Appendix

A Proofs and Derivations: Section 3

Proof of Lemma 1. (DS-weights: individual multiplicative decomposition; unique normalized decomposition)

Proof. By offering a constructive proof of part b), we automatically show that it is always possible to construct an individual multiplicative decomposition, in particular a normalized one. Let us start with a set of DS-weights $\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)>0$, defined for each individual, date, and history. After multiplying and dividing by $\sum_{s^t}\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)$, $\sum_{t=0}^{T}\sum_{s^t}\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)$, and $\int\sum_{t=0}^{T}\sum_{s^t}\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)di$, we reach the following identity:

$$\underbrace{\frac{\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)}{\int\sum_{t=0}^{T}\sum_{s^t}\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)di}}_{=\omega_t^i\left(s^t\middle|s_0\right)}=\underbrace{\frac{\sum_{t=0}^{T}\sum_{s^t}\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)}{\int\sum_{t=0}^{T}\sum_{s^t}\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)di}}_{=\tilde{\omega}^i(s_0)}\underbrace{\frac{\sum_{s^t}\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)}{\sum_{t=0}^{T}\sum_{s^t}\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)}}_{=\tilde{\omega}_t^i(s_0)}\underbrace{\frac{\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)}{\sum_{s^t}\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)}}_{=\tilde{\omega}_t^i\left(s^t\middle|s_0\right)},$$

which defines an individual multiplicative decomposition since $\omega_t^i\left(s^t\middle|\,s_0\right)$ and $\breve{\omega}_t^i\left(s^t\middle|\,s_0\right)$ are identical from the perspective of Definition 4, but for a normalization regarding the choice of units. It follows immediately that $\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)=1$, $\sum_{t=0}^{T}\tilde{\omega}_t^i\left(s_0\right)=1$, and $\int\tilde{\omega}^i\left(s_0\right)di=1$, which concludes the proof. □

Proof of Proposition 1. (Welfare assessments: aggregate additive decomposition)

Proof. Combining Equations (7) and (9), the definition of a desirable policy change for a DS-planner can be expressed as follows:

$$\frac{dW^{DS}\left(s_0\right)}{d\theta}=\int\tilde{\omega}^i\left(s_0\right)\frac{dV_i^{DS}\left(s_0\right)}{d\theta}di=\mathbb{E}_i\left[\tilde{\omega}^i\left(s^0\right)\frac{dV_i^{DS}\left(s_0\right)}{d\theta}\right],\tag{35}$$

where

$$\frac{dV_i^{DS}\left(s_0\right)}{d\theta}=\sum_{t=0}^{T}\tilde{\omega}_t^i\left(s_0\right)\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}.\tag{36}$$

Hence, we can first decompose $\frac{dW^{DS}(s_0)}{d\theta}$ as follows:

$$\frac{dW^{DS}\left(s_0\right)}{d\theta}=\underbrace{\mathbb{E}_i\left[\tilde{\omega}^i\left(s^0\right)\right]}_{=1}\mathbb{E}_i\left[\frac{dV_i^{DS}\left(s_0\right)}{d\theta}\right]+\underbrace{\mathbb{C}ov_i\left[\tilde{\omega}^i\left(s^0\right),\frac{dV_i^{DS}\left(s_0\right)}{d\theta}\right]}_{=\Xi_{RD}}\tag{37}$$

where we use the fact that — without loss of generality, but for the choice of units — we can set

$\mathbb{E}_i\left[\tilde{\omega}^i\left(s^0\right)\right]=\int \tilde{\omega}^i\left(s^0\right) di=1$, and where Ξ_{RD} satisfies

$$\Xi_{RD}=\mathbb{C}ov_i\left[\tilde{\omega}^i\left(s^0\right),\sum_{t=0}^{T}\tilde{\omega}_t^i\left(s_0\right)\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right].$$

Next, we can decompose $\mathbb{E}_i\left[\frac{dV_i^{DS}(s_0)}{d\theta}\right]$ as follows:

$$\begin{aligned}
\mathbb{E}_i\left[\frac{dV_i^{DS}\left(s_0\right)}{d\theta}\right] &= \mathbb{E}_i\left[\sum_{t=0}^{T}\tilde{\omega}_t^i\left(s_0\right)\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right] \\
&= \sum_{t=0}^{T}\mathbb{E}_i\left[\tilde{\omega}_t^i\left(s_0\right)\right]\mathbb{E}_i\left[\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]+\underbrace{\sum_{t=0}^{T}\mathbb{C}ov_i\left[\tilde{\omega}_t^i\left(s_0\right),\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]}_{=\Xi_{IS}} \\
&= \sum_{t=0}^{T}\mathbb{E}_i\left[\tilde{\omega}_t^i\left(s_0\right)\right]\sum_{s^t}\left(\mathbb{E}_i\left[\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\right]\mathbb{E}_i\left[\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]+\mathbb{C}ov_i\left[\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right),\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]\right)+\Xi_{IS} \\
&= \underbrace{\sum_{t=0}^{T}\mathbb{E}_i\left[\tilde{\omega}_t^i\left(s_0\right)\right]\sum_{s^t}\mathbb{E}_i\left[\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\right]\mathbb{E}_i\left[\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]}_{=\Xi_{AE}} \\
&\quad+\underbrace{\sum_{t=0}^{T}\mathbb{E}_i\left[\tilde{\omega}_t^i\left(s_0\right)\right]\sum_{s^t}\mathbb{C}ov_i\left[\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right),\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]}_{=\Xi_{RS}}+\Xi_{IS} \\
&= \Xi_{AE}+\Xi_{RS}+\Xi_{IS}. \qquad (38)
\end{aligned}$$

Proposition 1 follows immediately after combining Equations (37) and (38). □

Proof of Proposition 2. (Properties of aggregate additive decomposition: individual-invariant DS-weights)

Proof. a) If DS-weights $\omega_t^i\left(s^t\middle|\,s_0\right)$ do not vary across individuals, parts b), c), and d) below are valid. b) If the stochastic components, $\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)$, do not vary across individuals at all dates and histories, then

$$\mathbb{C}ov_i\left[\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right),\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]=0,\forall t,\forall s^t \implies \Xi_{RS}=0.$$

c) If the dynamic components, $\tilde{\omega}_t^i\left(s_0\right)$, do not vary across individuals at all dates, then

$$\mathbb{C}ov_i\left[\tilde{\omega}_t^i\left(s_0\right),\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]=0,\forall t \implies \Xi_{IS}=0.$$

d) If the individual components, $\tilde{\omega}^i\left(s_0\right)$, do not vary across individuals, then

$$\mathbb{C}ov_i\left[\tilde{\omega}^i\left(s^0\right),\sum_{t=0}^{T}\tilde{\omega}_t^i\left(s_0\right)\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]=0 \implies \Xi_{RD}=0.$$

□

Proof of Corollaries 1 through 4

Proof. Corollary 1 follows from part a). Corollary 2 follows from part b) since $\tilde{\omega}_t^i\left(s^t|s_0\right)=1$, $\forall t$, $\forall i$ in perfect foresight economies. Corollary 3 follows from part d). Corollary 4 follows from parts b) and c) since $\tilde{\omega}_t^i\left(s^t|s_0\right)=1$ and $\tilde{\omega}_t^i\left(s_0\right)=1$, $\forall s^t$, $\forall t$, $\forall i$ in static economies. □

Proof of Proposition 3. (Properties of aggregate additive decomposition: individual-invariant policies)

Proof. Note that $\sum_{t=0}^{T}\tilde{\omega}_t^i\left(s_0\right)$ and $\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)=1$ imply that $\sum_{t=0}^{T}\tilde{\omega}_t^i\left(s_0\right)\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)=1$.
a) If $\frac{du_{i|c}\left(s^t\right)}{d\theta}=g\left(\cdot\right)$, where $g\left(\cdot\right)$ does not depend on i, t, or s^t, then

$$\mathbb{C}ov_i\left[\tilde{\omega}^i\left(s^0\right),\sum_{t=0}^{T}\tilde{\omega}_t^i\left(s_0\right)\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\right]\frac{du_{i|c}\left(s^t\right)}{d\theta}=0\implies\Xi_{RD}=0.$$

And the results from parts b) and c) also apply.
b) If $\frac{du_{i|c}\left(s^t\right)}{d\theta}=g\left(t\right)$, where $g\left(t\right)$ may depend on t, but not on i or s^t, then

$$\mathbb{C}ov_i\left[\tilde{\omega}_t^i\left(s_0\right),\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right)\right]\frac{du_{i|c}\left(s^t\right)}{d\theta}=0\implies\Xi_{IS}=0.$$

And the result from part c) also applies.
c) If $\frac{du_{i|c}\left(s^t\right)}{d\theta}=g\left(t,s^t\right)$, where $g\left(t,s^t\right)$ may depend on t and s^t, but not on i, then

$$\mathbb{C}ov_i\left[\tilde{\omega}_t^i\left(s^t\middle|\,s_0\right),\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]=0\implies\Xi_{RS}=0.$$

□

Proof of Proposition 4. (Properties of aggregate additive decomposition: endowment economies)

Proof. In an endowment economy, Equation (11) simply corresponds to

$$\mathbb{E}_i\left[\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]=\int\frac{dc_t^i\left(s^t\right)}{d\theta}di=0,$$

where the last equality follows from the fact that aggregate consumption is equal to the aggregate endowment, and hence fixed and invariant to θ, that is, $\frac{d\int c_t^i\left(s^t\right)di}{d\theta}=0$. □

B Proofs and Derivations: Section 4

Proof of Proposition 5. (Normalized welfarist planners: individual multiplicative decomposition)

Proof. Starting from Equation (6), note that we can express $\frac{dV_i(s_0)}{d\theta}$ as follows:

$$\begin{aligned}
\frac{dV_i(s_0)}{d\theta} &= \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i} \frac{du_{i|c}\left(s^t\right)}{d\theta} \\
&= \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i} \sum_{t=0}^{T} \frac{(\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i} \frac{du_{i|c}(s^t)}{d\theta}}{\sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i}} \\
&= \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i} \sum_{t=0}^{T} \underbrace{\frac{(\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i}}{\sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i}}}_{=\tilde{\omega}_t^i(s_0)} \sum_{s^t} \underbrace{\frac{(\beta_i)^t \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i}}{(\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i}}}_{=\tilde{\omega}_t^i(s^t|s_0)} \frac{du_{i|c}\left(s^t\right)}{d\theta} \\
&= \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i} \sum_{t=0}^{T} \tilde{\omega}_t^i(s_0) \sum_{s^t} \tilde{\omega}_t^i\left(s^t \middle| s_0\right) \frac{du_{i|c}\left(s^t\right)}{d\theta},
\end{aligned} \tag{39}$$

where we define dynamic and stochastic components of DS-weights as in Equations (13) and (14). Hence, we can express $\frac{dW^{\mathcal{W}}(s_0)}{d\theta}$ — with appropriately normalized units — as follows:

$$\frac{\frac{dW^{\mathcal{W}}(s_0)}{d\theta}}{\int \lambda_i(s_0) \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i} di} = \int \tilde{\omega}^i(s_0) \sum_{t=0}^{T} \tilde{\omega}_t^i(s_0) \sum_{s^t} \tilde{\omega}_t^i\left(s^t \middle| s_0\right) \frac{du_{i|c}\left(s^t\right)}{d\theta} di,$$

where we define the individual component as in Equation (15):

$$\tilde{\omega}^i(s_0) = \frac{\lambda_i(s_0) \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i}}{\int \lambda_i(s_0) \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i} di}.$$

It is straightforward to verify that $\sum_{s^t} \tilde{\omega}_t^i\left(s^t \middle| s_0\right) = 1, \forall t, \forall i$; that $\sum_{t=0}^{T} \tilde{\omega}_t^i(s_0) = 1, \forall i$; and that $\int \tilde{\omega}^i(s_0) di = 1$, which concludes the proof. Note that by multiplying and dividing the dynamic and stochastic components of a given individual by his marginal utility of consumption at 0, we recover Equations (16) and (17):

$$\begin{aligned}
\tilde{\omega}_t^i(s_0) &= \frac{(\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i} / \frac{\partial u_i(s^0)}{\partial c_0^i}}{\sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i} / \frac{\partial u_i(s^0)}{\partial c_0^i}} = \frac{q_t^i\left(s^t|s_0\right)}{\sum_{s^t} q_t^i\left(s^t|s_0\right)} \\
\tilde{\omega}_t^i\left(s^t \middle| s_0\right) &= \frac{\pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i} / \frac{\partial u_i(s^0)}{\partial c_0^i}}{\sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i} / \frac{\partial u_i(s^0)}{\partial c_0^i}} = \frac{\sum_{s^t} q_t^i\left(s^t|s_0\right)}{\sum_{t=0}^{T} \sum_{s^t} q_t^i\left(s^t|s_0\right)}.
\end{aligned}$$

□

Proof of Proposition 6. (Properties of normalized welfarist planners: complete markets)

Proof. When markets are complete, there is a unique stochastic discount factor, which implies that $q_t^i\left(s^t|s_0\right) = q_t\left(s^t|s_0\right)$, $\forall i$. From Equations (16) and (17), it follows immediately that $\tilde{\omega}_t^i\left(s_0\right)$ and $\tilde{\omega}_t^i\left(s^t\middle|\, s_0\right)$ are invariant across all individuals at all dates and histories. Hence, parts b) and c) of Proposition 2 guarantee that $\Xi_{RS} = \Xi_{IS} = 0$. □

Proof of Proposition 7. (Properties of normalized welfarist planners: riskless borrowing/saving)

Proof. When individuals can freely borrow and save, it must be the case that the valuation of a riskless bond is identical for all individuals, which implies that $\sum_{s^t} q_t^i\left(s^t|s_0\right)$ is identical across individuals. Hence, from Equation (17), it follows immediately that $\tilde{\omega}_t^i\left(s_0\right)$ is invariant across all individuals at all dates. Hence, Part c) Proposition 2 guarantees that $\Xi_{IS} = 0$. □

Proof of Proposition 8. (Properties of normalized welfarist planners: welfarist planners only disagree about redistribution)

Proof. Note that Equations (13) and (14) do not depend on $\mathcal{W}\left(\cdot\right)$, while Equation (15) does. This fact, along with Proposition 1, immediately imply that Ξ_{AE}, Ξ_{RS}, and Ξ_{IS} identical for all welfarist planner, but Ξ_{RD} is not. □

Proof of Proposition 9. (Properties of normalized welfarist planners: invariance of efficiency components to utility transformations)

Proof. It follows immediately from Equations (13) and (14) that $\tilde{\omega}_t^i\left(s_0\right)$ and $\tilde{\omega}_t^i\left(s^t\middle|\, s_0\right)$ are invariant to the transformations considered, which multiply numerator and denominator by constant factors. □

Proof of Proposition 10. (Properties of normalized welfarist planners: Pareto improvements increase efficiency)

Proof. From Equation (38), it immediately follows that

$$\Xi_{AE} + \Xi_{RS} + \Xi_{IS} = \mathbb{E}_i\left[\frac{\frac{dV_i(s_0)}{d\theta}}{\sum_{t=0}^{T}\left(\beta_i\right)^t \sum_{s^t} \pi_t\left(s^t\middle|\, s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i}}\right],$$

where $\frac{dV_i(s_0)}{d\theta}$ is defined in Equation (39). If a policy is a strict Pareto improvement, $\frac{dV_i(s_0)}{d\theta} > 0$, which implies that $\Xi_{AE} + \Xi_{RS} + \Xi_{IS}$ must be strictly positive, since $\sum_{t=0}^{T}\left(\beta_i\right)^t \sum_{s^t} \pi_t\left(s^t\middle|\, s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i} > 0$. The same logic applies to weak Pareto improvements, since at least one individual must have $\frac{dV_i(s_0)}{d\theta} > 0$. □

C Proofs and Derivations: Section 5

Proof of Proposition 11 (AE/AR/NR DS-planners: properties)

Proof. a) This result follows from part a) of Proposition 3, since $\tilde{\omega}_t^i\left(s^t\middle|s_0\right)$, $\tilde{\omega}_t^i\left(s_0\right)$, and $\tilde{\omega}^i\left(s_0\right)$ do not vary across individuals. Note that Ξ_{AE} is identical for the pseudo-welfarist AE DS-planner and its associated normalized welfarist planner, since

$$\mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W},AE}\left(s_0\right)\right]=\mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W}}\left(s_0\right)\right] \text{ and } \mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W},AE}\left(s^t\middle|s_0\right)\right]=\mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right)\right].$$

b) This result follows from parts c) and d) of Proposition 3, since $\tilde{\omega}_t^i\left(s^t\middle|s_0\right)$ and $\tilde{\omega}_t^i\left(s_0\right)$ do not vary across individuals. Note that Ξ_{AE} and Ξ_{RS} are identical for the pseudo-welfarist AR DS-planner and its associated normalized welfarist planner, since

$$\mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W},AR}\left(s_0\right)\right]=\mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W}}\left(s_0\right)\right] \text{ and } \tilde{\omega}_t^{i,\mathcal{W},AR}\left(s^t\middle|s_0\right)=\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right).$$

c) This result follows from part d) of Proposition 3, since the individual components $\tilde{\omega}^i\left(s_0\right)$ do not vary across individuals. Note that Ξ_{AE}, Ξ_{RS}, and Ξ_{IS} are identical for the pseudo-welfarist NR DS-planner and its associated normalized welfarist planner, since

$$\tilde{\omega}_t^{i,\mathcal{W},NR}\left(s_0\right)=\tilde{\omega}_t^{i,\mathcal{W}}\left(s_0\right) \text{ and } \tilde{\omega}_t^{i,\mathcal{W},NR}\left(s^t\middle|s_0\right)=\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right).$$

□

Online Appendix

Section D of this Online Appendix includes proofs and derivations for Section 6. Section E includes additional results for Application 1. Section F includes several extensions and Section G contains additional results.

D Proofs and Derivations: Section 6

Proof of Proposition 12. (Aggregate efficiency component: stochastic decomposition)

Proof. Starting from the definition of the aggregate efficiency component in Equation (11), we can express Ξ_{AE} as follows:

$$\begin{aligned}\Xi_{AE} &= \sum_{t=0}^{T} \mathbb{E}_i \left[\tilde{\omega}_t^i (s_0)\right] \sum_{s^t} \mathbb{E}_i \left[\tilde{\omega}_t^i \left(s^t \middle| s_0\right)\right] \mathbb{E}_i \left[\frac{du_{i|c}(s^t)}{d\theta}\right] \\ &= \sum_{t=0}^{T} \overline{\omega}_t \sum_{s^t} \overline{\omega}_t \left(s^t \middle| s_0\right) \frac{d\overline{u}_{i|c}(s^t)}{d\theta},\end{aligned}$$

where we define $\overline{\omega}_t (s_0) = \mathbb{E}_i \left[\tilde{\omega}_t^i (s_0)\right]$, $\overline{\omega}_t \left(s^t \middle| s_0\right) = \mathbb{E}_i \left[\tilde{\omega}_t^i \left(s^t \middle| s_0\right)\right]$, and $\frac{d\overline{u}_{i|c}(s^t)}{d\theta} = \mathbb{E}_i \left[\frac{du_{i|c}(s^t)}{d\theta}\right]$. Multiplying and dividing by $\pi_t \left(s^t \middle| s_0\right)$ at every history, we can express and decompose Ξ_{AE} as follows:

$$\begin{aligned}\Xi_{AE} &= \sum_{t=0}^{T} \overline{\omega}_t (s_0) \sum_{s^t} \pi_t \left(s^t \middle| s_0\right) \frac{\overline{\omega}_t \left(s^t \middle| s_0\right)}{\pi_t \left(s^t \middle| s_0\right)} \frac{d\overline{u}_{i|c}(s^t)}{d\theta} = \sum_{t=0}^{T} \overline{\omega}_t (s_0) \mathbb{E}_0 \left[\overline{\omega}_t^{\pi} \left(s^t \middle| s_0\right) \frac{d\overline{u}_{i|c}(s^t)}{d\theta}\right] \\ &= \underbrace{\sum_{t=0}^{T} \overline{\omega}_t (s_0) \mathbb{E}_0 \left[\overline{\omega}_t^{\pi} \left(s^t \middle| s_0\right)\right] \mathbb{E}_0 \left[\frac{d\overline{u}_{i|c}(s^t)}{d\theta}\right]}_{=\Xi_{EAE}} + \underbrace{\sum_{t=0}^{T} \overline{\omega}_t (s_0) \mathbb{C}ov_0 \left[\overline{\omega}_t^{\pi} \left(s^t \middle| s_0\right), \frac{d\overline{u}_{i|c}(s^t)}{d\theta}\right]}_{=\Xi_{AM}},\end{aligned}$$

which corresponds to Equation (23) in the text. □

Proof of Proposition 13. (Risk-sharing/intertemporal-sharing components: alternative cross-sectional decompositions)

Proof. Here we make use of the following property of covariances (Bohrnstedt and Goldberger, 1969):

$$\mathbb{C}ov [X, YZ] = \mathbb{E}[Y] \mathbb{C}ov [X, Z] + \mathbb{E}[Z] \mathbb{C}ov [X, Y] + \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])(Z - \mathbb{E}[Z])],$$

where X, Y, and Z denote random variables. Applying this property to $\mathbb{C}ov_i \left[\tilde{\omega}_t^i (s_0), \tilde{\omega}_t^i \left(s^t \middle| s_0\right) \frac{du_{i|c}(s^t)}{d\theta}\right]$, we find that

$$\mathbb{C}ov_i\left[\tilde{\omega}_t^i(s_0),\tilde{\omega}_t^i\left(s^t\middle|s_0\right)\frac{du_{i|c}(s^t)}{d\theta}\right]=\underbrace{\mathbb{E}_i\left[\tilde{\omega}_t^i\left(s^t\middle|s_0\right)\right]\mathbb{C}ov_i\left[\tilde{\omega}_t^i(s_0),\frac{du_{i|c}(s^t)}{d\theta}\right]}_{\sim\Xi_{PIS}}$$
$$+\underbrace{\mathbb{E}_i\left[\frac{du_{i|c}(s^t)}{d\theta}\right]\mathbb{C}ov_i\left[\tilde{\omega}_t^i(s_0),\tilde{\omega}_t^i\left(s^t\middle|s_0\right)\right]}_{\sim\Xi_{WC}}$$
$$+\underbrace{\mathbb{E}_i\left[\left(\frac{du_{i|c}(s^t)}{d\theta}-\mathbb{E}_i\left[\frac{du_{i|c}(s^t)}{d\theta}\right]\right)\left(\tilde{\omega}_t^i(s_0)-\mathbb{E}_i\left[\tilde{\omega}_t^i(s_0)\right]\right)\left(\tilde{\omega}_t^i\left(s^t\middle|s_0\right)-\mathbb{E}_i\left[\tilde{\omega}_t^i\left(s^t\middle|s_0\right)\right]\right)\right]}_{\sim\Xi_{PC}},$$

which immediately yields Equation (24) in the text after adding up over dates and histories. Equation (25) follows immediately after using once again the same property of covariances on $\mathbb{C}ov_i\left[\tilde{\omega}_t^i(s_0)\,\tilde{\omega}_t^i(s^t|\,s_0),\frac{du_{i|c}(s^t)}{d\theta}\right]$. □

Proof of Proposition 14. (Redistribution component: stochastic decomposition)

Proof. We can express $\frac{dV_i^{DS}(s_0)}{d\theta}$, defined in Equation (36), as follows:

$$\frac{dV_i^{DS}(s_0)}{d\theta}=\sum_{t=0}^{T}\tilde{\omega}_t^i(s_0)\,\mathbb{E}_0\left[\frac{\tilde{\omega}_t^i(s^t|\,s_0)}{\pi_t(s^t|\,s_0)}\frac{du_{i|c}(s^t)}{d\theta}\right]$$
$$=\underbrace{\sum_{t=0}^{T}\tilde{\omega}_t^i(s_0)\,\mathbb{E}_0\left[\tilde{\omega}_t^{i,\pi}\left(s^t\middle|s_0\right)\right]\mathbb{E}_0\left[\frac{du_{i|c}(s^t)}{d\theta}\right]}_{=\frac{dV_i^{DS,ER}(s_0)}{d\theta}}+\underbrace{\sum_{t=0}^{T}\tilde{\omega}_t^i(s_0)\,\mathbb{C}ov_0\left[\left[\tilde{\omega}_t^{i,\pi}\left(s^t\middle|s_0\right)\right],\frac{du_{i|c}(s^t)}{d\theta}\right]}_{=\frac{dV_i^{DS,RM}(s_0)}{d\theta}}.$$

Hence, we can express Ξ_{RD} as follows:

$$\Xi_{RD}=\mathbb{C}ov_i\left[\tilde{\omega}^i\left(s^0\right),\frac{dV_i^{DS}(s_0)}{d\theta}\right]=\underbrace{\mathbb{C}ov_i\left[\tilde{\omega}^i\left(s^0\right),\frac{dV_i^{DS,ER}(s_0)}{d\theta}\right]}_{\Xi_{ER}}+\underbrace{\mathbb{C}ov_i\left[\tilde{\omega}^i\left(s^0\right),\frac{dV_i^{DS,RM}(s_0)}{d\theta}\right]}_{\Xi_{RM}},$$

which corresponds to Equation (15) in the text. □

Proof of Proposition 15. (Cross-sectional dispersion bounds)

Proof. Equations (26) through (28) follow from applying the Cauchy-Schwarz inequality, which states that $|\mathbb{C}ov[X,Y]|\leq\sqrt{\mathbb{V}ar[X]}\sqrt{\mathbb{V}ar[Y]}$ for any pair of square integrable random variables X and Y.

When applied to the relevant elements of Ξ_{RS}, Ξ_{IS}, and Ξ_{RD}, we find that

$$\mathbb{C}ov_i\left[\tilde{\omega}_t^i\left(s^t\middle|s_0\right), \frac{du_{i|c}\left(s^t\right)}{d\theta}\right] \leq \sqrt{\mathbb{V}ar_i\left[\tilde{\omega}_t^i\left(s^t\middle|s_0\right)\right]}\sqrt{\mathbb{V}ar_i\left[\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]}$$

$$\mathbb{C}ov_i\left[\tilde{\omega}_t^i\left(s_0\right), \sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right] \leq \sqrt{\mathbb{V}ar_i\left[\tilde{\omega}_t^i\right]}\sqrt{\mathbb{V}ar_i\left[\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]}$$

$$\mathbb{C}ov_i\left[\tilde{\omega}^i\left(s^0\right), \sum_{t=0}^{T}\tilde{\omega}_t^i\left(s_0\right)\sum_{s^t}\tilde{\omega}_t^i\left(s^t\middle|s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right] \leq \sqrt{\mathbb{V}ar_i\left[\tilde{\omega}^i\left(s^0\right)\right]}\sqrt{\mathbb{V}ar_i\left[\sum_{t=0}^{T}\sum_{s^t}\tilde{\omega}_t^i\tilde{\omega}_t^i\left(s^t\middle|s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]}.$$

These three inequalities, when combined with the definitions of Ξ_{RS}, Ξ_{IS}, and Ξ_{RD} in Equation (11), immediately imply Equations (26) through (28) in the text. □

Proof of Proposition 16. (Recursive formulation)

Proof. Starting from Equation (35), note that we can express $\frac{dW^{DS}(s_0)}{d\theta}$ as follows:

$$\begin{aligned}
\frac{dW^{DS}\left(s_0\right)}{d\theta} &= \int \tilde{\omega}^i\left(s_0\right)\frac{dV_i^{DS}\left(s_0\right)}{d\theta}di \\
&= \int \underbrace{\tilde{\omega}^i\left(s_0\right)\tilde{\omega}_0^i\left(s_0\right)\tilde{\omega}_0^i\left(s^0\middle|s_0\right)}_{=\omega_0^i\left(s^0|s_0\right)}\underbrace{\frac{\frac{dV_i^{DS}(s_0)}{d\theta}}{\tilde{\omega}_0^i\left(s_0\right)\tilde{\omega}_0^i\left(s^0\middle|s_0\right)}}_{=\frac{d\hat{V}_{i,0}^{DS}(s_0)}{d\theta}}di \\
&= \int \omega_0^i\left(s^0\middle|s_0\right)\frac{d\hat{V}_{i,0}^{DS}\left(s_0\right)}{d\theta}di.
\end{aligned}$$

Note that we can also express $\frac{d\hat{V}_{i,0}^{DS}(s_0)}{d\theta}$ as follows:

$$\begin{aligned}
\frac{d\hat{V}_{i,0}^{DS}\left(s_0\right)}{d\theta} &= \sum_{t=0}^{T}\frac{\tilde{\omega}_t^i\left(s_0\right)}{\tilde{\omega}_0^i\left(s_0\right)}\sum_{s^t}\frac{\tilde{\omega}_t^i\left(s^t\middle|s_0\right)}{\tilde{\omega}_0^i\left(s^0\middle|s_0\right)}\frac{du_{i|c}\left(s_t\right)}{d\theta} \\
&= \underbrace{\frac{\tilde{\omega}_0^i\left(s_0\right)}{\tilde{\omega}_0^i\left(s_0\right)}}_{=1}\underbrace{\frac{\tilde{\omega}_0^i\left(s^0\middle|s_0\right)}{\tilde{\omega}_0^i\left(s^0\middle|s_0\right)}}_{=1}\frac{du_{i|c}\left(s_0\right)}{d\theta} + \sum_{t=1}^{T}\frac{\tilde{\omega}_t^i\left(s_0\right)}{\tilde{\omega}_0^i\left(s_0\right)}\sum_{s^t}\frac{\tilde{\omega}_t^i\left(s^t\middle|s_0\right)}{\tilde{\omega}_0^i\left(s^0\middle|s_0\right)}\frac{du_{i|c}\left(s_t\right)}{d\theta} \\
&= \frac{du_{i|c}\left(s_0\right)}{d\theta} + \frac{\tilde{\omega}_1^i\left(s_0\right)}{\tilde{\omega}_0^i\left(s_0\right)}\left(\underbrace{\frac{\tilde{\omega}_1^i\left(s_0\right)}{\tilde{\omega}_1^i\left(s_0\right)}}_{=1}\sum_{s^1}\frac{\tilde{\omega}_1^i\left(s^1\middle|s_0\right)}{\tilde{\omega}_0^i\left(s^0\middle|s_0\right)}\frac{du_{i|c}\left(s_1\right)}{d\theta} + \sum_{t=2}^{T}\frac{\tilde{\omega}_t^i\left(s_0\right)}{\tilde{\omega}_1^i\left(s_0\right)}\sum_{s^t}\frac{\tilde{\omega}_t^i\left(s^t\middle|s_0\right)}{\tilde{\omega}_0^i\left(s^0\middle|s_0\right)}\frac{du_{i|c}\left(s_t\right)}{d\theta}\right) \\
&= \frac{du_{i|c}\left(s_0\right)}{d\theta} + \frac{\tilde{\omega}_1^i\left(s_0\right)}{\tilde{\omega}_0^i\left(s_0\right)}\left(\sum_{s^1}\frac{\tilde{\omega}_1^i\left(s^1\middle|s_0\right)}{\tilde{\omega}_0^i\left(s^0\middle|s_0\right)}\underbrace{\left(\frac{du_{i|c}\left(s_1\right)}{d\theta} + \sum_{t=2}^{T}\frac{\tilde{\omega}_t^i\left(s_0\right)}{\tilde{\omega}_1^i\left(s_0\right)}\sum_{s^t|s^1}\frac{\tilde{\omega}_t^i\left(s^t\middle|s_0\right)}{\tilde{\omega}_1^i\left(s^1\middle|s_0\right)}\frac{du_{i|c}\left(s_t\right)}{d\theta}\right)}_{=\frac{d\hat{V}_{i,1}^{DS}(s_1)}{d\theta}}\right),
\end{aligned}$$

which immediately implies Equation (30) in the text, since this derivation is valid starting from any state s_0.

The definitions of $\hat{\beta}_{i,t}^{\mathcal{W}}$ and $\hat{\pi}_{i,t}^{\mathcal{W}}$ follow immediately after combining Equations (13) and (14) with Equation (31). Note that the product $\hat{\beta}_{i}^{\mathcal{W}}(s) \cdot \hat{\pi}_{i}^{\mathcal{W}}(s' | s)$ corresponds to the state-price assigned at state s by individual i to state s':

$$\hat{\beta}_{i,t}^{\mathcal{W}} \cdot \hat{\pi}_{i,t}^{\mathcal{W}}(s' | s) = \beta_i \pi(s' | s) \frac{\partial u_i(s')}{\partial c^i} / \frac{\partial u_i(s)}{\partial c^i},$$

and that this state-price is time-independent. This observation, combined with the definition of the pseudo-utilitarian NR planner, implies the claim that Equation (30) is time invariant for welfarist and pseudo-welfarist NR planners. □

Proof of Proposition 17 (Linear instantaneous SWF formulation)

Proof. Note that, for a planner with a linear instantaneous SWF, it must be that

$$\frac{d\mathcal{I}(\cdot)}{d\theta} = \int \sum_{t=0}^{T} \sum_{s^t} \lambda_t^i \left(s^t\right) \frac{\partial u_i(s^t)}{\partial c_t^i} \frac{du_{i|c}(s^t)}{d\theta} di, \tag{OA1}$$

where $\frac{du_{i|c}(s^t)}{d\theta}$ is defined in Equation (3). The results for both the marginal welfare assessment and the optimum follow immediately by comparing Equation (7) to Equation (OA1), where the following relation must be satisfied:

$$\lambda_t^i \left(s^t\right) = \frac{\omega_t^i(s^t)}{\frac{\partial u_i(s^t)}{\partial c_t^i}}.$$

□

E Application: Additional Figures

Figures OA-1 and OA-2 are the counterparts of Figure 3 in the text when $\rho = 0.999$ and $\rho = 0.5$. When $\rho = 0.999$, the components of the individual multiplicative decompositions evolve extremely slowly. Given the extreme persistence of the shocks, all of the welfare gains from increasing θ arise from redistribution (Ξ_{RD}). When $\rho = 0.5$, endowments shocks are fully transitory, and the components of the individual multiplicative decomposition barely have any time-dependence. In this case, the welfare gains from increasing θ arise mostly from risk-sharing. The gains from redistribution are nonzero, but very small, since they are only driven by marginal utility differences at $t = 0$.

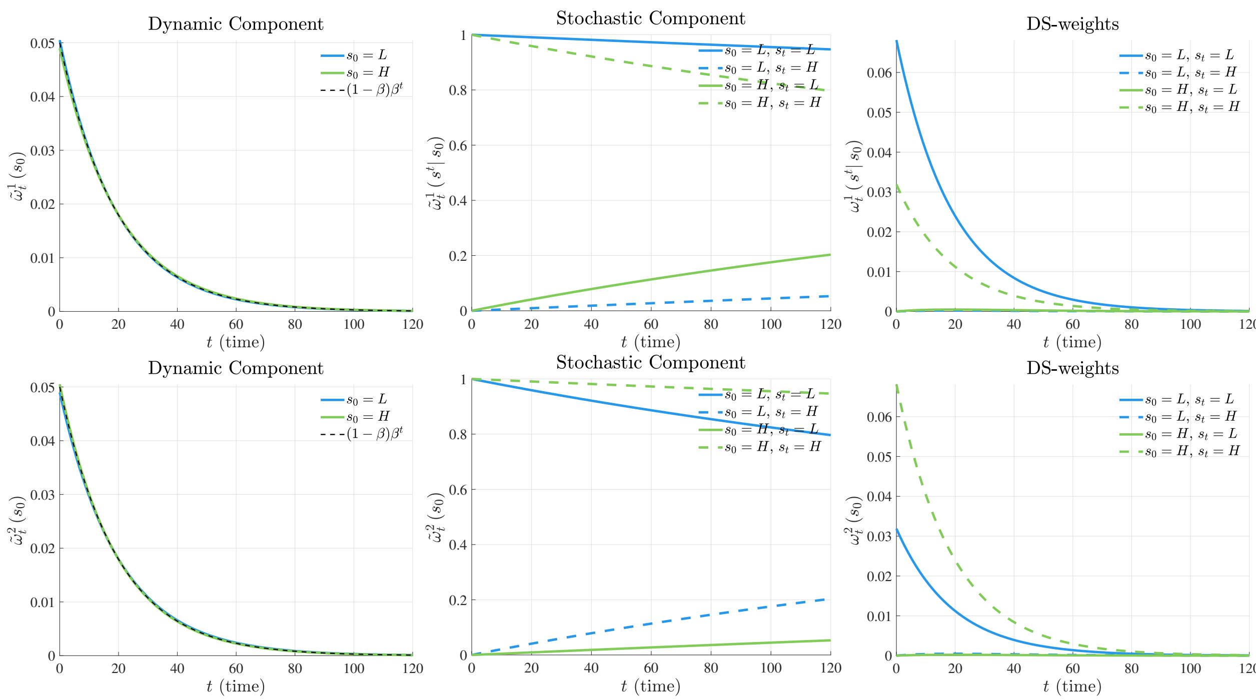

Figure OA-1: Individual multiplicative decomposition of DS-weights (Scenario 1; $\rho = 0.999$)

Note: Figure OA-1 is the counterpart of Figure 3 in the text when endowment shocks are extremely persistent ($\rho = 0.999$). The individual component of DS-weights in this case are $\tilde{\omega}^1\,(s_0 = L) = 1.349$ and $\tilde{\omega}^2\,(s_0 = L) = 0.651$ when an assessment takes place at $s_0 = L$; and $\tilde{\omega}^1\,(s_0 = H) = 0.651$ and $\tilde{\omega}^2\,(s_0 = H) = 1.349$ when the assessment takes place at $s_0 = H$.

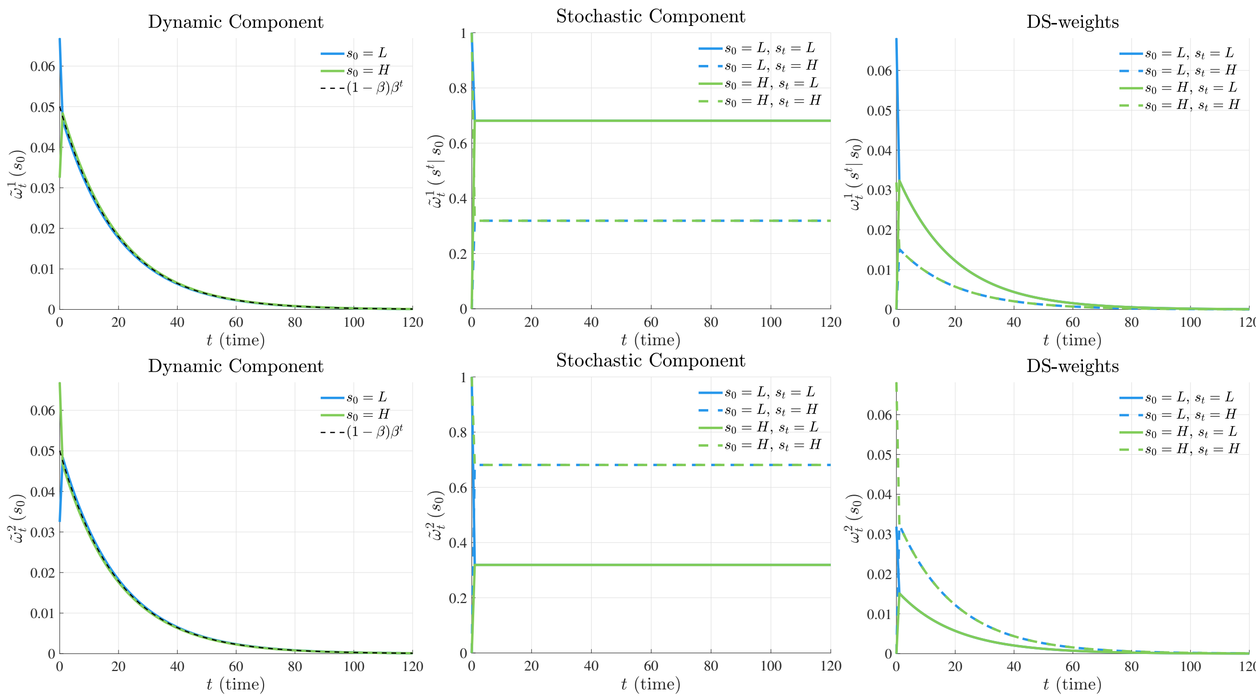

Figure OA-2: Individual multiplicative decomposition of DS-weights (Scenario 1; $\rho = 0.5$)

Note: Figure OA-2 is the counterpart of Figure 3 in the text when endowment shocks are fully temporary ($\rho = 0.5$). The individual component of DS-weights in this case are $\tilde{\omega}^1(s_0 = L) = 1.018$ and $\tilde{\omega}^2(s_0 = L) = 0.928$ when an assessment takes place at $s_0 = L$; and $\tilde{\omega}^1(s_0 = H) = 0.982$ and $\tilde{\omega}^2(s_0 = H) = 1.018$ when the assessment takes place at $s_0 = H$.

F Extensions

F.1 Heterogeneous beliefs

In this section, we show how to use DS-weights to make paternalistic and non-paternalistic welfare assessments in environments with heterogeneous beliefs.[47] Note that the notion of paternalism used here is fully consistent with the formal definition given in Footnote 32. To model heterogeneous beliefs, instead of Equation (1), we assume instead that individual preferences take the form

$$V_i\left(s_0\right)=\sum_{t=0}^{T}\left(\beta_i\right)^t\sum_{s^t}\pi_t^i\left(s^t\middle|\,s_0\right)u_i\left(c_t^i\left(s^t\right),n_t^i\left(s^t\right)\right), \tag{OA2}$$

where $\pi_t^i\left(s^t\middle|\,s_0\right)$, denotes the beliefs held by individual i over histories, which are now individual-specific.

In this case, a non-paternalistic planner would substitute $\pi_t^i\left(s^t\middle|\,s_0\right)$ for $\pi_t\left(s^t\middle|\,s_0\right)$ whenever it appears in Equations (8) through (22). Alternatively, a paternalistic planner who imposes a single-belief would substitute some planner's belief, $\pi_t^P\left(s^t\middle|\,s_0\right)$, which is invariant across individuals, for $\pi_t\left(s^t\middle|\,s_0\right)$ whenever it appears in Equations (8) through (22).[48]

F.2 Recursive utility: Epstein-Zin preferences

In this section, we show how to use DS-weights in the context of economies with recursive preferences. In particular, we consider the widely used Epstein-Zin preferences, which we define recursively as follows:

$$V_i\left(s\right)=\left(\left(1-\beta_i\right)\left(u_i\left(c^i\left(s\right),n^i\left(s\right)\right)\right)^{1-\frac{1}{\psi_i}}+\beta_i\left(\sum_{s'}\pi\left(s'|s\right)\left(V^i\left(s'\right)\right)^{1-\gamma_i}\right)^{\frac{1-\frac{1}{\psi_i}}{1-\gamma_i}}\right)^{\frac{1}{1-\frac{1}{\psi_i}}},$$

where γ_i modulates risk aversion and ψ_i modulates intertemporal substitution. We use s and s' to denote any two recursive states (Ljungqvist and Sargent, 2018).

In this case, we can recursively express the welfare effect of a policy change, measured in lifetime

[47] A recent literature has explored how to make normative assessments in environments with heterogeneous beliefs. See, among others, Brunnermeier, Simsek and Xiong (2014), Gilboa, Samuelson and Schmeidler (2014), Dávila (2020), Blume et al. (2018), Caballero and Simsek (2019), and Dávila and Walther (2021).

[48] At times, it makes sense to reinterpret heterogeneous beliefs as state-dependent preferences. In that case, $V_i\left(s_0\right)$ can be expressed as

$$V_i\left(s_0\right)=\sum_{t=0}^{T}\left(\beta_i\right)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i\left(c_t^i\left(s^t\right),n_t^i\left(s^t\right);s^t\right).$$

All our results remain valid in the case of state-dependent preferences.

utils (utility units), as follows:

$$\frac{dV_i(s)}{d\theta} = \frac{\partial V_i(s)}{\partial c^i(s)} \frac{du_{i|c}(s)}{d\theta} + \sum_{s'} \frac{\partial V_i(s)}{\partial V^i(s')} \frac{dV_i(s')}{d\theta}, \tag{OA3}$$

where

$$\frac{\partial V_i(s)}{\partial c^i(s)} = (1-\beta_i)(V_i(s))^{\frac{1}{\psi}} (u_i(s))^{-\frac{1}{\psi_i}} \frac{\partial u_i(s)}{\partial c^i}$$

$$\frac{\partial V_i(s)}{\partial V^i(s')} = \beta_i (V_i(s))^{\frac{1}{\psi}} \left(\sum_{s'} \pi(s'|s) \left(V^i(s')\right)^{1-\gamma_i} \right)^{\frac{\gamma_i - \frac{1}{\psi_i}}{1-\gamma_i}} \pi(s'|s) \left(V^i(s')\right)^{-\gamma_i},$$

and where $\frac{du_{i|c}(s)}{d\theta}$ is defined as in Equation (3). The structure of Equation (OA3) immediately implies that $\frac{dV_i(s)}{d\theta}$ can be expressed as a linear transformation of instantaneous consumption-equivalent effects, $\frac{du_{i|c}(s)}{d\theta}$, which in turn guarantees that the definition of a DS-planner in Equation (6) can also be used in the context of economies with recursive preferences.

Note that it is straightforward to define normalized DS-weights when considering normalized welfarist planners, as in Section 4. In particular, Equations (16), (17), and (19) remain valid, and the one-period version of Equation (18), from which it is straightforward to compute state-prices for any date and state, becomes

$$q^i(s'|s) = \frac{\frac{\partial V_i(s)}{\partial c^i(s')}}{\frac{\partial V_i(s)}{\partial c^i(s)}} = \frac{\frac{\partial V_i(s)}{\partial V^i(s')} \frac{\partial V^i(s')}{\partial c^i(s')}}{\frac{\partial V_i(s)}{\partial c^i(s)}} = \beta_i \pi(s'|s) \left(\frac{V_i(s')}{H(s)}\right)^{\frac{1}{\psi}-\gamma_i} \left(\frac{c^i(s')}{c^i(s)}\right)^{-\frac{1}{\psi_i}} \frac{\frac{\partial u_i(s')}{\partial c^i}}{\frac{\partial u_i(s)}{\partial c^i}},$$

where $H(s) = \left(\sum_{s'} \pi(s'|s)(V^i(s'))^{1-\gamma_i}\right)^{\frac{1}{1-\gamma_i}}$. It is straightforward to define DS-weights for even more general preferences, including preferences that are not time-separable or recursive, as we do next.

F.3 General utility with multiple commodities

In the baseline model, we already illustrate how to make welfare assessments when there are multiple goods/commodities, since we consider an environment with two commodities: consumption and hours. Here we consider a more abstract scenario, in which $i \in I$ individuals have general preferences over a set of commodities $\ell \in L$, which can also be indexed by dates $t \in T$ and histories s^t. In this case, the lifetime utility of individual i is given by

$$V_i(s_0) = U_i\left(\left\{x_t^{i,\ell}\left(s^t\right)\right\}_{t,s^t,\ell}\right).$$

At this level of generality, the different commodities can represent hours worked, as in the baseline environment, different consumption goods, flow utility from housing, or any variable that directly impacts instantaneous utility. Hence, we can express the lifetime utility effect of a policy change for individual i as follows:

$$\frac{dV_i\left(s_0\right)}{d\theta}=\sum_{t=0}^{T}\sum_{s^t}\sum_{\ell}\frac{\partial U_i}{\partial x_t^{i,\ell}\left(s^t\right)}\frac{dx_t^{i,\ell}\left(s^t\right)}{d\theta}.$$

Consequently, we can generalize the definition of DS-planner (in Definition 3) to a general environment, by assuming that $\frac{dW^{DS}(s_0)}{d\theta}$ takes the form

$$\frac{dW^{DS}\left(s_0\right)}{d\theta}=\int\sum_{t=0}^{T}\sum_{s^t}\sum_{\ell}\omega_t^{i,\ell}\left(s^t\middle|\,s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}di,$$

where $\omega_t^{i,\ell}\left(s^t\middle|\,s_0\right)$ is a DS-weight defined for each specific commodity ℓ, for each date t, at each history s^t, and for each individual i.[49]

Hence, paralleling Lemma 1, we can define a multiplicative decomposition of the form

$$\omega_t^{i,\ell}\left(s^t\middle|\,s_0\right)=\underbrace{\tilde{\omega}^i\left(s_0\right)}_{\text{individual}}\underbrace{\tilde{\omega}_t^i\left(s_0\right)}_{\text{dynamic}}\underbrace{\tilde{\omega}_t^i(s^t|s_0)}_{\text{stochastic}}\underbrace{\tilde{\omega}_t^{i,\ell}(s^t|s_0)}_{\text{commodity}},$$

where the choice of $\tilde{\omega}_t^{i,\ell}(s^t|s_0)$ is shaped by the choice of numeraire. Throughout the paper, we assume that consumption is the numeraire good, so with $\ell\in\{c,n\}$, we have that

$$\tilde{\omega}_t^{i,c}(s^t|s_0)=1$$
$$\tilde{\omega}_t^{i,n}(s^t|s_0)=\frac{\frac{\partial u_i\left(s^t\right)}{\partial n_t^i}}{\frac{\partial u_i\left(s^t\right)}{\partial c_t^i}}.$$

By doing this, we guarantee that $\sum_\ell\omega_t^{i,\ell}\left(s^t\middle|\,s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}$ is measured in units of consumption good at history s^t. These results highlight how welfare assessments also rely on the choice of numeraire. However, more generally we can consider any bundle of goods $\left\{\psi^\ell\right\}_{\ell\in L}$ as numeraire, that is we could set

$$\tilde{\omega}_t^{i,\ell}(s^t|s_0)=\frac{\frac{\partial U_i}{\partial x_t^{i,\ell}\left(s^t\right)}}{\sum_\ell\psi^\ell\frac{\partial U_i}{\partial x_t^{i,\ell}\left(s^t\right)}},$$

where by choosing a unit vector for some commodity ℓ we choose a single commodity as numeraire. For the purpose of making meaningful welfare assessments, this normalization/choice of numeraire must be consistent across all individuals. Welfare assessments are typically not invariant to the choice of numeraire, but there are good reasons to choose some numeraires (e.g., consumption, some particular consumption bundle, or dollars) over others.[50]

[49] In this case, the generalization of the lifetime and instantaneous Social Welfare Functions is a "commodity Social Welfare Function", given by

$$W\left(\left\{x_t^{i,\ell}\left(s^t\right)\right\}_{t,s^t,\ell,i}\right).$$

[50] One could potentially pick different numeraires in different dates or histories, but it seems natural to choose a consistent numeraire to yield easily interpretable results.

In the case of a normalized welfarist planner, it is straightforward to characterize commodity-DS-weights. Using the first commodity ($\ell = 1$) as numeraire, and defining $\lambda_i(s_0) = \frac{\partial \mathcal{W}\left(\{V_i(s_0)\}_{i \in I}\right)}{\partial V_i}$, it follows that

$$
\begin{aligned}
\frac{dW^{DS}(s_0)}{d\theta} &= \int \lambda_i(s_0) \sum_{t=0}^{T} \sum_{s^t} \sum_{\ell} \frac{\partial U_i}{\partial x_t^{i,\ell}(s^t)} \frac{dx_t^{i,\ell}\left(s^t\right)}{d\theta} di \\
&= \int \lambda_i(s_0) \sum_{t=0}^{T} \sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)} \sum_{\ell} \frac{\frac{\partial U_i}{\partial x_t^{i,\ell}(s^t)}}{\frac{\partial U_i}{\partial x_t^{i,1}(s^t)}} \frac{dx_t^{i,\ell}\left(s^t\right)}{d\theta} di \\
&= \int \lambda_i(s_0) \sum_{t=0}^{T} \sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)} \sum_{s^t} \frac{\frac{\partial U_i}{\partial x_t^{i,1}(s^t)}}{\sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)}} \sum_{\ell} \frac{\frac{\partial U_i}{\partial x_t^{i,\ell}(s^t)}}{\frac{\partial U_i}{\partial x_t^{i,1}(s^t)}} \frac{dx_t^{i,\ell}\left(s^t\right)}{d\theta} di \\
&= \int \lambda_i(s_0) \sum_{t} \sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)} \sum_{t=0}^{T} \frac{\sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)}}{\sum_t \sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)}} \sum_{s^t} \frac{\frac{\partial U_i}{\partial x_t^{i,1}(s^t)}}{\sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)}} \sum_{\ell} \frac{\frac{\partial U_i}{\partial x_t^{i,\ell}(s^t)}}{\frac{\partial U_i}{\partial x_t^{i,1}(s^t)}} \frac{dx_t^{i,\ell}\left(s^t\right)}{d\theta} di \\
&= \int \lambda_i(s_0) \sum_{t} \sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)} di \int \underbrace{\frac{\lambda_i(s_0) \sum_t \sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)}}{\int \lambda_i(s_0) \sum_t \sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)} di}}_{=\tilde{\omega}^i(s_0)} \sum_{t=0}^{T} \underbrace{\frac{\sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)}}{\sum_t \sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)}}}_{=\tilde{\omega}_t^i(s_0)} \sum_{s^t} \underbrace{\frac{\frac{\partial U_i}{\partial x_t^{i,1}(s^t)}}{\sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)}}}_{=\tilde{\omega}_t^i(s^t|s_0)} \underbrace{\sum_{\ell} \underbrace{\frac{\frac{\partial U_i}{\partial x_t^{i,\ell}(s^t)}}{\frac{\partial U_i}{\partial x_t^{i,1}(s^t)}}}_{=\tilde{\omega}_t^{i,\ell}(s^t|s_0)} \frac{dx_t^{i,\ell}\left(s^t\right)}{d\theta}}_{=\frac{du_{i|c}(s^t)}{d\theta}} di.
\end{aligned}
$$

so we can write

$$
\frac{\frac{dW^{DS}(s_0)}{d\theta}}{\int \lambda_i(s_0) \sum_t \sum_{s^t} \frac{\partial U_i}{\partial x_t^{i,1}(s^t)} di} = \int \tilde{\omega}^i(s_0) \sum_{t=0}^{T} \tilde{\omega}_t^i(s_0) \sum_{s^t} \tilde{\omega}_t^i\left(s^t | s_0\right) \frac{du_{i|c}(s^t)}{d\theta} di.
$$

This derivation highlights that once we choose a numeraire, the dynamic, stochastic, and individuals components of DS-weights are expressed in terms of such numeraire — it is straightforward to use a bundle-numeraire of the form $\sum_\ell \psi^\ell \frac{\partial U_i}{\partial x_t^{i,\ell}(s^t)}$. Hence, for Proposition 6 to be valid, the natural commodity to choose as numeraire is the commodity on which financial claims are written on.

Finally, note that it is also possible to introduce multiple commodities in the baseline model with time-separable expected utility preferences. To do so, we define a generalized version of Equation (1), which includes multiply commodities, indexed by $\ell \in L$, as follows:

$$
V_i(s_0) = \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) u_i\left(\left\{c_t^{i,\ell}\left(s^t\right)\right\}_{\ell \in L}\right).
$$

Without loss, we treat commodity 1 as the numeraire for the purpose of making welfare assessments, so we can express $\frac{dV_i(s_0)}{d\theta}$ as follows:

$$
\frac{dV_i(s_0)}{d\theta} = \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^{i,1}} \sum_{\ell \in L} \frac{du_{i|c^1}(s^t)}{d\theta},
$$

where $\frac{\partial u_i(s^t)}{\partial c_t^{i,\ell}} = \frac{\partial u_i\left(\left\{c_t^{i,\ell}(s^t)\right\}_{\ell \in L}\right)}{\partial c_t^{i,\ell}(s^t)}$ and where the instantaneous commodity-1-equivalent effect of the policy at history s^t, is given by $\frac{du_{i|c^1}(s^t)}{d\theta}$, where

$$\frac{du_{i|c^1}\left(s^t\right)}{d\theta} = \frac{\frac{du_i\left(\left\{c_t^{i,\ell}(s^t)\right\}_{\ell \in L}\right)}{d\theta}}{\frac{\partial u_i(s^t)}{\partial c_t^{i,1}}} = \frac{dc_t^{i,1}\left(s^t\right)}{d\theta} + \sum_{\ell \in L} \frac{\frac{\partial u_i(s^t)}{\partial c_t^{i,\ell}}}{\frac{\partial u_i(s^t)}{\partial c_t^{i,1}}} \frac{dc_t^{i,\ell}\left(s^t\right)}{d\theta}.$$

Once again, when there are multiple commodities, it is necessary to account for the marginal rates of substitutions between those commodities and the commodity chosen as numeraire. Note that the choice of numeraire will not change the directional welfare assessment of a welfarist planner, but it can have an impact on the units of such assessment, as well as on the value of the components of the aggregate additive decomposition.

F.4 Policy changes that affect probabilities

In this section, we describe how to use DS-weights in environments in which policy changes affect probabilities. Starting from Equation (2), note that we can express $\frac{dV_i(s_0)}{d\theta}$ as follows

$$\frac{dV_i\left(s_0\right)}{d\theta} = \sum_{t=0}^{T} \left(\beta_i\right)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i} \left(\frac{du_{i|c}\left(s^t\right)}{d\theta} + \frac{d \ln \pi_t\left(s^t \middle| s_0\right)}{d\theta} \frac{u_i\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right)\right)}{\frac{\partial u_i(s^t)}{\partial c_t^i}} \right).$$

Hence, we can use the following definition of a DS-planner in this case.

Definition 6. *(Desirable policy change for a DS-planner) A DS-planner, that is, a planner who adopts DS-weights, finds a policy change desirable in an environment in which policies can also affect probabilities if and only if* $\frac{dW(s_0)}{d\theta} > 0$*, where*

$$\frac{dW^{DS}\left(s_0\right)}{d\theta} = \int \sum_{t=0}^{T} \sum_{s^t} \omega_t^i\left(s^t \middle| s_0\right) \left(\frac{du_{i|c}\left(s^t\right)}{d\theta} + \frac{d \ln \pi_t\left(s^t \middle| s_0\right)}{d\theta} \frac{u_i\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right)\right)}{\frac{\partial u_i(s^t)}{\partial c_t^i}} \right) di,$$

where $\frac{d \ln \pi_t\left(s^t | s_0\right)}{d\theta} = \frac{\frac{d\pi_t\left(s^t | s_0\right)}{d\theta}}{\pi_t\left(s^t | s_0\right)}$.

Identical results apply in the case in which policy changes directly affect preferences. See Dávila and Goldstein (2021) for an application of the results of this paper to an environment in which policy changes have a discontinuous impact on payoffs.

F.5 Intergenerational considerations

In this section, we describe how to use DS-weights in environments with births, deaths, bequest motives, and related considerations, which non-trivially affect welfare assessments — see Calvo and

Obstfeld (1988), Farhi and Werning (2010), Heathcote, Storesletten and Violante (2017), or Phelan and Rustichini (2018). The most direct way of extending our baseline environment, is to interpret the set of individuals I considered in the baseline model as the set all individuals i) alive or ii) yet-to-be-born from the perspective of s_0. Under that interpretation, $\frac{du_{i|c}(s^t)}{d\theta}$ is non-zero only for those alive at a given history, so Definition 3 applies unchanged.[51]

Bequest motives, altruism, warm-glow preferences, social discounting or similar considerations only impact welfare assessments via the choice of DS-weights. For instance, a welfarist planner who values future generations directly placing a positive weight on their welfare and that in turn perceives an effective social discount rate lower than the private one, can be modeled by choosing a particular set of DS-weights. While do not explore that possibility in this paper, there is scope to use the law of total covariance to internationally decompose the cross-sectional components of the aggregate additive decomposition.

[51] An important practical consideration is that Proposition 7 will never apply to economies with births, since yet-to-be-born individuals cannot freely trade with alive individuals. These ideas deserve further exploration.

G Additional Results

G.1 Dimensional analysis

This paper puts great emphasis on the units in which different variables are defined. In this section, we carefully describe the units of the different components of individual multiplicative decomposition for a normalized welfarist planner and for a general DS-planner

Welfarist planners. As we discuss in the text, the units of our formulation of DS-weights for the case of the normalized welfarist planner have a clear interpretation in terms of dollars at different dates and histories. Here, we provide a systematic dimensional analysis (de Jong, 1967) of the welfare assessments made by a normalized welfarist planner. We denote the units of a specific variable by $\dim(\cdot)$, where, for instance, $\dim\left(c_t^i\left(s^t\right)\right) =$ dollars at history s^t ,where we interchangeably use dollars and units of the consumption good.

First, note that the units of $\tilde{\omega}_t^i\left(s^t \middle| s_0\right)$, $\tilde{\omega}_t^i\left(s_0\right)$, and $\tilde{\omega}^i\left(s_0\right)$ for a welfarist planner, as defined in Equations (13), (14), and (15), are respectively given by

$$\dim\left(\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t \middle| s_0\right)\right) = \frac{\frac{\text{instantaneous utils at } s_0 \text{ for individual } i}{\text{dollars at history } s^t}}{\frac{\text{instantaneous utils at } s_0 \text{ for individual } i}{\text{dollars at date } t}} = \frac{\text{dollars at date } t}{\text{dollars at history } s^t}$$

$$\dim\left(\tilde{\omega}_t^{i,\mathcal{W}}\left(s_0\right)\right) = \frac{\frac{\text{instantaneous utils at } s_0 \text{ for individual } i}{\text{dollars at date } t}}{\frac{\text{instantaneous utils at } s_0 \text{ for individual } i}{\text{dollars at all dates and histories}}} = \frac{\text{dollars at all dates and histories}}{\text{dollars at date } t}$$

$$\dim\left(\tilde{\omega}^{i,\mathcal{W}}\left(s_0\right)\right) = \frac{\frac{\text{instantaneous utils at } s_0 \text{ for individual } i}{\text{dollars at all dates and histories}}}{\frac{\text{instantaneous utils at } s_0}{\text{dollars at all dates and histories for all individuals}}} = \frac{\text{dollars at all dates and histories for all individuals}}{\text{dollars at all dates and histories}},$$

where the last cancellation accounts for the implicit comparability of utility units among individuals.[52] The term $(\beta_i)^t \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i}$, which defines the numerator of $\tilde{\omega}_t^i\left(s^t \middle| s_0\right)$, is measured in instantaneous utils at s_0 per dollars at history s^t for individual i, since

$$\dim\left((\beta_i)^t\right) = \frac{\text{instantaneous utils at } s_0 \text{ for individual } i}{\text{instantaneous utils at history } s^t \text{ for individual } i}$$

$$\dim\left(\frac{\partial u_i\left(s^t\right)}{\partial c_t^i}\right) = \frac{\text{instantaneous utils at history } s^t \text{ for individual } i}{\text{dollars at history } s^t},$$

and probabilities, like $\pi_t\left(s^t \middle| s_0\right)$, are unitless. The same logic applies to the remaining elements of $\tilde{\omega}_t^i\left(s^t \middle| s_0\right)$, $\tilde{\omega}_t^i\left(s_0\right)$, and $\tilde{\omega}^i\left(s_0\right)$.

[52] From the perspective of aggregation of lifetime utilities, which takes places through the individual component $\tilde{\omega}^i\left(s_0\right)$, any welfarist planner has $|I|+1$ degrees of freedom: the planner can give different weights to each of the $|I|$ individual assessments, and can further normalize the units of aggregate welfare.

Consequently, it follows that

$$\dim\left(\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle| s_0\right)\right) = \dim\left(\tilde{\omega}^i\left(s_0\right)\tilde{\omega}_t^i\left(s_0\right)\tilde{\omega}_t^i\left(s^t\middle| s_0\right)\right) \tag{OA4}$$
$$= \frac{\text{dollars at all dates and histories for all individuals}}{\text{dollars at history } s^t}.$$

Hence, the DS-weights $\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle| s_0\right)$ translates dollars at history s^t into $\lambda_i\left(s_0\right)$ dollars at all dates and histories for all individuals.

Second, note that the units of $\frac{du_{i|c}\left(s^t\right)}{d\theta}$ are given by

$$\dim\left(\frac{du_{i|c}\left(s^t\right)}{d\theta}\right) = \frac{\frac{\text{instantaneous utils at history } s^t \text{ for individual } i}{\text{unit of policy change}}}{\frac{\text{instantaneous utils at history } s^t \text{ for individual } i}{\text{dollars at history } s^t}} = \frac{\text{dollars at history } s^t}{\text{unit of policy change}}, \tag{OA5}$$

which follows directly from Equation (14).

Finally, combining Equations (OA4) and (OA5), it follows that

$$\dim\left(\frac{dW^{\mathcal{W}}\left(s_0\right)}{d\theta}\right) = \dim\left(\omega_t^i\left(s^t\middle| s_0\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right) = \frac{\text{dollars at all dates and histories for all individuals}}{\text{unit of policy change}}. \tag{OA6}$$

Hence, the units of $W^{\mathcal{W}}$ for a normalized welfarist planner are dollars paid to all individuals at all dates and histories. That is, if $\frac{dW^{NU}}{d\theta} = 7$, the welfare gain associated with a marginal policy change is equivalent to paying 7 dollars to all individuals in the economy at all dates and histories.

General DS-planners. The dimensional analysis in the case of general planners is similar. In this case, the welfare units of $\tilde{\omega}_t^{i,DS}\left(s^t\middle| s_0\right)$ can be directly computed as

$$\dim\left(\tilde{\omega}_t^{i,DS}\left(s^t\middle| s_0\right)\right) = \frac{\text{units of } W^{DS}}{\text{dollars at history } s^t}.$$

In this case, it is also possible to compute the units of each of the components of the individual multiplicative decomposition as we just did for welfarist planners. By doing so, it becomes clear that the units of each of the components of the individual multiplicative decomposition for any normalized DS-planner (including those who are not welfarist) are identical.

Undesirable properties of unnormalized decompositions. As briefly explained in the text, using unnormalized individual multiplicative decompositions of DS-weights can be problematic in the context of the aggregate additive decomposition, since unnormalized decompositions are expressed in utils. This is not the case for normalized decompositions since these always make tradeoffs in dollar units.

For instance, if one were to set $\lambda_i\left(s_0\right) = 1$, $\forall i$, in the decomposition presented in Equation (10), the redistribution component of the aggregate additive decomposition would be zero, $\Xi_{RD} = 0$. This result captures the fact that an unnormalized equal-weighted utilitarian planner is indifferent between redistribution across individuals in utility terms. Hence, by directly adding up utils, we would fail to capture the idea that a utilitarian planner wants to redistribute resources (in consumption units) towards individuals with low marginal utility — see e.g., Salanie (2011). Similarly, if individual discount factors are identical, that is, $\beta_i = \beta$, $\forall i$, a welfarist planner under the decomposition presented in Equation (10) will conclude that intertemporal-sharing is zero, that is, $\Xi_{IS} = 0$, regardless of the form of the policy under consideration. Equally important, the dynamic and stochastic weights for a welfarist planner defined as in Equation (10) need not add up to 1. Hence, according to Proposition 3, even when the instantaneous consumption-equivalent effect of a policy change is identical across individuals at all dates and histories, an unnormalized utilitarian planner would typically find non-zero intertemporal-sharing components and redistribution components of the aggregate additive decomposition. This is is another undesirable property of the unnormalized utilitarian welfare criterion.

An alternative date-0 normalization. One of the contributions of this paper is to introduce the notion of a normalized planner — see Lemma 1 — as one for which the stochastic, dynamic, and individual components of the multiplicative decomposition add up to 1 across specific dimensions. However, these is an alternative normalization that seems reasonable: one may consider normalizing the individual welfare effect of a policy change by date-0 marginal utility. In that case, it is possible to decompose the DS-weights of a welfarist planner as follows:

$$
\begin{aligned}
\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t \middle| s_0\right) &= \frac{\pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i}}{\sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i}} = \frac{q_t^i\left(s^t | s_0\right)}{\sum_{s^t} q_t^i\left(s^t | s_0\right)} \\
\tilde{\omega}_t^{i,\mathcal{W}}\left(s_0\right) &= \frac{\left(\beta_i\right)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i\left(s^t\right)}{\partial c_t^i}}{\frac{\partial u_i\left(s^0\right)}{\partial c_t^i}} = \sum_{t=0}^{T} \sum_{s^t} q_t^i\left(s^t | s_0\right) \\
\tilde{\omega}^{i,\mathcal{W}}\left(s_0\right) &= \frac{\lambda_i\left(s_0\right) \frac{\partial u_i\left(s^0\right)}{\partial c_t^i}}{\int \lambda_i\left(s_0\right) \frac{\partial u_i\left(s^0\right)}{\partial c_t^i} di}.
\end{aligned}
$$

This decomposition satisfies $\sum_{s^t} \tilde{\omega}_t^i\left(s^t \middle| s_0\right) = 1$, $\forall t$, $\forall i$, and $\int \tilde{\omega}^i\left(s_0\right) di = 1$, but it is clear that $\sum_{t=0}^{T} \tilde{\omega}_t^i\left(s_0\right) \neq 1$. Instead, in this decomposition, $\tilde{\omega}_0^{i,\mathcal{W}}\left(s_0\right) = 1$, for all individuals. In terms of units, this decomposition adds up individual welfare effects according to $\tilde{\omega}^{i,\mathcal{W}}\left(s_0\right)$, once they are expressed in date-0 dollars, which may seem reasonable or even desirable in some circumstances. However, in this case Proposition 3a) will not be valid if using this normalization. In particular, the redistribution component of the aggregate decomposition will not be zero for policies that are invariant across all individuals at all dates and histories. In this case, the component Ξ_{RD} captures redistribution from a date-0 perspective, note a lifetime perspective.

G.2 $\boldsymbol{\alpha}$-DS-Planners

After substituting the definition of the components of the DS-weights, we can explicitly express welfare assessments for a $\boldsymbol{\alpha}$-DS-planner as follows:

$$\begin{aligned}\frac{dW^{\mathcal{W},\boldsymbol{\alpha}}(s_0)}{d\theta} = & \underbrace{\sum_{t=0}^{T}\mathbb{E}_i\left[\left(1-\alpha_3^i\right)\tilde{\omega}_t^{i,\mathcal{W},AE}(s_0)+\alpha_3^i\tilde{\omega}_t^{i,\mathcal{W}}(s_0)\right]\sum_{s^t}\mathbb{E}_i\left[(1-\alpha_2)\,\tilde{\omega}_t^{i,\mathcal{W},AE}\left(s^t\middle|s_0\right)+\alpha_2\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right)\right]\mathbb{E}_i\left[\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]}_{=\Xi_{AE}\text{ (Aggregate Efficiency)}}\\
&+\underbrace{\sum_{t=0}^{T}\mathbb{E}_i\left[\left(1-\alpha_3^i\right)\tilde{\omega}_t^{i,\mathcal{W},AE}(s_0)+\alpha_3^i\tilde{\omega}_t^{i,\mathcal{W}}(s_0)\right]\sum_{s^t}\mathbb{C}ov_i\left[(1-\alpha_2)\,\tilde{\omega}_t^{i,\mathcal{W},AE}\left(s^t\middle|s_0\right)+\alpha_2\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right),\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]}_{=\Xi_{RS}\text{ (Risk-sharing)}}\\
&+\underbrace{\sum_{t=0}^{T}\mathbb{C}ov_i\left[\left(1-\alpha_3^i\right)\tilde{\omega}_t^{i,\mathcal{W},AE}(s_0)+\alpha_3^i\tilde{\omega}_t^{i,\mathcal{W}}(s_0),\sum_{s^t}\left((1-\alpha_2)\,\tilde{\omega}_t^{i,\mathcal{W},AE}\left(s^t\middle|s_0\right)+\alpha_2\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right)\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}\right]}_{=\Xi_{IS}\text{ (Intertemporal-sharing)}}\\
&+\underbrace{\mathbb{C}ov_i\left[(1-\alpha_4)\,\tilde{\omega}^{i,\mathcal{W},AE}(s_0)+\alpha_4\tilde{\omega}^{i,\mathcal{W}}(s_0),X\right]}_{=\Xi_{RD}\text{ (Redistribution)}},\end{aligned} \tag{OA7}$$

where

$$X=\sum_{t=0}^{T}\left((1-\alpha_3)\,\tilde{\omega}_t^{i,\mathcal{W},AE}(s_0)+\alpha_3\tilde{\omega}_t^{i,\mathcal{W}}(s_0)\right)\sum_{s^t}\left((1-\alpha_2)\,\tilde{\omega}_t^{i,\mathcal{W},AE}\left(s^t\middle|s_0\right)+\alpha_2\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right)\right)\frac{du_{i|c}\left(s^t\right)}{d\theta}.$$

Note that the notion of $\boldsymbol{\alpha}$-DS-planner introduced in Definition 4 is designed so that the following properties are satisfied:

$$\begin{aligned}\mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W},\boldsymbol{\alpha}}\left(s^t\middle|s_0\right)\right]&=\tilde{\omega}_t^{i,\mathcal{W},AE}\left(s^t\middle|s_0\right)=\mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right)\right]\\
\mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W},\boldsymbol{\alpha}}(s_0)\right]&=\tilde{\omega}_t^{i,\mathcal{W},AE}(s_0)=\mathbb{E}_i\left[\tilde{\omega}_t^{i,\mathcal{W}}(s_0)\right]\\
\mathbb{E}_i\left[\tilde{\omega}^{i,\mathcal{W},\boldsymbol{\alpha}}(s_0)\right]&=\tilde{\omega}^{i,\mathcal{W},AE}(s_0)=\mathbb{E}_i\left[\tilde{\omega}^{i,\mathcal{W}}(s_0)\right].\end{aligned}$$

Hence, Equation (OA7) implies that when $\boldsymbol{\alpha}=(0,0,0)$, we have an AE pseudo-welfarist DS-planner; when $\boldsymbol{\alpha}=(1,0,0)$, we have an AR pseudo-welfarist DS-planner; when $\boldsymbol{\alpha}=(1,1,0)$, we have a NR pseudo-welfarist DS-planner; and when $\boldsymbol{\alpha}=(1,1,1)$, we have a welfarist planner. We summarize this results in Table OA-1.

Table OA-1: $\boldsymbol{\alpha}$-DS-planner: Special cases

$(\alpha_2,\alpha_3,\alpha_4)$	$\tilde{\omega}_t^i\left(s^t\middle|s_0\right)$	$\tilde{\omega}_t^i(s_0)$	$\tilde{\omega}^i\left(s^0\right)$	Planner
$(1,1,1)$	$\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right)$	$\tilde{\omega}_t^{i,\mathcal{W}}(s_0)$	$\tilde{\omega}^{i,\mathcal{W}}(s_0)$	$\mathcal{W}$
$(1,1,0)$	$\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right)$	$\tilde{\omega}_t^{i,\mathcal{W}}(s_0)$	$\tilde{\omega}^{i,\mathcal{W},AE}(s_0)$	NR
$(1,0,0)$	$\tilde{\omega}_t^{i,\mathcal{W}}\left(s^t\middle|s_0\right)$	$\tilde{\omega}_t^{i,\mathcal{W},AE}(s_0)$	$\tilde{\omega}^{i,\mathcal{W},AE}(s_0)$	NS
$(0,0,0)$	$\tilde{\omega}_t^{i,\mathcal{W},AE}\left(s^t\middle|s_0\right)$	$\tilde{\omega}_t^{i,\mathcal{W},AE}(s_0)$	$\tilde{\omega}^{i,\mathcal{W},AE}(s_0)$	AE

Note: Note that all the $\boldsymbol{\alpha}$-DS-planners considered in this table are pseudo-welfarist.

However, note that there are other possible extreme combinations of $\boldsymbol{\alpha}$ that one may want to consider, these are the following:

$$\{(1,0,1),(0,1,0),(0,1,1),(0,0,1)\}. \tag{OA8}$$

The problem with the $\boldsymbol{\alpha}$'s in Equation (OA8) is that, as long as one of the first two elements of $\boldsymbol{\alpha}$ are 0, the redistribution component will be different from the redistribution component of the relevant welfarist planner. Hence, these choices of $\boldsymbol{\alpha}$ are not pseudo-welfarist. Hence, those $\boldsymbol{\alpha}$-DS-planners will not be pseudo-welfarist, despite being perfectly valid DS-planners.

G.3 Relation to existing work

G.3.1 Welfarist Social Welfare Functions

In addition to the utilitarian SWF, defined in Equation (5), there are other welfarist SWF's that are at times used in specific applications — see e.g., Mas-Colell, Whinston and Green (1995), Kaplow (2011), or Adler and Fleurbaey (2016) for details. Here we briefly described those.

The *isoelastic* SWF, commonly traced back to Atkinson (1970), is given by

$$\mathcal{W}\left(\{V_i(s_0)\}_{i\in I}\right)=\left(\int a_i\left(V_i(s_0)\right)^{\phi}di\right)^{1/\phi},$$

where the (inequality) coefficient ϕ is typically restricted to lie in $[-\infty,1]$, so that $\mathcal{W}(\cdot)$ is concave when $V_i(s_0)>0$, and where it is typically assumed that $\int a_i di=1$, and that $a_i\geq 0$, $\forall i$.[53] Limiting cases of the isoelastic SWF correspond to the other four widely used SWF's. First, when $\phi\to 1$, the isoelastic SWF becomes the conventional *utilitarian* SWF. In that case:

$$\mathcal{W}\left(\{V_i(s_0)\}_{i\in I}\right)=\int a_i V_i(s_0)\,di.$$

Second, when $\phi\to 0$, the isoelastic SWF becomes the *Nash* (Cobb-Douglas) SWF. In that case:

$$\mathcal{W}\left(\{V_i(s_0)\}_{i\in I}\right)=\int\left(V_i(s_0)\right)^{a_i}di.$$

Third, when $\phi\to-\infty$, the isoelastic SWF becomes the *Rawlsian/maximin* (Leontief) SWF. In that case:

$$\mathcal{W}\left(\{V_i(s_0)\}_{i\in I}\right)=\min\left\{\ldots,\frac{V_i(s_0)}{a_i},\ldots\right\}.$$

Finally, when the isoelastic SWF gives positive weight to a single individual, it can be interpreted

[53] Note that, for an isoelastic SWF, $\frac{\partial\mathcal{W}}{\partial V_i}=a_i\left(\frac{V_i}{\mathcal{W}}\right)^{\phi-1}$. More importantly $\frac{\frac{\partial\mathcal{W}}{\partial V_i}}{\frac{\partial\mathcal{W}}{\partial V_j}}=\frac{a_i}{a_j}\left(\frac{V_i}{V_j}\right)^{\phi-1}$. When lifetime utilities are negative, it is possible to define an isoelastic SWF of the form

$$\mathcal{W}\left(\{V_i(s_0)\}_{i\in I}\right)=\left(\int a_i\left(-V_i(s_0)\right)^{\phi}di\right)^{1/\phi},$$

by considering $\phi\in[1,\infty]$.

as a *dictatorial* SWF. In that case:

$$\mathcal{W}\left(\{V_i\left(s_0\right)\}_{i\in I}\right) = V_1\left(s_0\right).$$

Note that all of these SWF are Paretian, although the Rawlsian/maximin and the dictatorial SWF's are not strictly Paretian.[54]

G.3.2 Relation to Saez and Stantcheva (2016)

It is straightforward to define welfare assessments in our framework that are based on the approach introduced by Saez and Stantcheva (2016).

Definition 7. *(Desirable policy change for a planner who uses generalized social marginal welfare weights (Saez and Stantcheva, 2016)) A planner who uses generalized social marginal welfare weights finds a policy change desirable if and only if* $\frac{dW^{SS}(s_0)}{d\theta} > 0$*, where*

$$\frac{dW^{SS}\left(s_0\right)}{d\theta} = \int h_i\left(\cdot\right)\frac{dV_i\left(s_0\right)}{d\theta}di, \tag{OA9}$$

where $h_i\left(\cdot\right) > 0$*,* $\forall i \in I$*, are a collection of individual-specific positive functions, and where* $\frac{dV_i(s_0)}{d\theta}$ *is defined in Equation (2).*

By comparing Equation (OA9) with Equation (6), it is evident that the approach based on generalized social marginal welfare weights introduced in Saez and Stantcheva (2016) is more general than the welfarist approach. The key difference between the two approaches is that for welfarist planners the functions $h_i\left(\cdot\right)$ are restricted to take the form

$$h_i\left(\cdot\right) = \frac{\partial\mathcal{W}\left(\{V_i\left(s_0\right)\}_{i\in I}\right)}{\partial V_i},$$

while $h_i\left(\cdot\right)$ can take many other values under the Saez and Stantcheva (2016) approach. Saez and Stantcheva (2016) show that their approach can capture alternatives to welfarism, such as libertarianism or equality of opportunity. It is also evident from definition 7 that a planner who uses generalized social marginal welfare weights is not paternalistic, since welfare assessments are based on individual lifetime welfare effects, $\frac{dV_i(s_0)}{d\theta}$.

In static economies, the individual component of the individual multiplicative decomposition of DS-weights introduced in Lemma 1 exactly corresponds to the notion of generalized welfare weights introduced in Saez and Stantcheva (2016). In other words, in static environments, the contribution of our paper is only to introduce the aggregate additive decomposition of welfare assessments in aggregate efficiency and redistribution, but not to introduce the notion of generalized individual weights for particular individuals, which is already in Saez and Stantcheva (2016).

[54] A planner with an isoelastic SWF is strictly Paretian when $\phi > -\infty$ if $a_i > 0$, $\forall i$.

G.3.3 Relation to Kaldor/Hicks principle

The classic Kaldor/Hicks (Kaldor, 1939; Hicks, 1939) compensation principle can be formalized in marginal form in static environments by equal individual generalized weights among individuals, see e.g., Hendren (2020). This observation implies that the Kaldor/Hicks welfare criterion can also be formalized as a particular DS-planner.

In dynamic environments, there is some ambiguity on when and how to compensate different individuals. When the Kaldor/Hicks compensation is defined in permanent dollars (dollars across all dates and histories), the Kaldor/Hicks welfare criterion exactly correspond to the pseudo-welfarist NR planner introduced in Section 5, in which

$$\tilde{\omega}^i(s_0) = 1, \ \forall i. \quad \text{(Kaldor-Hicks)}$$

Intuitively, if a welfarist planner has access to permanent lump-sum transfers among individuals, an optimality condition for such a planner is that

$$\lambda_i(s_0) \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) \frac{\partial u_i(s^t)}{\partial c_t^i}$$

must be equal across all agents, implying that $\tilde{\omega}^i(s_0) = 1$.[55] This is the sense in which $\tilde{\omega}^i(s_0) = 1$ has the interpretation of a Kaldor-Hicks planner. However, while allowing for lump-sum transfers implies that $\tilde{\omega}^i(s_0) = 1$, the converse is not true, that is, it is possible to make welfare assessments using $\tilde{\omega}^i(s_0) = 1$ as individual weights even when no transfers at all are made in the background. We further elaborate on the role of transfers in Section G.3.5.

G.3.4 Relation to Lucas (1987) and Alvarez and Jermann (2004)

It is common in papers that study the welfare consequences of policies in dynamic and stochastic environments to compute welfare gains or losses of policies as in Lucas (1987), who measures the welfare gains associated with a policy change — specifically, the welfare gains associated with eliminating business cycles. Since our approach is built on marginal arguments, we connect instead our results to those in Alvarez and Jermann (2004), who provide a marginal formulation of the approach in Lucas (1987).

While the Lucas (1987) and Alvarez and Jermann (2004) approach is easily interpretable in representative agent economies, it has the pitfall that it cannot be meaningfully aggregated when there are heterogeneous individuals. See, for instance, how Atkeson and Phelan (1994), Krusell and Smith (1999), or Krusell et al. (2009) carefully avoid aggregating welfare gains or losses for different individuals.

To illustrate these arguments, here we consider a policy change for a given individual i, who

[55] Alternatively, as discussed in Footnote 24, a date-0 Kaldor-Hicks normalization, so that $\lambda_i(s_0) \frac{\partial u_i(s^0)}{\partial c_0^i} = 1$, is equivalent to assigning a higher individual weight to individuals with higher willingness to pay for T-consol bonds.

could be a representative agent or not. Formally, we consider a special case of the environment laid out in Section 3, in which an individual i has preferences given by

$$V_i(s_0) = \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) u_i\left(c_t^i\left(s^t\right)\right).$$

We suppose that the consumption of individual i at date t and history s^t can be written as

$$c_t^i\left(s^t\right) = (1-\theta)\underline{c_t^i}\left(s^t\right) + \theta\overline{c_t^i}\left(s^t\right),$$

where both $\underline{c_t^i}(s^t)$ and $\overline{c_t^i}(s^t)$ are sequences measurable with respect to history s^t. The sequence $\underline{c_t^i}(s^t)$ can be interpreted as a given initial consumption path (when $\theta = 0$) and the sequence $\overline{c_t^i}(s^t)$ can be interpreted as a final consumption path (when $\theta = 1$). In the case of Lucas (1987), $\theta = 1$ corresponds to fully eliminating business cycles.

First, we compute the marginal gains from marginally reducing business cycles, as in Alvarez and Jermann (2004). Next, we compute the marginal gains from marginally reducing business cycles using an additive compensation.

Multiplicative compensation. Lucas (1987) proposes using a time-invariant equivalent variation, expressed multiplicatively as a constant fraction of consumption at each date and history as follows

$$\sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) u_i\left(c_t^i\left(s^t\right)(1+\lambda(\theta))\right) = \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t\left(s^t \middle| s_0\right) u_i\left((1-\theta)\underline{c_t^i}\left(s^t\right) + \theta\overline{c_t^i}\left(s^t\right)\right), \tag{OA10}$$

where $\lambda(\theta)$ implicitly defines the welfare gains associated with a policy indexed by θ; the exact definition in Lucas (1987) exactly corresponds to solving for $\lambda(\theta = 1)$.[56]

Following Alvarez and Jermann (2004), we can compute the derivative of the RHS of Equation (OA10) as follows:

$$\frac{d\left((\beta_i)^t \sum_{s^t} \pi_t(s^t|s_0)\, u_i\left((1-\theta)\underline{c_t^i}(s^t) + \theta\overline{c_t^i}(s^t)\right)\right)}{d\theta} = \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t(s^t|s_0)\, u_i'\left((1-\theta)\underline{c_t^i}(s^t) + \theta\overline{c_t^i}(s^t)\right)\frac{dc_t^i(s^t)}{d\theta} \tag{OA11}$$

where here $\frac{dc_t^i(s^t)}{d\theta} = \overline{c_t^i}(s^t) - \underline{c_t^i}(s^t)$.

[56] Note that one could also define an alternative compensating variation as

$$\sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t(s^t|s_0)\, u_i\left(c_t^i(s^t)\right) = \sum_{t=0}^{T} (\beta_i)^t \sum_{s^t} \pi_t(s^t|s_0)\, u_i\left(\left(\overline{c_t^i}(s^t) + \theta\overline{\Delta c_t^i}(s^t)\right)(1+\lambda(\theta))\right).$$

Analogously, we can also compute the derivative of the LHS of Equation (OA10) as follows:

$$\frac{d\left(\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i\left(c_t^i\left(s^t\right)(1+\lambda(\theta))\right)\right)}{d\theta}=\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(c_t^i\left(s^t\right)(1+\lambda(\theta))\right)c_t^i\left(s^t\right)\lambda'(\theta)\,. \tag{OA12}$$

Hence, combining Equations (OA11) and (OA12) and solving for $\frac{d\lambda}{d\theta}=\lambda'(\theta)$, yields the marginal cost of business cycles, as defined in Alvarez and Jermann (2004). Formally, we can express $\frac{d\lambda}{d\theta}$ as follows

$$\begin{aligned}\frac{d\lambda}{d\theta}=\lambda'(\theta)&=\frac{\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i'\left((1-\theta)\underline{c_t^i}\left(s^t\right)+\theta\overline{c_t^i}\left(s^t\right)\right)\frac{dc_t^i\left(s^t\right)}{d\theta}}{\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(c_t^i\left(s^t\right)(1+\lambda(\theta))\right)c_t^i\left(s^t\right)}\\&=\sum_{t=0}^{T}\sum_{s^t}\omega_t^i\left(s^t\middle|\,s_0\right)\frac{dc_t^i\left(s^t\right)}{d\theta},\end{aligned} \tag{OA13}$$

where the second line shows how to reformulate $\frac{d\lambda}{d\theta}$ in terms of DS-weights given by

$$\omega_t^i\left(s^t\middle|\,s_0\right)=\frac{(\beta_i)^t\,\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(\overline{c_t^i}\left(s^t\right)+\theta\overline{\Delta c_t^i}\left(s^t\right)\right)}{\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(c_t^i\left(s^t\right)(1+\lambda(\theta))\right)c_t^i\left(s^t\right)}. \tag{OA14}$$

Additive compensation. Here, we would like to contrast the approach in Lucas (1987) to one that relies on a time-invariant compensating variation, expressed additively in terms of consumption at each date and history as follows:

$$\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i\left(c_t^i\left(s^t\right)+\lambda(\theta)\right)=\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i\left((1-\theta)\underline{c_t^i}\left(s^t\right)+\theta\overline{c_t^i}\left(s^t\right)\right).$$

In this case, we can follow the same steps as above to find the counterpart of Equation (OA13), which is given by

$$\begin{aligned}\frac{d\lambda}{d\theta}=\lambda'(\theta)&=\frac{\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i'\left((1-\theta)\underline{c_t^i}\left(s^t\right)+\theta\overline{c_t^i}\left(s^t\right)\right)\frac{dc_t^i\left(s^t\right)}{d\theta}}{\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(c_t^i\left(s^t\right)+\lambda(\theta)\right)c_t^i\left(s^t\right)}\\&=\sum_{t=0}^{T}\sum_{s^t}\omega_t^i\left(s^t\middle|\,s_0\right)\frac{dc_t^i\left(s^t\right)}{d\theta},\end{aligned} \tag{OA15}$$

where the second line shows how to reformulate $\frac{d\lambda}{d\theta}$ in terms of DS-weights given by

$$\omega_t^i\left(s^t\middle|\,s_0\right)=\frac{(\beta_i)^t\,\pi_t\left(s^t\middle|\,s_0\right)u_i'\left((1-\theta)\underline{c_t^i}\left(s^t\right)+\theta\overline{c_t^i}\left(s^t\right)\right)}{\sum_{t=0}^{T}(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(c_t^i\left(s^t\right)+\lambda(\theta)\right)}. \tag{OA16}$$

Comparison and implications. We focus on comparing Equations (OA13) and (OA15) in the case of $\theta=0$ — similar insights emerge when $\theta\neq0$. When $\theta=0$, Equations (OA14) and (OA16) become

$$\omega_t^i\left(s^t\middle|\,s_0\right)=\frac{(\beta_i)^t\,\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(\underline{c}_t^i\left(s^t\right)\right)}{\sum_{t=0}^T(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(\underline{c}_t^i\left(s^t\right)\right)\underline{c}_t^i\left(s^t\right)}\quad\text{(multiplicative}\Rightarrow\text{Lucas/Alvarez-Jermann)}\tag{OA17}$$

$$\omega_t^i\left(s^t\middle|\,s_0\right)=\frac{(\beta_i)^t\,\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(\underline{c}_t^i\left(s^t\right)\right)}{\sum_{t=0}^T(\beta_i)^t\sum_{s^t}\pi_t\left(s^t\middle|\,s_0\right)u_i'\left(\underline{c}_t^i\left(s^t\right)\right)}.\quad\text{(additive}\Rightarrow\text{normalized welfarist DS-planner)}\tag{OA18}$$

Two major insights emerge from Equations (OA17) and (OA18). First, the DS-weights defined for the additive case in Equation (OA18) exactly correspond to the dynamic and stochastic components of DS-weights for a normalized utilitarian planner, as defined in Equations (13) and (14). Second, note that the denominator of the DS-weights in the multiplicative case includes $\underline{c}_t^i\left(s^t\right)$ at all dates and histories. This captures the fact that the welfare assessment is computed as a fraction of consumption at each date and history, not in units of the consumption good. The presence of $\underline{c}_t^i\left(s^t\right)$ in the denominator is what complicates the aggregation of welfare assessments using the Lucas (1987) approach.

Relation to EV, CV, and CS. Finally, note that the analysis in this section illustrates how the marginal approach relates to the conventional approaches in classic demand theory: equivalent variation (EV), compensating variation (CV), and consumer surplus (CS).

The approach of Lucas (1987) and Alvarez and Jermann (2004), and the alternative version described in Footnote 56 are the dynamic counterpart of compensating and equivalent variations, expressed in proportional terms, in a dynamic stochastic environment. Hence, the analysis of this section shows that a DS-planner can be used to operationalize the counterpart of all three notions — either proportionally or additively — in dynamic stochastic environments. As expected, these considerations only matter away from the $\theta=0$ case. However, the consumer surplus approach yields the most straightforward approach to making global assessments, as explained in Section G.5.

G.3.5 Relation to welfare assessments that involve transfers

Finally, it is worth discussing how having the ability to costlessly transfer resources across individuals impact the welfare assessments of a DS-planner. To do so, we consider an environment in which a DS-planner has access to a set of transfers $T_i^i\left(s^t\right)$, so that individual's budget constraints have the form

$$c_t^i\left(s^t\right)=T_i^i\left(s^t\right)+\ldots.$$

In that case, it follows immediately that

$$\frac{dW^{DS}\left(s_0\right)}{dT_i^i\left(s^t\right)}=\omega_t^i\left(s^t\middle|\,s_0\right).$$

Hence, having transfers available will endogenously impose restrictions across the DS-weights of different individuals. For instance, a welfarist planner who can transfer resources freely across all individuals, at all dates and histories will equalize the DS-weights across all individuals, at all dates and histories. Given Proposition 2, this implies that this planner will only value aggregate efficiency. Similar conclusions can be reached when a DS-planner only has access to a subset of transfers.

G.3.6 Relation to existing welfare decompositions

Our paper is not the first to introduce a decomposition of welfare assessments in different components. In fact, most of the existing literature that applies welfare decompositions to specific environments follows versions of the decompositions introduced by Benabou (2002) and Floden (2001). There is also the more recent decomposition introduced by Bhandari et al. (2021). We discussed how our approach is related to both of these next.

Benabou (2002)/Floden (2001) The starting point for the Benabou (2002)/Floden (2001) approach is the (incorrect) presumption that the welfarist approach cannot distinguish the effects of policy that operate via efficiency, missing markets, and redistribution. Benabou (2002) explicitly writes:[57]

> "*I will also compute more standard social welfare functions, which are aggregates of (intertemporal) utilities rather than risk-adjusted consumptions. These have the clearly desirable property that maximizing such a criterion ensures Pareto efficiency. On the other hand, it will be seen that they cannot distinguish between the effects of policy that operate through its role as a substitute for missing markets, and those that reflect an implicit equity concern.*"

In this paper, we have shown that it is possible to distinguish — using standard Social Welfare Functions — the effects of policy that operate through efficiency, including in economies with missing markets, and redistribution/equity. As Benabou (2002) points out, his non-welfarist approach may conclude that Pareto-improving policies are undesirable. When staying within the welfarist class, our approach is trivially Paretian. When consider DS-planners outside of the welfarist class, our approach is precise in the way in which specific departures take place.

In terms of properties, it is evident that the Benabou (2002)/Floden (2001) approach does not satisfy Proposition 6, in which we show that all normalized welfarist planners conclude that the risk-sharing and intertemporal-sharing components are zero when markets are complete; Proposition 7, in which we show that all normalized welfarist planners conclude that intertemporal-sharing component is zero when individuals can freely trade a riskless bond; and Proposition 8, in which we show that different normalized welfarist planners exclusively disagree on the redistribution component. Their approach satisfies Proposition 9, in which we show that the efficiency components (aggregate

[57]The Benabou (2002)/Floden (2001) approach is based on first computing certainty-equivalent consumption levels for individuals and then building measures of inequality from the distribution of such certainty-equivalents.

efficiency, risk-sharing, and intertemporal-sharing) of the aggregate additive decomposition are invariant to monotonically increasing transformations of individual's lifetime utilities and positive affine (increasing linear) transformations of individual's instantaneous utilities. However, the Benabou (2002)/Floden (2001) approach satisfies Proposition 9 only because it is defined for environments in which all individuals have identical preferences, which are highly restrictive.

Bhandari et al. (2021) The approach introduced by Bhandari et al. (2021), considers the case of a utilitarian planner with arbitrary weights α_i. although it seems obvious to apply to general Social Welfare Functions. In contrast to Benabou (2002)/Floden (2001), the approach of Bhandari et al. (2021) is defined for a general dynamic stochastic economies in which individuals may have different preferences.

For simplicity, we consider a scenario in which there is a single consumption good. In this environment, Bhandari et al. (2021) propose to first decompose the consumption of a given individual at a given date and history as

$$c_t^i\left(s^t\right) = C \times w_i \times \left(1 + \varepsilon_t^i\left(s^t\right)\right), \tag{OA19}$$

where C captures aggregate consumption, w_i captures the share of individual i's consumption relative to the aggregate and $1+\varepsilon_t^i\left(s^t\right)$ captures any residual variation. While Equation (OA19) may resemble the individual multiplicative decomposition introduced in Lemma 1, it is conceptually different. First, and most importantly, the decomposition in Equation (OA19) decomposes consumption, $c_t^i\left(s^t\right)$, while the individual multiplicative decomposition introduced in Lemma 1 decomposes DS-weights, i.e., social marginal valuations, $\omega_t^i\left(s^t\right)$. Second, the term w_i in Equation (OA19) can heuristically be mapped to the individual component of our individual multiplicative decomposition, while the term $1+\varepsilon_t^i\left(s^t\right)$ can be heuristically mapped to both the dynamic and stochastic components.

Bhandari et al. (2021) then introduce a second-order Taylor expansion around a midpoint to write welfare differences (partially adopting the notation in that paper) as follows:

$$\mathcal{W}^B - \mathcal{W}^A \simeq \underbrace{\int \phi_i \Gamma di}_{\text{agg. efficiency}} + \underbrace{\int \phi_i \Delta_i di}_{\text{redistribution}} + \underbrace{\int \phi_i \gamma_i \Lambda_i di}_{\text{insurance}}, \tag{OA20}$$

where $\phi_i = \alpha_i \sum_t \sum_{s^t} \frac{\partial u_i\left(s^t\right)}{\partial c_t^i} c_t^i\left(s^t\right)$ denotes quasi-weights — using the terminology in Bhandari et al. (2021) — and γ_i is a measure of risk-aversion, $-c_t^i\left(s^t\right) \frac{\partial^2 u_i\left(s^t\right)}{\partial\left(c_t^i\right)^2} / \frac{\partial u_i\left(s^t\right)}{\partial c_t^i}$, and where $\Gamma = \ln C^B - \ln C^A$, $\Delta_i = \ln w_i^B - \ln w_i^A$, and $\Lambda_i = -\frac{1}{2}\left[\mathbb{V}ar_i\left[\ln c_i^B\right] - \mathbb{V}ar_i\left[\ln c_i^A\right]\right]$. It is then possible to decompose $\mathcal{W}^B - \mathcal{W}^A$ into three terms as follows:

$$1 = \underbrace{\frac{\int \phi_i \Gamma di}{\mathcal{W}^B - \mathcal{W}^A}}_{\text{agg. efficiency}} + \underbrace{\frac{\int \phi_i \Delta_i di}{\mathcal{W}^B - \mathcal{W}^A}}_{\text{redistribution}} + \underbrace{\frac{\int \phi_i \gamma_i \Lambda_i di}{\mathcal{W}^B - \mathcal{W}^A}}_{\text{insurance}}. \tag{OA21}$$

Bhandari et al. (2021) establish three properties of the decomposition in Equation (OA21): a) a welfare change that affects aggregate consumption C but not $\{w_i, \varepsilon_i\}_i$ is exclusively attributed to aggregate efficiency; b) a welfare change that affects expected shares $\{w_i\}_i$ but not C and $\{\varepsilon_i\}_i$ is exclusively attributed to redistribution; c) a welfare change that affects $\{\varepsilon_i\}_i$ but not C and $\{w_i\}_i$ is exclusively attributed to insurance.[58] These properties are conceptually the counterpart of Proposition 3, since they consider properties of a decomposition for particular policy changes. However, it should be evident that properties a), b), and c) in Bhandari et al. (2021) neither imply nor are implied by the properties that we establish in Proposition 3. This occurs because properties a), b), and c) consider proportional changes while Proposition 3 considers changes in levels of consumption, with both the proportional and level approaches being different but reasonable.[59]

However, more importantly, the decomposition of Bhandari et al. (2021) does not satisfy the counterparts of Proposition 6, in which we show that all normalized welfarist planners conclude that the risk-sharing and intertemporal-sharing components are zero when markets are complete; Proposition 7, in which we show that all normalized welfarist planners conclude that intertemporal-sharing component is zero when individuals can freely trade a riskless bond; Proposition 8, in which we show that different normalized welfarist planners exclusively disagree on the redistribution component; and Proposition 9, in which we show that the efficiency components (aggregate efficiency, risk-sharing, and intertemporal-sharing) of the aggregate additive decomposition are invariant to monotonically increasing transformations of individual's lifetime utilities and positive affine (increasing linear) transformations of individual's instantaneous utilities.

That is, it is possible to consider complete market economies in which the decomposition of Bhandari et al. (2021) attributes welfare changes to their insurance component. Also, it should be evident from Equation (OA21) that changing the Pareto weights α_i that a utilitarian planner assigns to an individual or simply multiplying the lifetime utility of a single individual by a constant factor — a transformation that has no impact on allocations — will change all three elements (aggregate efficiency, redistribution, insurance) of the decomposition introduced by Bhandari et al. (2021).[60] The are two choices that explain why the decomposition in Equation (OA21) does not

[58]The insurance component in Bhandari et al. (2021) is heuristically related to the risk-sharing and intertemporal-sharing components in our paper. Bhandari et al. (2021) also establish a fourth property, reflexivity, which our approach also satisfies.

[59]Formally, note that by writing $c_t^i\left(s^t\right) = C \times w_i \times \left(1 + \varepsilon_t^i\left(s^t\right)\right)$, we can express $\frac{du_{i|c}\left(s^t\right)}{d\theta}$ as follows:

$$\frac{du_{i|c}\left(s^t\right)}{d\theta} = \frac{dc_t^i\left(s^t\right)}{d\theta} = \frac{dC}{d\theta} \times w_i \times \left(1 + \varepsilon_t^i\left(s^t\right)\right) + C \times \frac{dw_i}{d\theta} \times \left(1 + \varepsilon_t^i\left(s^t\right)\right) + C \times w_i \times \frac{d\left(1 + \varepsilon_t^i\left(s^t\right)\right)}{d\theta}.$$

In this case, even when $\frac{dw_i}{d\theta} = \frac{d\left(1+\varepsilon_t^i\left(s^t\right)\right)}{d\theta} = 0$, a change in $\frac{dC}{d\theta}$, by virtue of being *proportional* to existing consumption, does not imply a uniform change in $\frac{du_{i|c}\left(s^t\right)}{d\theta}$ across individuals, dates, and histories, which are the changes considered in Proposition 3a). A similar logic applies to changes in $\frac{dw_i}{d\theta}$ and $\frac{d\varepsilon_t^i\left(s^t\right)}{d\theta}$. More generally, the decompositions yield different conclusions. For instance, the decomposition in Bhandari et al. (2021) attributes welfare gains associated to smoothing business cycles in a representative agent economy — as in Lucas (1987) — to insurance, while our decomposition attributes such gains to the aggregate insurance subcomponent of aggregate efficiency, as described in Section 6.1.

[60]Formally, it follows from the definition of ϕ_i above that a change in α_i or a linear transformation of utilities will

satisfy Propositions 6 through 9, which are central properties of our aggregate additive decomposition. First, the decomposition in Equation (OA19) does not ensure that the insurance component vanishes when individuals marginal rates of substitution are equalized across dates/states. Second, $\mathcal{W}^B - \mathcal{W}^A$ in Equation (OA20) (as well as ϕ_i) is expressed in utils, not consumption units.[61] Hence, changes in Pareto weights or utility transformations directly affect all the components of the decomposition, including aggregate efficiency and insurance in Equation (OA21). By introducing normalized DS-weights for welfarist planners, our approach confines the impact of varying SWF's or considering utility transformation to the redistribution component. Alternatively, directly specifying the individual component of DS-weights allows a DS-planner to directly modulate how the redistribution component is determined.

G.4 Optimal policy problems using DS-weights

Throughout most of the paper we have focused on how to make welfare assessments. Here, we show how it is straightforward to use DS-weights in the context of optimal policy problems, both in primal and in dual forms. To do so, we consider an environments in which a planner chooses a set of policy instruments $\boldsymbol{\tau}$ to maximize social welfare, which depends on allocations $\boldsymbol{X}(\boldsymbol{\tau})$. We consider two possibilities.

First, we consider a primal problem, in which a planner maximizes social welfare $W(\boldsymbol{X}(\boldsymbol{\tau}))$, subject to a set of implementability conditions, $\boldsymbol{H}(\boldsymbol{X}, \boldsymbol{\tau})$.[62] Consistent with Section 6.4, we assume that $W(\boldsymbol{X}(\boldsymbol{\tau}))$ corresponds to an instantaneous SWF. In this case, the planner solves

$$\min_{\boldsymbol{\lambda}} \max_{\boldsymbol{X}, \boldsymbol{\tau}} W(\boldsymbol{X}) + \boldsymbol{\lambda} \boldsymbol{H}(X, \boldsymbol{\tau}),$$

with optimality conditions for $\boldsymbol{\tau}$ given by

$$\frac{\partial W}{\partial \boldsymbol{X}} + \boldsymbol{\lambda} \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{X}} = 0. \tag{OA22}$$

Second, we consider a dual problem, in which a planner maximizes social welfare $W(\boldsymbol{X}^{\star}(\boldsymbol{\tau}))$, where $\boldsymbol{X}^{\star}(\tau)$ denotes the equilibrium mapping implicitly defined as $\boldsymbol{H}(\boldsymbol{X}^{\star}(\boldsymbol{\tau}), \boldsymbol{\tau}) = 0$. In this case, the planner solves

$$\max_{\boldsymbol{\tau}} W(\boldsymbol{X}^{\star}(\boldsymbol{\tau})),$$

with optimality conditions for $\boldsymbol{\tau}$ given

$$\frac{\partial W}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}^{\star}}{d\boldsymbol{\tau}} = 0. \tag{OA23}$$

change ϕ^i and consequently each of the three elements on the right-hand side of Equation (OA20).

[61] Bhandari et al. (2021) explain how $\mathcal{W}^B - \mathcal{W}^A$ is measured in utils as follows:

> *"Quasi-weights $\{\phi_i\}_i$ convert percent changes $\{\Gamma, \Delta_i, \Lambda_i\}_i$ that into a welfare change $\mathcal{W}^B - \mathcal{W}^A$, measured in utils."*

[62] While social welfare is a scalar, bold variables can be vectors/matrices.

In both cases, it is necessary to characterize $\frac{\partial W}{\partial \boldsymbol{X}}$ to find optimal policies. Hence, by defining $\frac{\partial W}{\partial \boldsymbol{X}}$ as in Definition 3, it is straightforward to find optimal policies for different DS-planners. As a final remark, note that, consistently with Section 6.4, it is important to understand that one cannot define a conventional SWF from the onset, DS-weights must be introduced at the marginal level in Equations (OA22) and (OA23).

G.5 Global welfare assessments

In the body of the paper, we have focused on marginal welfare assessments because there is no ambiguity about the welfare gains or losses of a policy when measured in units of a particular numeraire — see Schlee (2013) for a formal proof.[63] But one may still be interested in exploring the impact of non-marginal welfare assessments. It is well understood that even for a single individual there is no unambiguous approach to measure welfare gains or losses for non-marginal changes — see e.g., Silberberg (1972) or Mas-Colell, Whinston and Green (1995) — with the same logic extending to every component of the aggregate additive decomposition. This phenomenon is typically illustrated by the discrepancy between consumer surplus, equivalent variation, and compensating variation in classic demand theory. Despite this unavoidable hurdle, it is possible to make judicious global welfare assessments.

In practice, the easiest approach to study global policy changes is to parameterize policies using a line integral, as we illustrate in Scenarios 1 and 2 in Section 7. Assuming that policy changes can be scaled by $\theta \in [0, 1]$, where $\theta = 0$ corresponds to the status-quo and $\theta = 1$ corresponds to a global non-marginal change, it is possible to define a non-marginal welfare change as follows:

$$W^{DS}(s_0; \theta = 1) - W^{DS}(s_0; \theta = 0) = \int_0^1 \frac{dW^{DS}(s_0; \theta)}{d\theta} d\theta,$$

where θ is an explicit argument of $\frac{dW^{DS}(s_0;\theta)}{d\theta}$, which is given by

$$\frac{dW^{DS}(s_0; \theta)}{d\theta} = \int \sum_{t=0}^{T} \sum_{s^t} \omega_t^i \left(s^t \middle| s_0; \theta \right) \frac{du_{i|c}(s^t; \theta)}{d\theta} di. \tag{OA24}$$

That is, by recomputing $\frac{dW^{DS}(s_0;\theta)}{d\theta}$ along a particular path, it is possible to come up with a social welfare measure that is akin to consumer surplus, with the same logic applying to each of the components of the aggregate additive decomposition. While in principle using different paths may yield different answers when considering multidimensional policies even for identical start and end points, in practice it is often possible to find monotonic paths of integration, as defined by Zajac (1979) and Stahl (1984), which guarantees that the approach laid in Equation (OA24) yields globally consistent welfare assessments.

Two additional remarks are worth making. First, while the approach outlined here is the

[63] Schlee (2013) shows that the measures of consumer surplus, equivalent variation, and compensating variation are identical for marginal changes in a classical demand setup.

easiest to implement, it is possible to use the same methodology as Alvarez and Jermann (2004) to consider equivalent/compensating variation-like global assessments for welfarist planners within the DS-weights framework, although this will only be valid for aggregate assessments, not necessarily each of the components of the aggregate additive decomposition. Second, the potential for ambiguity of global assessments is not relevant if one is interested in using DS-planners to solve optimal policy problems, since $\frac{dW^{DS}(s_0)}{d\theta}$ is unambiguously defined for any policy perturbation. Hence, if there is a point at which $\frac{dW^{DS}(s_0)}{d\theta} = 0$ given the set of policy instruments, this will be a critical point and, under suitable second-order conditions, a local optimum. If there is single local optimum and it is possible to establish that the optimum is interior, this optimum will be global. If there are multiple local optima, one could use the value of the SWF to rank them in the welfarist case. So welfarist planners can unambiguously rank any two policies globally. Outside of the welfarist case, one can look for monotonic paths of integration (Zajac, 1979; Stahl, 1984) to rank different local optima, so it is only when this is not possible to find such paths that there may be some global ambiguity when ranking two particular policies.[64] In general, one can choose a set of reasonable policy paths (e.g., linear paths or bounded paths) and compare the predictions for the associated welfare assessments both in aggregate and for each of the elements of the aggregate additive decompositions.

G.6 Welfare assessments in economics with idiosyncratic/aggregate states

Until now, we have introduced our results in a canonical dynamic-stochastic model, following closely the notation of Chapter 8 in Ljungqvist and Sargent (2018). However, at times — in particular in Bewley-style economies — it is more convenient to work with a different notation that differentiates between idiosyncratic and aggregate states. We explain how to extend our framework to these environments, in which it is possible to derive new results. Our notation follows Krueger and Lustig (2010) whenever possible.

Environment We consider an economy populated by individuals that can be different for two different reasons at any point in time. First, we assume that individuals may be ex-ante heterogeneous, and we index this heterogeneity by i.[65] This form of heterogeneity is meant to capture immutable heterogeneity, for instance in terms of preferences. Second, we assume that individuals have different idiosyncratic states, so at a given point in time individuals that have in principle identical preferences may be different because they have a different idiosyncratic state.

In our economy there are aggregate and idiosyncratic states. We denote aggregate states by $z_t \in Z$ and idiosyncratic states by$y_t \in Y$. For simplicity, both Z and Y are assumed to be finite. We let $z^t = (z_0, \ldots, z_t)$ and $y^t = (y_0, \ldots, y_t)$ denote the history of aggregate and idiosyncratic states.

[64] Stahl (1984) proves that there always exist monotonic paths of integration in a classical demand context. While a formal proof of existence of such paths for the general framework considered here is outside of the scope of this paper, there is no reason to believe this result cannot be extended to natural applications.

[65] Importantly, the index i in this section, which indexes ex-ante heterogeneity, is completely different from the index i in the body of the paper, in which i indexes individuals. Formally, s^t in the body of the paper maps to z^t in this section, while i maps to the triple $\left\{i, y_0, y^t\right\}$.

States can be exogenous, in which case we refer to them as shocks, are they can be endogenous state variables (e.g., wealth or asset holdings). We denote the unconditional probability of transitioning from state y_0 given an initial aggregate state z_0 to a state (y^t, z^t) for an individual of ex-ante type i by $\pi_t^i(y^t, z^t | y_0, z_0)$. We assume that the economy starts at an initial aggregate state z_0, which a cross-sectional distribution of individuals represented by $dG(y_0, i)$, where $\iint dG(y_0, i) = 1$. Given our assumptions, at a give date t, there is a single aggregate state (of any dimension), but there are as many idiosyncratic states (of any dimension) of individuals in the economy.

The lifetime utility of an individual of type i, with initial idiosyncratic state y_0, given an aggregate state z_0, is given by

$$V_i(y_0, z_0) = \sum_{t=0}^{T} (\beta_i)^t \sum_{z^t} \sum_{y^t} \pi_t^i \left(y^t, z^t \middle| y_0, z_0 \right) u_i \left(c_t^i \left(y^t, z^t \right), n_t^i \left(y^t, z^t \right) \right),$$

where, for simplicity, we assume that β_i and $u_i(\cdot)$ are not functions of y^t and z^t. It is straightforward to extend our results to environments in which β_i and $u_i(\cdot)$ can be directly functions of y^t and z^t. Hence, we can express the change in the lifetime utility of an individual i with initial idiosyncratic state at a given initial aggregate state z_0 induced by a marginal policy change as follows:

$$\frac{dV_i(y_0, z_0)}{d\theta} = \sum_{t=0}^{T} (\beta_i)^t \sum_{z^t} \sum_{y^t} \pi_t^i \left(y^t, z^t \middle| y_0, z_0 \right) \frac{\partial u_i(y^t, z^t)}{\partial c_t^i} \frac{du_{i|c}(y^t, z^t)}{d\theta}.$$

In the case of a welfarist planner, the counterpart of Equation (6) is now

$$\begin{aligned} \frac{dW^{\mathcal{W}}(z_0)}{d\theta} &= \iint \lambda_i(y_0, z_0) \frac{dV_i(y_0, z_0)}{d\theta} dG(y_0, i) \\ &= \iint \lambda_i(y_0, z_0) \sum_{t=0}^{T} (\beta_i)^t \sum_{z^t} \sum_{y^t} \pi_t \left(y^t, z^t \middle| y_0, z_0 \right) \frac{\partial u_i(y^t, z^t)}{\partial c_t^i} \frac{du_{i|c}(y^t, z^t)}{d\theta} dG(y_0, i), \end{aligned}$$

where $\lambda_i(y_0, z_0) = \frac{\partial \mathcal{W}\left(\{V_i(y_0, z_0)\}_{i, y_0}\right)}{\partial V_i}$. Hence, a desirable policy change for a DS-planner, that is, the counterpart of Definition 2, is now based on

$$\frac{dW^{DS}(z_0)}{d\theta} = \iint \sum_{t=0}^{T} \sum_{z^t} \sum_{y^t} \omega_t^i \left(y^t, z^t \middle| y_0, z_0 \right) \frac{du_{i|c}(y^t, z^t)}{d\theta} dG(y_0, i),$$

where $\omega_t^i(y^t, z^t | y_0, z_0)$ denotes the DS-weight assigned to an individual of type i, whose idiosyncratic state at the time of the assessment is y_0, when the aggregate state at the time of the assessments is z_0, for a date t in which the idiosyncratic state of such individual is y^t and the aggregate state is z^t.

In this case, note that it is possible to define an individual multiplicative decomposition — the counterpart of Lemma 1 — that takes the form:

$$\omega_t^{i,y_0}\left(y^t, z^t\middle| z_0\right) = \underbrace{\tilde{\omega}^{i,y_0}(z_0)}_{\text{individual}}\underbrace{\tilde{\omega}_t^{i,y_0}(z_0)}_{\text{dynamic}}\underbrace{\tilde{\omega}_t^{i,y_0}\left(y^t, z^t|z_0\right)}_{\text{stochastic}}.$$

In this case, the individual multiplicative decomposition of a normalized welfarist planner — the counterpart of Proposition 5 — takes the form:

$$\tilde{\omega}_t^{i,y_0,\mathcal{W}}\left(y^t, z^t|z_0\right) = \frac{(\beta_i)^t \pi_t^i\left(y^t, z^t\middle| y_0, z_0\right)\frac{\partial u_i\left(y^t,z^t\right)}{\partial c_t^i}}{\sum_{z^t}\sum_{y^t}(\beta_i)^t \pi_t^i\left(y^t, z^t\middle| y_0, z_0\right)\frac{\partial u_i\left(y^t,z^t\right)}{\partial c_t^i}} = \frac{\pi_t^i\left(y^t, z^t\middle| y_0, z_0\right)\frac{\partial u_i\left(y^t,z^t\right)}{\partial c_t^i}}{\sum_{z^t}\sum_{y^t}\pi_t^i\left(y^t, z^t\middle| y_0, z_0\right)\frac{\partial u_i\left(y^t,z^t\right)}{\partial c_t^i}}$$

$$\tilde{\omega}_t^{i,y_0,\mathcal{W}}(z_0) = \frac{(\beta_i)^t\sum_{z^t}\sum_{y^t}\pi_t^i\left(y^t, z^t\middle| y_0, z_0\right)\frac{\partial u_i\left(y^t,z^t\right)}{\partial c_t^i}}{\sum_{t=0}^{T}\sum_{z^t}\sum_{y^t}(\beta_i)^t \pi_t^i\left(y^t, z^t\middle| y_0, z_0\right)\frac{\partial u_i\left(y^t,z^t\right)}{\partial c_t^i}}$$

$$\tilde{\omega}^{i,y_0,\mathcal{W}}(z_0) = \frac{\lambda_i(y_0,z_0)\sum_{t=0}^{T}\sum_{z^t}\sum_{y^t}(\beta_i)^t \pi_t^i\left(y^t, z^t\middle| y_0, z_0\right)\frac{\partial u_i\left(y^t,z^t\right)}{\partial c_t^i}}{\iint\lambda_i(y_0,z_0)\sum_{t=0}^{T}\sum_{z^t}\sum_{y^t}(\beta_i)^t \pi_t^i\left(y^t, z^t\middle| y_0, z_0\right)\frac{\partial u_i\left(y^t,z^t\right)}{\partial c_t^i}dG(y_0,i)}.$$

In this case, note that $\sum_{z^t}\sum_{y^t}\tilde{\omega}_t^{i,y_0,\mathcal{W}}\left(y^t, z^t\middle| z_0\right) = 1$, $\forall i$, $\forall y_0$; $\sum_t \tilde{\omega}_t^{i,y_0,\mathcal{W}}(z_0) = 1$, $\forall i$, $\forall y_0$; and $\iint \tilde{\omega}^{i,y_0,\mathcal{W}}(z_0)\,dG(y_0,i) = 1$. Interestingly, under mild assumptions, note that there is scope to further decompose the individual and stochastic components as follows:

$$\tilde{\omega}^{i,y_0}(z_0) = \underbrace{\tilde{\omega}^{i}(z_0)}_{\text{ex-ante}}\ \underbrace{\tilde{\omega}^{y_0|i}(z_0)}_{\text{state variable}} \qquad \text{(individual)} \qquad \text{(OA25)}$$

$$\tilde{\omega}_t^{i,y_0}\left(y^t, z^t|z_0\right) = \underbrace{\tilde{\omega}_t^{i,y_0}\left(z^t|z_0\right)}_{\text{aggregate}}\underbrace{\tilde{\omega}_t^{i,y_0}\left(y^t|z^t, z_0\right)}_{\text{idiosyncratic}}. \qquad \text{(stochastic)} \qquad \text{(OA26)}$$

The two sub-components of the individual component capture redistribution towards immutable ex-ante heterogeneity (indexed by i) and initial idiosyncratic state-variable heterogeneity (indexed by y_0). The two sub-components of the stochastic component will allow us to decompose the risk-sharing component into pure risk-sharing of idiosyncratic and risk-transfer of aggregate risk.

Proposition 19. *(Welfare assessments: aggregate additive decomposition) The aggregate welfare assessment of a DS-planner, $\frac{dW^{DS}(z_0)}{d\theta}$, can be decomposed into four components: i) an aggregate efficiency component, ii) a risk-sharing component, iii) an intertemporal-sharing component, and iv) a redistribution component, as follows:*

$$\frac{dW^{DS}(s_0)}{d\theta} = \underbrace{\sum_{t=0}^{T} \mathbb{E}_{i,y_0}\left[\tilde{\omega}_t^{i,y_0}(z_0)\right] \sum_{z^t} \mathbb{E}_{i,y^0,y^t}\left[\frac{\tilde{\omega}_t^{i,y_0}(y^t, z^t|z_0)}{G_t(y^t, z^t|y_0, z_0, i)}\right] \mathbb{E}_{i,y^0,y^t}\left[\frac{du_{i|c}(y^t, z^t)}{d\theta}\right]}_{=\Xi_{AE} \textit{ (Aggregate Efficiency)}}$$

$$+ \underbrace{\sum_{t=0}^{T} \mathbb{E}_{i,y_0}\left[\tilde{\omega}_t^{i,y_0}(z_0)\right] \sum_{z^t} \mathbb{C}ov_{i,y^0,y^t}\left[\frac{\tilde{\omega}_t^{i,y_0}(y^t, z^t|z_0)}{G_t(y^t, z^t|y_0, z_0, i)}, \frac{du_{i|c}(y^t, z^t)}{d\theta}\right]}_{=\Xi_{RS} \textit{ (Risk-sharing)}}$$

$$+ \underbrace{\sum_{t=0}^{T} \mathbb{C}ov_{i,y_0}\left[\tilde{\omega}_t^{i,y_0}(z_0), \sum_{z^t}\sum_{y^t} \tilde{\omega}_t^{i,y_0}\left(y^t, z^t|z_0\right) \frac{du_{i|c}(y^t, z^t)}{d\theta}\right]}_{=\Xi_{IS} \textit{ (Intertemporal-sharing)}}$$

$$+ \underbrace{\mathbb{C}ov_{i,y_0}\left[\tilde{\omega}^{i,y_0}(z_0), \sum_{t=0}^{T} (\beta_i)^t \sum_{z^t}\sum_{y^t} \pi_t^i\left(y^t, z^t \middle| y_0, z_0\right) \frac{\partial u_i(y^t, z^t)}{\partial c_t^i} \frac{du_{i|c}(y^t, z^t)}{d\theta}\right]}_{=\Xi_{RD} \textit{ (Redistribution)}},$$

where we denote by $G_t(y^t, z^t|y_0, z_0, i)$ the transition likelihood with which an individual i that starts at states y_0 and z_0 transitions to histories y^t and z^t at date t.

Typically, in applications, $G_t(y^t, z^t|y_0, z_0, i)$ will equal $\pi_t^i(y^t, z^t| y_0, z_0)$, but not always, for instance when agents have heterogeneous beliefs. The definition of intertemporal-sharing and redistribution are exactly identical to those in Proposition 1. The definitions of risk-sharing and aggregate efficiency, which crucially hinge on taking cross-sectional average and covariances conditional on the values of idiosyncratic states, need to be slightly adjusted to account for the fact that agents transition between different states.

Finally, note that by combining Equation (OA25) with the definition of Ξ_{RD}, it is possible to provide a subdecomposition of the redistribution term into three terms:

$$\Xi_{RD} = \mathbb{C}ov_{i,y_0}\left[\tilde{\omega}^i(z_0)\,\tilde{\omega}^{y_0|i}(z_0), \frac{dV_i^{DS}(y_0, z_0)}{d\theta}\right]$$

$$= \underbrace{\mathbb{E}_{i,y_0}\left[\tilde{\omega}^i(z_0)\right] \mathbb{C}ov_{i,y_0}\left[\tilde{\omega}^i(z_0), \frac{dV_i^{DS}(y_0, z_0)}{d\theta}\right]}_{\text{ex-ante redistribution}} + \underbrace{\mathbb{E}_{i,y_0}\left[\tilde{\omega}^{y_0|i}(z_0)\right] \mathbb{C}ov_{i,y_0}\left[\tilde{\omega}^{y_0|i}(z_0), \frac{dV_i^{DS}(y_0, z_0)}{d\theta}\right]}_{\text{state-variable redistribution}}$$

$$+ \underbrace{\mathbb{E}_{i,y_0}\left[\left(\tilde{\omega}^i(z_0) - \mathbb{E}_{i,y_0}\left[\tilde{\omega}^i(z_0)\right]\right)\left(\tilde{\omega}^{y_0|i}(z_0) - \mathbb{E}_{i,y_0}\left[\tilde{\omega}^{y_0|i}(z_0)\right]\right)\left(\frac{dV_i^{DS}(y_0, z_0)}{d\theta} - \mathbb{E}_{i,y_0}\left[\frac{dV_i^{DS}(y_0, z_0)}{d\theta}\right]\right)\right]}_{\text{ex-ante/state-variable coskewness redistribution}}.$$

A similar subdecomposition emerges combining Equation (OA26) with the definition of Ξ_{RS}. In this case

$$\Xi_{RS} = \sum_{t=0}^{T} \mathbb{E}_{i,y_0}\left[\tilde{\omega}_t^{i,y_0}\left(z_0\right)\right] \sum_{z^t} \mathbb{C}ov_{i,y^0,y^t}\left[\frac{\tilde{\omega}_t^{i,y_0}\left(y^t, z^t|z_0\right)}{G_t\left(y^t, z^t|y_0, z_0, i\right)}, \frac{du_{i|c}\left(y^t, z^t\right)}{d\theta}\right]$$

$$= \underbrace{\sum_{t=0}^{T} \mathbb{E}_{i,y_0}\left[\tilde{\omega}_t^{i,y_0}\left(z_0\right)\right] \sum_{z^t} \mathbb{C}ov_{i,y^0,y^t}\left[\frac{\tilde{\omega}_t^{i,y_0}\left(z^t|z_0\right)}{G_t\left(z^t|z_0\right)}, \frac{du_{i|c}\left(y^t, z^t\right)}{d\theta}\right]}_{\text{risk transfer}}$$

$$+ \underbrace{\sum_{t=0}^{T} \mathbb{E}_{i,y_0}\left[\tilde{\omega}_t^{i,y_0}\left(z_0\right)\right] \sum_{z^t} \mathbb{C}ov_{i,y^0,y^t}\left[\frac{\tilde{\omega}_t^{i,y_0}\left(y^t|z^t, z_0\right)}{G_t\left(y^t|z^t, z_0, y_0, i\right)}, \frac{du_{i|c}\left(y^t, z^t\right)}{d\theta}\right]}_{\text{idiosyncratic risk sharing}}$$

$$+ \underbrace{\sum_{t=0}^{T} \mathbb{E}_{i,y_0}\left[\tilde{\omega}_t^{i,y_0}\left(z_0\right)\right] \mathbb{E}_{i,y^0,y^t}\left[\begin{array}{c}\left(\frac{\tilde{\omega}_t^{i,y_0}\left(z^t|z_0\right)}{G_t\left(z^t|z_0\right)} - \mathbb{E}_{i,y^0,y^t}\left[\frac{\tilde{\omega}_t^{i,y_0}\left(z^t|z_0\right)}{G_t\left(z^t|z_0\right)}\right]\right) \times \left(\frac{\tilde{\omega}_t^{i,y_0}\left(y^t|z^t,z_0\right)}{G_t\left(y^t|z^t,z_0,y_0,i\right)} - \mathbb{E}_{i,y^0,y^t}\left[\frac{\tilde{\omega}_t^{i,y_0}\left(y^t|z^t,z_0\right)}{G_t\left(y^t|z^t,z_0,y_0,i\right)}\right]\right) \\ \times \left(\frac{du_{i|c}\left(y^t,z^t\right)}{d\theta} - \mathbb{E}_{i,y^0,y^t}\left[\frac{du_{i|c}\left(y^t,z^t\right)}{d\theta}\right]\right)\end{array}\right]}_{\text{risk coskewness}}.$$

where, under mild assumptions, we can define $G_t\left(y^t, z^t|y_0, z_0, i\right) = G_t\left(z^t|z_0\right) G_t\left(y^t|z^t, z_0, y_0, i\right)$.