Online Appendix for

Strategic Complementarities in a Dynamic Model of Techonology Adoption: P2P Digital Payments

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Appendix for Online Publication

A Discretization and Computation of Equilibrium

In this section, we describe an algorithm to compute the equilibrium. It is based on finding a fixed point of the finite difference approximation of the HBJ equation and the Kolmogorov forward equation.

We define the discretization of the model as follows:

Definition 3. A discretized version of the model is defined by positive integers I, J which determine the time and space step sizes: $\Delta_t = \frac{T}{J-1}$ and $\Delta_x = \frac{U}{I-1}$. Thus $t \in \mathbb{T} \equiv \{\Delta_t(j-1) : j = 1, \ldots, J\}$ and $x(t) \in \mathbb{X} \equiv \{\Delta_x(i-1) : i = 1, \ldots, I\}$. The reflecting BM is replaced by a process with: $x(t + \Delta_t) = x(t) \pm \Delta_x$ each with probability $q = \frac{1}{2}\frac{\sigma^2 \Delta_t}{(\Delta_x)^2}/(1 - \nu\Delta_t)$, and $x(t + \Delta_t) = x(t)$ with probability $1 - 2q$ for $0 < x(t) < U$. If $x(t) = 0$ or $x(t) = U$, then $x(t + \Delta_t) = x(t)$, with prob. $1 - q$, and $x(t + \Delta_t) = \Delta_x$, or $x(t + \Delta_t) = U - \Delta_x$ with probability q. Agents die with probability $\nu\Delta_t$, and use a discount factor $(1 - \Delta_t r)$. The period flow of those that adopted the technology is $[\theta_0 + \theta_n N(t)]\, x(t)\Delta_t$. Agents that die are replaced by other whose x is drawn from a uniform discrete distribution with probabilities Δ_x/U for each x. For any $0 < \Delta_t < 1/(r + \nu)$, the value of J, and hence Δ_x must be chosen so that $0 < q \leq 1/2$. In this case the value functions v and a can be represented as a vector on $v \in \mathbb{R}^{I\times J}$, the distribution of non-adopters $m \in \mathbb{R}_+^{I\times J}$, threshold path $\bar{x} : \mathbb{T} \to \mathbb{X}$, and the path of the measure of adopters $N : \mathbb{T} \to [0, 1]^J$. The initial condition is given by $m_0 \in \mathbb{R}_+^I$ and the terminal value by $v_T \in \mathbb{R}_+^I$.

Next we derive and describe the decision problem in discrete time using HBJ, and later derive and describe the discrete time version of the Kolmogorov forward equation.

A.1 Finite Difference Computation of HJB for v, a Given N

In this section we derive the finite difference approximation for $a(x, t)$ given the path $N = \{N_j\}_{j=1}^J$.

$$\rho a_{ij} = x_i\left(\theta_0 + \theta_n N_j\right) + \frac{\sigma^2}{2}\left[\frac{a_{i+1,j} - 2a_{i,j} + a_{i-1,j}}{(\Delta_x)^2}\right] + \frac{a_{i,j} - a_{i,j-1}}{\Delta_t}$$

for $i = 2, 3, \dots, I-1$ and $j = 2, 3, \dots, J-1$, which can be rearranged to give:

$$a_{i,j-1} = \Delta_t \, x_i \, (\theta_0 + \theta_n N_j) + \frac{\sigma^2 \Delta_t}{2(\Delta_x)^2} \left[a_{i+1,j} - 2a_{i,j} + a_{i-1,j} \right] + a_{i,j} - \rho \Delta_t a_{i,j}$$

Thus we define:

$$p = \frac{\sigma^2}{2} \frac{\Delta_t}{(\Delta_x)^2} \frac{1}{(1 - \rho \Delta_t)} \tag{38}$$

and write:

$$a_{i,j-1} = \Delta_t \, x_i \, (\theta_0 + \theta_n N_j) + (1 - \rho \Delta_t) \left[p a_{i-1,j} + (1-2p) a_{i,j} + p a_{i+1,j} \right] \tag{39}$$

for $i = 2, 3, \dots, I-1$, and $j = 2, 1, J-1$, and

$$a_{1,j-1} = \Delta_t \, x_1 \, (\theta_0 + \theta_n N_j) + (1 - \rho \Delta_t) \left[(1-p) a_{1,j} + p a_{2,j} \right] \tag{40}$$

$$a_{I,j-1} = \Delta_t \, x_I \, (\theta_0 + \theta_n N_j) + (1 - \rho \Delta_t) \left[p a_{I-1,j} + (1-p) a_{I,j} \right] \tag{41}$$

for $j = 2, \dots, J-1$ and at the terminal time we impose:

$$a_{i,J} = a_{i,T} \text{ for } i = 1, 2, \dots, I \tag{42}$$

If we require that $p \in (0,1)$ and $1 - 2p \in (0,1)$ then

$$\begin{aligned} &\frac{1}{\Delta_t} = \frac{J-1}{T} > \rho \text{ and} \\ &\sigma \frac{\sqrt{\Delta_t}}{\sqrt{1 - \rho \Delta_t}} = \sigma \frac{\sqrt{T}}{\sqrt{J - 1 - \rho T}} < \Delta_x = \frac{U}{I-1} \end{aligned}$$

We will use $a_T = \tilde{a}$, i.e., the stationary equilibria $\tilde{a}$ given N_{ss} as:

$$\tilde{a}_i = \Delta_t \, x_i \, (\theta_0 + \theta_n N_{ss}) + (1 - \rho \Delta_t) \left[p \tilde{a}_{i-1} + (1-2p) \tilde{a}_i + p \tilde{a}_{i+1} \right] \tag{43}$$

for $i = 2, 3, \dots, I-1$ and

$$\tilde{a}_1 = \Delta_t \, x_1 \, (\theta_0 + \theta_n N_{ss}) + (1 - \rho \Delta_t) \left[(1-p) \tilde{a}_1 + p \tilde{a}_2 \right] \tag{44}$$

$$\tilde{a}_I = \Delta_t \, x_I \, (\theta_0 + \theta_n N_{ss}) + (1 - \rho \Delta_t) \left[p \tilde{a}_{I-1} + (1-p) \tilde{a}_I \right] \tag{45}$$

Now we set the equations for v using a. Following a similar derivation we get:

$$v_{i,j-1} = \max\left\{-c + a_{i,j}\,,\ (1-\rho\Delta_t)\left[pv_{i-1,j} + (1-2p)v_{i,j} + pv_{i+1,j}\right]\right\} \tag{46}$$

for $i = 2, 3, \ldots, I-1$, and $j = 2, 1, J-1$, and

$$v_{1,j-1} = \max\left\{-c + a_{1,j}\,,\ (1-\rho\Delta_t)\left[(1-p)v_{1,j} + pv_{2,j}\right]\right\} \tag{47}$$
$$v_{I,j-1} = \max\left\{-c + a_{I,j}\,, (1-\rho\Delta_t)\left[pv_{I-1,j} + (1-p)v_{I,j}\right]\right\} \tag{48}$$

for $j = 2, \ldots, J-1$ and at the terminal time we impose:

$$v_{i,J} = v_{i,T} \text{ for } i = 1, 2, \ldots, I$$

Given v and a we can compute $\bar{x}$, which correspond to an J dimensional array as:

$$\begin{aligned}
\bar{x}_j &= \min_{\{i=1,\ldots,I\}} \{x_i : v_{i,j} = -c + a_{i,j}\} \text{ for all } j = 1, 2, \ldots, J \\
\bar{i}_j &= \min_{\{i=1,\ldots,I\}} \{i : v_{i,j} = -c + a_{i,j}\} \text{ for all } j = 1, 2, \ldots, J \text{ so that} \\
\bar{x}_j &= x_{\bar{i}_j} \text{ for all } j = 1, 2, \ldots, J
\end{aligned}$$

We let $\mathbb{X}$ be the set:

$$\mathbb{X} = \left\{\{x_j\}_{j=1}^J : x_j = (i-1)\Delta_x \text{ each } i = 1, 2, \ldots I \text{ and } j = 1, 2, \ldots, J\right\}$$

We will use $v_T = \tilde{v}$, the stationary equilibria $\tilde{v}$ given $\tilde{a}$ as:

$$\tilde{v}_i = \max\left\{-c + \tilde{a}_i\,,\ (1-\rho\Delta_t)\left[p\tilde{v}_{i-1} + (1-2p)\tilde{v}_i + p\tilde{v}_{i+1}\right]\right\} \tag{49}$$

for $i = 2, 3, \ldots, I-1$ and

$$\tilde{v}_1 = \max\left\{-c + \tilde{a}_1\,,\ (1-\rho\Delta_t)\left[(1-p)\tilde{v}_1 + p\tilde{v}_2\right]\right\} \tag{50}$$
$$\tilde{v}_I = \max\left\{-c + \tilde{a}_I\,,\ (1-\rho\Delta_t)\left[p\tilde{v}_{I-1} + (1-p)\tilde{v}_I\right]\right\} \tag{51}$$

A.2 Finite Difference Approximation of KFE for m Given $\bar{x}$

In this section we derive the finite difference approximation for $m(x,t)$ given the path $\bar{x} = \{\bar{x}_j\}_{j=1}^J$. We let $\bar{i}_j$ the index for which $\bar{x}_j = x_{\bar{i}_j}$ for all j.

$$\frac{m_{i,j+1}-m_{i,j}}{\Delta_t} = \frac{\sigma^2}{2}\left[\frac{m_{i+1,j}-2m_{i,j}+m_{i-1,j}}{(\Delta_x)^2}\right] - \nu\left(m_{i,j}-\frac{1}{U}\right) \text{ for } i=2,3,\ldots,\bar{i}_j-1$$
$$m_{i,j+1} = 0 \text{ for } i=\bar{i}_j,\ldots,I$$

and $j = 1,2,\ldots,J$. We can rewrite the first equation as:

$$m_{i,j+1} = \frac{\sigma^2}{2}\frac{\Delta_t}{(\Delta_x)^2}\left[m_{i+1,j}-2m_{i,j}+m_{i-1,j}\right] - \nu\Delta_t\left(m_{i,j}-\frac{1}{U}\right) + m_{i,j} \text{ for } i=2,3,\ldots,\bar{i}_j-1$$
$$m_{i,j+1} = 0 \text{ for } i=\bar{i}_j,\ldots,I$$

Defining q as

$$q = \frac{\sigma^2}{2}\frac{\Delta_t}{(\Delta_x)^2}\frac{1}{(1-\nu\Delta_t)} \tag{52}$$

we can write it as:

$$m_{1,j+1} = (1-\nu\Delta_t)\left(qm_{2,j}+(1-q)m_{1,j}\right)+\nu\Delta_t\frac{1}{U} \tag{53}$$
$$m_{i,j+1} = (1-\nu\Delta_t)\left(qm_{i+1,j}+(1-2q)m_{i,j}+qm_{i-1,j}\right)+\nu\Delta_t\frac{1}{U} \text{ for } i=2,3,\ldots,\bar{i}_j-1 \tag{54}$$
$$m_{i,j+1} = 0 \text{ for } i=\bar{i}_j,\ldots,I \tag{55}$$

and $j = 1,2,\ldots,J$,

$$m_{i,1} = m_0(x_i) \text{ and } i=1,2,\ldots,I \tag{56}$$

Given m we can compute the corresponding N, i.e.:

$$N_j = 1-\left(\sum_{i=1}^{I} m_{i,j}\Delta_x - m_{1,j}\Delta_x/2 - m_{\bar{i}_j-1,j}\Delta_x/2\right) \text{ for } j=1,2,\ldots,J \tag{57}$$

This gives $\mathcal{N}(\bar{x};m_0)$.

There is also the corresponding stationary distribution for $\tilde{m}$, given the index $\bar{i}^{ss}$:

$$\tilde{m}_1 = (1-\nu\Delta_t)\left(q\tilde{m}_2+(1-q)\tilde{m}_1\right)+\nu\Delta_t\frac{1}{U}$$
$$\tilde{m}_i = (1-\nu\Delta_t)\left(q\tilde{m}_{i+1}+(1-2q)\tilde{m}_i+q\tilde{m}_{i-1}\right)+\nu\Delta_t\frac{1}{U} \text{ for } i=2,3,\ldots,\bar{i}^{ss}$$
$$\tilde{m}_i = 0 \text{ for } i=\bar{i}^{ss},\ldots,I$$

and

$$N_{ss} = 1 - \left(\sum_{i=1}^{I} \tilde{m}_i \Delta_x - \tilde{m}_1 \Delta_x / 2 - \tilde{m}_{\bar{i}^{ss}-1} \Delta_x / 2 \right)$$

A.3 Computing Equilibrium Set

In this section we set up the fixed point given an initial condition m_0 and terminal value functions $v_T = \tilde{v}$, $a_T = \tilde{a}$ and $D_T = a_T - v_T$ for some stationary equilibruim. Recall that $\mathcal{F} : [0,1]^J \to [0,1]^J$ is defined as in equation (6). Thus, successive paths for N are indexed by k and computed as

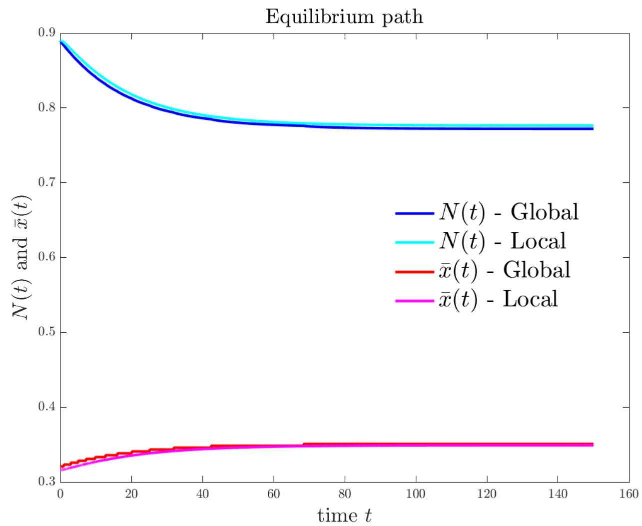
$$N^{k+1} = \mathcal{F}\left(N^k; m_0, D_T\right) \equiv \mathcal{N}\left(\mathcal{X}\left(N^k; D_T\right); m_0\right) \text{ for } k = 0, 1, 2, \ldots$$

for some initial condition N^0. To compute the equilibrium with the lowest path for N we start with the initial condition $N^0 = \{0, 0, \ldots, 0\}$. To compute the equilibrium with the highest path for N we start with the initial condition $N^0 = \{1, 1, \ldots, 1\}$. The convergence of N^k for large k is ensured by Tarski's theorem.

In Figure A1 we compare the computation that follows from discretizing time and state space with the one that comes from linearizing the model, i.e., our perturbation. Both computations start with the same initial conditions. For this figure we take as terminal value function corresponding to the stationary value function corresponding to the high adoption equilibrium, i.e., high value of N_{ss} and low value of $\bar{x}_{ss}$. The common initial condition is one where $m_0(x) = \tilde{m}(x)/2$. We make two remarks about the initial condition. First, it amounts to starting the economy with more agents with the technology than in corresponding stationary distribution (recall that $\tilde{m}$ is the density of the stationary distribution of agents without the technology). Second, the shock (deviation from the stationary distribution) is not a small one, hence the local perturbation might lose accuracy in principle.

The figure contains four lines. The two top lines display the computation of the path of N based on discretization (label as Global) with the one based on the perturbation (label as local). The two bottom lines display the computation of the path of $\bar{x}$ based on discretization (label as Global) with the one based on the perturbation (label as local). It is apparent that both methods gives very similar answer, i.e that the linearization is accurate for initial conditions far away from the stationary distribution. The other feature apparent with these computations is that the stationary equilibrium is stable even starting far away from the stationary distribution.

Figure A1: Global vs Local Solutions



B Proofs

Proof. (of Proposition 1).

As a preliminary step we establish a correspondence and inequality between sample paths of a Brownian Motion with reflected barriers 0 and U but with different initial conditions. In particular, we can write $x(t, \alpha)$ for each sample path α:

$$x(t, \alpha) = x(0, \alpha) + \sigma [W(\omega, t) - W(\omega, 0)] + u(t, \alpha) - d(t, \alpha)$$

where ω are the sample path of the standard Brownian Motion denoted by W, where $u(\cdot, \alpha)$ and $d(\cdot, \alpha)$ are increasing processes in each sample path, where $u(s, \alpha)$ only increases when $x(s, \alpha) = 0$, and where $d(s, \alpha)$ only increases when $x(s, \alpha) = U$ for $s \in [0, t]$. Consider any sample path α for which $x(0, \alpha) = x_1$ with a corresponding sample path ω for the standard Brownian Motion W. Then there is a corresponding sample path α' where $x(0, \alpha') = x_2$, and with $\omega = \omega'$ for W, i.e., the two sample paths correspond to the same path of W. Thus, these two sample paths occur with the same probability. From the last observation it follows that we can represent the sample path α by the pair $\omega, x(0)$, where $x(0) = x(0, \alpha)$. Finally, if $x_1 < x_2$, comparing these two sample paths we obtain $x(t, \alpha') \geq x(t, \alpha)$, i.e., we can pair the sample paths that start with different initial conditions and that occur with the same probability, and obtain that the one that starts at a higher value is (weakly) higher for all future times, and strictly higher for t small enough.

Now we turn to the main result. We proceed by contradiction, assuming that while it is optimal to adopt at (x_1, t), it is not optimal to adopt for (x_2, t) with $x_2 > x_1$. Without loss of generality we assume that $t = 0$. Our hypothesis imply that for all stopping times with $\tau_1 > 0$ it is not convenient to wait if $x(0) = x_1$, and thus

$$-c + \mathbb{E}\left[\int_0^\infty e^{-\rho t} x(t)\,(\theta_0 + \theta_n N(t))\,dt \,|\, x(0) = x_1\right] \geq \tag{58}$$
$$\mathbb{E}\left[-ce^{-\rho\tau_1} + \int_{\tau_1}^\infty e^{-\rho t} x(t)\,(\theta_0 + \theta_n N(t))\,dt \,|\, x(0) = x_1\right].$$

or equivalently that

$$-c + \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} x(t)\,(\theta_0 + \theta_n N(t))\,dt \,|\, x(0) = x_1\right] + c\,\mathbb{E}\left[e^{-\rho\tau_1} \,|\, x(0) = x_1\right] \geq 0\,.$$

Likewise, for $x(0) = x_2$ there exists a $\tau^* > 0$ for which it is optimal to wait:

$$-c + \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho t} x(t)\,(\theta_0 + \theta_n N(t))\,dt \,|\, x(0) = x_2\right] + c\,\mathbb{E}\left[e^{-\rho\tau^*} \,|\, x(0) = x_2\right] \leq 0\,.$$

We use the characterization for the sample paths described above, to construct a stopping time that only depends on the path ω as: $\tau_1(\omega, x_1) = \tau^*(\omega, x_2)$ for all ω. Using this equality, we immediately obtain $\mathbb{E}\left[e^{-\rho\tau_1} \,|\, x(0) = x_1\right] = \mathbb{E}\left[e^{-\rho\tau^*} \,|\, x(0) = x_2\right]$. Furthermore, using our characterization above for each path ω, we obtain:

$$\mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} x(t)\,(\theta_0 + \theta_n N(t))\,dt \,|\, x(0) = x_1\right] < \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} x(t)\,(\theta_0 + \theta_n N(t))\,dt \,|\, x(0) = x_2\right]$$
$$= \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho t} x(t)\,(\theta_0 + \theta_n N(t))\,dt \,|\, x(0) = x_2\right]$$

Using this strict inequality we get a contradiction with equation (58), and hence we establish the desired result. □

Proof. (of Lemma 1).

The proof is readily obtained by using the definitions $\hat{a}\,(z, t) \equiv \theta_0 a\,(zU, t)$ and $\hat{v}\,(z, t) \equiv \theta_0 v\,(zU, t)$. It is straightforward to verify that these functions satisfy the partial differential equations for $\hat{a}(z)$ and $\hat{v}(z)$ for $z \in (0, 1)$, including smooth pasting, value matching and boundary conditions. □

Proof. (of Proposition 2).

For this proof we set up the problem as a stopping time problem. We first prove a useful result in Lemma 4, showing that $\tau(N') \leq \tau(N)$ if $N' \geq N$. To convert the result on the monotonicity of the stopping times, into a result of the threshold $\bar{x}$, we note that the optimal decision rule is of the threshold type, as established in Proposition 1. We also show that exactly the same argument holds for the monotonicity with respect to θ. These results allow us to apply Topkis's (1978) theorem, which immediately establishes the proposition's result.

Next we set up the problem in terms of stopping times, and then state and prove Lemma 4.

$\square$

Decision Problem as Stopping Times. Fix $x_0 \in [0, U]$ and $t_0 \in [0, T]$. Let $N \in C([t_0, T]) = \{N : [t_0, T] \to [0, 1]\}$ and τ denote a stopping time. Let Ω denote the sample paths that start at time t_0 with $x(t_0) = x_0$. A set $\mathbb{L}^{t_0, x_0} = \{\tau : \Omega \to [t_0, T]\}$ is a lattice since $\min\{\tau_1, \tau_2\}$ and $\max\{\tau_1, \tau_2\}$ are stopping times.

Let $\omega \in \Omega$ be a sample path that corresponds to a continuation of (x_0, t_0) with measure $\mu(\cdot|x_0, t_0)$. We denote by $x(\cdot, \omega) : [t_0, T] \to [0, U]$ the sample path of the process for x that starts at $x(t) = x_0$. Then the objective function can be written as

$$F(\tau, N; x_0, t_0) = \int f(\tau(\omega), x(\cdot, \omega), N)\mu(d\omega|x_0, t_0)$$

where

$$f(\tau, x(\cdot, \omega), N; x_0, t_0) = \left[\int_\tau^T e^{-\rho t} x(t, \omega) \left[\theta_0 + \theta_n N(t)\right] dt - e^{-\rho\tau} c\right]$$

where $F : \mathbb{L}^{t_0, x_0} \times C([t_0, T]) \to \mathbb{R}$. We have the following important lemma:

Lemma 4. Let $\theta \equiv (\theta_0, \theta_n) \geq 0$ and fix (x_0, t_0). We establish three properties of $F(\tau, N; x_0, t_0)$: (i) it is submodular in τ; (ii) it has decreasing differences in (τ, N); (iii) it has decreasing differences in (τ, θ).

Proof. (of Lemma 4). Result (i): Submodularity in τ follows because F is additive across sample paths for all τ and τ'. We omit x_0, t_0 to simplify the notation. Fixing N we want to show:

$$F(\max\{\tau, \tau'\}, N) - F(\tau, N) \leq F(\tau', N) - F(\min\{\tau, \tau'\}, N)$$

which follows because for each sample path ω we have:

$$f(\max\{\tau, \tau'\}, N) - f(\tau, N) \leq f(\tau', N) - f(\min\{\tau, \tau'\}, N).$$

which holds since: $0 = f(\max\{\tau, \tau'\}, N) - f(\tau, N) - f(\tau', N) + f(\min\{\tau, \tau'\}, N)$.

Result (ii): We prove the submodularity of F, namely that given $\tau' > \tau$ and $N' > N$ we have

$$F(\tau', N') - F(\tau, N') \leq F(\tau', N) - F(\tau, N)$$

To this end consider $\tau'(\omega) \geq \tau(\omega)$ and compute:

$$F(\tau', N) - F(\tau, N) = \int \left(f(\tau', N) - f(\tau, N)\right) \mu(d\omega)$$

and for each ω

$$\begin{aligned} f(\tau', N, \omega) - f(\tau, N, \omega) &= \int_{\tau'}^{T} e^{-\rho t} \left[\theta_0 + \theta_n N(t)\right] x(t, \omega) dt - e^{-\rho \tau'} c \\ &- \left(\int_{\tau}^{T} e^{-\rho t} \left[\theta_0 + \theta_n N(t)\right] x(t, \omega) dt - e^{-\rho \tau} c \right) \\ &= - \int_{\tau}^{\tau'} e^{-\rho t} \left[\theta_0 + \theta_n N(t)\right] x(t, \omega) dt - e^{-\rho \tau'} c + e^{-\rho \tau} c. \end{aligned}$$

Thus, for all $N'(t) \geq N(t)$ and all t

$$\begin{aligned} &\left(f(\tau', N', \omega) - f(\tau, N', \omega)\right) - \left(f(\tau', N, \omega) - f(\tau, N, \omega)\right) \\ &= - \int_{\tau}^{\tau'} e^{-\rho t} \left[\theta_0 + \theta_n N'(t)\right] x(t, \omega) dt + \int_{\tau}^{\tau'} e^{-\rho t} \left[\theta_0 + \theta_n N(t)\right] x(t, \omega) dt \\ &= -\theta_n \int_{\tau}^{\tau'} e^{-\rho t} \left[N'(t) - N(t)\right] x(t, \omega) dt \leq 0 \end{aligned}$$

Thus

$$F(\tau', N') - F(\tau, N') - (F(\tau', N) - F(\tau, N)) = -\theta_n \int \left(\int_{\tau(\omega)}^{\tau'(\omega)} e^{-\rho t} \left[N'(t) - N(t)\right] x(t, \omega) dt \right) \mu(d\omega) \leq 0$$

Result (iii): Following the same steps followed in (ii) assuming $\theta' > \theta$ gives:

$$F(\tau', \theta') - F(\tau, \theta') - (F(\tau', \theta) - F(\tau, \theta)) = - \int \left(\int_{\tau(\omega)}^{\tau'(\omega)} e^{-\rho t} \left[(\theta_0' - \theta_0) + (\theta_n' - \theta_n) N(t)\right] x(t, \omega) dt \right) \mu(d\omega) \leq 0$$

□

Proof. (of Proposition 3) The fraction of agents that have not adopted at time t can be written as

$$M(t) \equiv \int_0^{\bar{x}(t)} m(z,t)dz = \int_0^U m_0(x)P(x,0,t)dx + \int_0^U \frac{\nu}{U} \int_0^t P(x,s,t)ds\,dx$$

where

$$P(x,s,t) = Pr\left[X(r) \leq \bar{x}(r), \text{ for all } r \in [s,t] \mid X(s) = x\right] e^{-\nu(t-s)} \tag{59}$$

where $X(\cdot)$ is a Brownian motion with reflecting barriers in $[0,U]$. Thus $P(x,s,t)$ is the fraction of agents that at time s have $X(s) = x$, survive until t, and also have had $X(r)$ below the threshold $\bar{x}(r)$ at all times $r \in [s,t]$. The first term in equation (59) is the fraction of those that have not adopted at in the initial distribution, and still have not adopted, and survive, at time t. The second term keeps tract of those cohort that have died at time s, and replaced by new agents, and themselves survive and not adopt up to time t.

Consider two paths $\bar{x}' \geq \bar{x}$ and the corresponding probabilities and measure of non-adopters $P'(x,s,t)$ and $M'(t)$ computed with $\bar{x}'$, and $P(x,s,t)$ and $M(t)$ computed with $\bar{x}$. The set of events $\{X(r) \leq \bar{x}(r), \text{ for all } r \in [s,t]\}$ is included in the set of events $\{X(r) \leq \bar{x}'(r), \text{ for all } r \in [s,t]\}$, since $\bar{x}(r) \leq \bar{x}'(r)$, and hence $P'(x,s,t) \geq P(x,s,t)$. Thus $M'(t) \geq M(t)$. Since $N'(t) = 1 - M'(t)$ and $N(t) = 1 - M(t)$, obtaining the desired result that $N'(t) \leq N(t)$.

The monotonicity with respect to m_0 follows immediately, since $\int_0^U m_0(x)P(x,0,t)dx$ is increasing in m_0 because $P(x,0,t)$ is non-negative.

□

Proof. (of Theorem 1) The proof uses Tarski's fixed point theorem for the function $\mathcal{F}$ as defined in equation (6). We restrict attention to the discrete time, discrete state version of the model so that we can we apply Tarski in a complete lattice.

We note that $\{N : \{0, \Delta_t, \ldots, T\} \to [0,1]\} = [0,1]^J$ where J is the integer that defines Δ_t. This set is a complete lattice. This function is monotone by virtue of Proposition 2 and Proposition 3. Then, Tarski's fixed point theorem implies that the set of fixed points is a lattice.

The comparative static result follows from the properties of the mapping $\mathcal{X}$ and $\mathcal{N}$ established in Proposition 2 and Proposition 3. □

Proof. (of Proposition 4) If an equilibrium without adoption exists, then $N(t) = N(0)e^{-\nu t}$, and hence if someone will adopt, it will adopt at time $t = 0$. Moreover, if someone will adopt it will be the one with $x = U$. Thus, we compute the value of $\underline{N}$ such that:

$$\begin{aligned}
c &= \mathbb{E}\left[\int_0^\infty e^{-\rho t} x(t)\left[\theta_0 + \theta_n N(t)\right] dt | x(0) = U\right] \\
&= \theta_0 \mathbb{E}\left[\int_0^\infty x(t) e^{-\rho t} dt | x(0) = U\right] + \theta_n N(0) \mathbb{E}\left[\int_0^\infty x(t) e^{-(\rho+\nu)t} dt | x(0) = U\right]
\end{aligned}$$

We note that $\tilde{a}(x;q) = \mathbb{E}\left[\int_0^\infty x(t) e^{-qt} dt | x(0) = x\right]$ solves the o.d.e. $q\tilde{a}(x) = 1 + \tilde{a}''(x)$ with boundary conditions $\tilde{a}'(0) = \tilde{a}'(U) = 0$. The solution of this o.d.e. is:

$$\begin{aligned}
\tilde{a}(x;q) &= \frac{1}{q}\left[x + \bar{A}_1 e^{\eta x} + \bar{A}_2 e^{-\eta x}\right] \\
\bar{A}_1 &\equiv \frac{1}{\eta} \frac{\left(1 - e^{-\eta U}\right)}{\left(e^{-\eta U} - e^{\eta U}\right)}, \ \bar{A}_2 \equiv \frac{1}{\eta} \frac{\left(1 - e^{\eta U}\right)}{\left(e^{-\eta U} - e^{\eta U}\right)} \text{ and } \eta \equiv \sqrt{2q/\sigma^2}
\end{aligned}$$

Evaluating $\tilde{a}(x;q)$ at $x = U$ we get:

$$\tilde{a}(U;q) = \frac{1}{q}\left[U - \frac{\coth(\eta U)}{\eta} + \frac{\operatorname{csch}(\eta U)}{\eta}\right]$$

Using this in the expression for $\underline{N}$ we obtain the desired expression. □

Proof. (of Proposition 5) First note that $x = U$ is a (non-interior) stationary equilibrium if, in case nobody adopts ($N = 0$), then those with $x = U$ find it optimal not to adopt, which is equivalent to $\theta_0 U < \rho c$.

An interior stationary equilibrium is equivalent to the zero of $q(x) \equiv (\theta_0 + \theta_n)x - (\rho c + x^2\theta_n/U)$ which belongs to $(0, U)$. Note that $q(0) = -\rho c < 0$. In case (i), we have $q(U) = \theta_0 U - \rho c > 0$. Thus there is only one interior solution belonging to $(0, U)$. In case (ii), we have $q(U) = \theta_0 U - \rho c < 0$. In this case, since $q(x)$ is quadratic it can have zero, one, or two solutions. Note that fixing an x we have three properties: (1) $\partial q(x)/\partial\theta_n = x\left(1 - x/U\right) > 0$ if $x \in (0, U)$, (2) $\theta_n = 0$ then, $q(x) = x\theta_0 - \rho c = U\left(\theta_0 x/U - \rho c/U\right) < U\left(\theta_0 - \rho c/U\right) < 0$, where the last inequality holds in case (ii), and (3) that for large enough θ_n then $q(x) = \theta_0 x - \rho c + x\theta_n(1 - x/U) > 0$ for $x \in (0, U)$. Hence, we can find a θ_n^* such that for $\theta_n \in [0, \theta_n^*)$ there is no interior root, for $\theta_n = \theta_n^*$ there is exactly one interior root, and for $\theta_n > \theta_n^*$ there are two interior roots. □

Proof. (of Lemma 2) The monotonicity of $\mathcal{X}_{ss}$ with respect to the parameters $\bar{\theta}_{ss} \equiv (\theta_0 + \theta_n N)/\rho$ is established in Appendix I.1. It is obtained by solving the o.d.e. for the value functions, and using the boundary conditions. It is clear that the optimal threshold, fixing η, solves an implicit equation $\psi(\gamma\bar{x}_{ss}) = \eta c/\bar{\theta}_{ss}$, where the function ψ is derived in Appendix I.1.

This function is strictly increasing, and satisfies $\psi(0) = 0$. Thus $\mathcal{X}_{ss}$ is strictly decreasing in $\bar{\theta}_{ss}$ and strictly increasing in c. A first order approximation of ψ gives the expansion used in the lemma. $\square$

Proof. (of Lemma 3) That $\mathcal{N}_{ss}$ is decreasing in $\bar{x}$ follows immediately since $\tanh(z)$ is, for positive z, concave and has $\tanh'(0) = 1$. Thus $\mathcal{N}_{ss}(\bar{x}) = \frac{1}{U}(-1 + \tanh(\bar{x}\gamma)) < 0$ if $\bar{x} > 0$.

That $\mathcal{N}_{ss}$ is strictly decreasing in γ follows from differentiating $\tanh(\bar{x}\gamma)/\gamma$ with respect to γ. This derivative is proportional to $-(\tanh(\bar{x}\gamma) - \bar{x}\gamma \text{sech}^2(\bar{x}\gamma)) = -(\tanh(\bar{x}\gamma) - \bar{x}\gamma \tanh'(\bar{x}\gamma)) < 0$, where we used that $\tanh(z)$ is strictly concave for $z > 0$. $\square$

Proof. (of Proposition 6). In the deterministic case, i.e., when $\sigma = 0$, there are at most two interior stationary equilibrium (the case we focus on). To simplify the notation let $N^o(\bar{x}_{ss}) \equiv \mathcal{X}_{ss}^{-1}(\bar{x}_{ss})$ and $N^a(\bar{x}_{ss}) \equiv \mathcal{N}_{ss}(\bar{x}_{ss})$. In each of the stationary equilibrium we write

$$N^a\left(\bar{x}^j(c)\right) = N^o\left(\bar{x}^j(c), c\right) \tag{60}$$

where $j = \{H, L\}$ (for high and low adoption, with $\bar{x}^H < \bar{x}^L$).

The functions N^a and N^o and their derivatives are continuous functions of $\bar{x}_{ss}, \sigma, c, \theta_0$. In each of the stationary equilibrium the functions N^a and N^o have strictly different slopes. Some analysis shows that the functions N^a, N^o intersect twice, and the derivative of $N^a - N^o$ with respect to $\bar{x}_{ss}$ is positive when the curves intersect at $\bar{x}_{ss}^H$ and negative when the curves intersect at the $\bar{x}_{ss}^L$. We summarize this by writing $N_{\bar{x}}^a(\bar{x}_{ss}^H) - N_{\bar{x}}^o(\bar{x}_{ss}^H) > 0$ while the derivative is negative at $\bar{x}_{ss}^L$.

Note that c does not enter in N^a. Differentiating equation (60) with respect to c:

$$\left[N_{\bar{x}}^a\left(\bar{x}(c)\right) - N_{\bar{x}}^o\left(\bar{x}(c), c\right)\right] \frac{\partial \bar{x}(c)}{\partial c} = N_c^o\left(\bar{x}(c), c\right) > 0$$

and again using the properties of each stationary equilibrium:

$$\frac{\partial \bar{x}_{ss}^H}{\partial c} > 0 > \frac{\partial \bar{x}_{ss}^L}{\partial c}$$

Following exactly the same steps we get:

$$\frac{\partial \bar{x}_{ss}^L}{\partial \theta_0} > 0 > \frac{\partial \bar{x}_{ss}^H}{\partial \theta_0}$$

$\square$

C Perturbation of the Stationary Equilibrium

We study the evolution of the MFG where the initial condition is given by a small perturbation ϵ of the stationary distribution:

$$m_0(x) = \tilde{m}(x) + \epsilon\, \omega(x) \ . \tag{61}$$

We consider an equilibrium with $\{\bar{x}(t,\epsilon), N(t,\epsilon), D(x,t,\epsilon), m(x,t,\epsilon)\}$. We will linearize this equilibrium with respect to ϵ and evaluate it at $\epsilon = 0$. For all $t \in [0,T]$, we denote these derivatives as follows:

$$\begin{aligned}
p(x,t) &\equiv \frac{\partial}{\partial \epsilon} m(x,t,\epsilon)\bigg|_{\epsilon=0} \\
d(x,t) &\equiv \frac{\partial}{\partial \epsilon} D(x,t,\epsilon)\bigg|_{\epsilon=0} \\
n(t) &\equiv \frac{\partial}{\partial \epsilon} N(t,\epsilon)\bigg|_{\epsilon=0} \\
\bar{y}(t) &\equiv \frac{\partial}{\partial \epsilon} \bar{x}(t,\epsilon)\bigg|_{\epsilon=0}
\end{aligned}$$

C.1 Linearization and Solution of the KB Equation

We differentiate $D(x,t,\epsilon)$ with respect to ϵ at each (x,t) to obtain $d(x,t)$ which solves the following p.d.e

$$\rho d(x,t) = x\theta_n n(t) + \frac{\sigma^2}{2} d_{xx}(x,t) + d_t(x,t) \tag{62}$$

for $x \in [0, \bar{x}_{ss}]$ and $t \in [0,T]$. The boundary conditions are obtained by differentiating the boundaries in equation (10) with respect to ϵ. This gives:

$$\begin{aligned}
d(\bar{x}_{ss}, t) &= 0 \\
\tilde{D}_{xx}(\bar{x}_{ss})\bar{y}(t) + d_x(\bar{x}_{ss}, t) &= 0 \\
d_x(0,t) &= 0
\end{aligned} \tag{63}$$

for $t \in [0,T]$ and $d(x,T) = 0$ for $x \in [0, \bar{x}_{ss}]$. Note that equation (63) defines $\bar{y}(t)$ and that $\tilde{D}_{xx}(\bar{x}_{ss}) = \tilde{a}_{xx}(\bar{x}_{ss}) - \tilde{v}_{xx}(\bar{x}_{ss}) < 0$.

Taking the derivative of the solution for $d(x,t)$ in equation (62) with respect to x and combining it with equation (63) we find

$$\bar{y}(t) = \frac{\theta_n}{\tilde{D}_{xx}(\bar{x}_{ss})} \int_t^T G(\tau - t) n(\tau) d\tau \tag{64}$$

where $G(s) \equiv \sum_{j=0}^{\infty} c_j e^{-\psi_j s} \geq 0$ for $s \geq 0$, $\psi_j \equiv \rho + \frac{\sigma^2}{2} \left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}} \right)^2$, and $c_j \equiv 2\left(1 - \frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right)$. An important property of this is that, since $G(s) \geq 0$ and $\tilde{D}_{xx}(\bar{x}_{ss}) < 0$, an increase in future adoption of the technology (i.e., future values of $n(\tau) > 0$ for $\tau > t$), then the threshold for adoption is smaller (i.e., more people will adopt today). Next we provide details of the solution of the p.d.e. for d. We have

Lemma 5. The solution for the KBE equation for d, satisfying the p.d.e. in equation (62), and the boundary conditions in equation (63), is given by

$$d(x,t) = \sum_{j=0}^{\infty} \varphi_j(x) \hat{d}_j(t) \qquad \text{for } x \in [0, \bar{x}_{ss}] \text{ and } t \in [0,T]$$

where for all $j = 1, 2, ...$ we have:

$$\begin{aligned}
\varphi_j(x) &\equiv \sin\left(\left(\frac{1}{2} + j\right)\pi\left(1 - \frac{x}{\bar{x}_{ss}}\right)\right) && \text{for } x \in [0, \bar{x}_{ss}] \\
\hat{d}_j(t) &\equiv \int_t^T e^{-\psi_j(\tau - t)} \hat{z}_j(\tau) d\tau && \text{for } t \in [0,T] \\
\hat{z}_j(t) &\equiv \theta_n n(t) \frac{\langle \varphi_j, x \rangle}{\langle \varphi_j, \varphi_j \rangle} = \theta_n n(t) \frac{2\bar{x}_{ss}}{\left(\frac{1}{2}+j\right)\pi} \left(1 - \frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right) && \text{for } t \in [0,T] \\
\text{where } \psi_j &\equiv \rho + \frac{\sigma^2}{2}\left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2 \quad \text{and} \quad \hat{d}_j(T) = 0
\end{aligned}$$

where $\langle \varphi_j, h \rangle \equiv \int_0^{\bar{x}_{ss}} h(x) \varphi_j(x) dx$. The proof can be done by verifying that the equation holds at the boundaries, and that for $t > 0$ the p.d.e in equation (62) holds in the interior since $\partial_{xx}\varphi_j(x) = -\left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2 \varphi_j(x)$, and $\partial_t \hat{d}_j(t) = \psi_j \hat{d}_j(t) - \hat{z}_j(t)$ for $t \in [0,T]$ and $j = 1, 2, ...$, and since the $\{\varphi_j(x)\}$ form an orthogonal basis for functions. Note finally that the boundary holds at $t = 0$ for $x \in [0, \bar{x}_{ss}]$, and that the derivative of the solution for d, used to solve for $\bar{y}$ in equation (63), is

$$d_x(\bar{x}_{ss}, t) = -\theta_n \int_t^T \sum_{j=0}^{\infty} c_j e^{-\psi_j(s-t)} n(s) ds \quad \text{where } \; c_j \equiv 2\left(1 - \frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right) \; .$$

C.2 Linearization and Solution of the KF Equation

We differentiate the KFE for $m(x,t,\epsilon)$ with respect to ϵ at each (x,t) to obtain:

$$p_t(x,t) = \frac{\sigma^2}{2} p_{xx}(x,t) - \nu p(x,t) \tag{65}$$

for $x \in [0, \bar{x}_{ss}]$ and $t \in [0,T]$.

Differentiating the boundary conditions $m(\bar{x}(t,\epsilon),t,\epsilon) = 0$ and $m_x(0,t,\epsilon) = 0$ with respect to ϵ we get

$$\begin{aligned} \tilde{m}_x(\bar{x}_{ss})\bar{y}(t) + p(\bar{x}_{ss},t) &= 0 \\ p_x(0,t) &= 0 \end{aligned} \tag{66}$$

The initial condition comes from differentiating $m_0(x)$ with respect to ϵ

$$p(0,x) = \omega(x) \tag{67}$$

The solution for p satisfies the p.d.e given in equation (65), its boundary conditions in equation (66), and the initial condition in equation (67). We have

Lemma 6. The solution for the KFE equation for p, satisfying the p.d.e given in equation (65), the boundary conditions in equation (66), and the initial condition in equation (67), is given by

$$\begin{aligned} p(x,t) &= \sum_{j=0}^{\infty} \varphi_j(x)\hat{p}_j(t) + r(t) && \text{for } x \in [0,\bar{x}_{ss}] \text{ and } t \in [0,T] \\ r(t) &\equiv -\tilde{m}_x(\bar{x}_{ss})\,\bar{y}(t) && \text{for } t \in [0,T] \end{aligned}$$

where for all $j = 1, 2, \ldots$ we have:

$$\begin{aligned} \hat{p}_j(t) &\equiv \hat{p}_j(0)e^{-\mu_j t} + \int_0^t e^{-\mu_j(t-\tau)}\hat{q}_j(\tau)d\tau && \text{for } t \in [0,T] \\ \hat{q}_j(t) &\equiv -(r'(t) + \nu r(t))\frac{\langle 1, \varphi_j\rangle}{\langle \varphi_j, \varphi_j\rangle} && \text{for } t \in [0,T] \\ \varphi_j(x) &\equiv \sin\left(\left(\frac{1}{2} + j\right)\pi\left(1 - \frac{x}{\bar{x}_{ss}}\right)\right) && \text{for } x \in [0,\bar{x}_{ss}] \end{aligned}$$

$$\text{where } \hat{p}_j(0) = \frac{\langle \varphi_j, \omega - r(0)\rangle}{\langle \varphi_j, \varphi_j\rangle} \quad \text{and} \quad \mu_j \equiv \nu + \frac{\sigma^2}{2}\left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2$$

where $\langle \varphi_j, h \rangle \equiv \int_0^{\bar{x}_{ss}} h(x)\varphi_j(x)dx$. The proof can be done by verifying that the equations hold at the boundaries, that for $t > 0$ the p.d.e holds in the interior since

$$\hat{p}_j'(t) = -\mu_j \hat{p}_j(t) + \hat{q}_j(t) \qquad\qquad \text{for } t \in [0, T] \text{ and } j = 1, 2, ...$$

and since $\{\varphi_j(x)\}$ form an orthogonal bases for functions, and finally that the boundary holds at $t = 0$ for $x \in [0, \bar{x}_{ss}]$, and it holds at $x = \bar{x}_{ss}$ for every $0 < t < T$

Given $p(x,t)$ we can compute $n(t)$ as:

$$\begin{aligned} n(t) &= -\int_0^{\bar{x}_{ss}} p(x,t)dx \\ &= n_0(t) + \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2}{\bar{x}_{ss}} \int_0^t J(t-\tau)\bar{y}(\tau)d\tau \end{aligned} \tag{68}$$

where $J(s) = \sum_{j=0}^{\infty} e^{-\mu_j s}$ with $\mu_j = \nu + \frac{1}{2}\sigma^2 \left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2$ and $n_0(t) \equiv -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}}{\pi(\frac{1}{2}+j)} \frac{\langle \varphi_j, \omega \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t}$.

C.3 Equilibrium in the Perturbed MFG

Recall that from equation (64), $\bar{y}(t)$ is equal to

$$\bar{y}(t) = \frac{\theta_n}{\tilde{D}_{xx}(\bar{x}_{ss})} \int_t^T G(\tau - t)n(\tau)d\tau$$

where $G(s) \equiv \sum_{j=0}^{\infty} c_j e^{-\psi_j s}$ for $s \geq 0$. From equation (68) we also know that $n(t)$ is

$$n(t) = n_0(t) + \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2}{\bar{x}_{ss}} \int_0^t J(t-\tau)\bar{y}(\tau)d\tau$$

where $J(s) = \sum_{j=0}^{\infty} e^{-\mu_j s}$ and $n_0(t) \equiv -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}}{\pi(\frac{1}{2}+j)} \frac{\langle \varphi_j, \epsilon \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t}$. Combining equation (64) and equation (68) we get

$$\begin{aligned} n(t) &= n_0(t) + \Theta(\bar{x}_{ss}) \int_0^t \int_\tau^T J(t-\tau)\bar{G}(s-\tau)n(s)dsd\tau \\ &= n_0(t) + \Theta(\bar{x}_{ss}) \int_0^T \int_0^{\min\{s,t\}} J(t-\tau)G(s-\tau)n(s)dsd\tau \\ &= n_0(t) + \Theta(\bar{x}_{ss}) \int_0^T K(t,s)n(s)ds \end{aligned}$$

where $K(t,s) = \int_0^{\min\{s,t\}} J(t-\tau)\bar{G}(s-\tau)d\tau$ and $\Theta(\bar{x}_{ss}) \equiv \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2\theta_n}{\bar{x}_{ss}\tilde{D}_{xx}(\bar{x}_{ss})}$. Using the definitions of $J(s)$ and $G(s)$ we find

$$
\begin{aligned}
K(t,s) &= \int_0^{\min\{s,t\}} J(t-\tau)G(s-\tau)d\tau \\
&= \int_0^{\min\{s,t\}} \left(\sum_{j=0}^{\infty} e^{-\mu_j(t-\tau)}\right)\left(\sum_{j=0}^{\infty} c_j e^{-\psi_j(s-\tau)}\right) d\tau \\
&= \sum_{j=0}^{\infty}\sum_{j=0}^{\infty} c_j e^{-\mu_i t-\psi_j s} \int_0^{\min\{s,t\}} e^{(\mu_i+\psi_j)\tau} d\tau \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j e^{-\mu_i t-\psi_j s} \left[\frac{e^{(\mu_i+\psi_j)\min\{t,s\}}-1}{\mu_i+\psi_j}\right].
\end{aligned}
$$

Note that $K(t,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j \left[\frac{1-e^{-(\mu_i+\psi_j)t}}{\mu_i+\psi_j}\right]$.

To calculate the Lipschitz bound $\text{Lip}_K \equiv \sup_{t\in[0,T]} \int_0^T |K(t,s)|ds$, let

$$
\kappa_{ij}(t) \equiv \int_0^T e^{-\mu_i t-\psi_j s}(e^{(\mu_i+\psi_j)\min\{t,s\}}-1)
$$

so that

$$
\int_0^T K(t,s)ds = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j \frac{\kappa_{ij}(t)}{\mu_i+\psi_j}.
$$

Computing the integrals in $\kappa_{ij}(t)$ we get

$$
\begin{aligned}
\kappa_{ij}(t) &= \int_0^t e^{-\mu_i t-\mu_i s}ds + \int_t^T e^{-\psi_j t-\psi_j s}ds - \int_0^T e^{-\mu_i t-\psi_j s}ds \\
&= \frac{e^{-\mu_i t}(e^{\mu_i t}-1)}{\mu_i} + \frac{e^{\psi_j t}(e^{-\psi_j T}-e^{-\psi_j t})}{-\psi_j} - \frac{e^{-\mu_i t}(e^{-\psi_j T}-1)}{-\psi_j} \\
&= \left(\frac{\psi_j+\mu_i}{\psi_j\mu_j}\right)(1-e^{-\mu_i t}) + e^{-\psi_j T}(e^{-\mu_i t}-e^{\psi_j t})
\end{aligned}
$$

and as $T\to\infty$

$$
\begin{aligned}
\kappa_{ij}(t) &= \left(\frac{\psi_j+\mu_i}{\psi_j\mu_j}\right)(1-e^{-\mu_i t}) \\
&\leq \frac{\psi_j+\mu_i}{\psi_j\mu_i}.
\end{aligned}
$$

Using that $\int_0^T |K(t,s)|ds \leq \int_0^\infty |K(t,s)|ds$ we get

$$\begin{aligned}\int_0^T K(t,s)ds &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j \frac{\kappa_{ij}(t)}{\mu_i+\psi_j} \\ &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j \frac{1}{\mu_i\psi_j} \\ &= \left(\sum_{i=0}^{\infty}\frac{1}{\mu_i}\right)\left(\sum_{j=0}^{\infty}\frac{c_j}{\psi_j}\right).\end{aligned}$$

We can use the definitions of μ_j, ψ_j, and c_j to further simplify this expression. First note that

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{1}{\mu_i} &= \sum_{i=0}^{\infty}\frac{1}{\nu+\frac{1}{2}\sigma^2\left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2} \\ &\leq \frac{2\bar{x}_{ss}^2}{\sigma^2}\sum_{i=0}^{\infty}\frac{1}{\left(\pi(\frac{1}{2}+j)\right)^2} \\ &= \frac{\bar{x}_{ss}^2}{\sigma^2}\end{aligned}$$

where we obtain the bound for $\nu = 0$. Notice also that

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{c_j}{\psi_j} &= \sum_{j=0}^{\infty}\frac{2\left(1-\frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right)}{\rho+\frac{1}{2}\sigma^2\left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2} \\ &\leq \frac{4\bar{x}_{ss}^2}{\sigma^2}\sum_{j=0}^{\infty}\frac{\left(1-\frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right)}{\left(\pi(\frac{1}{2}+j)\right)^2} \\ &= \frac{4\bar{x}_{ss}^2}{\sigma^2}\sum_{j=0}^{\infty}\left(\frac{1}{\left(\pi(\frac{1}{2}+j)\right)^2}-\frac{(-1)^j}{\left(\pi(\frac{1}{2}+j)\right)^3}\right) \\ &= \frac{4\bar{x}_{ss}^2}{\sigma^2}\sum_{j=0}^{\infty}\left(\frac{1}{2}-\frac{1}{4}\right) \\ &= \frac{\bar{x}_{ss}^2}{\sigma^2}\end{aligned}$$

where the bound is obtained for $\rho = 0$. Putting these together we find the Lipschitz bound

$$\begin{aligned}\mathrm{Lip}_K \equiv \sup_{t\in[0,T]} \int_0^T K(t,s)ds &\le \left(\sum_{i=0}^{\infty} \frac{1}{\mu_i}\right)\left(\sum_{j=0}^{\infty} \frac{c_j}{\psi_j}\right) \\ &= \left(\frac{\bar{x}_{ss}^2}{\sigma^2}\right)^2 .\end{aligned}$$

A sufficient condition for the existence and uniqueness of the equilibrium IRF, i.e., of the uniqueness and existence of a solution to equation (26) is that $|\Theta(\bar{x}_{ss})|\mathrm{Lip}_K < 1$. To establish a bound for $\Theta(\bar{x}_{ss})$, in terms of the fundamental model parameters, that ensures existence and uniqueness, we use the definition of $\Theta(\bar{x}_{ss})$ and the Lipschitz bound as follows:

$$\begin{aligned}\Theta(\bar{x}_{ss})\left(\frac{\bar{x}_{ss}}{\sigma^2}\right)^2 &= \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2\theta_n}{\bar{x}_{ss}\tilde{D}_{xx}(\bar{x}_{ss})}\left(\frac{\bar{x}_{ss}^2}{\sigma^2}\right)^2 \\ &= \frac{\tilde{m}_x(\bar{x}_{ss})\theta_n\bar{x}_{ss}^3}{\tilde{D}_{xx}(\bar{x}_{ss})\sigma^2} \\ &= \frac{\theta_n(\gamma\bar{x}_{ss})^2}{2U}\frac{\tanh(\gamma\bar{x}_{ss})}{\left(\theta_0+\theta_n\left(1-\frac{\gamma\bar{x}_{ss}}{\gamma U}+\frac{\tanh(\gamma\bar{x}_{ss})}{\gamma U}\right)\right)\gamma\bar{x}_{ss}-\rho c\gamma}\end{aligned}$$

where we obtained $D_{xx}(\bar{x}_{ss})$ evaluating equation (9) at $\bar{x}_{ss}$ and using equation (20), and we calculate $\tilde{m}_x(\bar{x}_{ss})$ from $\tilde{m}(x) = \frac{1}{U}\left(1-\frac{\cosh(\gamma x)}{\cosh(\gamma\bar{x}_{ss})}\right)$.

D Planning Problem

This section collects several results used to analyze the planning problem.

D.1 Planning Problem, stationary case

A stationary equilibrium is characterized by two constants N_{ss} and $\bar{x}_{ss}$ that solve the time invariant version of the p.d.e. stated in Section 6. The p.d.e. for non-adopters in the stationary case becomes the following o.d.e.:

$$
\begin{aligned}
\rho\tilde{\lambda}(x) &= x\big(\theta_0+\theta_n N_{ss}\big)+\theta_n Z_{ss}+\tfrac{\sigma^2}{2}\tilde{\lambda}_{xx}(x) \text{ if } x\leq \bar{x}_{ss} && \text{KBE}\\
\tilde{\lambda}(\bar{x}_{ss}) &= c && \text{FOC}\\
\tilde{\lambda}_x(\bar{x}_{ss}) &= 0 && \text{Smooth Pasting}\\
\tilde{\lambda}_x(0) &= 0 && \text{Reflecting}\\
0 &= -\nu\tilde{m}(x)+\nu f(x)+\frac{\sigma^2}{2}\tilde{m}_{xx}(xx) \text{ if } x\leq \bar{x}_{ss} && \text{KFE}\\
\tilde{m}(\bar{x}_{ss}) &= 0 \text{ and } \tilde{m}_x(0)=0
\end{aligned}
$$

and given $\tilde{m}$ and $\bar{x}_{ss}$, N_{ss} and Z_{ss} are defined as:

$$
\begin{aligned}
N_{ss} &= 1-\int_0^{\bar{x}_{ss}}\tilde{m}(x)dx\\
Z_{ss} &= U/2-\int_0^{\bar{x}_{ss}}x\tilde{m}(x)dx
\end{aligned}
$$

Recall that $\tilde{\lambda}(\bar{x}_{ss})$ is the Lagrange multiplier of the law of motion of the density of agents that have not adopted for the stationary case. The details of the solution can be found in Appendix D.4. The following proposition summarizes the solution of planning problem at a stationary distribution.

Proposition 11. Let $\tilde{\theta}_{ss}\equiv\frac{1}{\rho}(\theta_0+\theta_n N_{ss})$ and $\eta\equiv\sqrt{2\rho/\sigma^2}$. For fixed $0<\eta<\infty$ and small c, $\bar{x}_{ss}=2\left(\frac{c}{\tilde{\theta}_{ss}}-\frac{\theta_n Z_{ss}}{\rho\tilde{\theta}_{ss}}\right)$. For the case when σ is small (i.e., η is large), $\bar{x}_{ss}=\frac{c}{\tilde{\theta}_{ss}}-\frac{\theta_n Z_{ss}}{\rho\tilde{\theta}_{ss}}+\frac{\sigma}{\sqrt{2\rho}}$

Proposition 11 indicates that the solution of the stochastic version of the planning problem also has the option value present in the equilibrium. This proposition can be used to show that the stationary level of adoption in the planning problem is higher than the adoption level of the the high-activity stationary equilibrium.

D.2 Dynamics of N and Flow of Adoption Cost

Recall that

$$N(t)=1-\int_0^{\bar{x}(t)}m(x,t)dx.$$

Taking the derivative with respect to time

$$
\begin{aligned}
N_t(t) &= -\frac{d}{dt}\int_0^{\bar{x}(t)} m(x,t)dx \\
&= \underbrace{-m(\bar{x}(t),t)}_{=0}\frac{d\bar{x}(t)}{dt} - \int_0^{\bar{x}(t)} m_t(x,t)dx
\end{aligned}
$$

where the first term is zero from the exit point of the distribution of non-adopters. Using the law of motion of m

$$
\begin{aligned}
N_t(t) &= -\int_0^{\bar{x}(t)} \left(-\nu m(x,t) + \nu f(x) + \frac{\sigma^2}{2} m_{xx}(x,t)\right) dx \\
&= \nu \int_0^{\bar{x}(t)} m(x,t) - \frac{\nu \bar{x}(t)}{U} - \frac{\sigma^2}{2}\int_0^{\bar{x}(t)} m_{xx}(x,t)dx \\
&= \nu\left(1 - N(t)\right) - \frac{\nu\bar{x}(t)}{U} - \frac{\sigma^2}{2}\left(\underbrace{m_x(\bar{x}(t),t)}_{<0} - \underbrace{m_x(0,t)}_{=0}\right)
\end{aligned}
$$

where the last term is zero from our assumption of reflecting barriers. Let the adoption cost per unit of time $A(t)$ be defined as

$$
\begin{aligned}
A(t) &\equiv c\left(N_t(t) + \nu N(t)\right) \\
&= c\left(\nu\left(1 - N(t)\right) - \frac{\nu\bar{x}(t)}{U} - \frac{\sigma^2}{2} m_x(\bar{x}(t),t) + \nu N(t)\right) \\
&= c\left(\nu\left(1 - \frac{\bar{x}(t)}{U}\right) - \frac{\sigma^2}{2} m_x(\bar{x}(t),t)\right)
\end{aligned}
$$

where the first term are the agents that are replaced with $x \geq \bar{x}(t)$. The second term are the agents that hit $\bar{x}(t)$ from below per unit of time so they pay c and adopt the technology.

D.3 Derivation of the PDE's for the Planner's Problem

To derive the problem in continuous time, we write the adoption problem in a discrete-time discrete state setup. We do so by using finite-difference approximation and then we consider the planning problem in that set-up. We obtain the first order conditions for a problem in finite dimensions. Lastly, we take the limit to develop the corresponding p.d.e's.

First we derive the finite difference approximation for a Brownian motion reflected between two barriers. The time step Δ so that times are between $t = 0, \Delta, 2\Delta, \ldots$. The space step is Δ_x so that $x \in \{x_1, x_2, \ldots, x_I\}$, where $x_1 = 0, x_J = U$ and $x_{i+1} - x_i = \Delta_x$. The p.d.e. inside the barriers is

$$m_t(x,t) = -\nu m(x,t) + \nu f(x) + \frac{\sigma^2}{2} m_{xx}(x,t)$$

Its finite difference approximation is:

$$\frac{m_{i,t+\Delta} - m_{i,t}}{\Delta} = -\nu m_{i,t} + \nu f_i + \frac{\sigma^2}{2} \frac{(m_{i+1,t} - 2m_{i,t} + m_{i-1,t})}{(\Delta_x)^2}$$

for $i = 2, \ldots, I-1$. We can write the finite difference approximation as:

$$\begin{aligned} m_{i,t+\Delta} &= m_{i,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) + f_i \nu \Delta \\ &+ \frac{\sigma^2}{2} \frac{\Delta}{(\Delta_x)^2} m_{i+1,t} + \frac{\sigma^2}{2} \frac{\Delta}{(\Delta_x)^2} \Delta m_{i-1,t} \end{aligned}$$

For the finite approximation, we have that since the law of motion must be local, and mean preserving:

$$\begin{aligned} m_{1,t+\Delta} &= m_{1,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) + f_1 \nu \Delta \\ &+ \frac{\sigma^2}{2} \frac{\Delta}{(\Delta_x)^2} m_{2,t} + \frac{\sigma^2}{2} \frac{\Delta}{(\Delta_x)^2} m_{1,t} \\ m_{I,t+\Delta} &= m_{I,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) + f_I \nu \Delta \\ &+ \frac{\sigma^2}{2} \frac{\Delta}{(\Delta_x)^2} m_{I-1,t} + \frac{\sigma^2}{2} \frac{\Delta}{(\Delta_x)^2} m_{I,t} \end{aligned}$$

We can write the l.o.m. at the boundaries as:

$$\begin{aligned} m_{1,t+\Delta} &= m_{1,t}(1 - \nu\Delta) + f_1 \nu \Delta + \frac{\sigma^2}{2} \frac{\Delta}{\Delta_x} \frac{(m_{2,t} - m_{1,t})}{\Delta_x} \\ m_{I,t+\Delta} &= m_{I,t}(1 - \nu\Delta) + f_I \nu \Delta + \frac{\sigma^2}{2} \frac{\Delta}{\Delta_x} \frac{(m_{I-1,t} - m_{I,t})}{\Delta_x} \end{aligned}$$

At the reflecting boundaries $x = 0$ and $x = U$, the boundary conditions is $m_x(x,t) = 0$. Note that as $\Delta_x \to 0$ we require that

$$\frac{(m_{I-1,t} - m_{I,t})}{\Delta_x} = \frac{(m_{2,t} - m_{1,t})}{\Delta_x} \to 0$$

Now we get back to the planning problem. We will have two measures, $\{m_{i,t}\}$ and $\{g_{i,t}\}$. $m_{i,t}$ is the measures of those that have not adopted and $g_{i,t}$ the measure of those that have adopted. Let $\alpha_{it} \geq 0$ be the measure of adopting at t with $x = x_i$ at t. Thus at time t, the measure $\alpha_{i,t}$ is transferred from measure $m_{i,t}$ to measure $g_{i,t}$ Note that $m_{i,t} + g_{i,t} = \frac{1}{I}$ since the sum of the two is the invariant distribution. The initial condition are $g_{i,0} = 0 \ \forall i$ and $m_{i,0} = \frac{1}{I}$ all non-adopters. The law of motion of the state is then:

$$
\begin{aligned}
0 \leq m_{1,t+\Delta} &= m_{1,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) + f\nu\Delta \\
&+ \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2} m_{2,t} + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2} m_{1,t} - \alpha_{1,t} \\
0 \leq m_{i,t+\Delta} &= m_{i,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) + f\nu\Delta \\
&+ \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2} m_{i+1,t} + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2} \Delta m_{i-1,t} - \alpha_{i,t} \text{ for } i = 2, \ldots, I-1 \\
0 \leq m_{I,t+\Delta} &= m_{I,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) + f\nu\Delta \\
&+ \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2} m_{I-1,t} + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2} m_{I,t} - \alpha_{I,t}
\end{aligned}
$$

which can be written in vector notation as:

$$
m_{t+1} = L\, m_t - \alpha_t \geq 0
$$

where L is an $I \times I$ stochastic matrix which depends on I, ν, σ^2, Δ and Δ_x. We assume that $\Delta\left(\nu + (\sigma/\Delta_x)^2\right) < 1$ so that all implied probabilities are positive.

$$
\max_{\{\alpha_t, m_{t+\Delta}\}_{t=0}^{\infty}} \sum_{\{t=0,\Delta,2\Delta,\ldots\}} \left(\frac{1}{1+\Delta r}\right)^t \left\{\mathcal{U}(m_t)\,\Delta - \sum_{i=1}^{I} \alpha_{it} c\right\}
$$

where

$$
\mathcal{U}(m_t) \equiv \sum_{i=1}^{I} \left(\frac{1}{I} - m_{it}\right)\left(\theta_0 + \theta_n \left[1 - \sum_{j=1}^{I} m_{j,t}\right]\right) x_i
$$

subject to the law of motion:

$$
m_{t+1} = L\, m_t - \alpha_t \text{ for all } t = 0, \Delta, 2\Delta, \ldots
$$

and subject to non-negativity:

$$m_{j,t+1} \geq 0 \text{ and } \alpha_{j,t} \geq 0 \text{ for all } j = 1, \ldots, I, \text{ and for all } t = 0, \Delta, 2\Delta, \ldots$$

Let $\left(\frac{1}{1+\Delta r}\right)^t \lambda_{it}$ be Lagrange multiplier of the law of motion for m_{it}. Let L_i be the i^{th} row vector of the matrix L . The Lagrangian $\mathcal{L}$ becomes:

$$\begin{aligned}\mathcal{L} = &\sum_{\{t=0,\Delta,\ldots\}} \left(\frac{1}{1+\Delta r}\right)^t \left\{\mathcal{U}\left(m_t\right)\Delta - \sum_{i=1}^{I} \alpha_{it} c\right\} \\ &+ \sum_{\{t=0,\Delta,\ldots\}} \left(\frac{1}{1+\Delta r}\right)^t \left\{\sum_{i=1}^{I} \lambda_{it}\left(m_{i,t+\Delta} - L_i \cdot m_t + \alpha_{it}\right)\right\}\end{aligned}$$

Derivative of Lagrangian with respect to α_{it}:

$$\frac{\partial \mathcal{L}}{\partial \alpha_{jt}} = \left(\frac{1}{1+\Delta r}\right)^t \left[\lambda_{j,t} - c\right]$$

Derivative of Lagrangian with respect to m_{jt} for $2 \leq j \leq I-1$:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial m_{j,t}} = &\left(\frac{1}{1+\Delta r}\right)^t \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{j,t}} \Delta \\ &+ \left(\frac{1}{1+\Delta r}\right)^t \left[\lambda_{j,t-\Delta}(1+\Delta r) - \lambda_{j,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right)\right] \\ &- \left(\frac{1}{1+\Delta r}\right)^t \frac{\sigma^2}{2} \frac{\Delta}{(\Delta_x)^2}\left[\lambda_{j+1,t} + \lambda_{j-1,t}\right]\end{aligned}$$

where

$$\begin{aligned}\frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} &= -\, x_j\left(\theta_0 + \theta_n\left(1 - \sum_{i=1}^{I} m_{i,t}\right)\right) - \theta_n \sum_{i=1}^{I}\left(\frac{1}{I} - m_{it}\right) x_i \\ &= -x_j\left(\theta_0 + \theta_n N_t\right) - \theta_n\left(\frac{U}{2} - \sum_{i=1}^{I} m_{it} x_i\right)\end{aligned}$$

We can write m_{jt} for $2 \leq j \leq I-1$:

$$\begin{aligned}\left(\frac{1}{1+\Delta r}\right)^{-t} \frac{\partial \mathcal{L}}{\partial m_{jt}} = &\frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} \Delta + \lambda_{j,t-\Delta}(1+\Delta r) \\ &- \lambda_{j,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) - \frac{\sigma^2}{2} \frac{\Delta}{(\Delta_x)^2}\left[\lambda_{j+1,t} + \lambda_{j-1,t}\right]\end{aligned}$$

and rearranging:

$$(1+\Delta r)\lambda_{j,t-\Delta} = \left(\frac{1}{1+\Delta r}\right)^{-t} \frac{\partial \mathcal{L}}{\partial m_{jt}} - \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} \Delta$$
$$+ \lambda_{j,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}\left[\lambda_{j+1,t} + \lambda_{j-1,t}\right]$$

dividing by Δ and further rearranging the expressions:

$$(r+\nu)\lambda_{j,t-\Delta} = \left(\frac{1}{1+\Delta r}\right)^{-t} \frac{1}{\Delta}\frac{\partial \mathcal{L}}{\partial m_{jt}} - \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} - \nu\left(\lambda_{j,t} - \lambda_{j,t-\Delta}\right)$$
$$+ \left(\frac{\lambda_{j,t} - \lambda_{j,t-\Delta}}{\Delta}\right) + \frac{\sigma^2}{2}\left(\frac{\lambda_{j+1,t} - 2\lambda_{j,t} + \lambda_{j-1,t}}{(\Delta_x)^2}\right)$$

For the bottom boundary $j = 1$ we have:

$$\left(\frac{1}{1+\Delta r}\right)^{-t} \frac{\partial \mathcal{L}}{\partial m_{1t}} = \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{1t}} \Delta + \lambda_{1,t-\Delta}(1+\Delta r)$$
$$- \lambda_{1,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) - \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}\left[\lambda_{1,t} + \lambda_{2,t}\right]$$

$$(r+\nu)\lambda_{1,t-\Delta} = \left(\frac{1}{1+\Delta r}\right)^{-t} \frac{1}{\Delta}\frac{\partial \mathcal{L}}{\partial m_{1t}} - \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{1t}} - \nu\left(\lambda_{1,t} - \lambda_{1,t-\Delta}\right)$$
$$+ \left(\frac{\lambda_{1,t} - \lambda_{1,t-\Delta}}{\Delta}\right) + \frac{\sigma^2}{2}\frac{1}{\Delta_x}\left(\frac{\lambda_{2,t} - \lambda_{1,t}}{\Delta_x}\right)$$

For the top boundary $j = I$:

$$(r+\nu)\lambda_{I,t-\Delta} = \left(\frac{1}{1+\Delta r}\right)^{-t} \frac{1}{\Delta}\frac{\partial \mathcal{L}}{\partial m_{It}} - \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{It}} - \nu\left(\lambda_{I,t} - \lambda_{I,t-\Delta}\right)$$
$$+ \left(\frac{\lambda_{I,t} - \lambda_{I,t-\Delta}}{\Delta}\right) + \frac{\sigma^2}{2}\frac{1}{\Delta_x}\left(\frac{\lambda_{I-1,t} - \lambda_{I,t}}{\Delta_x}\right)$$

Thus the limit as $\Delta \downarrow 0$ and $\Delta_x \downarrow 0$ is that

$$\lambda_x(0,t) = \lambda_x(U,t) = 0$$

First order condition with respect to α_{it} for $t = 0, \Delta, \ldots$ and $j = 1, \ldots, I$::

$$\begin{aligned} \lambda_{j,t} - c &\leq 0\,,\ \alpha_{jt} \geq 0 \text{ and} \\ \alpha_{j,t}\left[\lambda_{j,t} - c\right] &= 0 \end{aligned}$$

First order condition with respect to m_{jt} for $t = \Delta, 2\Delta, \ldots$ and $j = 1, \ldots, I$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial m_{jt}} &\leq 0\,,\ m_{jt} \geq 0 \text{ and} \\ m_{jt}\,\frac{\partial \mathcal{L}}{\partial m_{jt}} &= 0 \end{aligned}$$

Note that as $\Delta \downarrow 0$ and $\Delta_x \downarrow 0$ and $x = x_j$ we have

$$\frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} \to x(\theta_0 + \theta_n N(t)) + \theta_n \left(\frac{U}{2} - \int_0^U m(z,t) z\, dz\right)$$

Consider a $x_j = x$ for $j = 2, \ldots, I-1$ or $0 < x < U$. Take the f.o.c. for $m_{j,t}$ derived above and assume that $\frac{\partial \mathcal{L}}{\partial m_{jt}} = 0$. Take the limit as $\Delta \downarrow 0$ and $\Delta_x \downarrow 0$:

$$\begin{aligned} (r+\nu)\lambda(x,t) = {} & x(\theta_0 + \theta_n N(t)) + \theta_n \left(\frac{U}{2} - \int_0^U m(z,t) z\, dz\right) \\ & + \lambda_t(x,t) + \frac{\sigma^2}{2}\lambda_{xx}(x,t) \end{aligned}$$

If instead $\frac{\partial \mathcal{L}}{\partial m_{jt}} \leq 0$, then

$$\begin{aligned} (r+\nu)\lambda(x,t) \leq {} & x(\theta_0 + \theta_n N(t)) + \theta_n \left(\frac{U}{2} - \int_0^U m(z,t) z\, dz\right) \\ & + \lambda_t(x,t) + \frac{\sigma^2}{2}\lambda_{xx}(x,t) \end{aligned}$$

We derive **smooth pasting** here. Suppose that at t we have $\lambda_{i,t} = c$ for all $i \geq j$, i.e., for all $x \geq \bar{x}(t)$, or $\lambda(x,t) < c$ for $x < \bar{x}(t)$ and $\lambda(x,t) = c$ for $x \geq \bar{x}(t)$. Assume also $m_{j,t} > 0$ and $m_{j-1,t} > 0$, so that $\partial \mathcal{L}/\partial m = 0$ for both. Then we can write the f. o.c. as:

$$\begin{aligned} (r+\nu)c = {} & -\frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} - \nu\left(c - \lambda_{j,t-\Delta}\right) \\ & + \left(\frac{c - \lambda_{j,t-\Delta}}{\Delta}\right) + \frac{\sigma^2}{2}\frac{1}{\Delta_x}\left(\frac{c - 2c + \lambda_{j-1,t}}{\Delta_x}\right) \end{aligned}$$

Taking the limit as $\Delta_x \downarrow 0$ we have: $\lambda_x(\bar{x}(t), t) = 0\,.$

In summary, a planner problem is given by $\{\bar{x}(t), \lambda(x,t), m(x,t)\}$ the path of optimal threshold so that adoption occurs for $x \geq \bar{x}(t)$, the Lagrange multiplier V, and the density of non-adopters m, respectively, such that the p.d.e. for the non-adopters is:

$$
\begin{aligned}
m_t(x,t) &= \nu\big(1/U - m(x,t)\big) + \tfrac{\sigma^2}{2} m_{xx}(x,t) \text{ for } x < \bar{x}(t) \text{ and } t \geq 0 \\
m(x,t) &= 0 \text{ for } x \geq \bar{x}(t) \text{ and } t \geq 0 \\
m_x(0,t) &= 0 \text{ for } t \geq 0
\end{aligned}
$$

The p.d.e. for the non-adopters:

$$
\begin{aligned}
\rho\lambda(x,t) &= x\big(\theta_0 + \theta_n[1 - \int_0^{\bar{x}(t)} m(z,t)dz]\big) + \theta_n\big(\tfrac{U}{2} - \int_0^{\bar{x}(t)} m(z,t)z\,dz\big) \\
&\quad + \tfrac{\sigma^2}{2}\lambda_{xx}(x,t) + \lambda_t(x,t) \text{ for } x \leq \bar{x}(t) \text{ and } t \geq 0 \\
\lambda(x,t) &= c \text{ for } x \geq \bar{x}(t) \text{ and } t \geq 0 \\
\lambda_x(\bar{x}(t),t) &= 0 \text{ for } t \geq 0 \\
\lambda_x(0,t) &= 0 \text{ for } t \geq 0
\end{aligned}
$$

The conditions for $\bar{x}$ are:

- We look for $\bar{x}(\cdot)$ to be continuous $t \geq 0$.

Conditions for m:

- We look for $m(\cdot,t)$ to be continuous for all $x \in [0,U]$ and $t \geq 0$.
- We look for $m(\cdot,t)$ to be C^2 for all $x \in [0,\bar{x}(t)]$, and $t \geq 0$.
- We look for $m(x,\cdot)$ to be C^1 for all $x \in [0,\bar{x}(t)]$, and $t \geq 0$.
- The initial boundary condition for m is $m(x,0) = 0$ for all $x \in [0,U]$

Conditions for λ:

- We look for $\lambda(\cdot,t)$ to be C^1 for all $x \in [0,U]$.
- We look for $\lambda(\cdot,t)$ to be C^2 for all $x \in [0,\bar{x}(t)]$, and $t \geq 0$.
- We look for $\lambda(x,\cdot)$ to be C^1 for all $x \in [0,\bar{x}(t)]$, and $t \geq 0$.
- The final boundary for λ is $\lambda(x,T) = 0$ for all $x \in [0,U]$ (T may be $+\infty$).

D.4 Solution of the Stationary Planning Problem

The solution for $\tilde{\lambda}$ of the form

$$\tilde{\lambda}(x) = x\frac{\theta_0 + \theta_n N_{ss}}{\rho} + \frac{\theta_n}{\rho} Z_{ss} x + C_1 e^{\eta x} + C_2 e^{-\eta x}$$

for $\eta = \sqrt{2\rho/\sigma^2}$, and

$$\frac{\theta_0 + \theta_n N_{ss}}{\rho} + \eta(C_1 e^{\eta \bar{x}_{ss}} - C_2 e^{-\eta \bar{x}_{ss}}) = 0$$
$$\frac{\theta_0 + \theta_n N_{ss}}{\rho} + \eta(C_1 - C_2) = 0$$

Thus, given $\theta_0 + \theta_n N_{ss}$, and $\bar{x}_{ss}$, the constants (C_1, C_2) are the solution of two linear equations. Moreover, the values of A_1, A_2 are proportional to $\tilde{\theta}_{ss}$ given by

$$\tilde{\theta}_{ss} \equiv \frac{\theta_0 + \theta_n N_{ss}}{\rho} = \eta(C_2 - C_1) = \eta(C_2 e^{-\eta \bar{x}_{ss}} - C_1 e^{\eta \bar{x}_{ss}})$$

Let $\tilde{C}_i \equiv C_i/\tilde{\theta}_{ss}$. We can write:

$$1 = \eta(\tilde{C}_2 - \tilde{C}_1) = \eta(\tilde{C}_2 e^{-\eta \bar{x}_{ss}} - \tilde{C}_1 e^{\eta \bar{x}_{ss}})$$

which has solution:

$$\tilde{C}_1 = \frac{1}{\eta} \frac{(1 - e^{-\eta \bar{x}_{ss}})}{(e^{-\eta \bar{x}_{ss}} - e^{\eta \bar{x}_{ss}})}$$
$$\tilde{C}_2 = \frac{1}{\eta} \frac{(1 - e^{\eta \bar{x}_{ss}})}{(e^{-\eta \bar{x}_{ss}} - e^{\eta \bar{x}_{ss}})}$$

Using value matching we get:

$$\eta \bar{x}_{ss} + \frac{\eta \theta_n}{\rho \tilde{\theta}_{ss}} Z_{ss} + \eta(\tilde{C}_1 e^{\eta \bar{x}_{ss}} + \tilde{C}_2 e^{-\eta \bar{x}_{ss}}) = \frac{\eta}{\tilde{\theta}_{ss}} c$$

Letting $y \equiv \eta \bar{x}_{ss}$ we can write

$$\tilde{\psi}(y) \equiv y + \eta(\tilde{C}_1 e^{y} + \tilde{C}_2 e^{-y}) + \eta \frac{\theta_n}{\rho \tilde{\theta}_{ss}} Z_{ss}$$

Using $\eta\tilde{C}_2 = 1 + \eta\tilde{C}_1$ and the definition of $\tilde{C}_1$ we get

$$\tilde{\psi}(y) \equiv y + e^{-y} - \frac{(1 - e^{-y})}{(e^{y} - e^{-y})}(e^{y} + e^{-y}) + \eta\frac{\theta_n}{\rho\tilde{\theta}_{ss}} Z_{ss}$$

We have the following properties:

1. $\tilde{\psi}(0) = \eta\frac{\theta_n}{\rho\tilde{\theta}_{ss}} Z_{ss}$
2. $\tilde{\psi}'(y) = \frac{e^{2y}+1}{(e^{y}+1)^2}$ so $\tilde{\psi}'(0) = \frac{1}{2}, \tilde{\psi}'(\infty) = 1$, and $\tilde{\psi}''(y) > 0$,
3. $\tilde{\psi}(y) = \frac{y}{2} + \frac{y^3}{24} + o(y^4) + \eta\frac{\theta_n}{\rho\tilde{\theta}_{ss}} Z_{ss}$ and $\lim_{y\to\infty} \frac{\tilde{\psi}(y) - y - \eta\frac{\theta_n}{\rho\tilde{\theta}_{ss}} Z_{ss}}{y} = 0$

For fixed $0 < \eta < \infty$ and small c using the first order approximation:

$$\bar{x}_{ss} = 2\left(\frac{c}{\tilde{\theta}_{ss}} - \frac{\theta_n}{\rho\tilde{\theta}_{ss}} Z_{ss}\right)$$

For the case when σ is small (i.e., η is large) we find:

$$\bar{x}_{ss} = \frac{c}{\tilde{\theta}_{ss}} + \frac{\sigma}{\sqrt{2\rho}} - \frac{\theta_n}{\rho\tilde{\theta}_{ss}} Z_{ss}$$

Defining $\gamma = \sqrt{2\nu/\sigma^2}$, for the uniform case we have:

$$\begin{aligned}
N_{ss} &= 1 - \int_0^{\bar{x}_{ss}(N_{ss})} \tilde{m}(s; N_{ss}) dx \\
&= 1 - \int_0^{\bar{x}_{ss}} \frac{1}{U}\left[1 - \frac{(e^{\gamma x} + e^{-\gamma x})}{(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})}\right] dx \\
&= 1 - \frac{\bar{x}_{ss}}{U} + \frac{(e^{\gamma\bar{x}_{ss}} - e^{-\gamma\bar{x}_{ss}})}{\gamma U\left(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}}\right)}
\end{aligned}$$

and

$$\begin{aligned}
Z_{ss} &= U/2 - \int_0^{\bar{x}_{ss}(N_{ss})} x\tilde{m}(s; N_{ss}) dx \\
&= U/2 - \int_0^{\bar{x}_{ss}} \frac{x}{U}\left[1 - \frac{(e^{\gamma x} + e^{-\gamma x})}{(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})}\right] dx \\
&= U/2 - \frac{\bar{x}_{ss}^2}{2U} + \frac{1}{U\left(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}}\right)} \int_0^{\bar{x}_{ss}} \left(xe^{\gamma x} + xe^{-\gamma x}\right) dx \\
&= U/2 - \frac{\bar{x}_{ss}^2}{2U} + \frac{\bar{x}}{\gamma U}\frac{(e^{\gamma\bar{x}_{ss}} - e^{-\gamma\bar{x}_{ss}})}{(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})} + \frac{1}{\gamma^2 U}\frac{2}{(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})} - \frac{1}{\gamma^2 U}
\end{aligned}$$

D.5 Perturbation and Stability of Invariant Distribution

In this section we analyze the linearization of the planning problem around its stationary distribution. This linearization is analogous to the one for the equilibrium in Section 5.

We approximate $\bar{x}(t) = \mathcal{X}^P(N, Z)(t)$ by taking the directional derivative (Gateaux) with respect to arbitrary perturbations n of a constant path N, and z of a constant path Z. In particular, we consider paths defined by $N(t) = N_{ss} + \epsilon\, n(t)$ and $Z(t) = Z_{ss} + \epsilon\, z(t)$ around the stationary value N_{ss} and Z_{ss}. We will denote this Gateaux derivative by $\bar{y}$.

Proposition 12. Let λ_T be equal to the stationary value function $\tilde{\lambda}$ corresponding to that invariant distribution. Let $n : [0, T] \to \mathbb{R}$ and $z : [0, T] \to \mathbb{R}$ be two arbitrary perturbations. Then

$$\begin{aligned}\bar{y}(t) &\equiv \lim_{\epsilon \downarrow 0} \frac{\mathcal{X}^{\mathcal{P}}(N_{ss} + \epsilon n, Z_{ss} + \epsilon z; \tilde{\lambda})(t) - \mathcal{X}^P(N_{ss}, Z_{ss}; \tilde{\lambda})(t)}{\epsilon} \\ &= \int_t^T G_{yn}(\tau - t) n(\tau) d\tau + \int_t^T G_{yz}(\tau - t) z(\tau) d\tau \end{aligned} \tag{69}$$

where

$$\begin{aligned} G_{yn}(\tau - t) &= \frac{\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})} \sum_{j=0}^{\infty} c_j e^{-\psi_j(\tau - t)} n(\tau) d\tau \\ G_{yz}(\tau - t) &= \frac{2\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss}) \bar{x}_{ss}} \sum_{j=0}^{\infty} c_j e^{-\psi_j(\tau - t)} z(\tau) d\tau \end{aligned}$$

and ψ_j, c_j, and γ are defined as in Proposition 7.

Now we turn to the perturbation for the inframarginal value Z as a function of the thresholds and of a perturbation of the initial condition. We approximate $Z(t) = \mathcal{Z}(\bar{x}, m_0)(t)$ by taking the directional derivative (Gateaux) with respect to an arbitrary perturbation y of a constant path $\bar{x}$ and a perturbation ω on the invariant distribution $\tilde{m}$. In particular, we consider paths defined by $\bar{x}(t) = \bar{x}_{ss} + \epsilon\, \bar{y}(t)$ around the invariant threshold x_{ss}, and around the invariant distribution $m_0(x) = \tilde{m}(x) + \epsilon \omega(x)$. We will denote this Gateaux derivative by z.

Proposition 13. Let $\tilde{m}$ be the corresponding invariant distribution of non-adopters for the planner. Let $\omega : [0, \bar{x}_{ss}] \to \mathbb{R}$ be an arbitrary perturbation to the distribution, and let

$\bar{y} : [0, T] \to \mathbb{R}$ be an arbitrary perturbation of the threshold. Then

$$
\begin{aligned}
z(t) &\equiv \lim_{\epsilon \downarrow 0} \frac{\mathcal{Z}(\bar{x}_{ss} + \epsilon y; \tilde{m} + \epsilon w)(t) - \mathcal{Z}(\bar{x}_{ss}; \tilde{m})(t)}{\epsilon} \\
&= z_0(\omega)(t) + \int_0^t H_{zy}(t-s)\bar{y}(s)ds \qquad (70)
\end{aligned}
$$

where

$$
z_0(\omega)(t) \equiv -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}^2(\pi j + \frac{1}{2} - \cos(j\pi))}{\pi(\frac{1}{2} + j)} \frac{\langle \varphi_j, \omega \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t} \text{ and} \qquad (71)
$$

$$
H_{zy}(q) = \tilde{m}_x(\bar{x}_{ss})\sigma^2 \sum_{j=0}^{\infty} \eta_j e^{-\mu_j q} \qquad (72)
$$

where $\varphi_j, \tilde{m}_x, \mu_j$ and γ are defined as in Proposition 8.

Thus we can write $Z(t) = Z_{ss} + \epsilon z(t) + o(\epsilon)$. This formula has the effect of two perturbations. One is the perturbation on the initial condition m_0 given by ω, whose effect is in the term $z_0(\omega)(t)$. Alternatively, $z_0(\omega)(t)$ is the effect at time t on the path $Z(t)$ of a perturbation of the initial condition keeping the threshold rule $\bar{x}$ fixed. As in the case of n_0 we can specialize ω by Dirac-delta function $\delta_{\hat{x}}$, so that we concentrate the perturbation around a value $x = \hat{x}$. The proof of this can be found in Appendix D.6.1.

Theorem 3. Let $\bar{x}_{ss}$ be the invariant threshold of the planner problem, with its corresponding N_{ss}, Z_{ss}, and let $\tilde{m}$ be the corresponding invariant distribution of non-adopters. Let $m_0(x) = \tilde{m}(x) + \epsilon\omega(x)$. Let λ_T be equal to the stationary value function $\tilde{\lambda}$. The linearized equilibrium must solve

$$
\bar{y}(t) = \bar{y}_0(t) + \tilde{\Theta} \int_0^T \tilde{K}(t,s)\bar{y}(s)ds \text{ where} \qquad (73)
$$

$$
\bar{y}_0(\omega)(t) \equiv \int_t^T G_{yn}(\tau - t)n_0(\omega)(\tau)d\tau + \int_t^T G_{yz}(\tau - t)z_0(\omega)(\tau)d\tau \qquad (74)
$$

where n_0 is derived in Proposition 8, z_0 is derived in Proposition 13, $\tilde{\Theta} \equiv \frac{\theta_n \tilde{m}_x(\bar{x}_{ss})\sigma^2}{\tilde{\lambda}_{xx}(\bar{x}_{ss})\bar{x}_{ss}}$ and where the kernel $\tilde{K}$ is given by

$$
\tilde{K}(t,s) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(c_j + c_i)e^{\psi_j t + \mu_i s}\left(\frac{e^{-(\psi_j+\mu_i)\max\{t,s\}} - e^{-(\psi_j+\mu_i)T}}{\psi_j + \mu_i}\right) > 0 \qquad (75)
$$

We have that $\text{Lip}_{\tilde{K}} \leq \left(\frac{\bar{x}_{ss}^2}{\sigma^2}\right)^2$. Furthermore, if $\tilde{\Theta}\,\text{Lip}_{\tilde{K}} < 1$ there exists a unique bounded solution to equation (73) which is the limit of

$$\bar{y}(t) = \left[I + \tilde{\Theta}\tilde{\mathcal{K}} + \tilde{\Theta}^2\tilde{\mathcal{K}}^2 + \dots\right] \bar{y}_0(\omega) \quad \text{where} \quad \tilde{\mathcal{K}}(g)(t) \equiv \int_0^T \tilde{K}(t,s)g(s)ds$$

and where $\tilde{\mathcal{K}}^{j+1}(g)(t) \equiv \int_0^T \tilde{K}(t,s)\,\tilde{\mathcal{K}}^j(g)(s)\,ds$ for any bounded $g : [0,T] \to \mathbb{R}$. The operator $\tilde{\mathcal{K}}$ is self-adjoint, and positive definite.

We again consider a perturbation to the invariant density of non-adopters. In this case, we let $m_0(x)$ be the invariant distribution of no-adopters of the decentralized problem, so that the shock resembles starting an equilibrium with lower adoption than that prescribed by the planning solution.

D.6 Perturbation of the Planning Problem

We consider the planning problem with $\{\bar{x}(t,\epsilon), N(t,\epsilon), \lambda(x,t,\epsilon), m(x,t,\epsilon)\}$. We again linearize this equilibrium with respect to ϵ and evaluate it at $\epsilon = 0$. We differentiate $\lambda(x,t,\epsilon)$ with respect to ϵ at each (x,t) to obtain $\ell(x,t) \equiv \frac{\partial}{\partial \epsilon}\lambda(x,t,\epsilon)\big|_{\epsilon=0}$ which solves the following p.d.e

$$\rho\ell(x,t) = x\theta_n n(t) + \theta_n z(t) + \frac{\sigma^2}{2}\ell_{xx}(x,t) + \ell_t(x,t) \tag{76}$$

for $x \in [0,\bar{x}_{ss}]$ and $t \in [0,T]$ and where $z(t) \equiv \frac{\partial}{\partial \epsilon}Z(t,\epsilon)\big|_{\epsilon=0}$ and $n(t) \equiv \frac{\partial}{\partial \epsilon}N(t,\epsilon)\big|_{\epsilon=0}$. The boundary conditions are:

$$\begin{aligned} \ell(x,T) &= 0 \\ \ell_x(0,t) &= 0 \\ \ell(\bar{x}_{ss},t) &= 0 \\ \tilde{\lambda}_{xx}(\bar{x}_{ss})\bar{y}(t) + \ell_x(\bar{x}_{ss},t) &= 0 \end{aligned} \tag{77}$$

Proposition 14. The solution for the KBE equation for ℓ is given by

$$\ell(x,t) = \sum_{j=0}^{\infty} \varphi_j(x)\hat{\ell}(t) \qquad \text{for } x \in [0,\bar{x}_{ss}] \text{ and } t \in [0,T]$$

where for all $j = 1, 2, ...$ we have:

$$
\begin{aligned}
\hat{\ell}(t) &= \int_t^T e^{-\psi_j(\tau - t))} \hat{s}_j(\tau) d\tau && \text{for } t \in [0, T] \\
\hat{s}_j(t) &= -\theta_n n(t) \frac{\langle \varphi_j, x \rangle}{\langle \varphi_j, \varphi_j \rangle} - \theta_n z(t) \frac{\langle \varphi_j, 1 \rangle}{\langle \varphi_j, \varphi_j \rangle} && \text{for } t \in [0, T] \\
\varphi_j(x) &= \sin\left(\left(\frac{1}{2} + j \right) \pi \left(1 - \frac{x}{\bar{x}_{ss}} \right) \right) && \text{for } x \in [0, \bar{x}_{ss}] \\
\langle \varphi_j, h \rangle &\equiv \int_0^1 h(x) \varphi_j(x) dx \\
\hat{\ell}(T) &= 0 \\
\psi_j &= \rho + \frac{1}{2} \sigma^2 \left(\frac{\pi(\frac{1}{2} + j)}{\bar{x}_{ss}} \right)^2
\end{aligned}
$$

The proof can be done by verifying that the equation hold at the boundaries, that for $t > 0$ the p.d.e holds in the interior since

$$
\hat{\ell}_j'(t) = \psi_j \hat{\ell}(t) + \hat{s}_j(t) \qquad \text{for } t \in [0, T] \text{ and } j = 1, 2, ...
$$

and since $\{\varphi_j(x)\}$ form an orthogonal bases for functions, and finally that the boundary holds at $t = 0$ for $x \in [0, \bar{x}_{ss}]$.

Note that the derivative of the solution for λ is

$$
\ell_x(\bar{x}_{ss}, t) = -\theta_n \int_t^T \sum_{j=0}^{\infty} c_j e^{-\psi_j(\tau - t)} n(\tau) d\tau - \theta_n \frac{2}{\bar{x}_{ss}} \int_t^T \sum_{j=0}^{\infty} e^{-\psi_j(\tau - t)} z(\tau) d\tau
$$

where $c_j = 2\left(1 - \frac{\cos(\pi j)}{\pi(j + \frac{1}{2})}\right)$.

D.6.1 Perturbation Analysis of the Planning Problem

Recall that from equation (77), $\bar{y}(t)$ is equal to

$$
\begin{aligned}
\bar{y}(t) &= \frac{-\ell_x(\bar{x}_{ss}, t)}{\tilde{\lambda}_{xx}(\bar{x}_{ss})} \\
&= \int_t^T \frac{\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})} \sum_{j=0}^{\infty} c_j e^{-\psi_j(\tau - t)} n(\tau) d\tau + \int_t^T \frac{2\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss}) \bar{x}_{ss}} \sum_{j=0}^{\infty} c_j e^{-\psi_j(\tau - t)} z(\tau) d\tau \\
&= \int_t^T G_{yn}(\tau - t) n(\tau) d\tau + \int_t^T G_{yz}(\tau - t) z(\tau) d\tau \qquad (78)
\end{aligned}
$$

The expression for $n(t)$ is given by equation (68) and can be written as

$$n(t) = n_0(t) + \int_0^t H_{ny}(t-s)\bar{y}(s)ds.$$

where as before $n_0(t) \equiv -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}}{\pi(\frac{1}{2}+j)} \frac{\langle \varphi_j, \omega \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t}$. We can obtain a similar expression for $z(t)$ using the solution for $p(x,t)$ as

$$\begin{aligned}
z(t) &= -\int_0^{\bar{x}_{ss}} x p(x,t) dx \\
&= -\sum_{j=0}^{\infty} \hat{p}_j(t) \int_0^{\bar{x}_{ss}} x \varphi_j(x) dx \\
&= -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}^2 (\pi(j+\frac{1}{2}) - \cos(j\pi)}{(\pi(\frac{1}{2}+j))^2} \frac{\langle \varphi_j, \omega \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t} + \tilde{m}_x(\bar{x}_{ss})\sigma^2 \int_0^t \sum_{j=0}^{\infty} \frac{\pi(j+\frac{1}{2}) - \cos(j\pi))}{\pi(j+\frac{1}{2})} e^{-\mu_j(t-\tau)} \bar{y}(\tau) d\tau \\
&= z_0(t) + \int_0^t H_{zy}(t-s)\bar{y}(s)ds
\end{aligned}$$

where $z_0(t) \equiv -\sum_{j=0}^{\infty} \frac{c_j}{2} \frac{\bar{x}_{ss}^2}{\pi(j+\frac{1}{2})} \frac{\langle \varphi_j, \omega \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t}$ and $c_j \equiv \left(1 - \frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right)$. Then, equation (78) can be written as

$$\begin{aligned}
\bar{y}(t) &= \int_t^T G_{yn}(\tau - t) \left(n_0(\tau) + \int_0^t H_{ny}(\tau - s)\bar{y}(s)ds \right) d\tau \\
&+ \int_t^T G_{yz}(\tau - t) \left(z_0(\tau) + \int_0^t H_{zy}(\tau - s)\bar{y}(s)ds \right) d\tau \\
&= \int_t^T G_{yn}(\tau - t) n_0(\tau) d\tau + \int_t^T \int_0^t G_{yn}(\tau - t) H_{ny}(\tau - s)\bar{y}(s)ds\, d\tau \\
&+ \int_t^T G_{yz}(\tau - t) z_0(\tau) d\tau + \int_t^T \int_0^t G_{yz}(\tau - t) H_{zy}(\tau - s)\bar{y}(s)ds\, d\tau \\
&= \bar{y}_0(t) + \int_0^T M(t,s)\bar{y}(s)ds
\end{aligned}$$

where

$$\bar{y}_0(t) \equiv \int_t^T G_{yn}(\tau - t) n_0(\tau) d\tau + \int_t^T G_{yz}(\tau - t) z_0(\tau) d\tau$$

and

$$\int_0^T M(t,s)\bar{y}(s)ds \equiv \int_t^T \int_0^t G_{yn}(\tau-t)H_{ny}(\tau-s)\bar{y}(s)ds\,d\tau + \int_t^T \int_0^t G_{yz}(\tau-t)H_{zy}(\tau-s)\bar{y}(s)ds\,d\tau$$
$$= \int\limits_0^T \int\limits_{\max\{t,s\}}^T G_{yn}(\tau-t)H_{ny}(\tau-s)\bar{y}(s)ds\,d\tau + \int\limits_0^T \int\limits_{\max\{t,s\}}^T G_{yz}(\tau-t)H_{zy}(\tau-s)\bar{y}(s)ds\,d\tau$$

with

$$G_{yn}(w) = \frac{\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})}\sum_{j=0}^{\infty} c_j e^{-\psi_j(w)}$$
$$G_{yz}(w) = \frac{2\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})\bar{x}_{ss}}\sum_{j=0}^{\infty} e^{-\psi_j(w)}$$
$$H_{zy}(q) = \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2}{2}\sum_{j=0}^{\infty} c_j e^{-\mu_j(q)}$$
$$H_{ny}(q) = \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2}{\bar{x}_{ss}}\sum_{j=0}^{\infty} e^{-\mu_j(q)}$$

where $e^{-rq}G_{yn}(w)H_{ny}(q) = G_{yz}(w)H_{zy}(q)e^{-rq}$. Using the definitions of $n_0(t)$ and $z_0(t)$ we first find the value of $\bar{y}_0(t)$ as

$$\begin{aligned}
\bar{y}_0(t) &\equiv \int_t^T G_{yn}(\tau-t)n_0(\tau)d\tau + \int_t^T G_{yz}(\tau-t)z_0(\tau)d\tau \\
&= \frac{-\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})}\int_t^T \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_j \frac{\bar{x}_{ss}}{\pi(\frac{1}{2}+i)}\frac{\langle\varphi_i,\omega\rangle}{\langle\varphi_i,\varphi_i\rangle}e^{-\psi_j(\tau-t)}e^{-\mu_i\tau}d\tau \\
&+ \frac{-\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})}\int_t^T \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_i \frac{\bar{x}_{ss}}{\pi(\frac{1}{2}+i)}\frac{\langle\varphi_i,\omega\rangle}{\langle\varphi_i,\varphi_i\rangle}e^{\psi_j t}e^{-\psi_j(\tau-t)}e^{-\mu_i\tau}d\tau \\
&= \frac{-\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(c_j+c_i)\frac{\bar{x}_{ss}}{\pi(\frac{1}{2}+i)}\frac{\langle\varphi_i,\omega\rangle}{\langle\varphi_i,\varphi_i\rangle}e^{\psi_j t}\left(\frac{e^{-(\psi_j+\mu_i)t}-e^{-(\psi_j+\mu_i)T}}{\psi_j+\mu_i}\right) \qquad (79)
\end{aligned}$$

Then, we find

$$\int_0^T M(t,s)\bar{y}(s)ds = \int\limits_0^T \left(\int\limits_{\max\{t,s\}}^T G_{yn}(\tau-t)H_{ny}(\tau-s)\bar{y}(s)d\tau + \int\limits_{\max\{t,s\}}^T G_{yz}(\tau-t)H_{zy}(\tau-s)\bar{y}(s)d\tau \right) ds$$
$$= \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \int_{\max\{t,s\}}^T \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_j e^{-\psi_j(\tau-t)} e^{-\mu_i(\tau-t)} \bar{y}(s) d\tau \, ds$$
$$+ \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \int_{\max\{t,s\}}^T \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_i e^{-\psi_j(\tau-t)} e^{-\mu_i(\tau-t)} \bar{y}(s) d\tau \, ds$$

where we let $\tilde{\Theta}(\bar{x}_{ss}) \equiv \frac{\theta_n \tilde{m}_x(\bar{x}_{ss})\sigma^2}{\tilde{\lambda}_{xx}(\bar{x}_{ss})\bar{x}_{ss}}$. Solving the integrals we get

$$\int_0^T M(t,s)\bar{y}(s)ds = \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_j e^{\psi_j t + \mu_i s} \left(\frac{e^{-(\psi_j+\mu_i)\max\{t,s\}} - e^{-(\psi_j+\mu_i)T}}{\psi_j + \mu_i} \right) \right) ds$$
$$+ \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_i e^{\psi_j t + \mu_i s} \left(\frac{e^{-(\psi_j+\mu_i)\max\{t,s\}} - e^{-(\psi_j+\mu_i)T}}{\psi_j + \mu_i} \right) \right) ds$$
$$= \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} (c_j + c_i) e^{\psi_j t + \mu_i s} \left(\frac{e^{-(\psi_j+\mu_i)\max\{t,s\}} - e^{-(\psi_j+\mu_i)T}}{\psi_j + \mu_i} \right) \right) ds$$
$$= \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \tilde{K}(t,s) ds.$$

Thus, equation (78) can be written as

$$\bar{y}(t) = \bar{y}_0(t) + \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \tilde{K}(t,s)\bar{y}(s)ds$$

Notice also that since $e^{-rt}M(t,s) = e^{-rs}M(t,s)$

$$\int_0^T e^{-rt} M(t,s)\bar{y}(s)ds = \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} (c_j + c_i) e^{\mu_j t + \mu_i s} \left(\frac{e^{-(r+\mu_j+\mu_i)\max\{t,s\}} - e^{-(r+\mu_j+\mu_i)T}}{\mu_j + \mu_i + r} \right) \right) ds$$

E A "Pure" Learning Model

In this section, we develop a model with random diffusion of the technology across agents. Agents can be either uninformed about the technology, or informed about it. If they are informed, they can decide to pay a cost c and adopt it. Newborn agents start as uninformed, and become informed by randomly matching with informed agents. Once an agent adopts the technology her flow benefit depends on the idiosyncratic value of the random variable x, but not on the size of the network, i.e., $\theta_n = 0$.

The main conclusions are that the pure learning model differs from the model with strategic complementarity in that:

1. it has a unique equilibrium, and a unique stable invariant distribution,
2. it has a logistic S shape adoption profile, provided the initial share of uninformed is small enough,
3. the use of the technology for those that adopt depends only on the cohort, and not the size of the network,
4. the equilibrium is constrained efficient: the optimal subsidy to use the technology is zero.

Learning Setup. We follow the canonical notation for an "SIR" model and assume that the population, normalized to have measure 1, is split between the uninformed, whose measure we denote by $S(t)$, and the informed, which have measure $I(t)$, so that $I(t)+S(t) = 1$. Those that are informed can be split in two groups, those that have adopted the technology, with measure $N(t)$, and those informed that have not adopted $M(t)$, so that $I(t) = M(t) + N(t)$.

The main assumption about learning about the technology is that agents do *not* need to use the technology to learn about it. In particular, agents that know about the technology will randomly meet agents that don't and transmit the information in such way. Recall that among the $I(t)$ informed agents, only a $N(t)$ have adopted, and $M(t)$ are informed but have decided not to adopt.

Optimal Adoption. Now we turn to the decision of agents. The uninformed agents have no decision to make. The decision problem of those that are informed is similar to the stationary problem in our model with strategic complementarities.

The value of an agent that already has adopted the technology is

$$\rho a(x) = \theta_0 x + \frac{\sigma^2}{2} a_{xx}(x) \text{ for } x \in [0, U]$$

with boundaries $a_x(0) = a_x(U) = 0$ The value function for an agent that is informed is:

$$\rho v(x) = \max \left\{ \frac{\sigma^2}{2} v_{xx}(x) \, , \, \rho(a(x) - c) \right\}$$

with time invariant threshold $\bar{x} < U$ solving, and boundary at zero:

$$v_x(\bar{x}) = a_x(\bar{x}) \text{ and } v(\bar{x}) = a(\bar{x}) - c \text{ and } v_x(0) = 0$$

The solution of v and a are identical to the stationary solutions of the baseline model $\tilde{v}$ and $\tilde{a}$ where we set $\theta_n = 0$. Likewise the solution for $\bar{x}$ is the same as the value $\bar{x}_{ss}$ for the model with $\theta_n = 0$.

Evolution of Distributions. Now we turn to the description of the distribution of agents across states. We let $s(x,t)$ the density of those uninformed at t with x, and $m(x,t)$ the density of those informed at t with x and that have not adopted yet. First we characterize g which satisfies:

$$s_t(x,t) = \frac{\sigma^2}{2} s_{xx}(x,t) - (\nu + \beta(S(t)))\, s(x,t) + \nu \frac{1}{U} \text{ all } t \geq 0 \text{ and } x \in [0, U]$$

with boundary conditions given by reflections at the boundary, i.e., $0 = s_x(0,t) = s_x(U,t)$ all $t \geq 0$ and initial condition independent of x:

$$s(x,0) = s_0 \text{ all } x \in [0, U]$$

In this case $S(t)$ is the total measure of uninformed agents at time t, and $\beta(\cdot)$ is a function that gives the probability per uninformed of becoming informed:

$$S(t) = \int_0^U s(x,t)dx$$

We assume that $\beta(\cdot)$ is given by

$$\beta(S) = \beta_0\,(1 - S) = \beta_0\, I \text{ for some constant } \beta_0 > \nu > 0$$

The interpretation is that each agent has β_0 meeting per unit of time, and that a fraction $1 - S$ are with those informed of the technology.

We will return to solve for S and I below. Now we turn to the law of motion for m is:

$$\begin{aligned} m_t(x,t) &= \frac{\sigma^2}{2} m_{xx}(x,t) + \beta(S(t)) s(x,t) - \nu m(x,t) \text{ all } t \geq 0 \text{ and } x \in [0, \bar{x}] \\ m(x,t) &= 0 \text{ all } t \geq 0 \text{ and } x \in [\bar{x}, U] \end{aligned}$$

Continuity of m implies that $m(\bar{x},t) = 0$ all $t \geq 0$. The reflecting barrier of x at zero implies

$0 = m_x(0,t)$ for all $t \geq 0$.

Comparing with the baseline model with constant $\bar{x}$, the evolution of the density m has one main difference. Instead of having the constant inflow ν/U, it has a time varying, and smaller, inflow $\beta(S(t))s(x,t)$. This smaller inflow, everything else the same, can substantially retard the adoption.

We define the total number that are uninformed as:

$$M(t) \equiv \int_0^{\bar{x}} m(x,t)dx \leq I(t) = 1 - S(t)$$

The initial condition that the density of those that have not adopted is smaller than the density of those that are informed, i.e.: $0 \leq M(0) \leq I(0)$ all $x \in [0,U]$. Note that by integrating across x and using the boundary conditions:

$$M_t(t) = \int_0^{\bar{x}} m_t(x,t)dx = \frac{\sigma^2}{2} m_x(\bar{x},t) + \beta(S(t))S(t)\frac{\bar{x}}{U} - \nu M(t) \text{ all } t \geq 0 \text{ and } x \in [0,\bar{x}]$$

We are interested in: $N(t) = 1 - S(t) - M(t)$, which using the previous equations gives:

$$N_t(t) = -\frac{\sigma^2}{2} m_x(\bar{x},t) - \nu N(t) + \beta(S(t))S(t)\left(1 - \frac{\bar{x}}{U}\right) \text{ for all } t \geq 0$$

with initial condition $N(0) = \left(1 - \frac{\bar{x}}{U}\right) I(0)$.

Note that since $m(x,t) > 0$ for $x < \bar{x}$ and $m(\bar{x},t) = 0$, then $m_x(\bar{x},t) < 0$. The next proposition rewrite this expression which it is useful to interpret the determinants of the dynamics of $N(t)$.

Proposition 15. Assume that $s_0(x) = S_0/U$ for all $x \in [0,U]$, and that $\beta(S) = \beta_0(1-S)$. Then we can write $N(t)$ as function of path $I(t)$ and $m(\bar{x},t)$ and the threshold $\bar{x}$:

$$N(t) = I(t)\left(1 - \frac{\bar{x}}{U}\right) + \int_0^t e^{-\nu(t-\tau)}\left[-\frac{\sigma^2}{2} m_x(\bar{x},\tau)\right] d\tau \tag{80}$$

The expression in the right hand side of $N(t)$ in Proposition 15 has the following interpretation. The term $I(t)\left(1 - \frac{\bar{x}}{U}\right)$ has the fraction of those informed with values of x above the threshold $\bar{x}$. The second term takes into account the past flows of agents that were informed, whose value of x went from below $\bar{x}$ to higher than $\bar{x}$.

Solving for Path of $N(t), M(t), I(t), S(t)$ Given $\bar{x}$. The solution is recursive: we first solve for $S(t)$ and $I(t)$, and then using the path of $I(t)$ we solve for $N(t)$. This is done in the next two propositions.

PROPOSITION 16. Assume that $\beta(S) = \beta_0(1-S)$ for $\beta_0 > \nu$. Furthermore assume that $s_0(x) = S_0/U$ for all $x \in [0, U]$. For a given $I(0)$ we have that the unique solution of

$$\dot{I}(t) = \beta_0 I(t) \left[\left(1 - \frac{\nu}{\beta_0}\right) - I(t) \right]$$

is given by

$$I(t) = 1 - S(t) = \left(1 - \frac{\nu}{\beta_0}\right) \frac{e^{(\beta_0-\nu)t}}{\frac{\left(1-\frac{\nu}{\beta_0}\right)}{I(0)} - 1 + e^{(\beta_0-\nu)t}} \tag{81}$$

Thus, if $0 < I(0) < 1 - \frac{\nu}{\beta_0}$, then $I(t)$ converges monotonically to $I_{ss} = 1 - \frac{\nu}{\beta_0} \in (0,1)$. If $I(0) < I_{ss}$, then

$$I(t) = \begin{cases} \text{is convex in } t & \text{if } t < \frac{\log((I_{ss}-I(0))/I(0))}{\beta_0-\nu} \text{ or } I(t) < \frac{I_{ss}}{2} \\ \text{is concave in } t & \text{if } t > \frac{\log((I_{ss}-I(0))/I(0))}{\beta_0-\nu} \text{ or } I(t) > \frac{I_{ss}}{2}\,. \end{cases}$$

As shown in Proposition 16, when $I(0)$ is small, then $I(t)$ displays a "logistic" type of path of technology adoption, but $I(t)$ is only the population that can adopt. We characterize the number of adopters in the next proposition.

PROPOSITION 17. Assume that $s_0(x) = S_0/U$ for all $x \in [0, U]$. Take the path $I(t)$ as given, and the optimal threshold $\bar{x} < U$. Then the unique solution of $m(x,t)$ is:

$$m(x,t) = \sum_{j=0}^{\infty} \varphi_j(x) \hat{b}_j(t) \text{ where } \varphi_j(x) = \sin\left((j+\tfrac{1}{2})\pi\left(1 - \frac{x}{\bar{x}}\right)\right)$$

$$\hat{b}_j(t) = \frac{2}{\pi(j+\frac{1}{2})}\left(e^{-\mu_j t}\frac{I(0)}{U} + \beta_0 \int_0^t e^{-\mu_j(t-\tau)} \frac{I(\tau)\,(1-I(\tau))}{U} d\tau\right) \text{ and } \mu_j = \nu + \left((j+\tfrac{1}{2})\tfrac{\pi}{\bar{x}}\right)^2$$

and thus $N(t) = I(t) - M(t)$ is given by:

$$N(t) = I(t) - \tfrac{\bar{x}}{U}\left(H(t)I(0) + \beta_0 \int_0^t H(t-\tau) I(\tau)\,(1 - I(\tau))\, d\tau\right) \text{ where}$$

$$H(z) \equiv \sum_{j=0}^{\infty} \omega_j e^{-\mu_j z} \text{ with } \omega_j \equiv \frac{2}{\left(\pi(j+\frac{1}{2})\right)^2} > 0 \text{ and } \sum_{j=0}^{\infty} \omega_j = 1\,.$$

Combining the expression for $N(t)$ in Proposition 17 with the path of $I(t)$ solved for in Proposition 16 we obtain an explicit solution to $N(t)$. Next we analyze the invariant distribution in this model, which is the value at which it tends as $t \to \infty$. We denote $\tilde{m}$ the density for m which satisfies: $\nu\tilde{m}(x) = \frac{\sigma^2}{2}\tilde{m}_{xx}(x) + \beta_0(1-\frac{\nu}{\beta_0})\frac{\nu}{\beta_0}\frac{\bar{x}}{U}$ for all $x \in [0, \bar{x}]$ and $\tilde{m}_x(\bar{x}) = 0$ and $\tilde{m}(\bar{x}) = 0$. The next proposition gives the solution for the distribution $\tilde{m}$, as well as the stationary number of adopters N_{ss}.

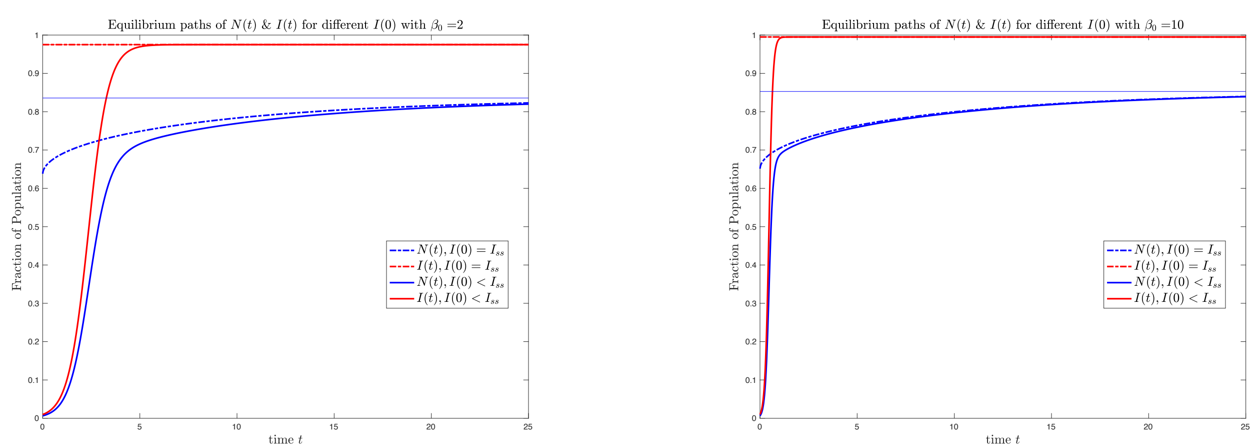
Proposition 18. Assume that $s_0(x) = S_0/U$ for all $x \in [0, U]$, that $\bar{x} < U$, $\beta(S) = \beta_0(1-S)$, and that $\beta_0 > \nu > 0$. Then the invariant density $\tilde{m}$ is given by:

$$\tilde{m}(x) = (1-\tfrac{\nu}{\beta_0})\tfrac{1}{U}\left(1 - \frac{\cosh(\gamma x)}{\cosh(\gamma \bar{x})}\right) \text{ where } \gamma = \sqrt{2\nu}/\sigma \text{ and thus}$$

$$N_{ss} = I_{ss} - \int_0^{\bar{x}} \tilde{m}(x)dx = (1-\tfrac{\nu}{\beta_0})\left[1 - \tfrac{\bar{x}}{U}\left(1 - \frac{\tanh(\gamma\bar{x})}{\gamma\bar{x}}\right)\right] \tag{82}$$

It is interesting to see that even if $I(0) = I_{ss} \equiv 1 - \frac{\nu}{\beta_0}$, then $N(0) < N_{ss}$, and convergence will take time. In words, even if all agents are informed about the technology it takes time for the selection process to yield N_{ss}. In particular equation (82) implies that $N_{ss} > I_{ss}(1-\frac{\bar{x}}{U})$, since among the adopters there are agents who had $x \geq \bar{x}$ in the past and currently have $x < \bar{x}$.

Figure E2: Equilibrium paths of N and I of Pure Learning Model



Slow learning, $\beta_0 = 2$. Fast Learning, $\beta_0 = 10$.

Figure E2 illustrates the main results of this section. The left and right panel differ in the value of β_0, with the left panel with a slow learning $\beta_0 = 2$, and the right panel a high value, $\beta_0 = 10$. In each panel we consider two initial condition for $I(0)$: one with $I(0) = I_{ss}$ (dotted lines), and with $I(0) = I_{ss}/100$ (solid lines). The remaining parameters are all the same. The paths for N are in blue, and the ones for A are in red. Focusing first in the slow learning case (left panel), note that when $I(0)$ is small, so that early on adoption is restricted by the information about the technology, the fraction that adopt $N(t)$ follows an approximate logistic path, as explained above. Instead, if $I(0) = I_{ss}$, then the path of $N(t)$ is concave in time, and starts at a high value at $t = 0$. In the case of fast learning, i.e., in the right panel, the same dynamics of learning are also present, but in a much abbreviated period of time.

Optimality of Equilibrium. The equilibrium path is constrained efficient. In particular, if the planner can only give a subsidy to those that use the technology, then the optimal subsidy is zero. This is because, given our assumptions about learning, such subsidy does not affect the fraction of people that learn about the application. Furthermore, since we assume that there is no complementary in the use of the technology, the individual decision will coincide with the planner decision for $\bar{x}$.

E.1 Proofs for the Learning Model

Proof. (Proposition 15) We start by integrating the differential equation for N to obtain

$$N(t) = e^{-\nu t}N(0) + \int_0^t e^{-\nu(t-s)} \left[-\frac{\sigma^2}{2} m_x(\bar{x}, s) + \beta(S(s))S(s) \left(1 - \frac{\bar{x}}{U}\right) \right] ds$$
$$N(0) = \left(1 - \tfrac{\bar{x}}{U}\right) I(0)$$

Using that $\dot{I}(t) = \beta(S(t))S(t) - \nu I(t)$, so

$$\int_0^t e^{-\nu(t-s)} \beta(S(s))S(s)ds = \int_0^t e^{-\nu(t-s)} \dot{I}(t)ds + \int_0^t e^{-\nu(t-s)} \nu I(t)ds$$

Integrating by parts:

$$\begin{aligned}\int_0^t e^{-\nu(t-s)} \beta(S(s))S(s)ds &= I(t) - I(0)e^{-\nu t} - \int_0^t \nu e^{-\nu(t-s)} I(s)ds + \int_0^t e^{-\nu(t-s)} \nu I(t)ds \\ &= I(t) - I(0)e^{-\nu t}\end{aligned}$$

Thus:

$$
\begin{aligned}
N(t) &= e^{-\nu t}\left(1-\tfrac{\bar{x}}{U}\right)I(0)+\int_0^t e^{-\nu(t-s)}\left[-\frac{\sigma^2}{2}m_x(\bar{x},s)\right]ds+\left[I(t)-I(0)e^{-\nu t}\right]\left(1-\frac{\bar{x}}{U}\right)\\
&= I(t)\left(1-\frac{\bar{x}}{U}\right)+\int_0^t e^{-\nu(t-s)}\left[-\frac{\sigma^2}{2}m_x(\bar{x},s)\right]ds
\end{aligned}
$$

□

Proof. (of Proposition 16) Integrating the p.d.e. for g we get:

$$
S_t(t) \equiv \int_0^U s_t(x,t)dx = \frac{\sigma^2}{2}\int_0^U s_{xx}(x,t)dx - (\nu+\beta(S(t)))\int_0^U s(x,t)dx + \nu\frac{\int_0^U dx}{U}
$$

and using its boundary conditions at $x=0$ and $x=U$:

$$
S_t(t) = -\left(\nu+\beta(S(t))\right)S(t)+\nu \text{ all } t\geq 0
$$

with initial condition:

$$
s(0) = S_0 \text{ for some constant } 0\leq S_0 = 1-I(0)\leq 1
$$

Since we assume that $s_0(x)$ is constant across x, i.e. if

$$
s_0(x) = \frac{S_0}{U} \text{ all } x\in[0,U]
$$

then the solution satisfies

$$
s(x,t) = \frac{S(t)}{U} \text{ all } t\geq 0 \text{ for all } x\in[0,U]
$$

Thus we obtain

$$
\begin{aligned}
S' &= -\left(\nu+\beta_0(1-S)\right)S+\nu = (1-S)\left(\nu-\beta_0 S\right)\\
&= \nu\left(1-S\right)\left(1-\frac{S}{S^*}\right)
\end{aligned}
$$

It is convenient to solve for the path of I, the fraction of agents informed of the technology, $I(t)+S(t)=1$ for all $t\geq 0$, so:

$$
I' = -I\left(\nu-\beta_0(1-I)\right) = \beta_0 I\left(I_{ss}-I\right) \text{ where } I_{ss} = 1-\frac{\nu}{\beta_0}
$$

Let $\tilde{I} = \beta_0 I$, so that:

$$\tilde{I}' = \tilde{I}\left(\tilde{I}_{ss} - \tilde{I}\right) = \tilde{I}_{ss}\tilde{I} - (\tilde{I})^2 \text{ where } \tilde{I}_{ss} = \beta_0 - \nu$$

Then we get that its solution is given by:

$$\tilde{I}(t) = \frac{\tilde{I}_{ss}e^{\tilde{I}_{ss}t}}{\frac{\tilde{I}_{ss}}{\tilde{I}(0)} - 1 + e^{\tilde{I}_{ss}t}}$$

Note that

$$\begin{aligned}
I_{ss}\frac{d}{dt}\frac{\tilde{I}_{ss}e^{\tilde{I}_{ss}t}}{\frac{\tilde{I}_{ss}}{\tilde{I}(0)} - 1 + e^{\tilde{I}_{ss}t}} &= \tilde{I}_{ss}\frac{\tilde{I}_{ss}e^{\tilde{I}_{ss}t}}{\frac{\tilde{I}_{ss}}{\tilde{I}(0)} - 1 + e^{\tilde{I}_{ss}t}} - \frac{\tilde{I}_{ss}e^{\tilde{I}_{ss}t}\tilde{I}_{ss}e^{\tilde{I}_{ss}t}}{\left(\frac{\tilde{I}_{ss}}{\tilde{I}(0)} - 1 + e^{\tilde{I}_{ss}t}\right)^2} \\
&= \tilde{I}_{ss}\tilde{I}(t) - (\tilde{I}(t))^2
\end{aligned}$$

which verifies the answer. Using $I = \tilde{I}/\beta_0$ we obtain the desired result.

□

Proof. (of Proposition 17) Given the path $\{S(t)\}$ define

$$B(t) \equiv \beta(S(t))S(t)\tfrac{1}{U}$$

We start with

$$m(x,t) = \sum_{j=0}^{\infty}\varphi_j(x)\hat{b}_j(t) \text{ where } \varphi_j(x) = \sin\left((j+\tfrac{1}{2})\pi\left(1 - \frac{x}{\bar{x}}\right)\right)$$

Note that each φ_j satisfies the lateral boundary conditions for $m(x,t)$ at $x = 0$ and $x = \bar{x}$ for all t. Then the p.d.e. can be written as:

$$\begin{aligned}
&0 = m_t(x,t) - \frac{\sigma^2}{2}m_{xx}(x,t) + \nu m(x,t) - B(t) \text{ or} \\
&0 = \sum_{j=0}^{\infty}\varphi_j(x)\left[\hat{b}_j'(t) + \nu\hat{b}_j(t) + \left((j+\tfrac{1}{2})\tfrac{\pi}{\bar{x}}\right)^2 b_j(t) - B(t)\tfrac{\langle\varphi_j,1\rangle}{\langle\varphi_j,\varphi_j,\rangle}\right]
\end{aligned}$$

or for each $j = 0, 1, \ldots$:

$$\hat{b}_j'(t) = -\left[\nu + \left((j+\tfrac{1}{2})\tfrac{\pi}{\bar{x}}\right)^2\right]b_j(t) + B(t)\tfrac{\langle\varphi_j,1\rangle}{\langle\varphi_j,\varphi_j,\rangle}$$

or letting $\mu_j = \left((j+\frac{1}{2})\frac{\pi}{\bar{x}}\right)^2$

$$\hat{b}_j(t) = \hat{b}_j(0)e^{-\mu_j t} + \tfrac{\langle \varphi_j, 1\rangle}{\langle \varphi_j, \varphi_j, \rangle} \int_0^t e^{-\mu_j(t-s)} B(s) ds$$

On the other hand $\{\hat{b}_j(0)\}$ are given so that

$$M(0) = \frac{\bar{x}}{U} I(0)$$

so that $M(0) = \int_0^{\bar{x}} m_0(x)dx$ and if $m_0(x)$ does not depend on x we have $M(0) = \bar{x}\, m_0(x)$:

$$m_0(x) = \frac{M(0)}{\bar{x}} = \frac{I(0)}{U}$$

$$\hat{b}_j(0) = \tfrac{\langle \varphi_j, 1\rangle}{\langle \varphi_j, \varphi_j\rangle} \frac{I(0)}{U}$$

which ensures:

$$\sum_{j=0}^{\infty} \hat{b}_j(0)\varphi_j(x) = \frac{I(0)}{U}$$

so

$$\hat{b}_j(t) = \tfrac{\langle \varphi_j, 1\rangle}{\langle \varphi_j, \varphi_j\rangle} \left(e^{-\mu_j t} \frac{I(0)}{U} + \int_0^t e^{-\mu_j(t-s)} B(s) ds \right)$$

Finally,

$$\langle \varphi_j, 1\rangle = \frac{\bar{x}}{\pi(j+\frac{1}{2})} \text{ and } \langle \varphi_j, \varphi_j\rangle = \frac{\bar{x}}{2}$$

Thus,

$$\hat{b}_j(t) = \frac{2}{\pi(j+\frac{1}{2})} \left(e^{-\mu_j t} \frac{I(0)}{U} + \int_0^t e^{-\mu_j(t-s)} B(s) ds \right)$$

Thus, if we compute:

$$M(t) = \int_0^{\bar{x}} m(x,t)dx = \sum_{j=0}^{\infty} \hat{b}_j(t) \int_0^{\bar{x}} \varphi_j(x)dx = \sum_{j=0}^{\infty} \hat{b}_j(t)\langle \varphi_j, 1\rangle$$

substituting the expression for $\hat{b}_j(t)$:

$$M(t) = \sum_{j=0}^{\infty} \tfrac{(\langle \varphi_j, 1\rangle)^2}{\langle \varphi_j, \varphi_j\rangle} \left(e^{-\mu_j t} \frac{I(0)}{U} + \int_0^t e^{-\mu_j (t-s)} B(s) ds \right)$$
$$= \sum_{j=0}^{\infty} \frac{2}{\left(\pi(j+\frac{1}{2})\right)^2} \left(e^{-\mu_j t} \frac{I(0)}{U} + \int_0^t e^{-\mu_j (t-s)} B(s) ds \right)$$

since

$$\frac{(\langle \varphi_j, 1\rangle)^2}{\langle \varphi_j, \varphi_j\rangle} = \left(\frac{\bar{x}}{\pi(j+\frac{1}{2})} \right)^2 \frac{1}{\bar{x}/2} = \bar{x} \frac{2}{\left(\pi(j+\frac{1}{2})\right)^2}$$

To check, note that at $t=0$:

$$M(0) = I(0) \frac{\bar{x}}{U} \sum_{j=0}^{\infty} \tfrac{(\langle \varphi_j, 1\rangle)^2}{\langle \varphi_j, \varphi_j\rangle} = I(0) \frac{\bar{x}}{U} \sum_{j=0}^{\infty} \frac{2}{\left(\pi(j+\frac{1}{2})\right)^2}$$

since $1 = \sum_{j=0}^{\infty} \frac{2}{\left(\pi(j+\frac{1}{2})\right)^2}$ Thus

$$N(t) = I(t) - \sum_{j=0}^{\infty} \tfrac{(\langle \varphi_j, 1\rangle)^2}{\langle \varphi_j, \varphi_j\rangle} \left(e^{-\mu_j t} \frac{I(0)}{U} + \int_0^t e^{-\mu_j (t-s)} B(s) ds \right)$$
$$= I(t) - \sum_{j=0}^{\infty} \bar{x} \frac{2}{\left(\pi(j+\frac{1}{2})\right)^2} \left(e^{-\mu_j t} \frac{I(0)}{U} + \int_0^t e^{-\mu_j (t-s)} B(s) ds \right)$$
$$= I(t) - \frac{\bar{x}}{U} \sum_{j=0}^{\infty} \frac{2}{\left(\pi(j+\frac{1}{2})\right)^2} \left(e^{-\mu_j t} I(0) + \beta_0 \int_0^t e^{-\mu_j (t-s)} I(s)(1-I(s)) ds \right)$$

So we can write:

$$N(t) = I(t) - \tfrac{\bar{x}}{U} \left(\sum_{j=0}^{\infty} \omega_j e^{-\mu_j t} I(0) + \beta_0 \int_0^t \sum_{j=0}^{\infty} \omega_j e^{-\mu_j (t-s)} I(s) \left(1 - I(s)\right) ds \right) \text{ where}$$
$$\omega_j \equiv \frac{2}{\left(\pi(j+\frac{1}{2})\right)^2} > 0 \text{ and } \sum_{j=0}^{\infty} \omega_j = 1\,.$$

Defining

$$H(z) \equiv \sum_{j=0}^{\infty} \omega_j e^{-\mu_j z}$$

we can write:

$$N(t) = I(t) - \tfrac{\bar{x}}{U}\left(H(t)I(t) + \beta_0 \int_0^t H(t-s)I(s)\left(1 - I(s)\right) ds\right) \text{ where}$$
$$\omega_j \equiv \frac{2}{\left(\pi(j+\frac{1}{2})\right)^2} > 0 \text{ and } \sum_{j=0}^{\infty} \omega_j = 1\,.$$

□

Proof. (of Proposition 18) We can rewrite the o.d.e. for $\tilde{m}$ as:

$$\tilde{m}(x) = \tfrac{\sigma^2}{2\nu}\tilde{m}_{xx}(x) + (1 - \tfrac{\nu}{\beta_0})\tfrac{1}{U} \text{ for all } x \in [0, \bar{x}]$$

The solution is given by a sum of particular solution, $(1-\frac{\nu}{\beta_0})\frac{1}{U}$, and two homogenous solutions. The homogenous solutions are exponentials $\exp(\pm\gamma x)$. The requirement that $\tilde{m}_x(0) = 0$ implies that the coefficient that multiplies each of the exponentials has the same absolute value but opposite sign, i.e., the two homogenous solutions combine into a cosh. Then, imposing that $\tilde{m}(\bar{x}) = 0$ we get:

$$\tilde{m}(x) = (1 - \tfrac{\nu}{\beta_0})\tfrac{1}{U}\left(1 - \frac{\cosh(\gamma x)}{\cosh(\gamma \bar{x})}\right) \text{ where } \gamma = \sqrt{2\nu}/\sigma$$

Thus, using that $\int_0^{\bar{x}} \frac{\cosh(\gamma x)}{\cosh(\gamma \bar{x})} = \frac{\tanh(\gamma \bar{x})}{\gamma}$ we obtain the desired result.

□

F HJB Equations for $a(x, t)$ and $v(x, t)$

Moreover, $a(x, t)$ solves the p.d.e. and boundary conditions for all $t \geq 0$:

$$\rho a(x,t) = x(\theta_0 + \theta_n N(t)) + \frac{\sigma^2}{2} a_{xx}(x,t) + a_t(x,t) \text{ if } x \in [0, U]$$
$$a_x(0,t) = a_x(U,t) = 0$$

where the boundary conditions arise from our assumption of reflecting barriers. Throughout, we assume $0 \leq a(x,t) \leq \frac{U(\theta_0+\theta_n)}{\rho}$ for all x, t, and $0 < c < \frac{U(\theta_0+\theta_n)}{\rho}$.

Adoption Decision: The value function of an agent that has not adopted solves the following variational inequality:

$$\rho v(x,t) = \max \left\{ \frac{\sigma^2}{2} v_{xx}(x,t) + v_t(x,t) \,,\, \rho\left(-c + a(x,t)\right) \right\}$$

for all $t \geq 0$ and $x \in [0, U]$. We conjecture that the optimal decision rule is given by a path for the threshold $\bar{x}(t) \in (0, U)$ such so that, for each $t \geq 0$, the following holds

$$\begin{aligned} \rho v(x,t) &= \frac{\sigma^2}{2} v_{xx}(x,t) + v_t(x,t) \text{ if } 0 \leq x \leq \bar{x}(t) \\ v(x,t) &= -c + a(x,t) \text{ if } \bar{x}(t) \leq x \leq U \end{aligned}$$

If $v(\cdot, t)$ is C^1 we have the following boundary conditions for all $t \geq 0$:

$$\begin{aligned} v(\bar{x}(t), t) &= a(\bar{x}(t), t) - c && \text{Value Matching} \\ v_x(\bar{x}(t), t) &= a_x(\bar{x}(t), t) && \text{Smooth Pasting} \\ v_x(0, t) &= 0 && \text{Reflecting} \end{aligned}$$

where the first one is the value matching condition, the second the smooth pasting condition, and the last one arises from the reflecting barrier at $x = 0$.

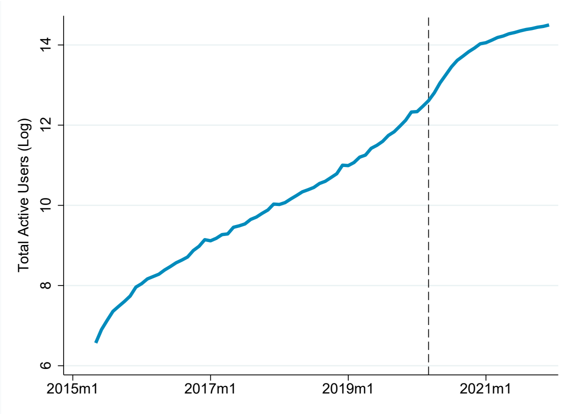
G Empirical Appendix

G.1 Descriptive Figures and Summary Statistics: SINPE

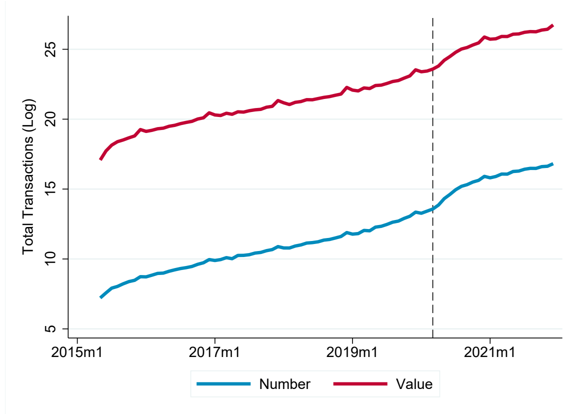
The technology diffused slowly. The aggregate adoption of SINPE has grown at a constant rate over time since its inception in 2015, as shown in Figure G1 using monthly data on the total number of adopters.[38] By 2021, close to 79% of the adult population in the country owned a bank account, and over 60% of adults were SINPE subscribers who had not deactivated their account. Moreover, the value of annual transactions in SINPE is approximately 10% of GDP. Thus, this setting has the unique feature of allowing us to study the adoption of mobile payments in the entire population of the country, across many years since the inception of the technology, and until it reached almost the universe of the country's adult population. The fact that adoption occurs gradually coincides with the dynamics of our dynamic stochastic model, and rules out the deterministic case in which adoption happens on impact.

[38]The figures include a vertical dashed line at the beginning of the COVID-19 pandemic (March 2020). As shown, it did not dramatically change the adoption rate.

Figure G1: Users, Transactions, and Value of Transactions



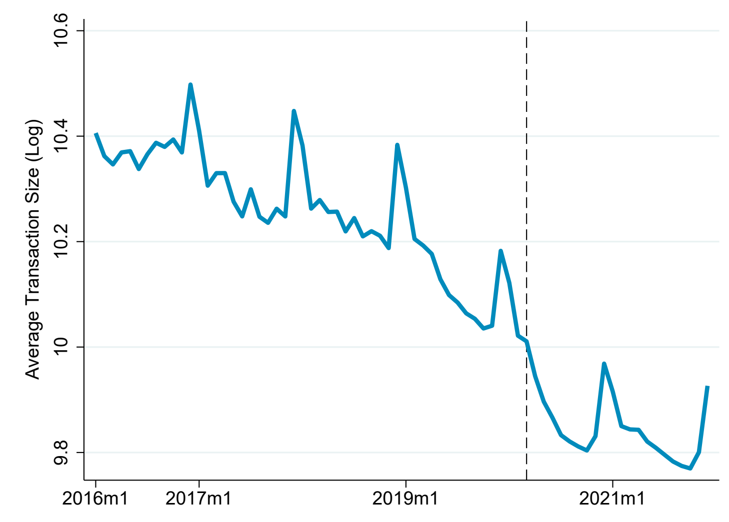
(a) Total Users



(b) Number and Value of Transactions

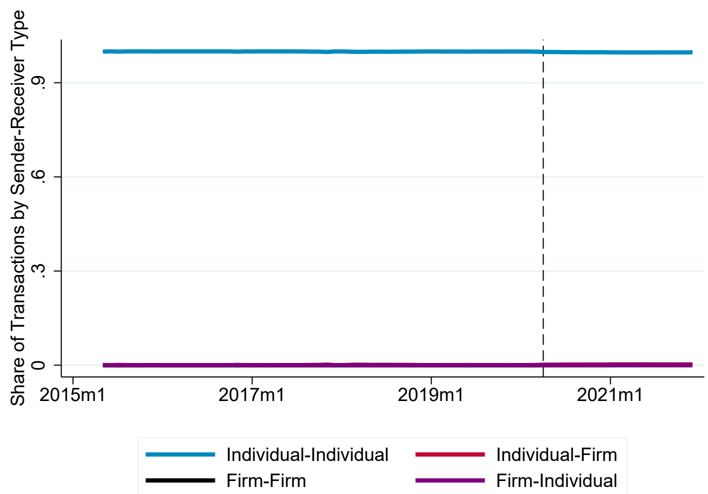
Notes: Panel (a) shows total active SINPE users. We include only active subscriptions by individuals, as users have the option of deactivating their account. Panel (b) shows both total transactions in the application and total value of transactions by individuals. Both figures include a vertical dashed line to mark the start of the COVID-19 pandemic (March 2020).

Figure G2: Average Transaction Size



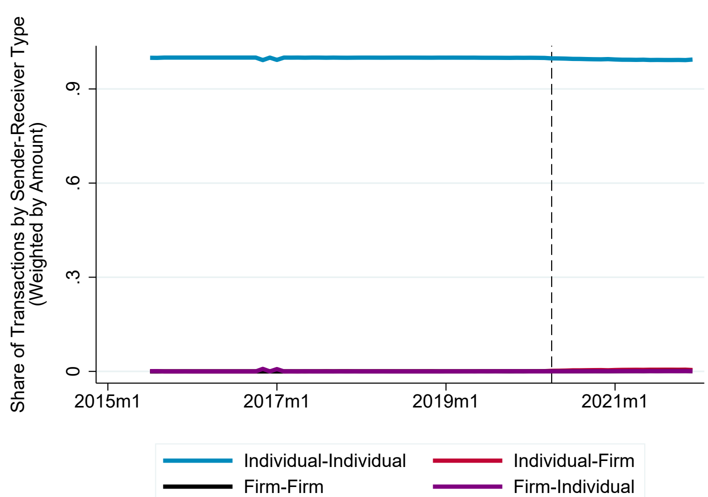
Notes: The figure shows the evolution of the average transaction size in SINPE.

Figure G3: Transactions by Sender-Receiver Type



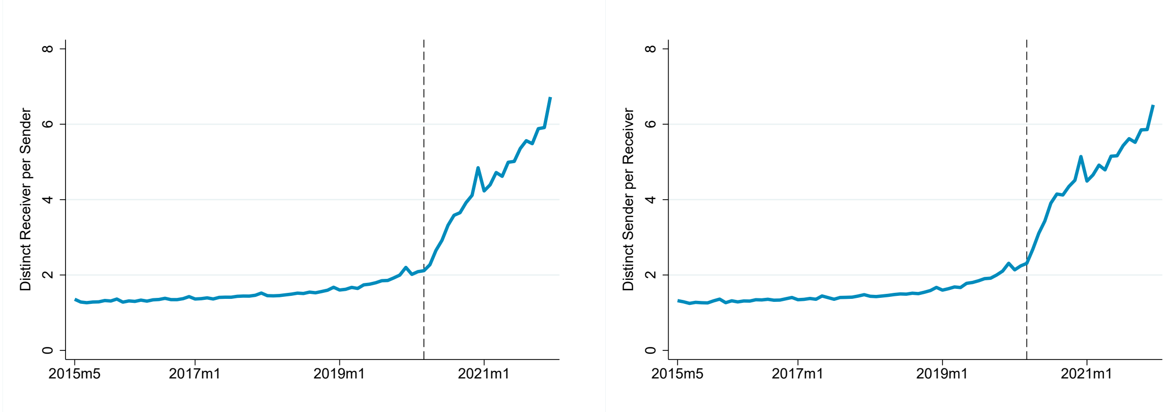
Notes: Transactions are classified according to the type of user. Individuals correspond with Costa Rican adult citizens. Firms correspond with formal enterprises.

Figure G4: Share of Transactions Between Types of Users (Weighted by Amount)



Notes: The figure shows total number of SINPE transactions between four different types of users, as a share of all of their transactions.

Figure G5: Mean Number of Connections per User



(a) Receivers (b) Senders

Figure G6: Average Age at the Time of Adoption

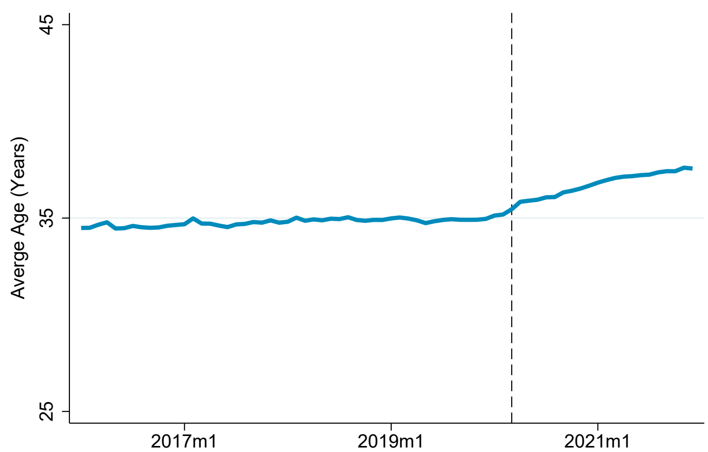


Table G1: Mean Share of Transactions Within Network (2015-2021)

	Neighborhood	Firm	Family	Union of all three
Neighborhood	0.39			
Firm	0.56	0.39		0.65
Family	0.50	0.58	0.25	

Notes: We construct average shares using data from May 2015, when the technology was introduced, to December 2021. Shares using data from the middle of the period (year 2018) only are shown in Table 1.

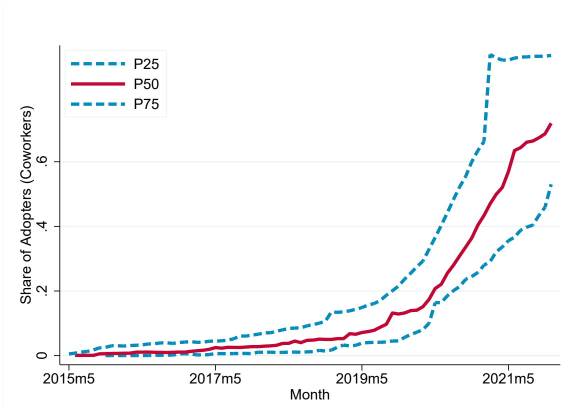
G.2 Evidence on Selection at Entry: Robustness

Table G2: Amount Transacted and Size of Network at Entry

	Dependent variable: Amount transacted (IHS)		
Size of Neighbors' Network at Entry	-5.805***		
	(0.014)		
Size of Coworkers' Network at Entry		-2.663***	
		(0.013)	
Size of Family Network at Entry			-2.077***
			(0.240)
Observations	7,135,126	163,050	6,742,411
R-squared	0.022	0.006	0.003
Network×Time/Cohort FE	Yes	Yes	Yes

Notes: The dependent variable in this estimation is the amount transacted each month for each user, which we transform using the inverse hyperbolic sine function. The coefficient describes the effect of increasing the share of an individual's network who had adopted the app at the time when she downloaded it. We run regressions using data from May 2015, when the technology was introduced, to December 2021.

Figure G7: Gradual Diffusion Across a Networks of Coworkers



(Coworkers)

Notes: The figure shows the dynamics of diffusion across networks defined as r coworkers. Percentiles are calculated in the period with highest adoption in the sample given the share of individuals that had adopted the technology.

G.3 Evidence on Strategic Complementarities: Robustness

Table G3: Changes in Number of Transactions and Network Changes

	Dependent variable: Δ Number of Transactions			
(a) Logs	(1)	(2)	(3)	(4)
Δ Share Neighborhood Adopters	0.597***			0.470***
	(0.023)			(0.032)
Δ Share Coworkers Adopters		0.205***		0.202***
		(0.007)		(0.007)
Δ (Log) Wag		0.046***		0.046***
		(0.001)		(0.001)
Δ Share Relatives Adopters			0.228***	0.250***
			(0.003)	(0.004)
Observations	24,025,266	12,374,020	22,775,723	11,727,213
Time/Cohort FE	Yes	Yes	Yes	Yes
Adjusted R-squared	0.020	0.025	0.021	0.026
(b) Davis & Haltiwanger				
Δ Share Neighborhood Adopters	1.286***			1.090***
	(0.027)			(0.037)
Δ Share Coworkers Adopters		0.302***		0.293***
		(0.009)		(0.009)
Δ (Log) Wage		0.047***		0.047***
		(0.001)		(0.001)
Δ Share Relatives Adopters			0.305***	0.339***
			(0.004)	(0.005)
Observations	28,160,145	14,311,886	26,663,615	13,549,708
Time/Cohort FE	Yes	Yes	Yes	Yes
Adjusted R-squared	0.014	0.018	0.015	0.019

Notes: The unit of observation is the individual. We run regressions using data from May 2015, when the technology was introduced, to December 2021. Standard errors (clustered by individual) are in parentheses. All regressions control for network size (in levels).

Table G4: Intensity of Usage and Network Changes (Value of Transactions)

	Dependent variable: Δ Value of Transactions (IHS)			
Δ Share Neighborhood Adopters	3.954***			3.573***
	(0.101)			(0.141)
Δ Share Coworkers Adopters		0.824***		0.796***
		(0.033)		(0.034)
Δ (Log) Wage		0.126***		0.125***
		(0.003)		(0.003)
Δ Share Relatives Adopters			0.867***	0.997***
			(0.014)	(0.019)
Observations	32,391,602	16,232,003	30,633,379	15,355,945
Time/Cohort FE	Yes	Yes	Yes	Yes
Adjusted R-squared	0.008	0.009	0.008	0.009

Notes: The unit of observation is the individual. Regressions control for firm size (in levels). We run regressions using data from May 2015, when the technology was introduced, to December 2021. Standard errors (clustered by individual) are in parentheses.

Table G5: Changes in Intensity of Usage and 2021 Network Changes

	Dependent variable: %Δ Number of Transactions		
	Logs	Davis & Haltiwanger	Inverse hyperbolic sine
Δ Share Adopters in 2021 Network	1.815***	1.950***	1.580***
	(0.007)	(0.008)	(0.006)
Observations	23,512,962	27,532,941	31,682,276
R-squared	0.022	0.017	0.017
Time/Cohort FE	Yes	Yes	Yes

Notes: The unit of observation is the individual. We run regressions using data from May 2015, when the technology was introduced, to December 2021. These regressions do not control for network size (in levels) as by construction in this exercise size is constant across time. We run regressions using data from May 2015, when the technology was introduced, to December 2021. Standard errors, clustered by individual, are in parentheses.

Table G6: Weighted Changes in Intensity of Usage and 2021 Network Changes)

Dependent variable: %Δ Value of Transactions			
	Logs	Davis & Haltiwanger	Inverse hyperbolic sine
Δ Share Adopters in 2021 Network	5.375***	1.802***	1.978***
	(0.031)	(0.012)	(0.009)
Observations	32,355,374	24,022,540	28,150,990
Time FE/Cohort	Yes	Yes	Yes
Adjusted R-squared	0.009	0.019	0.018

Notes: The unit of observation is the individual. We run regressions using data from May 2015, when the technology was introduced, to December 2021. These regressions do not control for network size (in levels) as by construction in this exercise size is constant across time. Standard errors, clustered by individual, are in parentheses.

Table G7: Leave-One-Out Instrument

First Stage. Dependent Variable: $\Delta N_{it}^{neighborhood}$	
$\Delta N_{-i,t}^{district}$	0.694***
	(0.054)
Observations	32,391,602
Clusters	1,987,052
Time/Cohort FE	Yes
F-statistic	31,717.94
Second Stage. Dependent variable: Δ Number of Transactions (IHS)	
$\Delta N_{-i,t}^{district}$	0.694***
	(0.054)
Observations	32,391,602
Clusters	1,987,052
Time/Cohort FE	Yes

Notes: The unit of observation is the individual. Robust standard errors are in parentheses in the first panel; standard errors clustered by individual are in parentheses in the second panel. Regressions control for network size (in levels).

Table G8: Leave-One-Out Instrument (Value of Transactions)

First Stage. Dependent Variable: $\Delta N_{it}^{neighborhood}$	
$\Delta N_{-i,t}^{district}$	1.596***
	(0.009)
Observations	32,391,602
Clusters	1,987,052
Time/Cohort FE	Yes
F-statistic	31,717.94
Second Stage. Dependent variable: Δ *Number of Transactions (IHS)*	
$\Delta N_{-i,t}^{district}$	0.694***
	(0.054)
Observations	32,391,602
Clusters	1,987,052
Time/Cohort FE	Yes

Notes: The unit of observation is the individual. Robust standard errors are in parentheses in the first panel; standard errors clustered by individual are in parentheses in the second panel. Regressions control for network size (in levels).

Table G9: Intensity of Usage and Network Changes: Balanced Panel of 2016 Users

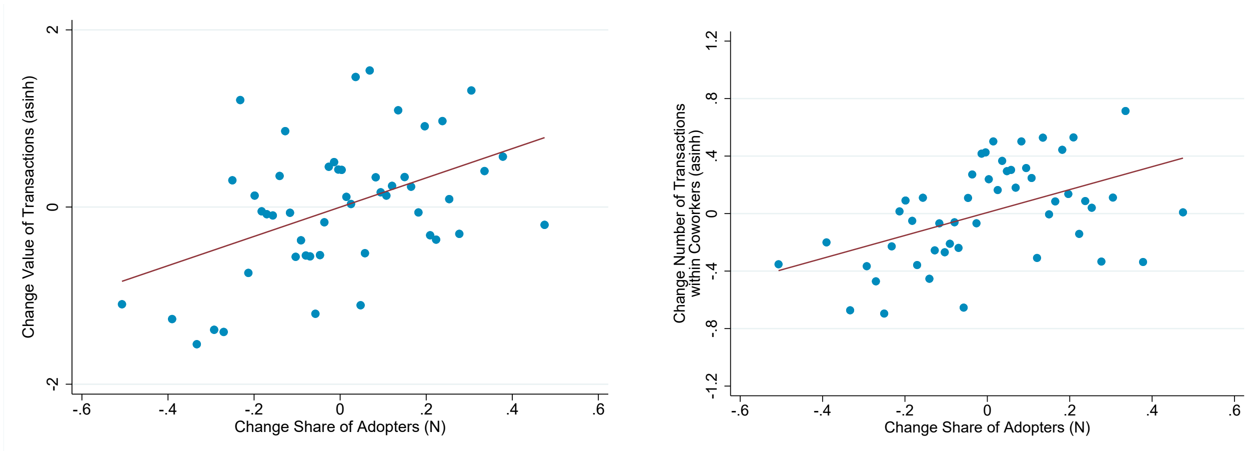
Dependent variable: Δ *Number of Transactions (IHS)*

Δ Share Neighborhood Adopters	0.768***			0.599***
	(0.134)			(0.157)
Δ Share Coworkers Adopters		0.157***		0.139***
		(0.031)		(0.031)
Δ (Log) Wage		0.035***		0.035***
		(0.003)		(0.003)
Δ Share Relatives Adopters			0.441***	0.491***
			(0.020)	(0.025)
Observations	1,073,880	743,321	1,026,384	710,142
Time/Cohort FE	Yes	Yes	Yes	Yes
Adjusted R-squared	0.014	0.016	0.015	0.017

Notes: The unit of observation is the individual, and we consider only a subsample of users who had already adopted by 2016, and follow them from then onwards, until December 2021. Regressions control for firm size (in levels). Standard errors (clustered by individual) are in parentheses.

G.4 Mass Layoffs and Adoption Changes: Robustness and Details

Figure G8: Marginal Effect of Network Changes on Usage Intensity (Value of Transactions)



(a) All transactions

(b) Transactions with coworkers only

Notes: Panel (a) plots the marginal effect of $\Delta N_i^{coworkers}$ in the specification described by Column (4) of Table 4. Bars denote 95% confidence intervals. The dependent variable in this estimation is the value of transactions (transformed using the inverse hyperbolic sine function) on each period for each user. Panel (b) is similar, but differs as the dependent variable in this estimation is the value of transactions *which have a coworker as a counterpart* (inverse hyperbolic sine function) on each period for each user.

Table G10: Intensity of Usage and Changes in Coworkers' Network After a Mass Layoff (Value of Transactions)

Dependent Variable: Δ Value of transactions (IHS)

$\Delta N_i^{coworkers}$	4.879***	3.134***	2.075***	1.519**
	(0.509)	(0.552)	(0.588)	(0.594)
$\Delta \ln wage_i$		0.833***	0.736***	0.866***
		(0.157)	(0.157)	(0.166)
$\Delta Covid_i$			0.317***	0.238***
			(0.065)	(0.076)
Observations	1,554	1,554	1,554	1,554
Time FE	No	Yes	Yes	Yes
Cohort FE	No	No	No	Yes
Adjusted R-squared	0.063	0.116	0.130	0.170

Notes: The unit of observation is the individual. We run regressions using data on mass layoffs that occurred between May 2015, when the technology was introduced, until December 2021. All regressions control for network size (in levels). Standard errors are in parentheses.

Table G11: Intensity of Usage *Among Coworkers* and Changes in Coworkers' Network After a Mass Layoff

Dependent Variable: Δ Number of transactions (IHS)

$\Delta N_i^{coworkers}$	1.338***	0.776***	0.690***	0.640***
	(0.138)	(0.172)	(0.185)	(0.192)
$\Delta \ln wage_i$		0.361***	0.352***	0.367***
		(0.041)	(0.042)	(0.046)
$\Delta Covid_i$			0.029	0.034
			(0.020)	(0.024)
Observations	1,554	1,554	1,554	1,554
Time FE	No	Yes	Yes	Yes
Cohort FE	No	No	No	Yes
Adjusted R-squared	0.057	0.099	0.099	0.086

Notes: The unit of observation is the individual. We run regressions using data on mass layoffs that occurred between May 2015, when the technology was introduced, until December 2021. Standard errors are in parentheses.

Table G12: Intensity of Usage *Among Coworkers* and Changes in Coworkers' Network After a Mass Layoff (Value of Transactions)

Dependent Variable: Δ Value of transactions (IHS)

$\Delta N_i^{coworkers}$	5.639***	4.235***	3.763***	3.564***
	(0.584)	(0.663)	(0.728)	(0.772)
$\Delta \ln wage_i$		1.628***	1.585***	1.589***
		(0.176)	(0.178)	(0.190)
$\Delta Covid_i$			0.141	0.198*
			(0.088)	(0.101)
Observations	1,554	1,554	1,554	1,554
Time FE	No	Yes	Yes	Yes
Cohort FE	No	No	No	Yes
Adjusted R-squared	0.058	0.108	0.109	0.100

Notes: The unit of observation is the individual. We run regressions using data on mass layoffs that occurred between May 2015, when the technology was introduced, until December 2021. All regressions control for network size (in levels). Standard errors are in parentheses.

G.4.1 Details on Mass Layoffs

This section provides additional details on the choices made to construct the variables and sample used in Section 7.3

Definition of a Mass Layoff To define a mass layoff, we follow Davis and Von Wachter (2011) and identify establishments with at least 50 workers that contracted their monthly employment by at least 30% *and* which did not recover in the following 12 months. We define a recovery as a firm which went back to its initial size (or above) within the following 12 months. Given this definition, the descriptive statistics of firms and workers impacted by a mass layoff are reported in Table G13.

Table G13: Mass Layoffs: Descriptive Statistics

Number of firms	856	
Number of displaced workers who had not adopted SINPE when fired	32,620	
Number of displaced workers who had adopted SINPE when fired	2,585	
Average firm size	529	(2147)
Average monthly wage pre-layoff, laid-off workers	\$504	(\$623)
Average monthly wage pre-layoff, all workers	\$663	(\$487)

Notes: Standard deviations for mean variables are reported in parenthesis. We consider layoffs that reduce in 30 workers or more the size of firms with at least 50 workers, and limit the analysis to workers with a period of unemployment of 6 months or less. Wages were calculated based on an exchange rate of 634 colones per dollar and the last month in which workers were employed. We include mass layoffs which occurred between May 2015, when the technology was introduced, and December 2021. The last row includes the average monthly wage pre-layoff for all workers who were employed at those firms at the time of the mass layoff.

Definition of Variables We construct several variables that are used in equation (36). We now provide more details on each of them.

- $Adopt_i$ equals one if individual i adopted SINPE within 6 months after arriving to her new firm, and zero otherwise. This variable is only computed for individuals who found a job within 6 months of being fired. Results are robust to considering shorter unemployment spells, including conducting the analysis using only job-to-job transitions.

- $\Delta N_i^{coworkers}$ is the change between the share of coworkers who had adopted at the old and the new employer. We compute this variable by calculating the difference between (i) the share of adopters at the old firm on the last month in which the individual was employed and (ii) the share of adopters at the new firm in month i, and considering only months i after the individual was hired at the new firm.

- $\Delta \ln wage_i$ corresponds with the change in the average wage (in logs) across 6 months before the layoff and after the rehiring.

- $\Delta \ln size_i$ is the change in the number of workers (in levels) at the new firm versus the old firm.

- $date\,hired_i$ controls for the month in which individual i was hired by the new firm.

- $\ln \sum_{t=0}^{move} \left(\tilde{\xi}_{t,\text{ new firm}} - \tilde{\xi}_{t,\text{ old firm}} \right)$ is the difference in the historical transactions made by workers at the new firm and the old firm prior to the move, which aims to control for factors, other than strategic complementarities, which might facilitate adoption at the new vs. the old firm.

- $\Delta Covid_i$ controls for the change in the cumulative COVID-19 cases (transformed using the inverse hyperbolic sine function) in the individual's neighborhood across the 6 months before the layoff and after the rehiring. This change is zero for pre-pandemic years, thus, this variable is introduced using an inverse hyperbolic sine transformation, as opposed to a logarithm.

The regression described in equation (37) relies on the same variables that we described above, but also includes additional ones which we now describe.

- $\Delta \ln \tilde{\xi}_i$ refers to the change in monthly intensity with which individual i used SINPE within 6 months *after* arriving to her new firm compared with 6 months *before* being fired. We only compute this variable for workers who had adopted SINPE more than 6 months before being fired, in order to attenuate any effect coming from a "learning curve." We transform $\tilde{\xi}_i$ using the inverse hyperbolic sine function, as zeros are common in the monthly data. Note that this inflates coefficients, particularly, for large values of intensity, which are likely to appear when the left-hand-side variable describes the total value (as opposed to the number) of transactions.

- $cohort_i$ controls for the month when individual i adopted SINPE. We include this variable to attenuate any effect coming from learning how to better use the app.

- $\ln \sum^t \tilde{\xi}_i$ is the sum of all historical transactions made by agent i since she adopted the app. This variable has no zeros by construction, as our definition of adoption is that the individual has used the app at least once. Similarly to $cohort_i$, the variable intends to control for learning how to use the app thanks to having more people in your network who have adopted it.

H Quantitative Exercises

H.1 Calibration

In this section, we describe the procedure to calibrate σ and θ_0 using simulated method of moments. Intuitively, we want to choose these two parameters so that they are consistent with the distribution of transactions in the data. To do so, in the data, we focus on a balanced sample of users that were active by 2019, and compute moments for the distribution of transactions over the years 2020-2021 in neighborhoods close to a stationary equilibria (i.e., the top 5 percentile of neighborhoods in terms of adoption). We simulate the model replicating the same characteristics of our empirical sample. This is, we start from the invariant distribution and simulate a panel of 5000 users for two years.[39] We then compute the distribution of transactions both in the data and the model and choose σ and θ_0 to minimize the distance between the two distributions. We provide further details of this procedure below.

We begin by simulating the model for a panel of agents for different values of σ and θ_0. Our simulation takes as given the values of ν, r, ρ, β_0, and $\widetilde{\theta}$, since they are calibrated either externally or using other reduced form evidence. Initial conditions $x(0)$ are drawn from the stationary distribution of adopters. To find this distribution, we first find $\bar{x}$ (given $N_{ss} = 0.93$) using the following equation:

$$N_{ss} = (1 - \tfrac{\nu}{\beta_0}) \left[1 - \tfrac{\bar{x}}{U} \left(1 - \frac{\tanh(\gamma\bar{x})}{\gamma\bar{x}}\right)\right].$$

Then, given $\bar{x}$, we find the distribution of adopters using the stationary distribution of non-adopters:

$$\tilde{m}(x) = (1 - \tfrac{\nu}{\beta_0})\tfrac{1}{U} \left(1 - \frac{\cosh(\gamma x)}{\cosh(\gamma\bar{x})}\right) \text{ where } \gamma = \sqrt{2\nu}/\sigma$$

using that $N_{ss} = I_{ss} - M_{ss}$ and $I_{ss} = (1 - \frac{\nu}{\beta_0})$. In the simulation, agents die at rate ν and they become inactive in the application just as in the data. The process of x follows a Brownian motion, independent across agents, with variance per unit of time σ, no drift, and reflecting barriers at $x = 0$ and $x = U$. We then interpret the flow benefit of agents who adopt the technology as being proportional to how intensively they use SINPE. Thus, we compute

$$\xi_t = [\theta_0(1 + \widetilde{\theta} N_{ss}) x_t]^{\frac{1}{1+p}}. \tag{83}$$

Given the discreteness of the number of transactions in the data, ξ_t is interpreted as the mean of a Poisson distribution; transactions each period are drawn from a Poisson probability distribution with mean ξ_t.

Parameters σ and θ_0 are chosen to minimize a sum of the percent deviations of simulated

[39] Our estimates are not sensitive to simulating a larger sample of users.

moments from target moments:

$$\min \sum_{i}^{4} \frac{|\text{Model}(i) - \text{Data}(i)|}{\text{Data}(i)}$$

where Model(i) is a simulated i-th moment and Data(i) is a target value of i-th moment. Table H1 reports the empirical and simulated moments.

Table H1: Moments: Distribution of Transactions

Moment	Data	Model
Mean Number of Transactions	8.76	8.69
Median Number of Transactions	7.97	8.48
Absolute Value Changes in Transactions	3.87	3.22
IQR Changes in Transactions	4.56	4.00

Intuitively, equation (83) implies

$$\sqrt{\text{Var}\left(\frac{\xi_{t+\Delta}^{1+p}}{1+\widetilde{\theta}N_{ss}} - \frac{\xi_{t}^{1+p}}{1+\widetilde{\theta}N_{ss}}\right)} = \theta_0\sqrt{\Delta}\sigma.$$

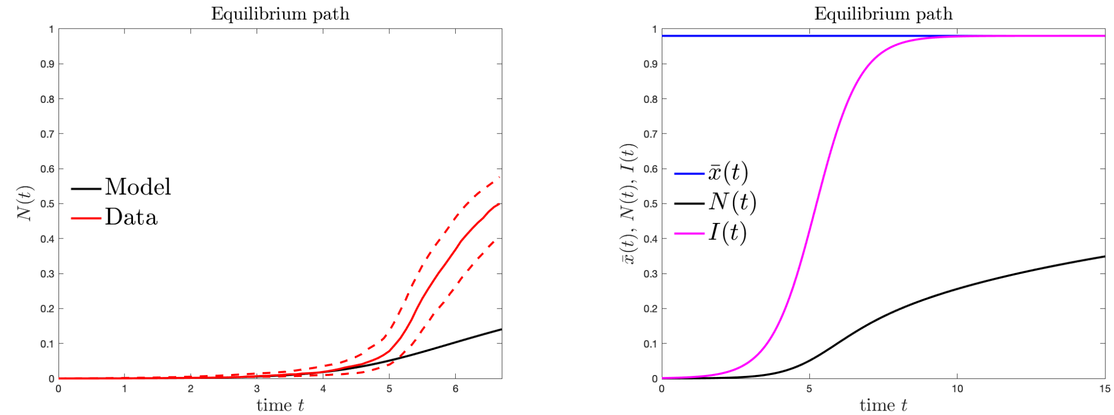
Thus, the dispersion in the changes of transactions contains relevant information to pin down σ. Similarly, the average number of transactions provides information relevant to pin down θ_0. This can be seen taking expectations of equation (83)

$$\mathbb{E}(\xi^{1+p}) = \theta_0(1+\widetilde{\theta}N_{ss})\mathbb{E}(x).$$

H.2 Only Learning: $\widetilde{\theta} = 0$

In this section, we examine the behavior of a model without strategic complementarities. Not surprisingly, if we keep all parameter at their baseline value and set $\theta_n = 0$, the model predicts lower adoption at its stationary equilibrium, $N_{ss} = 0.59$. The adoption in this model is purely determined by the idiosyncratic benefits of the technology. Panel (b) shows that convergence to the stationary equilibrium takes longer in a model without complementarities. Recall that the model matches the fraction of agents informed about the technology three years after it was launched. Panel (b) suggests that in a pure learning model, adoption would be much slower than that observed in the data. Panel (b) also shows that the path of $\bar{x}(t)$ in the model with only learning is flat, which indicates there is no selection in the adoption of the technology as observed in the data. Importantly, this version of the model is constrained efficient: the optimal subsidy to use the technology is zero.

Figure H1: Path of Adopters - Only Learning (Short-Run and Long-Run)



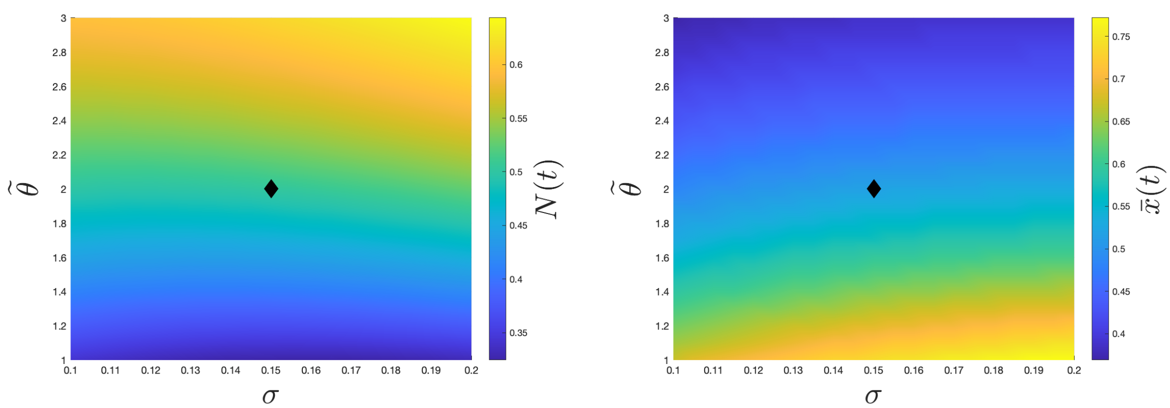
(a) Model vs Data (b) Long-Run Path

Notes: Panel(a) compares the path of adopters in the model with $\theta_n = 0$ and in the data. The solid red line shows the patterns of diffusion of the technology in the median neighborhood, where the percentile is calculated in the last period of the sample using the share of individuals that had adopted the technology. The dashed red lines show the 25^{th} and 75^{th} percentiles. Panel (b) shows the share of informed agents, $I(t)$, the share of adopters, $N(t)$, and the levels of $\bar{x}(t)$ predicted by the model under our baseline calibration but setting $\theta_n = 0$.

H.3 Comparative Statics

H.3.1 Stochastic Model: Short-Run

Figure H2: Adoption: $N(t)$ and $\bar{x}(t)$

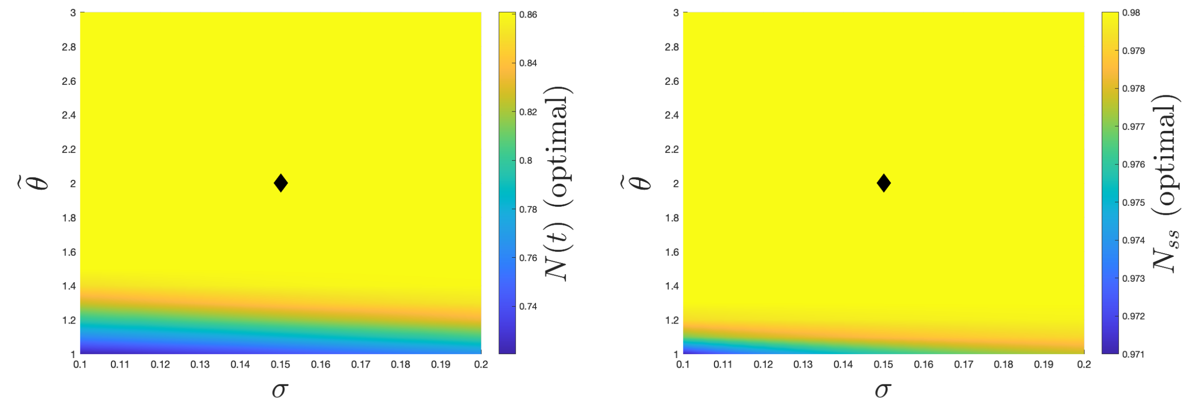


(a) Comparative Statics: $N(t)$ (b) Comparative Statics: $\bar{x}(t)$

Notes: Panel (a) and (b) show how $N(t)$ and $\bar{x}(t)$ change with $\widetilde{\theta}$ and σ, keeping the rest of the parameters constant 7 years after the technology was launched. The black diamonds indicate the levels of $\widetilde{\theta}$ and σ in our baseline calibration.

H.3.2 Stochastic Model: Planning Problem

Figure H3: Optimal Adoption: $N(t)$ and N_{ss}



(a) Comparative Statics: $N(t)$

(b) Comparative Statics: N_{ss}

Notes: Panel (a) shows how $N(t)$ changes 7 years after the technology was launched with $\widetilde{\theta}$ and σ, keeping the rest of the parameters constant. The black diamonds indicate the levels of $\widetilde{\theta}$ and σ in our baseline calibration. Panel (b) shows the same comparative static for N_{ss}.