Market Incompleteness and Exchange Rate Spill-over

Online Appendix

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# A Proof for Proposition 3, Conditions for Symmetric SDF-FX Pass-Through, Continuous Time

For expositional convenience, let us label the two conditions in the statement of the proposition as

**Assumption 1.** $[dZ_t^{(1)}, dY_t^{(1)}] = [dZ_t^{(2)}, dY_t^{(2)}] = 0.$

**Assumption 2.** $[dZ_t^{(1)}, dY_t^{(2)}] = [dZ_t^{(2)}, dY_t^{(1)}] = 0.$

*Proof.* (a) Substitute the pricing kernels into the first-order conditions. The first-order conditions for domestic bond holdings become:

$$\begin{aligned} 0 = \mathcal{A}[M_t^{(i)} P_t^{(i)}] &= P_t^{(i)} \mathcal{A}[M_t^{(i)}] + M_t^{(i)} \mathcal{A}[P_t^{(i)}] \\ 0 &= P_t^{(i)} M_t^{(i)} (-\mu_t^{(i)} + \frac{1}{2}(\sigma_t^{(i)})^2) + M_t^{(i)} P_t^{(i)} r_t^{(i)} \\ r_t^{(i)} &= \mu_t^{(i)} - \frac{1}{2}(\sigma_t^{(i)})^2. \end{aligned}$$

Note that:

$$\begin{aligned} dS_t^{i/0} &= d(\exp(s_t^{i/0})) = S_t^{i/0} ds_t^{i/0} + \frac{1}{2} S_t^{i/0} [ds_t^{i/0}, ds_t^{i/0}] dt \\ d(S_t^{i/0})^{-1} &= d(\exp(-s_t^{i/0})) = -(S_t^{i/0})^{-1} ds_t^{i/0} + \frac{1}{2}(S_t^{i/0})^{-1} [ds_t^{i/0}, ds_t^{i/0}] dt \end{aligned}$$

The first-order conditions for cross-country bond holdings become:

$$\begin{aligned} 0 &= \mathcal{A}[M_t^{(0)} (S_t^{i/0})^{-1} P_t^{(i)}] \\ &= M_t^{(0)} P_t^{(i)} \mathcal{A}[(S_t^{i/0})^{-1}] + M_t^{(0)} (S_t^{i/0})^{-1} \mathcal{A}[P_t^{(i)}] + P_t^{(i)} (S_t^{i/0})^{-1} \mathcal{A}[M_t^{(0)}] + P_t^{(i)} [dM_t^{(0)}, d(S_t^{i/0})^{-1}] \\ &= -\mu_t^{(0)} + \frac{1}{2}(\sigma_t^{(0)})^2 - \mathcal{A}[s_t^{i/0}] + \frac{1}{2}[ds_t^{i/0}, ds_t^{i/0}] + [dm_t^{(0)}, -ds_t^{i/0}] + r_t^{(i)} \end{aligned}$$

and

$$
\begin{aligned}
0 &= \mathcal{A}[M_t^{(i)} S_t^{i/0} P_t^{(0)}] \\
&= M_t^{(i)} S_t^{i/0} \mathcal{A}[P_t^{(0)}] + M_t^{(i)} P_t^{(0)} \mathcal{A}[S_t^{i/0}] + S_t^{i/0} P_t^{(0)} \mathcal{A}[M_t^{(i)}] + P_t^{(0)} [dM_t^{(i)}, dS_t^{i/0}] \\
&= -\mu_t^{(i)} + \frac{1}{2}(\sigma_t^{(i)})^2 + \mathcal{A}[s_t^{i/0}] + \frac{1}{2}[ds_t^{i/0}, ds_t^{i/0}] + [dm_t^{(i)}, ds_t^{i/0}] + r_t^{(0)}
\end{aligned}
$$

Take the sum and plug in the first set of first-order conditions,

$$
0 = [ds_t^{i/0}, ds_t^{i/0}] - [dm_t^{(0)} - dm_t^{(i)}, ds_t^{i/0}] \tag{OA1}
$$

Plug in the exchange rate dynamics

$$
ds_t^{i/0} = x_t^{(i)} dt + z_t^{(i)} (dm_t^{(0)} - dm_t^{(i)}) + y_t^{(i)} dY_t^{(i)},
$$

then Eq. (OA1) implies an equilibrium restriction on the relationship between $z_t^{(i)}$ and $y_t^{(i)}$

$$
0 = (y_t^{(i)})^2 + (2z_t^{(i)} - 1)\sigma_t^{(i)} \pi_{i,i,t} y_t^{(i)} + ((z_t^{(i)})^2 - z_t^{(i)})((\sigma_t^{(0)})^2 + (\sigma_t^{(i)})^2 - 2\rho_{0,i,t}\sigma_t^{(0)}\sigma_t^{(i)}). \tag{OA2}
$$

Hence, under symmetric pass-through, i.e., $z_t^{(1)} = z_t^{(2)} = z_t$, the cross-country Euler equations for country pairs (0, 1) and (0, 2) imply

$$
\begin{aligned}
0 &= (y_t^{(1)})^2 + (2z - 1)\sigma_t^{(1)} \pi_{1,1,t} y_t^{(1)} + (z^2 - z)((\sigma_t^{(0)})^2 + (\sigma_t^{(1)})^2 - 2\rho_{0,1,t}\sigma_t^{(0)}\sigma_t^{(1)}) \quad \text{(OA3)} \\
0 &= (y_t^{(2)})^2 + (2z - 1)\sigma_t^{(2)} \pi_{2,2,t} y_t^{(2)} + (z^2 - z)((\sigma_t^{(0)})^2 + (\sigma_t^{(2)})^2 - 2\rho_{0,2,t}\sigma_t^{(0)}\sigma_t^{(2)}) \quad \text{(OA4)}
\end{aligned}
$$

Similar to Eq. (OA1), the first-order conditions from the cross-country bond holdings between countries 1 and 2 imply

$$
0 = [ds_t^{1/2}, ds_t^{1/2}] - [dm_t^{(2)} - dm_t^{(1)}, ds_t^{1/2}]. \tag{OA5}
$$

Plug in the bilateral exchange rate dynamics

$$
ds_t^{1/2} = (x_t^{(1)} - x_t^{(2)})dt + (z_t^{(1)} - z_t^{(2)})dm_t^{(0)} - z_t^{(1)} dm_t^{(1)} + z_t^{(2)} dm_t^{(2)} + y_t^{(1)} dY_t^{(1)} - y_t^{(2)} dY_t^{(2)},
$$

then,

$$\begin{aligned}0 &= ((z_t^{(1)}-z_t^{(2)})(\sigma_t^{(0)}),-(z_t^{(1)}-1)\sigma_t^{(1)},(z_t^{(2)}-1)\sigma_t^{(2)})R_t((z_t^{(1)}-z_t^{(2)})(\sigma_t^{(0)}),-z_t^{(1)}\sigma_t^{(1)},z_t^{(2)}\sigma_t^{(2)})' \\ &+ (y_t^{(1)})^2+(y_t^{(2)})^2+(2z_t^{(1)}-1)\sigma_t^{(1)}y_t^{(1)}\pi_{1,1,t}+(2z_t^{(2)}-1)\sigma_t^{(2)}y_t^{(2)}\pi_{2,2,t}-2y_t^{(1)}y_t^{(2)}[dY_t^{(1)},dY_t^{(2)}],\end{aligned}$$

where $R_{i,j,t}=\rho_{i,j,t}$ denotes the matrix of SDF correlation. Under symmetric pass-through, that is

$$\begin{aligned}0 &= (0,-(z_t-1)(\sigma_t^{(1)}),(z_t-1)(\sigma_t^{(2)}))R(0,-z_t(\sigma_t^{(1)}),z_t(\sigma_t^{(2)}))' \\ &+ (y_t^{(1)})^2+(y_t^{(2)})^2+(2z_t-1)\sigma_t^{(1)}y_t^{(1)}\pi_{1,1,t}+(2z_t-1)\sigma_t^{(2)}y_t^{(2)}\pi_{2,2,t}-2y_t^{(1)}y_t^{(2)}[dY_t^{(1)},dY_t^{(2)}]\end{aligned}$$

Plug in Eq. (OA3) and Eq. (OA4), then we have

$$y_t^{(1)}y_t^{(2)}[dY_t^{(1)},dY_t^{(2)}]=(z_t-(z_t)^2)((\sigma_t^{(0)})^2+\rho_{1,2,t}\sigma_t^{(1)}\sigma_t^{(2)}-\rho_{0,1,t}\sigma_t^{(0)}\sigma_t^{(1)}-\rho_{0,2,t}\sigma_t^{(0)}\sigma_t^{(2)}) \tag{OA6}$$

Then, by Assumption 2,

$$\begin{aligned}[ds_t^{1/0},ds_t^{2/0}] &= [z_t(dm_t^{(0)}-dm_t^{(1)})+y_t^{(1)}dY_t^{(1)},z_t(dm_t^{(0)}-dm_t^{(2)})+y_t^{(2)}dY_t^{(2)}] \\ &= z_t^2[(-\sigma_t^{(0)}dZ_t^{(0)}+\sigma_t^{(1)}dZ_t^{(1)}),(-\sigma_t^{(0)}dZ_t^{(0)}+\sigma_t^{(2)}dZ_t^{(2)})]+y_t^{(1)}y_t^{(2)}[dY_t^{(1)},dY_t^{(2)}] \\ &= z_t((\sigma_t^{(0)})^2+\rho_{1,2,t}\sigma_t^{(1)}\sigma_t^{(2)}-\rho_{0,1,t}\sigma_t^{(0)}\sigma_t^{(1)}-\rho_{0,2,t}\sigma_t^{(0)}\sigma_t^{(2)}) \\ &= z_t[dm_t^{(0)}-dm_t^{(1)},dm_t^{(0)}-dm_t^{(2)}].\end{aligned}$$

Also note that

$$\begin{aligned}[ds_t^{i/0},ds_t^{i/0}] &= [z_t^{(i)}(dm_t^{(0)}-dm_t^{(i)})+y_t^{(i)}dY_t^{(i)},z_t^{(i)}(dm_t^{(0)}-dm_t^{(i)})+y_t^{(i)}dY_t^{(i)}] \\ &= (z_t^{(i)})^2((\sigma_t^{(0)})^2+(\sigma_t^{(i)})^2-2\rho_{0,i,t}\sigma_t^{(0)}\sigma_t^{(i)})+(y_t^{(i)})^2+2z_t^{(i)}(\sigma_t^{(i)})y_t^{(i)}\pi_{i,i,t}\end{aligned}$$

With Assumption 1 and symmetric pass-through,

$$\begin{aligned}[ds_t^{i/0},ds_t^{i/0}] &= z_t((\sigma_t^{(0)})^2+(\sigma_t^{(i)})^2-2\rho_{0,i,t}\sigma_t^{(0)}\sigma_t^{(i)}) \\ &= z_t[dm_t^{(0)}-dm_t^{(i)},dm_t^{(0)}-dm_t^{(i)}]\end{aligned}$$

Then, under Assumption 1, we have

$$\frac{[ds_t^{1/0},ds_t^{2/0}]}{\sqrt{[ds_t^{1/0},ds_t^{1/0}]}\sqrt{[ds_t^{2/0},ds_t^{2/0}]}}=\frac{[dm_t^{(0)}-dm_t^{(1)},dm_t^{(0)}-dm_t^{(2)}]}{\sqrt{[dm_t^{(0)}-dm_t^{(1)},dm_t^{(0)}-dm_t^{(1)}]}\sqrt{[dm_t^{(0)}-dm_t^{(2)},dm_t^{(0)}-dm_t^{(2)}]}}$$

(b) Without loss of generality, I focus on the case where $[dZ_t^{(1)}, dY_t^{(2)}] \neq 0$. I first decompose country 1's SDF shock $dZ_t^{(1)}$ into country 0's SDF shock $dZ_t^{(0)}$ and an residual component:

$$d\tilde{Z}_t^{(1)} = \frac{1}{\sqrt{1-\rho_{0,1,t}^2}}(dZ_t^{(1)} - \rho_{0,1,t} dZ_t^{(0)})$$

Then, I decompose the IM shock $dY_t^{(2)}$ into the residual shock $d\tilde{Z}_t^{(1)}$ and an orthogonal component:

$$d\tilde{Y}_t^{(2)} = \frac{1}{\sqrt{1-\kappa_t^2}}(dY_t^{(2)} - \kappa_t d\tilde{Z}_t^{(1)})$$

Next, I can rewrite the bilateral exchange rate movement between countries 0 and 2 as

$$ds^{2/0} = \tilde{x}_t^{(2)} dt - \frac{y_t^{(2)}\kappa_t}{\sigma_t^{(1)}\sqrt{1-\rho_{0,1,t}^2}} dm_t^{(1)} + \tilde{z}_t^{(0)} dm_t^{(0)} - z_t^{(2)} dm_t^{(2)} + \tilde{y}_t^{(2)} d\tilde{Y}_t^{(2)},$$

with

$$\begin{aligned}
\tilde{x}_t^{(2)} &= x_t^{(2)} - \frac{y_t^{(2)}\kappa_t}{\sigma_t^{(1)}\sqrt{1-\rho_{0,1,t}^2}}\mu_t^{(1)} + \frac{y_t^{(2)}\kappa_t\rho_{0,1,t}}{\sigma_t^{(0)}\sqrt{1-\rho_{0,1,t}^2}}\mu_t^{(0)} \\
\tilde{z}_t^{(0)} &= z_t^{(2)} + \frac{y_t^{(2)}\kappa_t\rho_{0,1,t}}{\sigma_t^{(0)}\sqrt{1-\rho_{0,1,t}^2}} \\
\tilde{y}_t^{(2)} &= y_t^{(2)}\sqrt{1-\kappa_t^2}
\end{aligned}$$

□