mapping defined by $\mathbf{x}(h_{t,\tau}) = x_n$ for $h_{t,\tau}$ as described in (57). Proceeding this way, the number of switches between two dates is finite by construction. The stochastic process $\mathbf{x}$ generates the σ-algebra $\mathcal{B}_t(x)$.

Equipped with these probabilities, one can define expectations and conditional expectations. For example, the expectation of the stochastic process $f(\tau)$ at some date $\tau > t$ and

[25]One could start from a description of the probability law, imposing this consistency condition and some other mild assumptions, allowing us to move beyond the Markov structure imposed in (59). We do not pursue this here.

[26]It is generated by the Cartesian products of open subsets of $\mathbf{R}^{n+1}$ with any subset of X.

$f \in \mathcal{S}(x)$ amounts to integration with respect to $P_{t,\tau}$ over the set $\mathcal{H}_{t,\tau}(x)$. It is given by

$$\begin{aligned} E[f(\tau) \mid x(t) = x] &= \int_{\mathcal{H}_{t,\tau}(x)} \mathbf{f}(h_{t,\tau}) dP_{t,\tau} \\ &= \sum_{n=0}^{\infty} \sum_{(x_1,\ldots,x_n)\in X^n} \int_{t_1=t}^{t_1=\tau} \int_{t_2=t_1}^{t_2=\tau} \cdots \int_{t_n=t_{n-1}}^{t_n=\tau} \mathbf{f}(\tau, n, t, t_1, t_2, \ldots, t_n, x, x_1, \ldots, x_n) \\ &\qquad dP_{t,\tau}(\tau, n, t, t_1, t_2, \ldots, t_n, x, x_1, \ldots, x_n) \end{aligned} \tag{61}$$

As an arbitrary example with $t = 0, \tau = 3$ and $x(0) = 0$, the term for $n = 2$ and $x_1 = 1, x_2 = 0$ becomes

$$\int_0^3 \int_{t_1}^3 \mathbf{f}(3, 2, 0, t_1, t_2, 0, 1, 0)\alpha_{0,1}\alpha_{1,0} \exp((2 - t_2 + t_1)\alpha_{0,0} + (t_2 - t_1)\alpha_{1,1})dt_2 dt_1,$$

integrating the value of the stochastic process $f(3)$ over all histories from $t = 0$ to $\tau = 3$ with exactly $n = 2$ switches from $x_0 = 0$ to $x_1 = 1$ and back to $x_2 = 0$ across all the switching dates.

D Analysis of the Optimal Risk-Sharing Contract

We assume $0 < r \leq \rho$ throughout this section.

D.1 Basic Properties

Given the apparatus of Appendix C, we can restate definition 1 more formally. We impose the condition that a consumption contract starting at date t is a mapping from histories in $\mathcal{H}_t$, starting state x and promised utility levels U to the real line, with $\mathbf{c}(h_{t,\tau}; x, U)$ the amount of consumption at date τ, after observing the history $h_{t,\tau}$. The resulting stochastic process[27] $c(\tau; x, U) = \mathbf{c}(h_{t,\tau}; x, U)$ is adapted to the stochastic process $(x_\tau)_{\tau \geq t}$. Note that this allows for processes that are functions of the last switch date t_n, the current date τ as well as the current state $x_n = x(\tau)$. These suffice to describe the contracts in the main body of the paper. Here we repeat this definition, but now we use the notation of Appendix C.

Definition 3 (contracts). *For fixed outside options $U^{out}(z)$, with $z \in Z$ and a starting date*

[27] Formally, given τ, x and U, $c(\tau; x, U)$ is a random variable, mapping $\mathcal{H}_{t,\tau}(x)$ into $\mathbf{R}$.

t, let $\mathcal{U}(x) = \left[U^{out}(\mathbf{z}(x)), \frac{\bar{u}}{\rho}\right)$ *for* $x \in X$ *and let* $\mathcal{C}$ *be the set of all consumption contracts,*

$$\mathcal{C} = \{\mathbf{c} \mid \mathbf{c} : \{(h_{t,\tau}; x, U) \mid x \in X, h_{t,\tau} \in \mathcal{H}_t(x), U \in \mathcal{U}(x)\} \to \mathbf{R}_+\} \tag{62}$$

Definition 4 (cost-minimizing contracts). *For a fixed wage* w *and rate of return on capital or interest rate* r*, an* **optimal consumption insurance contract** $\mathbf{c} \in \mathcal{C}$ *and the* **cost function** $V : \{(x, U) \mid x \in X, U \in \mathcal{U}(x)\} \to \mathbf{R}$ *solve*

$$V(x, U) = \min_{\mathbf{c} \in \mathcal{C}} \int_t^\infty \int_{\mathcal{H}_{t,\tau}(x)} e^{-r(\tau - t)} \left[w\mathbf{c}(h_{t,\tau}; x, U) - w\mathbf{z}(\mathbf{x}(h_{t,\tau}))\right] dP_{t,\tau} d\tau \tag{63}$$

subject to the **promise keeping constraint**

$$\int_t^\infty \int_{\mathcal{H}_{t,\tau}(x)} e^{-\rho(\tau - t)} u(w\mathbf{c}(h_{t,\tau}; x, U)) dP_{t,\tau} d\tau \quad \geq \quad U \tag{64}$$

and the **limited commitment constraints**

$$\int_s^\infty \int_{\mathcal{H}_{s,\tau}(x(h_{t,s}))} e^{-\rho(\tau - s)} u(w\mathbf{c}([h_{t,s}, h_{s,\tau}]; x, U)) dP_{s,\tau} d\tau \quad \geq \quad U^{out}(\mathbf{z}(\mathbf{x}(h_{t,s})))$$

$$\text{for all } s > t \text{ and } h_{t,s} \in \mathcal{H}_{t,s}(x) \tag{65}$$

for all $x \in X$ *and* $U \in \mathcal{U}(x)$*.*

Lemma 3 (monotonicity of the cost function). $V(x, U)$ *is increasing in* U*. It is strictly increasing in* U*, if (64) binds.*

Proof of Lemma 3. Consider two levels $U = U^A$ and $U = U^B$, with $U^A > U^B$. Any consumption contract that satisfies (64) for $U = U^A$ as well as (65) also satisfies (64) for $U = U^B$ as well as (65). This is true in particular for the cost-minimizing contract at $U = U^A$. Thus, the costs for U^B cannot be larger, that is, $V(x, U^B) \leq V(x, U^A)$.

Now suppose that $V(x, U^B) = V(x, U^A)$. That means that utility level $U = U^A$ could have been delivered for the same costs when a contract delivering $U = U^B$ is sought; that is, (64) could not have been binding at $U = U^B$, and thus $V(x, U^B) < V(x, U^A)$; that is, V is strictly increasing in U as along as (64) is binding at $U = U^B$. □

Lemma 4 (convexity of the cost function). $V(z, U)$ *is convex in* U*. It is strictly convex in* U*, if (64) binds.*

Proof of Lemma 4. Let $\lambda \in [0,1]$. Consider two levels $U = U^A$ and $U = U^B$. Suppose w.l.o.g. that $U^B < U^A$. Now consider the linear combination $\mathbf{c}^\lambda = \lambda \mathbf{c}^A + (1-\lambda)\mathbf{c}^B$. Since $u(\cdot)$ is concave, $\mathbf{c}^\lambda$ will satisfy constraint (65), since $\mathbf{c}^A$ and $\mathbf{c}^B$ both do. Let $V(\mathbf{c}^\lambda)$ be the costs of the contract $\mathbf{c}^\lambda$ per the right hand side of equation (92). Likewise, let $U(\mathbf{c}^\lambda)$ be the present value of the utility of the contract $\mathbf{c}^\lambda$ per the right hand side of equation (65). Note that $V(\mathbf{c}^\lambda) = \lambda V(x, U^A) + (1-\lambda)V(x, U^B)$. Since $u(\cdot)$ is strictly concave, $U(\mathbf{c}^\lambda) \geq U^\lambda$. The inequality is strict, if $\mathbf{c}^B \neq \mathbf{c}^A$ on a set of positive measure. Thus, $V(\mathbf{c}^\lambda) \geq V(x, U^\lambda)$ and strictly so, if (64) binds for U^A, per Lemma 3.

If (64) binds for U^B, it also binds for U^A and $U^\lambda = \lambda U^A + (1-\lambda)U^B$. Let $\mathbf{c}^A = \mathbf{c}(\cdot; x, U^A)$ and $\mathbf{c}^B = \mathbf{c}(\cdot; x, U^B)$ be the parts of the optimal consumption contract solving the cost minimization problems for (x, U^A) and (x, U^B). If (64) binds for U^B, then $\mathbf{c}^A$ cannot be the solution for U^B, i.e. $\mathbf{c}^B \neq \mathbf{c}^A$ on a set of positive measure. □

Lemma 5 (differentiability of the cost function). *The cost function $V(x, U)$ is continuous. It is differentiable in U on the right and left.*

Proof of Lemma 5. Continuity and right- as well as left-differentiability follow from the concavity of $V(x, \cdot)$ for interior points. Continuity and differentiability to the right at the lower bound $U^{out}(\mathbf{z}(x))$ follow, because $V(x, \cdot)$ is increasing and convex. □

We shall denote the right-hand side derivatives and the left-hand side derivatives of V with V'_+ and V'_-, Let

$$V'_+(x, U) = \lim_{h>0, h\to 0} \frac{V(x, U+h) - V(x, U)}{h}, \;\; V'_-(x, U) = \lim_{h<0, h\to 0} \frac{V(x, U+h) - V(x, U)}{h}. \tag{66}$$

For $U = U^{out}(\mathbf{z}(\mathbf{x}))$, we define the left-hand side derivative as $V'_-(U^{out}(\mathbf{z}(\mathbf{x}))) = \mathbf{0}$, since $V(x, \cdot)$ is an increasing function.

D.2 A Lagrangian Approach

The analysis of the optimal contract and the derivation of the relevant first-order conditions follows in spirit the approach of Marimon and Marcet (2019) (see also Golosov et al. (2016) as well as the generalization of Marcet-Marimon (2019) to continous-time heterogeneous-agent settings and the introduction of “timeless penalties” in Dàvila and Schaab (2022a)). We first restate the optimization problem as a Lagrangian in this subsection, provide a recursive perspective in subsection D.4 and then establish some key properties in subsection D.3.

As a first pass at the problem and for notational clarity, we shall drop the explicit history dependence and conditioning information in the constraint (3). Heuristically, let ζ be the Lagrange multiplier on (2) and let $\mu(s)$ be the Lagrange multiplier on (3). Integrating the constraints (3) discounted at $e^{-\rho(s-t)}$, the Lagrangian would then be

$$\begin{aligned} L \; = \; & \mathbf{E}\left[\int_t^\infty e^{-r(\tau-t)}\left(wc(\tau)-wz(\tau)\right)d\tau \right. \\ & -\zeta\left(\int_t^\infty e^{-\rho(\tau-t)}u(wc(\tau))d\tau - U\right) \\ & \left. -\int_t^\infty e^{-\rho(s-t)}\mu(s)\left(\int_s^\infty e^{-\rho(\tau-s)}u(wc(\tau))d\tau - U^{out}(z(s))\right)ds\right] \end{aligned} \tag{67}$$

This can be rewritten as

$$\begin{aligned} L \; = \; & \mathbf{E}\left[\int_t^\infty e^{-r(\tau-t)}\left[wc(\tau)-wz(\tau)\right]d\tau \right. \\ & -\int_t^\infty \lambda(\tau)e^{-\rho(\tau-t)}u(wc(\tau))d\tau \\ & \left. +\lambda(t)U+\int_t^\infty e^{-\rho(s-t)}U^{out}(z(s))d\lambda(s)\right] \end{aligned} \tag{68}$$

provided that

$$\lambda(\tau)=\zeta+\int_t^\tau \mu(s)ds \tag{69}$$

We shall call $\lambda(\cdot)$ the cumulative Lagrange multiplier. We proceed with (68) as the Lagrangian function without imposing that $\lambda(\tau)$ is differentiable or even continuous. We drop the multipliers $\mu(s)$ from the problem, though we keep the restriction from (69) that $\lambda(\tau)$ is a weakly increasing and nonnegative function of τ and that it is only increasing, if (3) binds. Given a path for λ up to date s, the integral with respect to $d\lambda(s)$ on the last line of (68) is a Riemann-Stieltjes integral, given the state history up to s.

$\lambda(\tau)$ is an adapted stochastic process and depends on the history of the state. To be more precise, we build on the formulation in definition 4. Let ζ be the Lagrange multiplier on (64) and let $\boldsymbol{\mu}(h_{t,s})$ be the Lagrange multiplier on (65). Integrating the constraints (65) discounted at $e^{-\rho(s-t)}$, the Lagrangian is

$$
\begin{aligned}
L &= \int_t^\infty \int_{\mathcal{H}_{t,\tau}} e^{-r(\tau-t)} \left(w\mathbf{c}(h_{t,\tau}) - w\mathbf{z}(\mathbf{x}(h_{t,\tau}))\right) dP_{t,\tau} d\tau \\
&\quad -\zeta \left(\int_t^\infty \int_{\mathcal{H}_{t,\tau}} e^{-\rho(\tau-t)} u(w\mathbf{c}(h_{t,\tau})) dP_{t,\tau} d\tau - U \right) \\
&\quad - \int_t^\infty \int_{\mathcal{H}_{t,s}} e^{-\rho(s-t)} \boldsymbol{\mu}(h_{t,s}) \left(\int_s^\infty \int_{\mathcal{H}_{s,\tau}} e^{-\rho(\tau-s)} u(w\mathbf{c}([h_{t,s}, h_{s,\tau}])) dP_{s,\tau} d\tau - U^{out}(\mathbf{z}(\mathbf{x}(h_{t,s}))) \right) dP_{t,s} ds
\end{aligned} \tag{70}
$$

This can be rewritten as

$$
\begin{aligned}
L &= \int_t^\infty \int_{\mathcal{H}_{t,\tau}} e^{-r(\tau-t)} \left[w\mathbf{c}(h_{t,\tau}) - w\mathbf{z}(\mathbf{x}(h_{t,\tau}))\right] dP_{t,\tau} d\tau \\
&\quad - \int_t^\infty \int_{\mathcal{H}_{t,\tau}} \boldsymbol{\lambda}(h_{t,\tau}) e^{-\rho(\tau-t)} u(w\mathbf{c}(h_{t,\tau})) dP_{t,\tau} d\tau \\
&\quad + \boldsymbol{\lambda}(h_{t,t}) U + \int_{s=t}^\infty \int_{\mathcal{H}_{t,s}} e^{-\rho(s-t)} U^{out}(\mathbf{z}(\mathbf{x}(h_{t,s}))) dP_{t,s} \times d\lambda
\end{aligned} \tag{71}
$$

provided[28] that **the cumulative Lagrange multiplier** is given by

$$
\boldsymbol{\lambda}(h_{t,\tau}) = \zeta + \int_t^\tau \boldsymbol{\mu}(h_{t,\tau}(t,s)) ds \tag{72}
$$

The cumulative Lagrange multiplier reformulation in (71) provides a version of Marcet and Marimon (2019) in continuous time. We proceed with (71) as the Lagrangian function without imposing that the mapping $\tau \mapsto \lambda(h_{t,s}[t,\tau])$ for some $h_{t,s}$ and $t \leq \tau \leq s$ is differentiable or even continuous, but keep the restriction from (72) that these mappings are weakly increasing and nonnegative and that they are only increasing, if (3) binds.

Differentiating the Lagrangian (68) with respect to $c(\tau)$ resp (71) with respect to $\mathbf{c}(h_{t,\tau}; x, U)$ yields the first-order condition

$$
e^{(\rho-r)(\tau-t)} = \lambda(\tau) u'(wc(\tau)) \tag{73}
$$

[28]Regarding the measure $dP_{t,s} \times d\lambda$ for the double integral $s \in [t,\infty)$, $h_{t,s} \in \mathcal{H}_{t,s}$, use (61), replace τ with s, and integrate over $s \in [t,\infty)$. As one of the terms, consider $n = 2$ and $x_0 = 2, x_1 = 0, x_2 = 1$. Exchange the order of integration and calculate $\int_{t_1=t}^\infty \int_{t_2=t_1}^\infty \int_{s=t_2}^\infty f(t_1,t_2,s) d\lambda(s,2,t,t_1,t_2,2,0,1) dt_2 dt_1$, where $f(t_1,t_2,s)$ collects the remaining terms and where the integral with respect to $d\lambda(s,2,t,t_1,t_2,2,0,1)$ is a Riemann-Stieltjes integral, using the weakly increasing and possibly discontinuous function $s \mapsto \lambda(s,2,t,t_1,t_2,2,0,1)$ of $s \geq t_2$. Proceed likewise with all other terms. We drop a further discussion, as the integral with respect to $dP_{t,s} \times d\lambda$ is a constant and drops out in the first-order conditions.

or

$$e^{(\rho-r)(\tau-t)} = \boldsymbol{\lambda}(h_{t,\tau})u'(w\mathbf{c}(h_{t,\tau};x,U)) \tag{74}$$

Lemma 6 (consumption is not increasing). $\mathbf{c}(h_{t,\tau};x,U)$ *is decreasing at* τ *for* $r < \rho$ *and constant for* $r = \rho$*, when the limited commitment constraint (65) does not bind at* τ*.*

Proof. In that case, the Lagrange multiplier $\boldsymbol{\lambda}(h_{t,\tau})$ remains constant. The claim now follows from (74) and $u'' < 0$. □

D.3 Key Properties of the Optimal Contract

From here on, we drop the dependency on x and U in the consumption contract, in order to save on notation.

For the next result, the following assumption is helpful.

Assumption 3 (bounded risk aversion). *The utility function* $u(\cdot)$ *satisfies*

$$0 < -\frac{u''(x)x}{u'(x)} < \bar{\sigma} < \infty \tag{75}$$

for all $x > 0$ *and some* $\bar{\sigma}$*.*

Assumption 3 is obviously satisfied for all nonlinear CRRA utility functions. For $\mathbf{c}$ and a given history $h_{t,s}$, $s > t$, define the left limit

$$\mathbf{c}_-(h_{t,s}) = \lim_{\Delta\to 0,\Delta>0} \mathbf{c}(h_{t,s}(t, s-\Delta))$$

and likewise the left derivative

$$\dot{\mathbf{c}}_-(h_{t,s}) = \lim_{\Delta\to 0,\Delta>0} \frac{\mathbf{c}(h_{t,s}) - \mathbf{c}(h_{t,s}(t, s-\Delta))}{\Delta},$$

provided they exist. For the right limit, observe that $\mathbf{c}(h_{t,s+\Delta})$ is stochastic, and a probabilistic limit needs to be taken. Since the probability of a state change within the time interval from s to $s+\Delta$ vanishes, the probabilistic limit is equal to the limit, when only the future histories without a state change are taken into account. Formally, let $x(s)$ be the current state, $x(s) = \mathbf{x}(h_{t,s}) = x_n$. The extension of the current history without a state change until Δ or **no-state-change history extension** can be written as the concatenated history

$$h_{t,s;\Delta} = [h_{t,s}, (s+\Delta, 0, s, x(s))].$$

Define[29]

$$\mathbf{c}_{+}(h_{t,s}) = \lim_{\Delta \to 0, \Delta \geq 0} \mathbf{c}(h_{t,s;\Delta})$$

and likewise the right derivative

$$\dot{\mathbf{c}}_{+}(h_{t,s}) = \lim_{\Delta \to 0, \Delta > 0} \frac{\mathbf{c}(h_{t,s;\Delta}) - \mathbf{c}(h_{t,s})}{\Delta},$$

provided they exist. Continuity as well as differentiability at a given history $h_{t,s}$, $s > t$ are defined, when the left and right limits exist and coincide. The derivative in that case will be denoted with $\dot{\mathbf{c}}(h_{t,s})$ or simply $\dot{c}(s)$. We proceed likewise for $\boldsymbol{\lambda}$.

Lemma 7 (key properties of the contract). *1. Suppose that the constraint* (65) *does not bind at history $h_{t,s}$. Then $\dot{\boldsymbol{\lambda}}(h_{t,s})_{+} = 0$ and the derivative $\dot{\mathbf{c}}_{+}(h_{t,s})$ exists. If the last jump occurred strictly before date s, i.e., if $t_n < s$, then $\dot{\boldsymbol{\lambda}}(h_{t,s}) = 0$ and $\mathbf{c}$ is differentiable at $h_{t,s}$.*

2. *Suppose $\mathbf{c}$ is differentiable at history $h_{t,s}$. Then $\boldsymbol{\lambda}$ is differentiable at history $h_{t,s}$ and*

$$\rho - r = \left(\frac{u''(wc)wc}{u'(w\mathbf{c}(h_{t,s}))} \right) \frac{\dot{\mathbf{c}}(h_{t,s})}{\mathbf{c}(h_{t,s})} + \frac{\dot{\boldsymbol{\lambda}}(h_{t,s})}{\boldsymbol{\lambda}(h_{t,s})} \tag{76}$$

The statement and equation likewise hold for the left-derivatives, if $\mathbf{c}$ is left-differentiable at history $h_{t,s}$, and for the right-derivatives, if $\mathbf{c}$ is right-differentiable at history $h_{t,s}$.

3. *Suppose that the limited commitment constraint (65) binds at history $h_{t,s}$. Suppose that $\rho = r$. Alternatively, suppose that $\rho > r$ and that Assumption 3 holds. Then $\mathbf{c}(h_{t,s;\Delta})$ is constant in $\Delta \geq 0$ and $\dot{\mathbf{c}}_{+}(h_{t,s}) = 0$.*

4. *$\boldsymbol{\lambda}_{-}(h_{t,s}) \neq \boldsymbol{\lambda}(h_{t,s})$, i.e. $\boldsymbol{\lambda}$ is discontinuous at history $h_{t,s}$, if and only if $\mathbf{c}_{-}(h_{t,s}) \neq \mathbf{c}(h_{t,s})$. In that case, $\mathbf{c}_{-}(h_{t,s}) < \mathbf{c}(h_{t,s})$, $\boldsymbol{\lambda}_{-}(h_{t,s}) < \boldsymbol{\lambda}(h_{t,s})$ and* (65) *binds at history $h_{t,s}$ with $t_n = s$, i.e. just when the state change occurred.*

The proof of the lemma builds on techniques and results for the subsequent subsection D.4 and the recursive formulations there. In terms of mathematical logic, that subsection precedes rather than builds on the material here. Since it is only necessary for understanding the proof here, we chose the current ordering in the interest of readability.

[29] Note that we use $\Delta \geq 0$ for defining $\mathbf{c}_{+}(h_{t,s})$, as the processes $c(s) = \mathbf{c}(h_{t,s})$ may generally often be cadlag, i.e. right-continuous, but with a left limit.

Proof. 1. If the constraint does not bind at $h_{t,s}$, then it will not bind either for the no-state-change history extensions $h_{t,s;\Delta}$, provided $\Delta > 0$ is sufficiently small. Thus, $\boldsymbol{\lambda}(h_{t,s}) = \boldsymbol{\lambda}(h_{t,s;\Delta})$ is locally constant[30] and thus $\dot{\boldsymbol{\lambda}}_+(h_{t,s}) = 0$. The existence of $\dot{\mathbf{c}}_+$ at $h_{t,s}$ now follows from (73). If $t_n < s$, the same argument applied to the truncated histories $h_{t,s}(t, s - \Delta)$ shows that $\dot{\boldsymbol{\lambda}}(h_{t,s}) = 0$ and the differentiability of $\mathbf{c}$ at $h_{t,s}$.

2. Differentiation of (74) with respect to τ shows that λ is also differentiable[31] at τ and delivers (76), when replacing τ with s.

3. By assumption, the limited commitment constraint (65) binds at $h_{t,s}$. For ease of notation, write x for $x(s)$ and z for $\mathbf{z}(x)$. Consider the no-state-change history extension $h_{t,s;\Delta} = [h_{t,s}, (s + \Delta, 0, s, x)]$ for $\Delta > 0$. The proof proceeds in two parts. For part A, suppose that the limited commitment constraint (65) binds again at $h_{t,s;\bar{\Delta}}$ for some $\bar{\Delta} > 0$. We use an averaging argument to establish that consumption must be the same at s and at $s + \bar{\Delta}$. With the help of the first two parts as well as some careful analysis, we then show that the limited commitment constraint (65) must bind for all $h_{t,s;Delta}$ and $0 < \Delta \leq \bar{\Delta}$ and thus establish the claim for part A. For part B, suppose that the constraint (65) never binds again for $h_{t,s;\Delta}$ at any $\Delta > 0$. We show that this leads to a contradiction.

   A. For the first part, suppose that the limited commitment constraint (65) binds at $h_{t,s}$ as well as at $h_{t,s;\bar{\Delta}}$ for some (possibly large) $\bar{\Delta} > 0$. Compare the contract going forward conditional on these two histories: we will argue that one can do better by averaging them, should they be different. To that end and to express this precisely, let $h_{s,\tau}$ be some continuation at the current state x of the history $h_{t,s}$ to $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$: for a graphical illustration, see Figure 7. Construct the corresponding continuation $h_{t,\tau+\bar{\Delta}} = [h_{t,s;\bar{\Delta}}, h^{\bar{\Delta}}_{s,\tau}]$ of $h_{t,s;\bar{\Delta}}$ with the $\bar{\Delta}$-time-shifted history $h^{\bar{\Delta}}_{s,\tau}$ (see equation (58)). This correspondence is one-one and measure preserving. Suppose now that the contract $\mathbf{c}(h_{t,\tau})$ differs from the corresponding $\mathbf{c}(h_{t,\tau+\bar{\Delta}})$ for a set $\mathcal{S}$ of extensions $h_{s,\tau}$ with positive measure, i.e., suppose that $\int_{\tau=s}^{\infty} \int_{\mathcal{H}_{s,\tau}} 1_{h_{s,\tau} \in \mathcal{S}} dP_{s,\tau} d\tau > 0$. Consider then a new contract, which is the average between the original contract and the contract following

[30] Returning to our original Lagrange multipliers, $\boldsymbol{\mu}(h_{t,s;\Delta}) = 0$ for $\Delta \geq 0$ sufficiently small.
[31] This is a standard calculus argument.

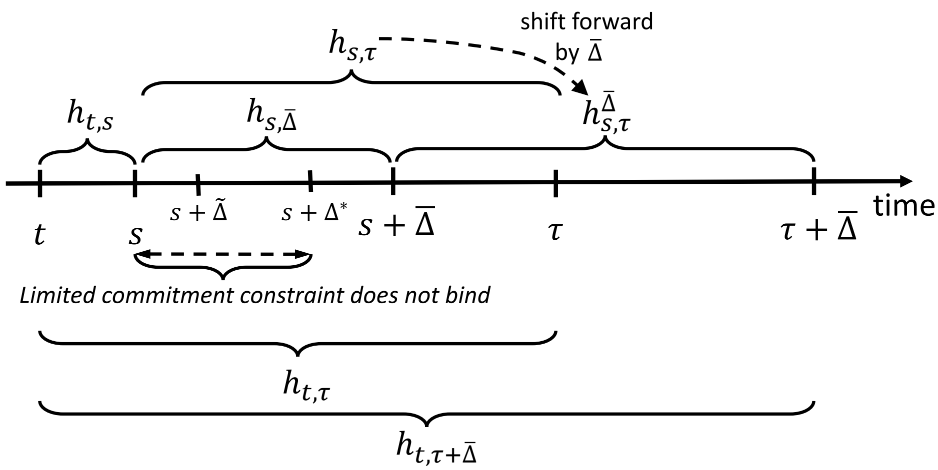

Figure 7: Timeline for part 3.A of the proof for Lemma 7. Starting point is the history $h_{t,s}$. Consider a time interval $\bar{\Delta}$ and the history extended to $h_{t,s;\Delta}$ without a state change between s and $s+\bar{\Delta}$. Suppose that the limited commitment constraint (65) binds at $h_{t,s}$ as well as at $h_{t,s;\bar{\Delta}}$. Consider some $\tau > s$ and history $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ coinciding with $h_{t,s}$ until s: there may be state changes at several points between s and τ. Shift the continuation piece $h_{s,\tau}$ forward by $\bar{\Delta}$ and append it to the history $h_{t,s;\bar{\Delta}}$. Consider the original contract for $h_{t,\tau}$ and the contract for this shifted-and-appended history. Averaging the original contract and the shifted contract as in (77) shows that consumption must be the same at s and $s+\bar{\Delta}$. Hence, if (65) binds for all $\bar{\Delta} > 0$, we'd be done: consumption would need to be constant, while the state does not change. Thus, suppose that for some Δ^* that the limited commitment constraint (65) does not bind at the no-change histories $h_{t,s;\Delta}$ for all $0 < \Delta < \Delta^*$. In part 3.A.i and 3.A.ii of the proof and illustrated in Figure 8, we show that this leads to a contradiction.

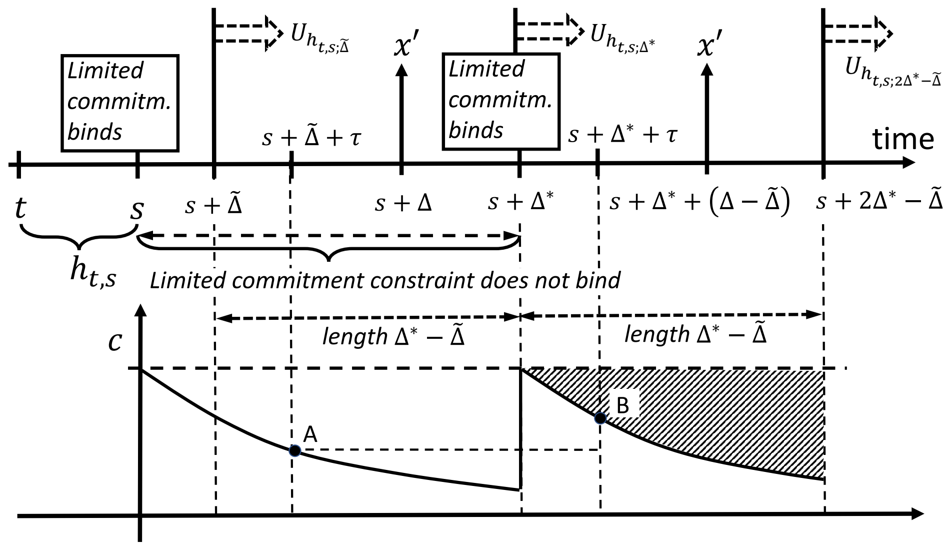

Figure 8: Timeline for part 3.A.ii of the proof for Lemma 7. This zooms in on a portion of Figure 7 and extends it with some additional detail. Suppose the limited commitment constraint (65) binds at $h_{t,s}$ as well as at $h_{t,s;\Delta^*}$, where Δ^* is chosen as small as possible. That means, that consumption must be declining between s and $s+\Delta^*$, when $r<\rho$, and jumps back up at $s+\Delta^*$ to the consumption level at s. Compare now the continuation utility $U_{h_{t,s;\tilde{\Delta}}}$ at some $s+\tilde{\Delta}$, $0<\tilde{\Delta}<\Delta^*$ to the continuation utility $U_{h_{t,s;\Delta^*}}$ at $s+\Delta^*$. Since the limited commitment constraint does not bind at $s+\tilde{\Delta}$, that continuation utility must be higher there than at $s+\Delta^*$. However, along the no-change-in-state, consumption at every $s+\tilde{\Delta}+\tau$ is smaller than at $s+\Delta^*+\tau$ as long as $\tau<\Delta^*-\tilde{\Delta}$. One can see this by comparing points A and B, where the shaded area indicates the range of consumption values beyond $s+\Delta^*$ and the lower bound results, if the outside option does not bind between $s+\Delta^*$ and $s+2\Delta^*-\tilde{\Delta}$. Furthermore, $U_{h_{t,s;\Delta^*}}=U^{out}(\mathbf{z}(x))<U_{h_{t,s;2\Delta^*-\tilde{\Delta}}}$. The principle of optimality of Lemma 9 delivers the result that this cannot be "compensated" for by the state change points, comparing $s+\Delta$ to $s+\Delta^*+(\Delta-\tilde{\Delta})$ or comparing the continuation utility at $s+\Delta^*$ (as a portion for the $s+\tilde{\Delta}$ calculation) to the continuation utility at $s+2\Delta^*-\tilde{\Delta}$ (as a portion for the $s+\Delta^*$ calculation). This is a contradiction.

$h_{t,s;\bar{\Delta}}$, i.e., consider

$$\tilde{\mathbf{c}}(h_{t,\tau}) = \left(\mathbf{c}([h_{t,s}, h_{s,\tau}]) + \mathbf{c}([h_{t,s;\bar{\Delta}}, h^{\bar{\Delta}}_{s,\tau}])\right)/2 \tag{77}$$

defined for all continuations $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ of $h_{t,s}$. In words, $\tilde{\mathbf{c}}$ is the average between the current contract as well as the contract portion following $h_{t,s;\bar{\Delta}}$ shifted backward by $\bar{\Delta}$. Since utility is strictly concave, this contract now delivers strictly higher continuation utility at history $h_{t,s}$, while its costs stay unchanged, a contradiction to the hypothesis, that the constraint (65) binds at $h_{t,s}$, i.e. a contradiction to the assertion that the original contract was cost-minimizing, see lemma 3. It follows that consumption at $s+\bar{\Delta}$ will be the same as at s for any $\bar{\Delta} > 0$, where (65) binds: let us denote that consumption level as c.

If the limited commitment constraint (65) binds for all $0 < \tilde{\Delta} < \bar{\Delta}$, we would be done with this part, since consumption would then be constant at $\mathbf{c}(h_{t,s;\tilde{\Delta}}) \equiv c$. Indeed, we would be done if this is true for some sufficiently small $\bar{\Delta} > 0$, since it must then be true for all $\bar{\Delta}$ per "shifting" the contract by $\bar{\Delta}/2$ into the future. Suppose thus that (65) binds at some[32] $\Delta^* \leq \bar{\Delta}$, but does not bind for all $0 < \tilde{\Delta} < \Delta^*$. According to the first part of the lemma, the derivative $\dot{\mathbf{c}}(h_{t,s;\tilde{\Delta}})$ exists and $\dot{\boldsymbol{\lambda}}(h_{t,s;\tilde{\Delta}}) = 0$.

i. Consider the case $\rho = r$. According to the second part of the lemma, $\dot{\mathbf{c}}(h_{t,s;\tilde{\Delta}}) = 0$. Thus, consumption is constant at $\mathbf{c}(h_{t,s;\tilde{\Delta}}) \equiv c$ for all $0 < \tilde{\Delta} < \bar{\Delta}$, regardless of whether the constraint (65) binds or does not bind at $\tilde{\Delta}$, establishing our claim.

ii. Consider the case $\rho > r$ and current state x. We will show that we arrive at a contradiction; see Figure 8. According to the second part of the lemma, $\dot{\mathbf{c}}(h_{t,s;\tilde{\Delta}}) < 0$ for $0 < \tilde{\Delta} < \Delta^*$. Fix such a $\tilde{\Delta}$. It follows that $\mathbf{c}(h_{t,s;\tilde{\Delta}}) < \mathbf{c}(h_{t,s;\Delta^*}) = c$; that is, consumption jumps up at Δ^*, even though $U_{h_{t,s;\tilde{\Delta}}} > U_{h_{t,s;\Delta^*}} = U^{out}(\mathbf{z}(x))$. We will show that the contract at history $h_{t,s;\tilde{\Delta}}$ can therefore not have been cost-minimizing. For $x' \neq x$ and $\Delta > 0$, let $h_{t,s;\Delta,x'} = [h_{t,s}, (s+\Delta, 1, s, x, x(s+\Delta) = x']$ be the extensions of the original history $h_{t,s}$ with a first state change to a new state x' occurring at date $s + \Delta$.

[32] Δ^* exists, because $U_{h_{t,s;\tilde{\Delta}}}$ is continuous in $\tilde{\Delta}$.

Define $U_{h_{t,s;\tilde{\Delta}}}$ as the continuation utility starting at the history $h_{t,s;\tilde{\Delta}}$. Deploying the construction of Appendix D.4 leading up to equation (82), it is given by

$$\begin{aligned} U_{h_{t,s;\tilde{\Delta}}} &= \int_0^{\Delta^*-\tilde{\Delta}} e^{(\alpha_{x,x}-\rho)\tau} u(w\mathbf{c}(h_{t,s;\tilde{\Delta}+\tau}))d\tau \\ &+e^{(\alpha_{x,x}-\rho)(\Delta^*-\tilde{\Delta})} U^{out}(\mathbf{z}(x)) \\ &+\sum_{x'\neq x}\int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\tilde{\Delta}+\tau,x'}} d\tau \end{aligned} \tag{78}$$

The first term captures the present discounted utility over the time interval from $t+s+\tilde{\Delta}$ to $t+s+\Delta^*$, conditional on no state change; the second term captures the associated continuation utility from $t+s+\Delta^*$ onward in that case; and the last term captures the expected continuation utility conditional on some state change from x to x' during the time interval $\Delta^*-\tilde{\Delta}$ following history $h_{t,s;\tilde{\Delta}}$.
Compare this to the similar continuation at $s+\Delta^*$,

$$\begin{aligned} U_{h_{t,s;\Delta^*}} &= \int_0^{\Delta^*-\tilde{\Delta}} e^{(\alpha_{x,x}-\rho)\tau} u(w\mathbf{c}(h_{t,s;\Delta^*+\tau}))d\tau \\ &+e^{(\alpha_{x,x}-\rho)(\Delta^*-\tilde{\Delta})} U_{h_{t,s;\Delta^*+(\Delta^*-\tilde{\Delta})}} \\ &+\sum_{x'\neq x}\int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\Delta^*+\tau,x'}} d\tau \end{aligned} \tag{79}$$

Note that $\mathbf{c}(h_{t,s;\tilde{\Delta}+\tau}) < \mathbf{c}(h_{t,s;\Delta^*+\tau})$ and that $U^{out}(z) \leq U_{h_{t,s;\Delta^*+(\Delta^*-\tilde{\Delta})}}$. Since $U_{h_{t,s;\tilde{\Delta}}} > U_{h_{t,s;\Delta^*}} = U^{out}(z)$, it must be the case that the inequality is reversed for the last term,

$$\begin{aligned} &\sum_{x'\neq x}\int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\tilde{\Delta}+\tau,x'}} d\tau \\ &\qquad > \sum_{x'\neq x}\int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\Delta^*+\tau,x'}} d\tau \end{aligned}$$

Recall, though, that the contract is cost-minimizing at history $h_{t,s;\tilde{\Delta}}$. Per the principle of optimality established in Lemma 9 of Appendix D.4 below, we thus arrive at a contradiction.

B. For the second part, suppose instead that (65) never binds for any $\Delta > 0$ at the no-state-change history extensions $h_{t,s;\Delta} = [h_{t,s}, (s+\Delta, 0, s, x(s))]$. The first part of the lemma shows that $\dot{\lambda}(h_{t,s;\Delta}) = 0$ for all $\Delta > 0$ as well as $\dot{\lambda}_+(h_{t,s}) = 0$. If $\rho = r$, then the second part of the lemma shows that $\dot{\mathbf{c}}_+(h_{t,s}) = 0$ and hence the claim. If $\rho > r$, then (76) shows that $\dot{\mathbf{c}}(h_{t,s;\Delta})/\mathbf{c}(h_{t,s;\Delta}) < (r-\rho)/\bar{\sigma} < 0$ and hence $\mathbf{c}(h_{t,s;\Delta}) \to 0$ as $\Delta \to \infty$.

   i. Suppose then that there is some $\underline{\Delta} > 0$, so that the continuation utility constraints upon a state change to $x' \neq x(s)$ bind at all extended histories

$$h_{t,s;\Delta,x'} = [h_{t,s}, (s+\Delta, 1, s, s+\Delta, x(s), x')]$$

   for $\Delta > \underline{\Delta}$ and any $x' \in X$. The continuation contract at the no-state-change history extension $h_{t,s;\underline{\Delta}}$ is feasible when shifted backward in time to s, i.e., consider the contract

$$\tilde{\mathbf{c}}(h_{t,\tau}) = \mathbf{c}([h_{t,s;\underline{\Delta}}, h^{\underline{\Delta}}_{s,\tau}])$$

   defined for all continuations $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ of $h_{t,s}$. Contract $\tilde{\mathbf{c}}$ is cheaper for the principal than contract $\mathbf{c}$, since consumption along the $h_{t,s;\Delta}$-histories keeps declining and since one cannot do better upon a state change than to achieve a binding constraint there. This is a contradiction to the assertion that the contract was cost-minimizing $\mathbf{c}$.

   ii. Suppose instead that for any $\underline{\Delta} > 0$, there is a positive measure of dates $s + \Delta$ with $\Delta > \underline{\Delta}$, at which the utility promised upon a state change is not binding. But then and with sufficiently large Δ and thus sufficiently small $\mathbf{c}(h_{t,s;\Delta})$ along the no-state-change path, the principal can achieve a higher promised utility for the agent by promising less consumption upon the state change for some positive interval of time and more consumption along the no-state-change path, again a contradiction to the contract being cost-minimizing.

4. Since λ is weakly increasing, $\lambda(h_{t,s}) - \lambda_-(h_{t,s}) > 0$. The claim now follows from (74) and, exploiting the fact that $u'(\cdot)$ is strictly decreasing, as well as from noting that $\lambda(h_{t,s})$ is only increasing if (65) binds. Furthermore, it must be the case that $t_n = s$,

i.e., that the state change just occurred on date s, since otherwise the derivative of consumption would have been zero per the third part of the lemma.

□

Using the simplified notation $c(s) = \mathbf{c}(h_{t,s})$ and $\lambda(s) = \boldsymbol{\lambda}(h_{t,s})$, note that (76) implies that

$$\frac{\dot{c}(s)}{c(s)} = -\frac{\rho - r}{\sigma} + \frac{1}{\sigma}\frac{\dot{\lambda}(s)}{\lambda(s)} \tag{80}$$

for the CRRA utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \tag{81}$$

The utility function $u(c) = \log(c)$ is the special case, where $\sigma = 1$.

D.4 A Recursive Approach

Consider some $\Delta > 0$. Using the principle of optimality, one can rewrite the cost minimization problem for the optimal insurance contract by examining $\tau \in [t, t + \Delta]$ and then use the minimized costs for $\tau \geq t + \Delta$. More precisely, let $x(t) = x \in X$ be the state at the beginning date t of the contract. The histories beyond t are of two kinds. There is the no-change-in-state path $h^0_{t,s} = (s, 0, t_0 = t, x_0 = x)$ all the way up to $s = t + \Delta$, where the superindex 0 indicates zero state changes.[33] This includes in particular the starting point $h^0_{t,t} = h_{t,t}$. Then there are paths with a jump to state $x' \neq x$ at some date $s \in [t, t + \Delta]$, starting with the histories $h^1_{t,s;x'} = (s, 1, t_0 = t, t_1 = s, x_0 = x, x_1 = x')$, where the superindex 1 indicates one state change. Consider the continuation costs and continuation utility promises following these histories discounted to the new starting dates. Writing $\mathbf{c}(h_{t,\tau})$ rather than $\mathbf{c}(h_{t,\tau}; x, U)$ to save on notation,

$$\begin{aligned}
V_{h^0_{t,t+\Delta}} &= \int_{t+\Delta}^{\infty} \int_{\mathcal{H}_{t+\Delta,\tau}(x)} e^{-r(\tau - t - \Delta)} \left(w\mathbf{c}([h^0_{t,t+\Delta}, h_{t+\Delta,\tau}]) - w\mathbf{z}([h^0_{t,t+\Delta}, h_{t+\Delta,\tau}]) \right) dP_{t+\Delta,\tau} d\tau \\
V_{h^1_{t,s;x'}} &= \int_{s}^{\infty} \int_{\mathcal{H}_{s,\tau}(x')} e^{-r(\tau - s)} \left(w\mathbf{c}([h^1_{t,s;x'}, h_{s,\tau}]) - w\mathbf{z}([h^1_{t,s;x'}, h_{s,\tau}]) \right) dP_{s,\tau} d\tau \\
U_{h^0_{t,t+\Delta}} &= \int_{t+\Delta}^{\infty} \int_{\mathcal{H}_{t+\Delta,\tau}(x)} e^{-\rho(\tau - t - \Delta)} u(w\mathbf{c}([h^0_{t,t+\Delta}, h_{t+\Delta,\tau}])) dP_{t+\Delta,\tau} d\tau \\
U_{h^1_{t,s;x'}} &= \int_{s}^{\infty} \int_{\mathcal{H}_{s,\tau}(x')} e^{-\rho(\tau - s)} u(w\mathbf{c}([h^1_{t,s;x'}, h_{s,\tau}])) dP_{s,\tau} d\tau
\end{aligned}$$

[33] The superindex notation was avoided in the proof of Lemma 7 in order to declutter the notation there.

With that and using the appropriate probabilities, the continuation utility after date $\tau \in [t, t+\Delta]$ and no-change-in-state between t and τ is

$$\begin{aligned} U_{h^0_{t,\tau}} &= \int_\tau^{t+\Delta} e^{(\alpha_{x,x}-\rho)(s-\tau)} u(w\mathbf{c}(h^0_{t,s}))ds \\ &\quad + e^{(\alpha_{x,x}-\rho)(t+\Delta-\tau)} U_{h^0_{t,t+\Delta}} \\ &\quad + \sum_{x' \neq x} \int_\tau^{t+\Delta} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)(s-\tau)} U_{h^1_{t,s;x'}} ds \end{aligned} \tag{82}$$

The cost function in definition 4 can likewise be rewritten. Formally,

Definition 5 (recursive cost-minimization). *For a fixed $\Delta > 0$, fixed outside options $U^{out}(z)$, with $z \in Z$, a starting date t, and a fixed wage w and rate of return on capital or interest rate r, a* **recursive cost function** *$V(x, U)$ optimally chooses $\mathbf{c}(h^0_{t,\tau})_{\tau \in [t,t+\Delta]} \geq 0$, $U_{h^0_{t,t+\Delta}}$ and $(U_{h^1_{t,s;x'}})_{s \in [t,t+\Delta], x' \in X/\{x\}}$ to solve*

$$\begin{aligned} V(x, U) &= \min \int_t^{t+\Delta} e^{(\alpha_{x,x}-r)(\tau-t)} \left[w\mathbf{c}(h^0_{t,\tau}) - w\mathbf{z}(x) \right] d\tau \\ &\quad + e^{(\alpha_{x,x}-r)\Delta} V(x, U_{h^0_{t,t+\Delta}}) \\ &\quad + \sum_{x' \neq x} \int_t^{t+\Delta} \alpha_{x,x'} e^{(\alpha_{x,x}-r)(s-t)} V(x', U_{h^1_{t,s;x'}}) ds \end{aligned} \tag{83}$$

subject to the **promise-keeping constraint**

$$U_{h_{t,t}} \geq U \tag{84}$$

and the **limited commitment constraints**

$$\frac{\bar{u}}{\rho} > U_{h^0_{t,\tau}} \geq U^{Out}(\mathbf{z}(x)) \text{ for all } \tau \in [t, t+\Delta] \tag{85}$$

$$\frac{\bar{u}}{\rho} > U_{h^1_{t,s;x'}} \geq U^{Out}(\mathbf{z}(x')) \text{ for all } s \in [t, t+\Delta] \tag{86}$$

for all $x(t) = x \in X$ and all $U \in \left[U^{out}(\mathbf{z}(x)), \frac{\bar{u}}{\rho} \right)$.

Lemma 8 (equivalence). *The two definitions 4 and 5 coincide.*

Proof. Clear from the calculations above. □

Lemma 9 (principle of optimality). *Fix the state x. Consider two utility levels $U^A > U^B$ and suppose that (84) binds at both. Suppose there is some $\Delta > 0$, so that (85) does not bind for all $\tau \in [t, t+\Delta]$ and the no-change histories $h^0_{t,s}$, starting from the promise*[34] *$U = U^B$. Consider the optimal recursive cost function choices $\mathbf{c}^\Psi(h^0_{t,\tau})_{\tau\in[t,t+\Delta]} \geq 0$, $\mathbf{c}^\Psi(h^0_{t,\tau})$ and $(U^\Psi_{h^1_{t,s;x'}})_{s\in[t,t+\Delta],x'\in X/\{x\}}$ for $\Psi \in \{A, B\}$. Then*

$$\begin{aligned} \mathbf{c}^A(h^0_{t,\tau}) &\geq \mathbf{c}^B(h^0_{t,\tau}) \text{ for almost all } \tau \in [t, t+\Delta] \\ U^A_{h^0_{t,t+\Delta}} &\geq U^B_{h^0_{t,t+\Delta}} \\ U^A_{h^1_{t,s;x'}} &\geq U^B_{h^1_{t,s;x'}} \text{ for almost all } s \in [t, t+\Delta] \text{ and all } x' \in X/\{x\} \end{aligned}$$

Proof of Lemma 9. This is due to the similar structure of the utility formula (82) and the cost function (83). With (83), only the constraints (84,85,86) have to be taken into account, using their Lagrange multipliers: the constraints beyond that are part of the continuation cost functions. Start at the promised utility U^B. Since (85) does not bind for all $\tau \in [t, t+\Delta]$ and the no-change histories $h^0_{t,s}$, $\lambda(h^0_{t,\tau}) \equiv \zeta$ for $\tau \in [t, t+\Delta]$ (where ζ is the Lagrange multiplier on (84)), utilizing that the Lagrange multipliers on the limited commitment constraint (85) for the no-state-change histories $h^0_{t,\tau}$ with $t \leq \tau \leq t+\Delta$ are zero, write the "original" Lagrangian (70) as

$$\begin{aligned} L = & \int_t^{t+\Delta} e^{(\alpha_{x,x}-r)(\tau-t)} \left(w\mathbf{c}(h^0_{t,\tau}) - w\mathbf{z}(x)\right) - \zeta e^{(\alpha_{x,x}-\rho)(\tau-t)} u(w\mathbf{c}(h^0_{t,\tau}))\, d\tau \\ & + e^{(\alpha_{x,x}-r)\Delta} V(x, U_{h^0_{t,t+\Delta}}) - \zeta e^{(\alpha_{x,x}-\rho)\Delta} U_{h^0_{t,t+\Delta}} + \text{const.} \\ & + \sum_{x'\neq x} \int_t^{t+\Delta} \alpha_{x,x'} e^{(\alpha_{x,x}-r)(s-t)} V(x', U_{h^1_{t,s;x'}}) - (\zeta + \mu(h^1_{t,s;x'}))\alpha_{x,x'} e^{(\alpha_{x,x}-\rho)(s-t)} U_{h^1_{t,s;x'}}\, ds \end{aligned} \tag{87}$$

This is legitimate at $U = U^B$: we will show that this is legitimate for all $U \in [U^B, U^A]$. Note that $\mu(h^1_{t,s;x'}) \neq 0$ only if (86) binds at $h^1_{t,s;x'}$: in that case, it must be the case that $U_{h^1_{t,s;x'}} = U^{out}(\mathbf{z}(\mathbf{x}'))$. Differentiate with respect to $\mathbf{c}(h^0_{t,\tau})_{\tau\in[t,t+\Delta]} \geq 0$, $\mathbf{c}(h^0_{t,\tau})$ and $(U_{h^1_{t,s;x'}})_{s\in[t,t+\Delta],x'\in X/\{x\}}$. Noting the dependency of the Lagrange multipliers on the

[34] Suppose that $U^B > U^{out}(\mathbf{z}(\mathbf{x}))$. Per continuity in τ of integrating the future consumption path starting at the lower bound τ, one can show that such a $\Delta > 0$ exists.

promised utility U by including it as an argument, the first-order conditions at $U = U^B$ are

$$\begin{aligned} e^{(\rho-r)(\tau-t)} &= \zeta(U) u'(w\mathbf{c}(h^0_{t,\tau})) && (88)\\ e^{(\rho-r)(\Delta-t)} V'_-(x, U_{h^0_{t,t+\Delta}}) &\leq \zeta(U) \leq e^{(\rho-r)(\Delta-t)} V'_+(x, U_{h^0_{t,t+\Delta}}) && (89)\\ e^{(\rho-r)(s-t)} V'_-(x', U_{h^1_{t,s;x'}}) &\leq \zeta(U) + \mu(h^1_{t,s;x'}; U) \leq e^{(\rho-r)(s-t)} V'_+(x', U_{h^1_{t,s;x'}}) && (90) \end{aligned}$$

Given the Lagrange multipliers $\zeta(U)$ and $\mu(h^1_{t,s;x'}; U)$, let $\mathbf{c}(h^0_{t,\tau}; U)$ for $\tau \in [t, t+\Delta]$, $U_{h^0_{t,t+\Delta};U}$ and $U_{h^1_{t,s;x'};U}$ for $s \in [t, t+\Delta]$, and $x' \in X/\{x\}$ be the solution to these equations. Given the strict concavity of $u(\cdot)$, $\mathbf{c}(h^0_{t,\tau}; U)$ for $\tau \in [t, t+\Delta]$ is strictly increasing in $\zeta(U)$. Given the convexity of $V(x, \cdot)$ according to Lemma 4, $U_{h^0_{t,t+\Delta};U}$ is weakly increasing in $\zeta(U)$ per equation (89). Likewise, equation (90) shows that $U_{h^1_{t,s;x'};U}$ is either weakly increasing in $\zeta(U)$ or constant and equal to the lower bound $U^{out}(\mathbf{z}(x'))$ of equation (86). It follows that $\zeta(U)$ is increasing in U. Note now that these statements are all correct at $U = U^B$ and that (85) does not bind for $\tau \in [t, t+\Delta]$ by assumption. Exploiting the local monotonicity of the solutions, equation (82) then shows that (85) does not bind for $\tau \in [t, t+\Delta]$ for all $U^B < U < U^B + \epsilon$, when $\epsilon > 0$ is sufficiently small, and that (87) is the appropriate Lagrangian for these U as well. Continuing that argument all the way to U^A shows that (85) does not bind for $\tau \in [t, t+\Delta]$ for any $U \in [U^B, U^A]$. Thus, $\mathbf{c}(h^0_{t,\tau}; U)$, $U_{h^0_{t,t+\Delta};U}$ and $U_{h^1_{t,s;x'};U}$ are weakly increasing functions of U. The statements comparing $\mathbf{c}^A(h^0_{t,\tau}) = \mathbf{c}(h^0_{t,\tau}; U^A)$ to $\mathbf{c}^B(h^0_{t,\tau}) = \mathbf{c}(h^0_{t,\tau}; U^B)$ now follow as do the others. □

As a consequence of (88), note that $\mathbf{c}(h^0_{t,\tau})$ is a weakly decreasing and continuous function of τ. As a consequence of (89), note that $U_{h^0_{t,t+\Delta}}$ is a weakly decreasing function of Δ. With some work, one can show that $U_{h^1_{t,s;x'}}$ for any $x' \in X$ is continuous in s at $s = t$. With that, we shall examine the limit, as $\Delta \to 0$.

Proposition 14 (the cost-minimizing HJB equation). *For fixed outside options $U^{out}(z)$, with $z \in Z$, a starting date t, and a fixed wage w and rate of return on capital or interest rate $r > 0$, a recursive cost function $V(x, U)$ solves the Hamilton-Jacobi-Bellman equation*

$$rV(x, U) = \min_{c, \dot{U}, (U(x'))_{x' \in X/\{x\}}} wc - w\mathbf{z}(x) + V'_-(x, U)\dot{U} + \sum_{x' \neq x} \alpha_{x,x'}(V(x', U(x')) - V(x, U))$$

subject to

$$\begin{aligned}\rho U &= u(wc) + \dot{U} + \sum_{x'\neq x} \alpha_{x,x'}(U(x') - U) \\ \dot{U} &\geq 0, \text{ if } U = U^{\boldsymbol{Out}}(\mathbf{z}(x)) \\ \frac{\bar{u}}{\rho} > U(x') &\geq U^{\boldsymbol{Out}}(\mathbf{z}(x'))\end{aligned}$$

for all $x(t) = x \in X$ and all $U \in \left[U^{out}(\mathbf{z}(x)), \frac{\bar{u}}{\rho}\right)$, provided that (84) binds.

Proof of Proposition 14. The continuity of $\mathbf{c}(h^0_{t,s})$ and $U_{h^1_{t,s;x'}}$ in s at $s = t$, together with equation (82), shows that that $U_{h^0_{t,\tau}}$ is continuous in τ at $\tau = t$. Equation (82) furthermore implies that $U_{h^0_{t,\tau}}$ is differentiable with respect to τ at $\tau = t$. Denote that derivative by $\dot{U} = \dot{U}_{h_{t,t}}$. The arguments preceding the proposition imply $\dot{U} \leq 0$. Equation (82) implies that

$$\dot{U}_{h_{t,t}} = -u(w\mathbf{c}(h_{t,t})) + (\rho - \alpha_{x,x})U_{h_{t,t}} - \sum_{x'\neq x} \alpha_{x,x'} \lim_{s\to t} U_{h^1_{t,s;x'}}$$

Use the cost function definition (83) evaluated at the minimizing choices as well as $\dot{U} \leq 0$ and calculate

$$\begin{aligned}V(x, U_{h_{t,t}}) &= [w\mathbf{c}(h_{t,t}) - w\mathbf{z}(x)]\,\Delta \\ &\quad + (1 + (\alpha_{x,x} - r)\Delta)V(x, U_{h_{t,t}}) + V'_-(x, U_{h_{t,t}})\dot{U}_{h_{t,t}}\Delta \\ &\quad + \sum_{x'\neq x} \alpha_{x,x'} V(x', U_{h^1_{t,s;x'}})\Delta + o(\Delta),\end{aligned}$$

Subtract $V(x, U_{h_{t,t}})$, divide by Δ and let $\Delta \to 0$. Write x and x' in place of $h_{t,t}$ and $h_{t,t+\Delta;x'}$. The lemma follows, noting that $\alpha_{x,x} + \sum_{x'\neq x} \alpha_{x,x'} = 0$. □

The following will be useful. Let $\dot{V}$ denote the derivative of $V(x, U_{h^0_{t,\tau}})$ with respect to τ at $\tau = t$. Then,

$$\dot{V} = V'_-(x, U)\dot{U} \tag{91}$$

D.5 The Dual Problem: Utility Maximization

The dual problem to the contractual cost minimization problem above is a utility maximization problem, subject to a budget constraint. The budget is the resources provided by the intermediary. The intermediary uses capital in order to fund the consumption claims by contracted agents, effectively maintaining an account for each agent denoted in units of capital to do so. Thus, write k rather than v for the budget available. Rather than provide the contract formulation for arbitrary levels of outside options, we note that these outside options are available to agents starting from scratch, i.e. when signing up with a new intermediary. Starting from scratch is thus the same as starting from zero capital. With this, the dual problem becomes one of choosing state contingent capital and consumption subject to the constraint that capital must be non-negative: negative amounts would trigger the selection of the outside option and a default on future obligations.

Definition 6 (utility-maximizing contract). *For a starting date t, a starting state x and an initial amount of capital $k_t \geq 0$, a fixed wage w and rate of return on capital or interest rate r, an optimal consumption plan $\mathbf{c} : \mathcal{H}_t(x) \to \mathbf{R}_+$ and optimal savings plan $\mathbf{k} : \mathcal{H}_t(x) \to \mathbf{R}_+$ with $\mathbf{k}(h_{t,t}) = k_t$ solve*

$$\max_{\mathbf{c},\mathbf{k}} U(k_t; x) = \int_t^\infty \int_{\mathcal{H}_{t,\tau}(x)} e^{-\rho(\tau-t)} u(w\mathbf{c}(h_{t,\tau})) dP_{t,\tau} d\tau \tag{92}$$

subject to

$$\begin{aligned} \mathbf{k}(h_{t,\tau}) \;&=\; \int_{\mathcal{H}_{\tau,s}(x(h_{t,\tau}))} \Bigg(e^{-r(s-t)} \mathbf{k}([h_{t,\tau}, h_{\tau,s}]) \\ &\quad + \int_\tau^s e^{-r(\tau-t)} \left(w\mathbf{c}([h_{t,\tau}, h_{\tau,s}(\tau,s')]) - w\mathbf{z}([h_{t,\tau}, h_{\tau,s}(\tau,s')]) \right) ds' \Bigg) dP_{t,s} \end{aligned} \tag{93}$$

for all $s \geq \tau \geq t$ and $h_{t,\tau} \in \mathcal{H}_{t,\tau}$.

Equation (93) is the budget constraint for the agent. For $\tau = t$ and $s \to \infty$, one obtains that k_t is the expected net present value of future consumption in excess of wage income,

$$k_t \;=\; \int_t^\infty \int_{\mathcal{H}_{t,\tau}(x)} e^{-r(\tau-t)} \left(w\mathbf{c}(h_{t,\tau}) - w\mathbf{z}(h_{t,\tau}) \right) dP_{t,\tau} d\tau, \tag{94}$$

changing the order of integration and index of integration and noting that $h_{t,s}[t,\tau] \in \mathcal{H}_{t,\tau}(x)$. However, (93) is a stricter constraint, since we have imposed the condition that

$\mathbf{k} : \mathcal{H}_t(x) \to \mathbf{R}_+$ is nonnegative. Conversely, it provides for more choices than the Aiyagari-style saving constraint

$$k_t = e^{-r(s-t)}\mathbf{k}(h_{t,s}) + \int_t^s e^{-r(\tau-t)} \left(w\mathbf{c}(h_{t,s}[t,\tau]) - w\mathbf{z}(h_{t,s}[t,\tau]) \right) d\tau \qquad (95)$$

and $\mathbf{k}(h_{t,s}) \geq 0$, for all s and $h_{t,s} \in \mathcal{H}_{t,s}(x)$, as (93) allows for state-contingent reallocation of capital and thus insurance against future state changes, subject to the constraint that capital cannot be negative. The next proposition establishes the equivalence between the cost-minimization problem in definition 4 and the utility-maximization problem in definition 6.

Proposition 15 (**equivalence of cost-minimizing and utility-maximizing**). *1. Given a cost-minimizing contract as defined in definition 4, suppose that* $V(x, U^{out}(\mathbf{z}(x))) = 0$ *for all* $x \in X$*. For* $x \in X$ *and some* $U \in \mathcal{U}(x)$*, define the continuation utilities,*

$$U(h_{t,\tau}) = \int_\tau^\infty \int_{\mathcal{H}_{\tau,s}(x(h_{t,\tau}))} e^{-\rho(s-\tau)} u(w\mathbf{c}([h_{t,\tau}, h_{\tau,s}]))dP_{\tau,s}ds \qquad (96)$$

Define

$$\mathbf{k}(h_{t,\tau}) = V(x(h_{t,\tau}), U(h_{t,\tau})) \text{ for all } \tau \text{ and } h_{t,\tau} \in \mathcal{H}_{t,\tau}(x) \qquad (97)$$

Then $\mathbf{c}(\cdot; x, U)$ *and* $\mathbf{k}$ *are utility-maximizing consumption and savings plans for the initial capital* $k_t = V(x, U)$*, as defined in definition 6, resulting in* $U(k_t; x) = U(h_{t,t}) \geq U$*.*

2. *Suppose that the optimal cost function of definition 4 is continuous in* U[35]*. For all* $x \in X$ *and* $k_t \geq 0$*, calculate* $U(k_t; x)$ *and the optimal consumption and savings plans per definition 6. Denote them by* $\mathbf{c}_{x,k_t}$ *and* $\mathbf{k}_{x,k_t}$*. Let* $U^{out}(\mathbf{z}(x)) = U(0; x)$ *for all* $x \in X$*. Define*

$$\mathbf{c}(h_{t,\tau}; x, U(k_t; x)) = \mathbf{c}_{x,k_t}(h_{t,\tau}) \text{ for all } \tau \text{ and } h_{t,\tau} \in \mathcal{H}_{t,\tau}(x) \qquad (98)$$

Then $\mathbf{c}$ *is an optimal consumption contract as defined in definition 4, resulting in the cost*

$$V(x, U(k_t; x)) = k_t \qquad (99)$$

[35]With some work, this can probably be shown to be true.

Proof. 1. Similar to the recursive construction for (83), note that $\mathbf{k}(h_{t,\tau})$, defined as $V(x(h_{t,\tau}), U(h_{t,\tau}))$ satisfies (93). Since $V(x, U^{out}(\mathbf{z}(\mathbf{x}))) = \mathbf{0}$ and since $V(x, U)$ is increasing in U, it follows that $\mathbf{k}(h_{t,\tau}) \geq 0$. Since $V(x, \cdot)$ is strictly increasing at the promise $U(h_{t,t})$, there is no other consumption and savings plan, resulting in a higher utility.

2. Note that $V(x, U(k_t; x)) = k_t$ satisfies (94) and thus the equation for the cost function in definition 4. Suppose that the optimal contract achieves $U(k_t; x)$ at a lower cost. Since the optimal cost function is increasing and continuous in U, there is some utility level $U > U(k_t; x)$ resulting still in a cost below k_t. Exploiting the preceding reverse construction of proceeding from the cost-minimizing contract to a utility maximizing plan in the previous step then shows that the plan for k_t cannot have been optimal.

□

Similar to Proposition 14 and with the same assumptions, we obtain

Proposition 16 (the utility-minimizing HJB equation). *For a fixed wage w and rate of return on capital or interest rate $r > 0$, a recursive utility function $U(k; x)$ solves the Hamilton-Jacobi-Bellman equation*

$$\rho U(k; x) = \max_{c, \dot{k}, (k(x'))_{x' \in X/\{x\}}} u(c) + \frac{\partial U(k; x)}{\partial k} \dot{k} + \sum_{x' \neq x} \alpha_{x,x'} (U(k(x'); x') - U(k; x))$$

subject to

$$\begin{aligned} c + \dot{k} + \sum_{x' \neq x} \alpha_{x,x'} (k(x') - k) &= rk + w\mathbf{z}(\mathbf{x}) \\ k(x') \geq 0 & \qquad \text{for all } x' \in X/\{x\} \\ \dot{k} \geq 0 & \qquad \text{if } k = 0 \end{aligned}$$

for all $x \in X$ and all $k \geq 0$.

The proof is analogous to the proof of Proposition 14. We skip the details.

E Proofs of Lemmas and Propositions in the Main Text

In this section we provide the proofs for propositions in the main text as well as provide further details of the mathematical derivations. These are straightforward but tedious manipulations that were therefore excluded from the main text.

We wish to formally compare contracts starting from high productivity to those starting from low productivity. Therefore and from here on, we assume that there is a three-state Markov process for an underlying state $x(t) \in X = \{0, 1, 2\}$ for each agent, evolving independently from each other. The transition rates $\alpha_{i,j}$ to transit from state $x = i$ to $x = j$ are $\alpha_{0,1} = \alpha_{2,1} = \nu$, $\alpha_{1,0} = \alpha_{2,0} = \xi$ and $\alpha_{0,2} = \alpha_{1,2} = 0$. Let $\alpha_{i,i} = -\sum_{j \neq i} \alpha_{i,j}$, so that α is an intensity matrix or infinitesimal generator matrix.

Additionally, we assume that there is a mapping $\mathbf{z} : X \to Z$ so that the implied Markov process $z(t) = \mathbf{z}(X(t))$ has the transition rates ξ for transiting from $z = \zeta$ to $z = 0$ and ν for transiting from $z = 0$ to $z = \zeta$, as stated in subsection 2.1 and given some initial productivity. There are two options in particular. For the first option, let the mapping $\mathbf{z} = \mathbf{z}_A$ be given by $\mathbf{z}_A(0) = \mathbf{z}_A(2) = 0$ and $\mathbf{z}_A(1) = \zeta$. The three-state process starting at $x(t) = 0$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z(t) = 0$: the exit rate out of zero productivity is ν, regardless of whether the underlying state is $x = 0$ or $x = 2$, and the transitions between these two states play no role. The transition out of high productivity only happens from state $x = 1$ at rate ξ, exactly as in the two-state formulation. For the second option, let the mapping $\mathbf{z} = \mathbf{z}_B$ be given by $\mathbf{z}_B = 0$ and $\mathbf{z}_B(1) = \mathbf{z}_B(2) = \zeta$. The three-state process starting at $x(t) = 1$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z_t = \zeta$: the exit rate out of $z = \zeta$ productivity is ξ, regardless of whether the underlying state is $x = 1$ or $x = 2$, and the transitions between these two states play no role. The transition out of low productivity only happens from state $x = 0$ at rate ν, exactly as in the two-state formulation.

E.1 Ordering of the Outside Utilities

Assume $r > 0$. Let the net present value of future income be defined as

$$NPV(z) = \mathbf{E}\left[\int_t^\infty e^{-r(\tau - t)} w z(\tau) d\tau \,\middle|\, z(t) = z \right]$$

conditional on the starting income z at date t.

Lemma 10. *$NPV(z)$ is increasing in z. Specifically,*

$$NPV(z) = \begin{cases} \frac{\nu}{r+\nu+\xi}\frac{\zeta}{r} & \textit{if } z = 0 \\ \frac{r+\nu}{r+\nu+\xi}\frac{\zeta}{r} & \textit{if } z = \zeta \end{cases} \tag{100}$$

Proof of Lemma 10. Using Bellman logic, the two NPVs satisfy

$$(r+\nu)NPV(0) = \nu NPV(\zeta) \tag{101}$$

$$(r+\xi)NPV(\zeta) = \zeta + \xi NPV(0) \tag{102}$$

Solve. □

Proof of Lemma 1. The key idea is that an agent currently at high productivity can be provided with the contract of the low-productivity agent, delivering the same utility and a profit to the principal, a contradiction to perfect competition between the principals. Some care needs to be taken to implement this idea, however. Contracts depend on the history of states. Thus, if the history was expressed only in terms of productivities, it would be meaningless to give an agent starting with high productivity "the same" contract as an agent starting with low productivity. The underlying state and the corresponding productivity need to be decoupled. It is here where the three-state construction described at the beginning of this section and the careful distinction between the state and the productivity at that state as described at the beginning of Appendix C pay off.

Suppose by contradiction to the claim (10) that

$$U^{out}(0) \geq U^{out}(\zeta) \tag{103}$$

Fix the productivity mapping $\mathbf{z} : X \to Z$ to be $\mathbf{z}_A$. Recall that $\mathbf{z}_A(0) = \mathbf{z}_A(2) = 0$ and $\mathbf{z}_A(1) = \zeta$, and that the three-state process starting at $x(t) = 0$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z(t) = 0$. Consider an optimal consumption contract $c(\tau; 0, U^{out}(0))$ given to an agent at date $t = 0$, say, and starting off with productivity $z(0) = 0$, delivering date-0 promised utility $U = U^{out}(0)$ in (2) and generating costs $V(0, U^{out}(0)) = 0$. Wlog, we shall impose the condition that $x(0) = 2$: any contract as defined per history dependence in Appendix C and starting at $x(0) = 0$ can be written[36] as a contract starting at $x(0) = 2$ delivering the same outcomes, per ignoring transitions from $x = 2$ to $x = 0$. Thus, the optimal

[36] This argument can be made precise with some tedious notation.

consumption contract $c(\tau; 0, U^{out}(0))$ is a mapping $\mathbf{c} : \mathcal{H}_0 \to \mathbf{R}_+$ from x-histories into consumption outcomes, where all $h_{s,0} \in \mathcal{H}_0$ satisfy $x(0) = 2$, and which satisfies the constraints (65).

Next, fix the productivity mapping $\mathbf{z} : X \to Z$ to be $\mathbf{z}_B$. Recall that $\mathbf{z}_B(0) = 0$ and $\mathbf{z}_B(1) = \mathbf{z}_B(2) = \zeta$, and that the three-state process starting at $x(t) = 1$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z(t) = \zeta$. The contract $\mathbf{c}$ delivers the same expected utility $U^{out}(0)$. The contract $\mathbf{c}$ satisfies the constraints (65) for states $x(s) = 0$ and states $x(s) = 1$, where $\mathbf{z}_A$ and $\mathbf{z}_B$ coincide. With equation (103), the constraints are also satisfied for the state $x(s) = 2$ and $\mathbf{z}_B(2) = \zeta$ rather than $\mathbf{z}_A(2) = 0$. The consumption portion generates the same costs for the principal, as nothing has changed regarding the consumption process, but the expected revenue from productivity income is now strictly higher per Lemma 10. It follows, that the contract $\mathbf{c}$ now delivers strictly negative costs[37] $V(\zeta, U^{out}(0))$. Per Lemma 3 and equation (103), $0 > V(\zeta, U^{out}(0)) \geq V(\zeta, U^{out}(\zeta))$. However, $V(\zeta, U^{out}(\zeta)) = 0$ per the definition of equilibrium. With that, we have arrived at a contradiction. □

E.1.1 The Case of $-\xi < r < 0$

The proof of Lemma 10, and thus the proof of Lemma 1, required that $r > 0$ since consumption insurance contracts last forever and thus discounting has to be positive to render present discounted values of future incomes and costs of the contract finite. However, since contracts will effectively end and reset every time a high income shock is realized, the same arguments as in the proofs of Lemma 10, and thus of Lemma 1 can be used as long as $r > -\xi$.

The expected net present value of income during such a contract that starts with high income $z = \zeta$, extends through a spell of low income $z = 0$, and ends the instant a new high-income spell starts solves

$$rNPV(\zeta) = \zeta + \xi(NPV(0) - NPV(\zeta)) \tag{104}$$

as in equation (102), but now $NPV(0) = 0$ since the contract ends the next time high productivity is reached. Thus $NPV(\zeta) = \frac{\zeta}{r+\xi}$, which is finite as long as $r > -\xi$. Thus, for the class of contracts that turn out to be optimal (for which the incentive constraint is

[37] In slight abuse of notation, we calculate the costs, given a contract, rather than insisting that $V(\cdot, \cdot)$ are the minimized costs.

binding every time high productivity is realized, effectively resetting the contract), only the restriction $r > -\xi$ rather than the restriction $r > 0$ has to be imposed.

Note that all calculations that lead to Proposition 2 go through under this weaker restriction. During the high-productivity spell (which has a length with exponential distribution with parameter ξ) the intermediary makes expected discounted profits

$$\int_0^\infty (\zeta - c_h)e^{-rt}e^{-\xi t}dt = \frac{\zeta - c_h}{r + \xi} \tag{105}$$

which are finite as long as $r > \xi$. Similarly, the expected discounted cost of the low-productivity spell (in which consumption drifts down at rate $-(\rho - r) < 0$) that starts at random start date t and lasts a random, exponentially distributed (with parameter ν) time τ is given by

$$e^{-rt}\int_0^\infty e^{-r\tau}c_h e^{-(\rho-r)t}e^{-\nu t}dt = \frac{e^{-rt}c_h}{\rho + \nu} \tag{106}$$

and taking expectation with respect to the random time t at which productivity switches from high to low gives the expected cost of the low-productivity spell as

$$\int_0^\infty \frac{e^{-rt}c_h}{\rho + \nu}\xi e^{-\xi t}dt = \frac{\xi c_h}{(\rho + \nu)(r + \xi)} \tag{107}$$

which again is finite as long as $r > -\xi$. Equating expected profits and cost on the contract spell yields

$$\frac{\zeta - c_h}{r + \xi} = \frac{\xi c_h}{(\rho + \nu)(r + \xi)} \tag{108}$$

which yields c_h in Proposition 2 from the main text and shows that the relevant net present value calculations are all finite as long as $r > -\xi$.

The fact that the Poisson rate ξ of a productivity drop is helpful in relaxing the constraint required to keep present discounted values finite is intuitive since it determines the expected length of the initial high-income spell. What is perhaps surprising is that ν does not play a role in keeping the present discounted value of the cost of the low-income spell finite. This is because a low-income spell consumption is discounted at rate r and consumption itself falls at rate $-(\rho - r)$ and the spell ends at rate ν, and thus the effective discount rate is $r + \rho - r + \nu = \rho + \nu$ and thus the present discounted value is finite independent of the interest rate r. This in turn is a reflection of the income effect and substitution effect canceling out with log-utility (and would not be the case for $\sigma \neq 1$).

E.2 Optimal Consumption Insurance Contract

E.2.1 Full Insurance: $r = \rho$

Proof of Proposition 1: If (3) does not bind, then the first two parts of Lemma 7 show that consumption is constant. If (3) does bind, then consumption is locally constant to the right of $\dot{\mathbf{c}}_+(h_{t,s})$, as the third part of Lemma 7 shows. Note that the argument there does not require Assumption 3 in the case that $r = \rho$. The fourth part of the lemma shows that consumption may jump upward upon a state transition. Lemma 1 implies that the jump may occur for a transition from $z = 0$ to $z = \zeta$, but not vice versa. □

The optimal consumption contract is fully characterized by the constant consumption level and associated insurance premium charged to high-income households:

$$\begin{aligned} c_h(\rho) &= \frac{\rho+\nu}{\rho+\nu+\xi}\zeta \\ v_{hl} &= \frac{c_h}{\rho+\nu} > 0 \end{aligned}$$

Corollary 3. *Impose the conditions of Proposition 14. Define the wage-deflated contract costs by* $v_h = V(x, U^{out}(\zeta))/w$*, if* $\mathbf{z}(x) = \zeta$ *and* $v_l = V(x, U^{out}(0))/w$*, if* $\mathbf{z}(x) = 0$ *and there never was a high income in the past and finally* $v_{hl} = V(x, U^{out}(\zeta))/w$*, if* $\mathbf{z}(x) = 0$*, if there was. Then*

$$\begin{aligned} rv_l &= c_l + \nu(v_h - v_l) \\ rv_h &= c_h - \zeta + \xi(v_{hl} - v_h) \\ rv_{hl} &= c_h + \nu(v_h - v_{hl}) \end{aligned}$$

Proof. The transition rates $\alpha_{x,x'}$ correspond to the transition rates from productivity $\mathbf{z}(x)$ to $\mathbf{z}(x')$ per the construction in Appendix C. Note that $\dot{U} = 0$ in Proposition 14 for all U. Rewriting the Hamilton-Jacobi-Bellman equation in Proposition 14 at the optimal consumption choices c_l and c_h yields the equations here. □

E.2.2 Partial Insurance: $r < \rho$

In order to prove Proposition 2, we prove the more general version

Proposition 17. *Suppose that the utility function satisfies Assumption 3.*

1. *Whenever a household has high productivity, it consumes a constant wage-deflated amount c_h.*
2. *When productivity switches to 0, consumption is continuous and subsequently drifts down according to the full-insurance Euler equation*

$$\frac{\dot{c}(t)}{c(t)} = -g < 0 \tag{109}$$

where the negative of the consumption growth rate g satisfies

$$g = \left(\frac{-u''(wc(t))wc(t)}{u'(wc(t))} \right)^{-1} (\rho - r)$$

If the utility function is of the CRRA variety (81), then $g = (\rho - r)/\sigma$ as in (80). *In that case, let τ be the time elapsed, since productivity last switched to 0. Then,*

$$c(t) = c_h e^{-g\tau} \tag{110}$$

Proof of Proposition 17. Let $c_h(0; \zeta, U^{out}(\zeta))$ be the consumption level at starting date $t = 0$ in a contract that just delivers the outside option $U^{out}(\zeta)$ at high productivity $z(0) = \zeta$. The limited commitment constraint (3) binds; see the proof of Lemma 3. Alternatively, note that it must bind, since otherwise consumption will drift down according to (76) with $\dot{\lambda} = 0$ and the outside option would be better shortly after $t = 0$, if no further state switch occurred. The third part of Lemma 7 thus implies that consumption is constant while productivity is high. Upon a switch to low productivity, the limited commitment constraint (3) never binds. Thus, the fourth part of Lemma 7 implies that consumption is continuous, that $\dot{\lambda}_+(h_{t,s}) = 0$ and that consumption drifts down according to (76) or (109), applied only to the right-derivatives at the date of the switch. The rest follows with some algebra. □

Corollary 4. *Impose the conditions of Proposition 14. Denote by τ the time elapsed since having had high productivity and by $v_{hl}(\tau)$ the remaining wage-deflated costs of the contract, at that point. The Hamilton-Jacobi-Bellman equations characterizing the wage-deflated costs in the high-productivity state, the low-productivity state prior to having had* *a high-productivity realization, and after time τ since having had high productivity read as*

$$rv_h = c_h - \zeta + \xi(v(0) - v_h) \tag{111}$$
$$rv_l = c_l + \nu(v_h - v_l) \tag{112}$$
$$rv_{hl}(\tau) = c(\tau) + \nu(v_h - v_{hl}(\tau)) + \dot{v}_{hl}(\tau) \tag{113}$$

with terminal condition

$$v_{hl}(\infty) = v_l = 0.$$

Proof. The transition rates $\alpha_{x,x'}$ correspond to the transition rates from productivity $\mathbf{z}(x)$ to $\mathbf{z}(x')$ per the construction in Appendix C. Note that $\dot{U} = 0$ in Proposition 14, if $U = U^{out}(\zeta)$ and $U = U^{out}(0)$. Rewriting the Hamilton-Jacobi-Bellman equation in Proposition 14 yields equations (111) and 112. For equation (113), suppose that $U = U(\tau) > U^{out}(0)$, but that $\mathbf{z}(x) = 0$. Rewriting the Hamilton-Jacobi-Bellman equation in Proposition 14 at the optimal consumption choice $c(\tau)$ and exploiting equation (91) yields equation (113) here. □

We proceed to provide the details for the cost calculations, allowing the utility function $u(c)$ to be of the CRRA form (81). Equation (21) is a standard linear ODE. It can be integrated, using the fact that $c(\tau) = c_h e^{-g\tau}$ with $g = (\rho - r)/\sigma$ to obtain

$$v_{hl}(\tau) = \int_\tau^\infty e^{-(r+\nu)(s-\tau)} c_h e^{-gs} ds = c_h e^{-g\tau} \int_\tau^\infty e^{-(r+\nu+g)(s-\tau)} ds = \frac{e^{-g\tau}}{r+\nu+g} c_h \tag{114}$$

either using standard formulas for ODEs or checking the result per differentiating the solution to back out the original differential equation.

We can evaluate (114) at $t = 0$ to obtain[38]

$$v_{hl}(0) = \frac{c_h}{r+\nu+g} \tag{115}$$

The optimal consumption contract has consumption declining at rate $-g = r - \rho$ from c_h toward $c_l = 0$, and asymptotically it reaches $c_l = 0$. Thus the consumption level c_h fully characterizes the consumption contract. Using equation (20) to substitute out $v_{hl}(0)$

[38]Note that this cost $v(0)$ is the counterpart to the insurance cost in equation (13) for the full-insurance case; if $r = \rho$ and thus $g = 0$, $v(0) = v_{hl}$ in (13).

in equation (115) yields

$$\frac{c_h}{r+\nu+g} = \frac{\zeta - c_h}{\xi}$$

or

$$c_h = \frac{r+\nu+g}{r+\nu+g+\xi}\zeta \tag{116}$$

With this, we obtain a generalization of Proposition 3 to the CRRA case.

Proposition 18. *If $\rho > r$, there exists a unique consumption level*

$$c_h = \frac{r+\nu+g}{r+\nu+g+\xi}\zeta$$

which is strictly increasing in ζ and with the following properties:

1. *Agents with currently high productivity receive the wage-deflated consumption c_h.*

2. *Agents with currently low productivity who switched from high productivity τ periods ago receive the wage-deflated consumption*

$$c(t) = c_h e^{-g\tau}$$

Households that never have had high income consume the nontradable endowment $c_l = \chi$ until the first time they receive high income and sign the consumption risk-sharing contract.

E.3 Goods Supply and Capital Demand

Proof of Proposition 6: Calculating the capital stock and wages for Cobb-Douglas production from the production first-order conditions (4) and (5) yields

$$\begin{aligned} K(r) &= \left(\frac{\theta A}{r+\delta}\right)^{\frac{1}{1-\theta}} \\ w(r) &= (1-\theta)AK^{\theta} \end{aligned}$$

and thus

$$\frac{\left[AF_K(K(r),1)-\delta\right]K(r)}{AF_L(K(r),1)} = \frac{r}{(1-\theta)AK(r)^{\theta-1}} = \frac{r\theta}{(1-\theta)\,(r+\delta)}.$$

With Euler's theorem,

$$\begin{aligned}
G(r) &= 1 + \frac{[AF_K(K(r),1)-\delta]\,K(r)}{AF_L(K(r),1)} \\
&= 1 + \frac{r\theta}{(1-\theta)\,(r+\delta)} \\
\kappa^d(r) &= \frac{K^d(r)}{w(r)} = \frac{\left[K^d(r)\right]^{1-\theta}}{(1-\theta)A} = \frac{\theta}{(1-\theta)(r+\delta)}
\end{aligned}$$

The properties of these functions stated in the main text follow directly from inspection.

□

E.4 Capital Supply and Consumption Demand for $r < \rho$

In this section we collect the details of the derivations about the properties of the capital supply function $\kappa^s(r)$ in the partial insurance case. Substitute ζ from equation (1) into equation (24) to obtain

$$c_h = \frac{\nu+\rho}{\xi+\nu+\rho}\,\frac{\xi+\nu}{\nu}$$

Direct calculations and exploiting the explicit functional form of ϕ_r in Proposition 5 reveal that wage-normalized aggregate consumption demand and capital supply are given by

$$\begin{aligned}
C(r) &= \frac{\nu}{\nu+\xi}c_h + \int_0^{c_h} c\frac{\xi\nu\,(c_h)^{-\frac{\nu}{\rho-r}}}{(\rho-r)(\nu+\xi)}c^{\frac{\nu}{\rho-r}-1}dc = \frac{\nu}{\nu+\xi}\frac{\nu+\rho-r+\xi}{\nu+\rho-r}c_h \\
&= \frac{\nu+\rho-r+\xi}{\nu+\rho-r}\frac{\nu+\rho}{\nu+\rho+\xi} \\
&= \left(1+\frac{\xi}{\nu+\rho-r}\right)\left(1-\frac{\xi}{\nu+\rho+\xi}\right) \\
&= 1+\frac{\xi}{\nu+\rho-r}-\frac{\xi}{\nu+\rho+\xi}-\frac{\xi^2}{(\nu+\rho+\xi)(\nu+\rho-r)} \\
&= 1+\frac{r\xi}{(\nu+\rho+\xi)(\nu+\rho-r)} \\
\kappa^s(r) &= \frac{\xi}{(\nu+\rho+\xi)\,(\nu+\rho-r)}
\end{aligned}$$

Proof of Proposition 7: It follows immediately from the equations above that the function $\kappa^s(r)$ is continuously differentiable and strictly increasing on $[-\delta,\rho)$. Aggregate consumption demand and capital supply are continuous in the interest rate at $r = \rho$ since

$$\lim_{r \nearrow \rho} C(r) = 1 + \frac{\rho\xi}{\nu(\rho + \nu + \xi)} = C(r = \rho) \tag{117}$$

$$\lim_{r \nearrow \rho} \kappa^s(r) = \frac{\xi}{\nu(\xi + \nu + \rho)} = \kappa^s(r = \rho) \tag{118}$$

coincide with the values in equations (31) and (32) for the full-insurance case $r = \rho$. $\square$

F General CRRA Utility

The analysis for the full-insurance case goes through completely unchanged, since at $r = \rho$ the growth rate of consumption and thus the aggregate consumption demand and capital supply are unaffected by the intertemporal elasticity of substitution $1/\sigma$. Here we focus on the case $\rho > r$.

F.1 Optimal Consumption Contract

As in the log-case, whenever a household has high income, it consumes c_h, and when income switches to 0, consumption drifts down according to the full-insurance Euler equation[39]

$$\frac{\dot{c}(t)}{c(t)} = -\frac{\rho - r}{\sigma} = -g < 0$$

where the growth rate of consumption is now defined as

$$g = \frac{\rho - r}{\sigma} > 0.$$

The log-utility case is of course just a special case where $\sigma = 1$ and thus $g = \rho - r$.

The steps of deriving the optimal consumption contract and associated cost then proceeds completely in parallel to the log-case. Consumption is given as

$$c(t) = c_h e^{-gt} \tag{119}$$

[39] The proof of Proposition 2 in Appendix E.2.2 is conducted for a general CRRA function and thus applies here.

and the cost of the contract is given by

$$v(t) = \frac{c_h e^{-gt}}{r + \nu + g} \tag{120}$$

Evaluating (22) at $t = 0$ gives

$$v(0) = \frac{c_h}{r + \nu + g} \tag{121}$$

Using equation (20), which continues to hold unchanged, to substitute out $v(0)$ into equation (23) yields

$$\frac{c_h}{r + \nu + g} = \frac{\zeta - c_h}{\xi}$$

or

$$c_h(r) = \frac{r + \nu + g}{r + \nu + g + \xi}\zeta = \frac{1}{1 + \frac{\xi}{r\left(1 - \frac{1}{\sigma}\right) + \nu + \frac{\rho}{\sigma}}}\zeta \tag{122}$$

Note that $c_h(r)$ together with $v(t) > 0$ requires $r + \nu + g > 0$. For $\sigma \leq 1$ this is satisfied for all $r \leq \rho$. For $\sigma > 1$, this is satisfied at $r = -\delta$ and thus for all $r \in [-\delta, \rho]$, if condition (45) of Proposition 10 holds. For $\sigma \to \infty$, condition (45) becomes $\delta < \nu$.

F.2 Invariant Consumption Distribution

As in the log-case, on $c \in (0, c_h)$ the consumption process follows a diffusion process with drift $-g$ (and no variance) and thus on $(0, c_h)$ the stationary consumption distribution satisfies the Kolmogorov forward equation (for the case of Poisson jump processes):

$$0 = -\frac{d\left[-gc\phi(c)\right]}{dc} - \nu\phi(c)$$

where the second term comes from the fact that with Poisson intensity ν the household has a switch to high income. Since

$$-\frac{d\left[-gc\phi(c)\right]}{dc} = -\left[-g\phi(c) - gc\phi'(c)\right] = g\left[\phi(c) + c\phi'(c)\right]$$

we find that on $c \in (0, c_h)$ the stationary distribution satisfies

$$\frac{c\phi'(c)}{\phi(c)} = \frac{\nu}{g} - 1$$

and thus on this interval the stationary consumption distribution is Pareto with tail parameter $\kappa = \frac{\nu}{g} - 1$, that is

$$\phi(c) = \phi_1 c^{\left(\frac{\nu}{g}-1\right)}$$

where ϕ_1 is a constant that needs to be determined. Now we need to determine the constant ϕ_1. Because of the mass point at c_h it is easier to think of the cdf for consumption on $(0, c_h)$ given by $\Phi(c) = \frac{\phi_1(c)^{\kappa+1}}{\kappa+1}$. The inflow mass into this range is given by the mass of individuals at c_h given by $\phi(c_h) = \frac{\nu}{\nu+\xi}$ times the probability ξ of switching to the low-income state, whereas the outflow is due to receiving the high-income shock, and thus the stationary cdf has to satisfy

$$\nu\Phi(c_h) = \frac{\xi\nu}{\nu+\xi}$$

and therefore

$$\nu\frac{\phi_1 \left(c_h\right)^{\kappa+1}}{\kappa+1} = \frac{\xi\nu}{\nu+\xi}$$

Exploiting the fact that $\kappa + 1 = \frac{\nu}{g}$ we find

$$\phi_1 g \left(c_h\right)^{\frac{\nu}{g}} = \frac{\xi\nu}{\nu+\xi}$$

and thus

$$\phi_1 = \frac{\xi\nu \left(c_h\right)^{-\frac{\nu}{g}}}{g(\nu+\xi)}$$

and therefore the density on $(0, c_h)$ is given by

$$\phi(c) = \frac{\xi\nu \left(c_h\right)^{-\frac{\nu}{g}}}{g(\nu+\xi)} c^{\frac{\nu}{g}-1}.$$

Therefore the stationary consumption distribution is now given by:

$$\phi_r(c) = \begin{cases} \frac{\xi\nu(c_h(r))^{-\frac{\nu}{g}}}{g(\nu+\xi)} c^{\frac{\nu}{g}-1} & \text{if} \quad c \in (0, c_h) \\ \frac{\nu}{\nu+\xi}\boldsymbol{\delta}_{c_h} & \text{if} \quad c = c_h \end{cases}$$

where $\boldsymbol{\delta}_{c_h}$ indicates a Dirac mass point at c_h. Thus, for a given interest rate r the invariant consumption distribution is completely characterized by the upper bound $c_h(r) = \frac{r+\nu+g}{r+\nu+g+\xi}\zeta$.

F.3 Equilibrium

We now determine the aggregate consumption demand $C(r)$ and the normalized capital supply function $\kappa^s(r)$. Direct calculations reveal that aggregate consumption demand and capital supply are given by:

$$\begin{aligned} C(r) &= \frac{\nu}{\nu+\xi}c_h(r) + \int_0^{c_h(r)} c\frac{\xi\nu\left(c_h(r)\right)^{-\frac{\nu}{g}}}{g(\nu+\xi)}c^{\frac{\nu}{g}-1}dc = \frac{\nu}{\nu+\xi}\frac{\xi+\nu+g(r)}{\nu+g(r)}c_h(r) \\ &= \frac{\xi+\nu+g(r)}{\nu+g(r)}\frac{r+\nu+g(r)}{\xi+r+\nu+g(r)} \\ &= \left(1+\frac{\xi}{\nu+g(r)}\right)\left(1-\frac{\xi}{\xi+\nu+g(r)+r}\right) \\ &= 1+\frac{\xi}{\nu+g(r)} - \frac{\xi}{\xi+\nu+g(r)+r} - \frac{\xi^2}{(\xi+\nu+g(r)+r)(\nu+g(r))} \\ &= 1+\frac{r\xi}{(\xi+\nu+g(r)+r)(\nu+g(r))} \\ \kappa^s(r) &= \frac{\xi}{\left(\xi+\nu+\frac{\rho-r}{\sigma}+r\right)\left(\nu+\frac{\rho-r}{\sigma}\right)} \end{aligned} \tag{123}$$

where we have repeatedly used $g(r) = \frac{\rho-r}{\sigma}$.

F.3.1 Proof of Proposition 10

Proof. The first step of the proof is to establish that the normalized capital supply function is well-defined and continuous on $r \in [-\delta, \rho]$. The previous section gave $\kappa^s(r)$ in closed form, and it is evidently continuous and well-defined on $[-\delta, \rho]$ as long as both terms of the denominator are strictly positive. Since $r \leq \rho$, the second term in the denominator of equation (123) is always strictly positive. The first term is always positive for $\sigma \leq 1$ and $r \leq \rho$. For $\sigma > 1$, it is positive for $r \geq \delta$ due to condition (45). That condition is also needed for $c_h > 0$ and $v(t) > 0$; see the remarks at the end of Appendix F.1.

Since by Assumption 2 we have $\kappa^s(r=\rho) > \kappa^d(r=\rho)$ and since $\kappa^s(r=-\delta) < \infty = \kappa^d(r=-\delta)$, it follows that κ^s and κ^d intersect at least once in $(-\delta, \rho)$. This establishes the existence of a stationary equilibrium.

The uniqueness of equilibrium follows if $\kappa^s(r)$ is increasing (given that $\kappa^d(r)$ is strictly decreasing). The derivative of $\kappa^s(r)$ is given by

$$\frac{d\kappa^s(r)}{dr} = \xi \frac{\left[\frac{2}{\sigma} - 1\right]\left[\frac{\rho - r}{\sigma} + \nu\right] + \frac{\xi + r}{\sigma}}{\left[\left(\xi + \nu + \frac{\rho - r}{\sigma} + r\right)\left(\nu + \frac{\rho - r}{\sigma}\right)\right]^2}$$

A sufficient condition for this expression to be positive is $\sigma < 1$ (part 1 of the proposition) or $\sigma \in (1, 2]$ and $\xi \geq \delta$ (part 2a of the proposition). Part 2b follows from the fact that equation (43) is a quadratic equation, and thus has at most two solutions (and we have already established that under the assumptions made it has at least one solution). The numerical example in the main text shows that the statement in 2b of the proposition is not vacuous. □

F.3.2 Proof of Corollary 2

Proof. In general equilibrium interest rates are real-valued solutions to the quadratic equation

$$0 = F(r) \equiv A_2 r^2 + A_1 r + A_0 \tag{124}$$

where

$$\begin{aligned}
A_0 &= (\sigma - 1)^2 \left[\xi\delta - \theta\nu^2 - \xi\theta(\delta + \nu)\right] + (\sigma - 1)\left[-2\theta\nu(\nu + \rho) - \xi\left(2\delta(\theta - 1) + \theta(2\nu + \rho)\right)\right] \\
&\quad -\theta(\nu + \rho)^2 - \xi\left(\delta(\theta - 1) + \theta(\nu + \rho)\right) \\
A_1 &= -(\sigma - 1)^2\left(\theta(\nu + \xi) - \xi\right) - (\sigma - 1)\left(\theta(\rho + \xi) - 2\xi\right) + \theta(\rho + \nu) + \xi \\
A_2 &= \theta(\sigma - 1)
\end{aligned}$$

The coefficients A_0, A_1, A_2 defined above are functions of the parameters. Note that

$$\begin{aligned}
A_0(\alpha\rho, \alpha\delta, \alpha\xi, \alpha\nu; \sigma, \theta) &= \alpha^2 A_0(\rho, \delta, \xi, \nu; \sigma, \theta) \\
A_1(\alpha\rho, \alpha\xi, \alpha\nu; \sigma, \theta) &= \alpha A_1(\rho, \xi, \nu; \sigma, \theta)
\end{aligned}$$

and $A_2(\sigma, \theta)$ does not depend on ρ, δ, ξ, ν. Define

$$F(r; \alpha) = A_2(\sigma, \theta) r^2 + A_1(\alpha\rho, \alpha\xi, \alpha\nu; \sigma, \theta) r + A_0(\alpha\rho, \alpha\delta, \alpha\xi, \alpha\nu; \sigma, \theta)$$

Then

$$\alpha^2 F(r;1) \quad = \quad F(\alpha r;\alpha)$$

Hence, if $\bar{r}$ solves $F(\bar{r};1)=0$, then $r=\alpha\bar{r}$ solves $F(r;\alpha)=0$. □

G Superinsurance

In this appendix we characterize the optimal consumption insurance contract when the interest rate r exceeds the rate of time preference ρ, that is, $r>\rho$, and then discuss the possibility of a stationary distribution associated with that consumption contract.

G.1 The Optimal Contract for Superinsurance: $\rho<r$

If the limited commitment constraint is not binding, as in the partial-insurance case, consumption grows at a constant rate,

$$c_h(t)=c_h(0)e^{(r-\rho)t}$$

but now $\rho>r$; that is, consumption grows at a positive rate. As in the full and partial-insurance case, households born with low income cannot obtain insurance until their income switches to ζ, at which point it jumps to $c_h(0)$, as in the partial and full-insurance cases. From that point on, the household obtains income insurance (as in the full insurance case), but now consumption grows at rate $r-\rho>0$ (rather than remaining constant), until the household dies. The level $c_h(0)$ is determined by the zero profit condition of the intermediary, equating the expected revenue from the household's income stream with the expected cost of the consumption contract.

To determine this level, $c_h(0)$, first calculate the present discounted revenue (a_l,a_h) for the intermediary from a currently productive and currently unproductive individual (normalized by the wage) as follows. These PDV revenues satisfy

$$\begin{aligned} ra_h(t) &= \zeta+\xi(a_l(t)-a_h(t))+\dot{a}_h(t) \\ ra_l(t) &= \nu(a_h(t)-a_l(t))+\dot{a}_l(t) \end{aligned}$$

Evidently these two functions do not depend on time and solve

$$\begin{aligned} ra_h &= \zeta + \xi(a_l - a_h) \\ ra_l &= \nu(a_h - a_l) \end{aligned}$$

Solving yields

$$\begin{aligned} a_h &= \frac{r+\nu}{r(r+\nu+\xi)}\zeta \\ a_l &= \frac{\nu}{r(r+\nu+\xi)}\zeta \end{aligned}$$

both of which are finite since $r > \rho > 0$. Now we derive the present discounted value for the cost of the contract that starts at entry consumption $c_h(0)$ and grows at rate $r - \rho > 0$ over time. This gives

$$\kappa = \int_0^\infty e^{-r\tau} c_h e^{(r-\rho)\tau} d\tau = \frac{c_h(0)}{\rho}$$

Equating $\kappa = a_h$ delivers

$$c_h(0) = \left(\frac{\rho}{r} \cdot \frac{r+\nu}{r+\nu+\xi} \right) \zeta < c_h(\rho) \tag{125}$$

Note that the entry-level consumption in this case is smaller than in the full-insurance case $r = \rho$ in order to compensate for the higher cost of growing consumption. The household pays an insurance premium $\zeta - c_h(0)$ in exchange for future consumption insurance and consumption growth. Note that since $r > \rho$ and consumption grows along the contract, the insurance premium must be larger (and initial consumption $c_h(0)$ smaller) than in the full-insurance case $(r = \rho)$ to finance future consumption growth, and as the interest rate r converges to the time discount rate ρ from above, the entry-level consumption and the insurance premium converge to the full-insurance consumption level $c_h(\rho)$ from below.

G.2 A Stationary Consumption Distribution?

Although we can characterize the optimal consumption insurance contract in this case, since, conditional on having received the high income, once the consumption of all individuals continues to drift up at the constant (and identical) rate $r - \rho$, there is no stationary consumption distribution for the case $r > \rho$, and thus we can discard this case as a possibility for a *stationary* equilibrium.

Formally, all households experiencing a jump to high income jump to $c_h(0)$ and immediately their consumption drifts up at rate $r - \rho > 0$, so there is no mass point at $c_h(0)$. Instead, there is a continuous consumption density on $[c_h(0), \infty)$ with power and scale parameters that need to determined in the same way as we did for the $r < \rho$ case.

In $c \in [c_h(0), \infty)$ the consumption process follows a diffusion process with drift $r - \rho > 0$ (and no variance) and thus on this interval the stationary consumption distribution satisfies the Kolmogorov forward equation

$$0 = -\frac{d\left[-gc\phi(c)\right]}{dc}$$

Since

$$-\frac{d\left[-gc\phi(c)\right]}{dc} = -\left[-g\phi(c) - gc\phi'(c)\right] = g\left[\phi(c) + c\phi'(c)\right]$$

we find that on $c \in (c_h(0), \infty)$ the stationary distribution satisfies

$$0 = g\left[\phi(c) + c\phi'(c)\right]$$

and thus the consumption distribution is Pareto on $[c_h(0), \infty)$ with power

$$-\frac{c\phi'(c)}{\phi(c)} = 1$$

But this implies that stationary aggregate consumption

$$\int_{c_h}^{\infty} c\phi(c)dc = \infty$$

(as a Pareto distribution with tail parameter 1 has infinite mean) and thus no stationary consumption distribution with finite aggregate consumption can exist in the case of $\rho < r$, ruling out the existence of a stationary equilibrium in this case.

H Welfare in Stationary Equilibrium for IES $\sigma \neq 1$

The wage-deflated consumption allocation in a partial-insurance stationary equilibrium is given by

$$\begin{aligned} c(t) &= c_h(r)e^{-g(r)t} \\ c_h(r) &= \frac{1}{1+\frac{\xi}{r+\nu+g(r)}}\zeta = \frac{1}{1+\frac{\xi}{r+\nu+\frac{\rho-r}{\sigma}}}\zeta = \frac{1}{1+\frac{\xi}{\rho+\nu+(\rho-r)\left(\frac{1}{\sigma}-1\right)}}\zeta \\ g(r) &= \frac{\rho-r}{\sigma} \end{aligned}$$

H.1 Lifetime Utility for Given Interest rate r

Expected lifetime utility is the weighted sum of lifetime utility from being born with low (no) income and being born with high income z. It is given, for interest rate r, by

$$EU(r) = \frac{\xi U_l(r) + \nu U_h(r)}{\xi+\nu}$$

where $U_i(r)$ is lifetime utility being born with income $i = l, h$. For the low-income state lifetime utility is given by

$$\rho U_l(r) = \underline{u} + \nu(U_h(r) - U_l(r))$$

and thus

$$U_l(r) = \frac{\underline{u} + \nu U_h(r)}{\rho+\nu}$$

Thus

$$EU(r) = \frac{\xi\frac{\underline{u}+\nu U_h(r)}{\rho+\nu} + \nu U_h(r)}{\xi+\nu} = \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)}U_h(r)$$

and lifetime utility is linear in lifetime utility conditional on being born with high income.

For being born with high income (for now, suppressing dependence on r), lifetime utility is given by

$$\rho U_h = u(w(r)c_h(r)) + \xi(U(0) - U_h)$$

where $U(t)$ is the lifetime continuation utility from the consumption contract after having had low income for t units of time. It is given by the differential equation

$$\rho U(t) = u(w(r)c_h(r)e^{-g(r)t}) + \nu(U_h - U(t)) + \dot{U}(t)$$

Now define

$$\begin{aligned} u(t) &= \frac{U(t)}{w(r)^{1-\sigma}} \\ u_h(r) &= \frac{U_h(r)}{w(r)^{1-\sigma}} \end{aligned}$$

as wage-deflated lifetime utility. Lifetime utility can be decomposed in this way since the period utility function is CRRA (and thus lifetime utility is homothetic), and the aggregate wage is constant over time in a stationary equilibrium, and can be expressed as a function of the interest rate r (and exogenous parameters) only. The so-defined deflated lifetime utility function follows the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} \rho u_h(r) &= u(c_h(r)) + \xi(u(0) - u_h(r)) \\ \rho u(t) &= u(c_h(r))e^{-(1-\sigma)g(r)t} + \nu(u_h - u(t)) + \dot{u}(t) \end{aligned}$$

or rewriting the second equation

$$\dot{u}(t) = (\rho + \nu)u(t) - u(c_h(r))e^{-(1-\sigma)g(r)t} - \nu u_h$$

Solving the differential equation (one can differentiate with respect to time t using Leibnitz' rule to check that the solution is correct) yields, for now suppressing the dependence of $u_h(r)$ on r:

$$u(t) = \int_t^{\infty} e^{-(\rho+\nu)(s-t)} \left[\nu u_h + u(c_h(r))e^{-(1-\sigma)g(r)s}\right] ds.$$

Evaluating at $t = 0$ one obtains

$$\begin{aligned} u(0) &= \int_0^\infty e^{-(\rho+\nu)s}\left[\nu u_h + u(c_h(r))e^{-(1-\sigma)g(r)s}\right] ds \\ &= \nu u_h \int_0^\infty e^{-(\rho+\nu)s} ds + u(c_h(r)) \int_0^\infty e^{-[\rho+\nu+(1-\sigma)g(r)]s} ds \\ &= -\frac{\nu u_h}{\rho+\nu}\, e^{-(\rho+\nu)s}\Big|_0^\infty - \frac{u(c_h(r))}{\rho+\nu+(1-\sigma)g(r)}\, e^{-[\rho+\nu+(1-\sigma)g(r)]s}\Big|_0^\infty \\ &= \frac{\nu u_h}{\rho+\nu} + \frac{u(c_h(r))}{\rho+\nu+(1-\sigma)g(r)} \end{aligned}$$

and thus the two equations

$$\begin{aligned} u(0) &= \frac{\nu u_h}{\rho+\nu} + \frac{u(c_h(r))}{\rho+\nu+(1-\sigma)g(r)} \\ (\rho+\xi)u_h &= u(c_h(r)) + \xi u(0) \end{aligned}$$

can be solved for $u_h, u(0)$. This delivers

$$\begin{aligned} (\rho+\xi)u_h &= u(c_h(r)) + \frac{\xi\nu u_h}{\rho+\nu} + \frac{\xi u(c_h(r))}{\rho+\nu+(1-\sigma)g(r)} \\ \left[1+\frac{\xi}{\rho+\nu}\right]\rho u_h &= \left[1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r)}\right] u(c_h(r)) \end{aligned}$$

and thus

$$u_h(r) = \frac{1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r)}}{1+\frac{\xi}{\rho+\nu}}\frac{u(c_h(r))}{\rho} = \frac{1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r)}}{\left(1+\frac{\xi}{\rho+\nu}\right)\rho}\frac{c_h(r)^{1-\sigma}}{1-\sigma}$$

and

$$EU(r) = \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)} w(r)^{1-\sigma} u_h(r)$$

Now suppose we scale consumption in all periods by a factor $\alpha > 0$. Expected lifetime utility from this scaled consumption process, denoted by $EU(r;\alpha)$, is given by

$$\begin{aligned} EU(r;\alpha) &= \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)} w(r)^{1-\sigma} u_h(r;\alpha) \\ &= \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \alpha^{1-\sigma}\frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)} w(r)^{1-\sigma} u_h(r;1) \end{aligned}$$

since

$$u_h(r;\alpha) = \frac{1 + \frac{\xi}{\rho+\nu+(1-\sigma)g(r)}}{\left(1+\frac{\xi}{\rho+\nu}\right)\rho} \frac{\left[\alpha c_h(r)\right]^{1-\sigma}}{1-\sigma} = \alpha^{1-\sigma} \frac{1 + \frac{\xi}{\rho+\nu+(1-\sigma)g(r)}}{\left(1+\frac{\xi}{\rho+\nu}\right)\rho} \frac{\left[c_h(r)\right]^{1-\sigma}}{1-\sigma} = \alpha^{1-\sigma} u_h(r;1)$$

H.2 Comparing Welfare Across Equilibria with Interest Rates r_1, r_2

Now we want to compare welfare across two interest rates. For that, we ask by what factor α we have to scale equilibrium consumption under interest rate r_1 so that the household is indifferent to living under interest rate $r_2 > r_1$. That is, we are looking for α such that

$$EU(r_1;\alpha) = EU(r_2;1)$$

where $\alpha < 1$ indicates that the low interest rate equilibrium is preferred, and $\alpha > 1$ indicates that the high interest rate equilibrium is preferred. Using the results from the previous section, we solve for α such that

$$\begin{aligned} & \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \alpha^{1-\sigma}\frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)} w(r_1)^{1-\sigma} u_h(r_1;1) \\ = \; & \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)} w(r_2)^{1-\sigma} u_h(r_2;1) \end{aligned}$$

and thus, using the expression for $c_h(r) = \frac{1}{1+\frac{\xi}{r\left(1-\frac{1}{\sigma}\right)+\nu+\frac{\rho}{\sigma}}} z$ from equation (24):

$$\alpha = \frac{w(r_2)}{w(r_2)} \cdot \left[\frac{u_h(r_2;1)}{u_h(r_1;1)}\right]^{\frac{1}{1-\sigma}} \tag{126}$$

$$= \frac{w(r_2)}{w(r_1)} \cdot \left[\frac{\left(1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r_2)}\right)}{\left(1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r_1)}\right)}\right]^{\frac{1}{1-\sigma}} \frac{c_h(r_2)}{c_h(r_1)} \tag{127}$$

$$= \frac{w(r_2)}{w(r_1)} \cdot \left[\frac{c_h(r_2)}{c_h(r_1)}\right]^{\frac{\sigma}{\sigma-1}} = \alpha_{wage} \cdot \alpha_{contract} \tag{128}$$

where

$$\alpha_{wage} = \frac{w(r_2)}{w(r_1)}$$
$$\alpha_{contract} = \left[\frac{c_h(r_2)}{c_h(r_1)}\right]^{\frac{\sigma}{\sigma-1}}$$

A higher interest rate means a lower capital stock and thus lower wages. Therefore, unambiguously,

$$\alpha_{wage} = \frac{w(r_2)}{w(r_1)} < 1.$$

The second term captures lifetime utility from the *wage-deflated* consumption contract:

$$\alpha_{contract} = \left[\frac{c_h(r_2)}{c_h(r_1)}\right]^{\frac{\sigma}{\sigma-1}} > 1$$

A higher interest rate leads to a better consumption contract, since a higher interest rate is associated with better consumption insurance (consumption starts higher and falls less slowly).[40]

Given that $\alpha_{contract} > 1$ and $\alpha_{wage} < 1$, the overall welfare term α can be smaller or larger than 1. Since both $\alpha_{contract}, \alpha_{wage}$ are closed-form expressions of the two equilibrium interest rates, and these in turn are closed-form solutions of a quadratic equation, we could in principle give conditions on parameters under which the low interest rate yields higher welfare, and alternative conditions under which the reverse is true. However, that these parameter sub-spaces are both nonempty can also be verified numerically.

Note that we could also have defined welfare as expected period utility in the steady

[40] For $\sigma > 1$, we have $c_h(r_2) > c_h(r_1)$ and for $\sigma < 1$ the reverse is true (but then the ratio is taken to a negative exponent.)

state. Doing so, we obtain, when scaling consumption by a constant α

$$
\begin{aligned}
W(r,\alpha) &= \int_0^{c_h(r)} \frac{(\alpha w(r)\, c)^{1-\sigma}}{1-\sigma}\phi(c,r)\, dc + \frac{(\alpha w(r)\, c_h(r))^{1-\sigma}}{1-\sigma}\phi(c_h,r) \\
&= \int_0^{c_h(r)} \frac{(\alpha w(r)\, c)^{1-\sigma}}{1-\sigma}\frac{\xi\nu}{(\xi+\nu)\, g(r)} c_h(r)^{-\frac{\nu}{g(r)}} c^{\frac{\nu}{g(r)}-1} dc \\
&\quad + \frac{(\alpha w(r)\, c_h(r))^{1-\sigma}}{1-\sigma}\frac{\nu}{\xi+\nu} \\
&= \frac{(\alpha w(r))^{1-\sigma}}{1-\sigma}\frac{\nu}{\xi+\nu} c_h(r)^{1-\sigma}\left(1+\frac{\xi}{\nu+(1-\sigma)\, g(r)}\right)
\end{aligned}
$$

Again comparing welfare across two equilibria yields

$$
\begin{aligned}
W(r_1,\alpha) &= W(r_2,1) \\
\frac{(\alpha w(r_1))^{1-\sigma}}{1-\sigma} c_h(r_1)^{1-\sigma}\left(1+\frac{\xi}{\nu+(1-\sigma)\, g(r_1)}\right) &= \\
\frac{(w(r_2))^{1-\sigma}}{1-\sigma} c_h(r_2)^{1-\sigma}\left(1+\frac{\xi}{\nu+(1-\sigma)\, g(r_2)}\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\alpha} &= \frac{w(r_2)}{w(r_1)}\left(\frac{1+\frac{\xi}{\nu+(1-\sigma)g(r_2)}}{1+\frac{\xi}{\nu+(1-\sigma)g(r_1)}}\right)^{\frac{1}{1-\sigma}} \frac{c_h(r_2)}{c_h(r_1)} \\
&= \alpha_{wage}\cdot\hat{\alpha}_{contract}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_{wage} &= \frac{w(r_2)}{w(r_1)} \\
\hat{\alpha}_{contract} &= \left[\frac{\left(1+\frac{\xi}{\nu+(1-\sigma)g(r_2)}\right)}{\left(1+\frac{\xi}{\nu+(1-\sigma)g(r_1)}\right)}\right]^{\frac{1}{1-\sigma}}\cdot\frac{c_h(r_2)}{c_h(r_1)}
\end{aligned}
$$

This alternative welfare measure therefore results in a similar decomposition. The aggregate wage factor α_{wage} is exactly the same, and the contract factor only differs by discounting. Thus both welfare measures give similar welfare comparisons; the second just contains an additional time discounting term, since in the first measure every contract starts with c_h, and in the other, the agent is randomly placed in the stationary consumption distribution.