Technical Appendix for

Strategic Complementarities in a Dynamic Model of Techonology Adoption: P2P Digital Payments

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Fernando Alvarez
University of Chicago and NBER

David Argente
Penn State University and NBER

Francesco Lippi
LUISS and EIEF

Esteban Méndez
Central Bank of Costa

Diana Van Patten
Yale University and NBER

I Solution of the Stationary Problem

I.1 Solution for $\tilde{a}(x)$ and $\tilde{v}(x)$

The solution to $\tilde{a}$ is of the form:

$$\tilde{a}(x) = x\frac{\theta_0 + \theta_n N_{ss}}{\rho} + A_1 e^{\eta x} + A_2 e^{-\eta x}$$

for $\eta = \sqrt{2\rho/\sigma^2}$, and

$$0 = \frac{\theta_0 + \theta_n N_{ss}}{\rho} + \eta(A_1 - A_2) = \frac{\theta_0 + \theta_n N_{ss}}{\rho} + \eta(A_1 e^{\eta U} - A_2 e^{-\eta U})$$

Thus, given $\theta_0 + \theta_n N_{ss}$, the constants (A_1, A_2) are the solution of two linear equations. Moreover, the values of A_1, A_2 are proportional to $\bar{\theta}_{ss}$ given by

$$\bar{\theta}_{ss} \equiv \frac{\theta_0 + \theta_n N_{ss}}{\rho} = \eta(A_2 - A_1) = \eta(A_2 e^{-\eta U} - A_1 e^{\eta U})$$

Let $\bar{A}_i \equiv A_i/\bar{\theta}_{ss}$, we can write:

$$1 = \eta(\bar{A}_2 - \bar{A}_1) = \eta(\bar{A}_2 e^{-\eta U} - \bar{A}_1 e^{\eta U})$$

which has solution:

$$\bar{A}_1 = \frac{1}{\eta}\frac{\left(1 - e^{-\eta U}\right)}{\left(e^{-\eta U} - e^{\eta U}\right)} \quad , \quad \bar{A}_2 = \frac{1}{\eta}\frac{\left(1 - e^{\eta U}\right)}{\left(e^{-\eta U} - e^{\eta U}\right)}$$

The solution for $\tilde{v}$ for $x \in [0, \bar{x}_{ss}]$ is of the form

$$\tilde{v}(x) = B_1 e^{\eta x} + B_2 e^{-\eta x}$$

Given the solution for $\tilde{a}$, then $B_1, B_2, \bar{x}_{ss}$ solve:

$$\begin{aligned} 0 &= \eta(B_1 - B_2) \\ \tilde{a}_x(\bar{x}_{ss}) &= \eta(B_1 e^{\eta \bar{x}_{ss}} - B_2 e^{-\eta \bar{x}_{ss}}) \\ \tilde{a}(\bar{x}_{ss}) - c &= B_1 e^{\eta \bar{x}_{ss}} + B_2 e^{-\eta \bar{x}_{ss}} \end{aligned}$$

Thus, using the first equation $B_1 = B_2 = B$ and taking the ratio of these equations:

$$\frac{\tilde{a}(\bar{x}_{ss}) - c}{\tilde{a}_x(\bar{x}_{ss})} = \frac{1}{\eta}\frac{e^{\eta\bar{x}_{ss}} + e^{-\eta\bar{x}_{ss}}}{(e^{\eta\bar{x}_{ss}} - e^{-\eta\bar{x}_{ss}})}$$

Replacing the expressions for $\tilde{a}(\bar{x}_{ss})$ and $\tilde{a}'(\bar{x}_{ss})$, we obtain:

$$\frac{\bar{x}_{ss} + \bar{A}_1 e^{\eta\bar{x}_{ss}} + \bar{A}_2 e^{-\eta\bar{x}_{ss}} - c/\bar{\theta}_{ss}}{1 + \eta\left(\bar{A}_1 e^{\eta\bar{x}_{ss}} - \bar{A}_2 e^{-\eta\bar{x}_{ss}}\right)} = \frac{1}{\eta}\frac{e^{\eta\bar{x}_{ss}} + e^{-\eta\bar{x}_{ss}}}{(e^{\eta\bar{x}_{ss}} - e^{-\eta\bar{x}_{ss}})}$$

Note that this is one equation for $\bar{x}_{ss}$ as a function of $\bar{\theta}_{ss}$ (recall that $\bar{A}_1, \bar{A}_2$ are known constants). The last expression can be written as

$$\eta\bar{x}_{ss} + \eta\bar{A}_1 e^{\eta\bar{x}_{ss}} + \eta\bar{A}_2 e^{-\eta\bar{x}_{ss}} - \frac{e^{\eta\bar{x}_{ss}} + e^{-\eta\bar{x}_{ss}}}{(e^{\eta\bar{x}_{ss}} - e^{-\eta\bar{x}_{ss}})}\left(1 + \eta\left(\bar{A}_1 e^{\eta\bar{x}_{ss}} - \bar{A}_2 e^{-\eta\bar{x}_{ss}}\right)\right) = \frac{\eta}{\bar{\theta}_{ss}}c$$

which gives equation (19) in the main text.

Letting $y \equiv \eta\bar{x}_{ss}$ and defining $\psi(y)$ we can write

$$\begin{aligned}\psi(y) &\equiv y + \eta\left(\bar{A}_1 e^{y} + \bar{A}_2 e^{-y}\right) - \frac{e^{y} + e^{-y}}{(e^{y} - e^{-y})}\left(1 + \eta\left(\bar{A}_1 e^{y} - \bar{A}_2 e^{-y}\right)\right) \\ &= \frac{\eta}{\bar{\theta}_{ss}}c\end{aligned}$$

We can approximate the left hand side around $\bar{x}_{ss} = 0$, which corresponds to $c = 0$. Using that $\eta\bar{A}_2 = \eta\bar{A}_1 + 1$, we have the following properties.

1. $\psi(0) = 0$, $\psi(y) > 0$ if $y > 0$
2. $\psi'(y) = \frac{e^{2y}+1}{(e^y+1)^2}$ so $\psi'(0) = \frac{1}{2}, \psi'(\infty) = 1$, and $\psi''(y) > 0$,
3. $\psi(y) = \frac{y}{2} + \frac{y^3}{24} + o(y^4)$ and $\lim_{y\to\infty}\frac{\psi(y)-y}{y} = 0$

Now we use ψ to solve for $\bar{x}_{ss} = \chi(\eta, c/\bar{\theta}_{ss})$ i.e. $\frac{\eta c}{\bar{\theta}_{ss}} = \psi(\eta\chi(\eta, c/\bar{\theta}_{ss}))$. $\bar{x}_{ss}$ is the unique solution of $\frac{\psi(\eta\bar{x}_{ss})}{\eta} = \frac{c}{\bar{\theta}_{ss}}$, which always exists. For fixed $0 < \eta < \infty$ and small c using the first order approximation:

$$y = \eta\bar{x}_{ss} = 2\frac{\eta c}{\bar{\theta}_{ss}} \text{ or } \bar{x}_{ss} = 2\frac{c}{\bar{\theta}_{ss}}$$

since $\eta = \frac{\sqrt{2\rho}}{\sigma}$ the option value for a fixed $\bar{\theta}$ is given by:

$$\lim_{c\to 0} \frac{\chi(\eta, c/\bar{\theta}_{ss})}{\chi(\infty, c/\bar{\theta}_{ss})} = 2$$

For fixed $0 < \eta < \infty$ and small c, using the third order approximation $y^3 + 12y = \hat{\kappa} \equiv \frac{24\eta c}{\bar{\theta}_{ss}}$ or:

$$\begin{aligned}\bar{x}_{ss} &= \frac{1}{\eta}\left(\frac{1}{2}\hat{\kappa} + \sqrt{\frac{1}{4}\hat{\kappa}^2 + \frac{12^3}{27}}\right)^{1/3} + \frac{1}{\eta}\left(\frac{1}{2}\hat{\kappa} - \sqrt{\frac{1}{4}\hat{\kappa}^2 + \frac{12^3}{27}}\right)^{1/3} \\ &= \frac{1}{\eta}\left(\frac{1}{2}\right)^{1/3}\left[\left(\hat{\kappa} + \sqrt{\hat{\kappa}^2 + 16}\right)^{1/3} + \left(\hat{\kappa} - \sqrt{\hat{\kappa}^2 + 16}\right)^{1/3}\right]\end{aligned}$$

For the case when σ is small (i.e., η is large), let $S(y) \equiv y - \psi(y) + 1$ and recall that $\lim_{y\to\infty} S(y) = 0$. Then, using the definitions of y and $\psi(y)$, this implies

$$\lim_{\sigma\to 0} \frac{\sqrt{2\rho}}{\sigma}\left(\chi\left(\infty, \frac{c}{\bar{\theta}_{ss}}\right) - \frac{c}{\bar{\theta}_{ss}} - \frac{\sigma}{\sqrt{2\rho}}\right) = 0$$

Thus, for σ small we can use:

$$\bar{x}_{ss} = \frac{c}{\bar{\theta}_{ss}} + \frac{\sigma}{\sqrt{2\rho}} + o(\sigma)$$

Alternatively, note that $\bar{x}_{ss} - \frac{\sigma}{\sqrt{2\rho}}$ is the derivative of $\frac{\psi(\eta\bar{x}_{ss})}{\eta}$ with respect to σ evaluated at $\sigma = 0$.

I.2 Solution for $\tilde{m}(x)$

We can write the solution of the KFE as the sum the two homogeneous and the particular solution m_p, given $\bar{x}_{ss}$, i.e.

$$\tilde{m}(x) = C_1 e^{\gamma x} + C_2 e^{-\gamma x} + m_p(x)$$

where $\gamma = \sqrt{2\nu/\sigma^2}$. The solution is

$$\tilde{m}(x) = \frac{1}{U}\left[1 - \frac{(e^{\gamma x} + e^{-\gamma x})}{(e^{\gamma \bar{x}_{ss}} + e^{-\gamma \bar{x}_{ss}})}\right] \text{ for } x \in [0, \bar{x}_{ss}]$$

Finally, we want to compute:

$$1 - N_{ss} = \int_0^{\bar{x}_{ss}} \tilde{m}(x)dx = \int_0^{\bar{x}_{ss}} \frac{1}{U}\left[1 - \frac{(e^{\gamma x} + e^{-\gamma x})}{(e^{\gamma \bar{x}_{ss}} + e^{-\gamma \bar{x}_{ss}})}\right] dx$$

This gives another equation for $\bar{x}_{ss}$ as function of $\bar{\theta}$.

J The Dynamics of the Deterministic Model

For each x, let $a(x,t) = x\,\alpha(t)$ where $\rho\alpha(t) = \theta_0 + \theta_n N(t) + \alpha_t(t)$. We fix an x and a path $\alpha(t)$ for $t \geq 0$. Let $t^*(x)$ be the optimal t that solves the adoption problem:

$$t^*(x) = \arg\max_{t \geq 0} G(t,x) \text{ with } G(x,t) \equiv e^{-\rho t}(\alpha(t)x - c)$$

The necessary first order conditions if $\alpha(t)$ is differentiable at $t = t^*(x) < \infty$ are:[40]

$$-\rho(\alpha(t^*)x - c) + x\alpha_t(t^*) = 0 \text{ if } t^*(x) > 0$$
$$-\rho(\alpha(0)x - c) + x\alpha_t(0) \leq 0 \text{ if } t^*(x) = 0$$

Furthermore, since not adopting is feasible (i.e., $t = \infty$) and yields a zero payoff, then

$$\alpha(t^*(x))x \geq c \text{ for all } x.$$

Given $t^*(x)$ we can define $\bar{x}(t)$ as the smallest value of x that makes $t = t^*(x)$ optimal for any $t \geq 0$.[41] We will look for an equilibrium where at any $t \geq 0$ someone adopts, so

$$\rho(\alpha(t)\bar{x}(t) - c) = \bar{x}(t)\alpha_t(t)$$

provided α is differentiable at t.[42] The following two lemmas are useful to characterize the solution for the deterministic case. The proof of both lemmas can be found in Appendix J.1.

[40] If $\alpha(t)$ is not differentiable at $0 < t = t^*(x) < \infty$:

$$-\rho(\alpha(t^*)x - c) + x\alpha_t^+(t^*) \leq 0 \leq -\rho(\alpha(t^*)x - c) + x\alpha_t^-(t^*)$$

where $\alpha_t^-(t^*)$ and $\alpha_t^+(t^*)$ are the left and right derivatives of $\alpha(t)$ at $t = t^*(x)$. Note that α is differentiable at t provided that $N(t)$ does not jump at t. If $N(t)$ jumps at t, then α has right and left derivatives.

[41] Since in equilibrium $0 \leq N(t) \leq 1$, then $\frac{\theta_0}{\rho} \leq \alpha(t) \leq \frac{\theta_0+\theta_n}{\rho}$ and $0 < \frac{\rho c}{\theta_0+\theta_n} \leq \bar{x}(t) \leq \frac{\rho c}{\theta_0}$. Note that we allow $\bar{x}(t) > U$, but in this case everybody is adopting at t. Thus, we assume that $\frac{(\theta_0+\theta_n)}{\rho} > \frac{c}{U}$, otherwise nobody can ever adopt, and $c > 0$, so that some type will never adopt.

[42] If α is not differentiable at t, then: $\bar{x}(t)\alpha_t^+(t) \leq \rho(\alpha(t)\bar{x}(t) - c) \leq \bar{x}(t)\alpha_t^-(t)$.

LEMMA 7. Assume that for $t > 0$ there is some x for which $0 < t^*(x) < \infty$, we denote the smallest of such x as $\bar{x}(t)$. Then if the first and second order necessary conditions holds, then $N(t)$ and $\alpha(t)$ are weakly increasing in time.

The fact that the threshold $\bar{x}(t)$ is weakly decreasing rules out a solution where the number of adopters is decreasing through time.

LEMMA 8. Assume that $\bar{x}(t)$ is continuously differentiable with respect to time, that $N(t)$ is weakly increasing in time, and that the initial condition satisfies $M_0(x) \equiv \int_0^x m_0(z)dz \leq F(x)$ for all x. Then, $\bar{x}(t)$ must be decreasing in time, and if in an interval $N(t)$ is strictly decreasing, then $\bar{x}(t)$ must be strictly decreasing. Thus,

$$M(x,t) \equiv \int_0^x m(z,t)dz = \begin{cases} (1-e^{-\nu t})F(x) + e^{-\nu t}M_0(x) & \text{for } x \leq \bar{x}(t) \\ (1-e^{-\nu t})F(\bar{x}(t)) + e^{-\nu t}M_0(\bar{x}(t)) & \text{for } x > \bar{x}(t) \end{cases}$$

The previous lemmas has the following immediate implication.

LEMMA 9. Consider the initial condition $m_0(x) = f(x)$ holding for all $x < \bar{x}(0)$. Then: $m(x,t) = f(x)$ and $N(t) = 1 - F(\bar{x}(t))$.

The last lemma states that if we start the economy with a threshold $\bar{x}(0)$ and no agent below that threshold has adopted, then all agents with $x > \bar{x}(0)$ will immediately adopt and the distribution of the non adopters is time invariant. This result is intuitive and it is at the heart of the lack of dynamics in the equilibrium of the deterministic problem.

Assuming Lemma 9 holds and that $m_0(x) = f(x)$, combining the first order conditions with the law of motion for $\alpha(t)$, $\rho\alpha(t) - \alpha_t(t) = \theta_0 + \theta_n(1 - F(\bar{x}(t)))$, we get

$$\bar{x}(t)\left[\theta_0 + \theta_n(1 - F(\bar{x}(t)))\right] - \rho c = 0 \tag{84}$$

Note that in equation (84) the solution for $\bar{x}(t)$ does not depend on t. Thus, we can construct equilibrium where $\bar{x}(0) = \bar{x}(t)$ for all $t \geq 0$. We summarize this result in the following proposition

PROPOSITION 19. Consider the initial condition $m_0(x) = f(x)$ for all $x < \bar{x}(0)$. Then the solution implies a time invariant threshold $\bar{x}(t) = \bar{x}$ solving equation (84), immediate adoption for all agents with $x \geq \bar{x}$, and a time-invariant fraction of adopters $N(t) = N = 1 - F(\bar{x})$.

Note that equation (84) may have multiple solutions. Given that from Lemma 7 we know

that $\bar{x}(t)$ is weakly decreasing and $\alpha(t)$ must be strictly increasing in time, then the lower root is the stable solution in the sense that, if the economy is at that point it will remain there forever. We show below that other paths are also possible in the presence of multiple solutions, with the fraction of adopters $N(t)$ ratcheting up at discrete moments in time.

Let us consider the case with two possible stationary equilibria, denoted by $\bar{x}_H > \bar{x}_L$, with associated adoption rates $N_H < N_L$, and the stationary value function $\rho\alpha(t) = \theta_0 + \theta_n N(t)$ (recall $\alpha_t = 0$) with solution $\bar{\alpha}_H = (\theta_0 + \theta_n N_H)/\rho$ and $\bar{\alpha}_L = (\theta_0 + \theta_n N_L)/\rho$, where $\alpha_H < \alpha_L$ since a low threshold yields higher utility due to the larger adoption rate.

We can now check that indeed $t^*(x)$ are optimal for a stationary equilibrium. Since $\alpha(t) = \bar{\alpha}_i$ for $i = L, H$, then the adoption problem becomes:

$$t^*(x) = \arg\max_{t \geq 0} e^{-\rho t} (\bar{\alpha}_i x - c)$$

and the solution is:

$$t^*(x) = \begin{cases} \infty & \text{if } x < \frac{c}{\bar{\alpha}_i} \\ 0 & \text{if } x \geq \frac{c}{\bar{\alpha}_i} \end{cases}$$

Hence, there are no dynamics in the deterministic case. Nonetheless, an equilibrium can be constructed where $\bar{x}(t)$ is piecewise continuous and jumps from $\bar{x}_H$ to $\bar{x}_L$ at some arbitrary time T and where the value of $N(t)$ also jumps. For instance, let $\bar{x}(t) = \bar{x}_H$ for $t \in [0, T)$ and let $\bar{x}(t) = \bar{x}_L$ for $t \in [T, \infty)$, where $T > 0$ is arbitrary. For $t \geq T$, set $\alpha(t) = \bar{\alpha}_L$ and for $t \in [0, T)$, solve $\alpha_t(t) = \rho(\alpha(t) - \alpha_H)$ with boundary condition $\alpha(T) = \bar{\alpha}_L$. This gives

$$\alpha(t) = \bar{\alpha}_H + (\bar{\alpha}_L - \bar{\alpha}_H)\ e^{-\rho(T-t)}$$

Note that $\alpha'(t) > 0$ for $t \in [0, T)$ and $\alpha(0) > \bar{\alpha}_H$. The equilibrium so constructed satisfies the first and second order condition for $t^*(x)$. The following proposition describes such "ratcheting" equilibria:

PROPOSITION 20. Assume that $m_0(x) = f(x)$ for all $x \in [0, U]$. Let $\bar{X}$ be the set of stationary equilibria, i.e.

$$\bar{X} \equiv \{0 < \bar{z}_i \leq U : \ \rho c = \bar{z}_i \left[\theta_0 + \theta_n \left(1 - F(\bar{z}_i)\right)\right]\}$$

An equilibrium is described by a path $\bar{x}(t)$ that at times $0 = t_0 \leq t_1 < t_2 < t_m < \infty$:

$$\bar{x}(t) = \bar{z}_i \in \bar{X} \text{ for } t_{i-1} \leq t < t_i \text{ for all } i = 1, 2, \ldots, m$$

and where $\bar{z}_i > \bar{z}_{i+1}$ for all $i = 1, 2, \ldots, m$.

In words, an equilibrium is given by a piece-wise constant path for $\bar{x}(t)$, such that at each discontinuity point $\bar{x}(t)$ jumps down to a value that is one of the stationary solutions of equation (84). The set of equilibria thus includes the fully static one where $\bar{x}(0) = \bar{z}_m$, as well as several other time-varying paths where the elements of $\bar{X}$ (the stationary solutions) and the time sequence t_i are arbitrarily selected subject to the constraint that the path for $\bar{x}(t)$ must be weakly decreasing.

J.1 Proofs of the Deterministic Model

Proof. (of Lemma 7) The necessary second order condition for $0 < t^*(x) < \infty$ is:

$$G_{tt}(t,x)|_{t=t^*(x)} = e^{-\rho t^*}\left(-\rho\alpha_t(t^*)x + \alpha_{tt}(t^*)x\right)$$

Differentiating with respect to time the law of motion for α (i.e., $\rho\alpha(t) = \theta_0 + \theta_n N(t) + \alpha_t(t)$) we have:

$$\rho\alpha_t(t) = \theta_n N_t(t) + \alpha_{tt}(t)$$

Evaluating the second order condition at $(t, \bar{x}(t))$ and replacing $\alpha_{tt}(t)$:

$$\begin{aligned} G_{tt}(t,\bar{x}(t)) &= e^{-\rho t}\left(-\rho\alpha_t(t)\bar{x}(t) + \bar{x}(t)\alpha_{tt}(t)\right) \\ &= e^{-\rho t}\left(-\rho\alpha_t(t)\bar{x}(t) + \bar{x}(t)\left(\rho\alpha_t(t) - \theta_n N_t(t)\right)\right) = -e^{-\rho t}\bar{x}(t)\theta_n N_t(t) \end{aligned}$$

Thus, if the necessary first order condition holds, i.e if $G_{tt}(t,\bar{x}(t)) \leq 0$, then $N_t(t) \geq 0$ and hence it is weakly increasing.

Furthermore, using the first order condition at $t > 0$

$$\rho(\alpha(t)\bar{x}(t) - c) = \alpha_t(t)\bar{x}(t)$$

Note that if $\alpha_t(t)$ is strictly decreasing then $\alpha(t)\bar{x}(t) - c < 0$, which is a contradiction with $\alpha(t)\bar{x}(t) - c \geq 0$. Thus, for a t where α is differentiable (no jump on N), then $\alpha(t)$ must be weakly increasing.

Lastly, notice that

$$\begin{aligned} N(t) &= 1 - \int_0^{\bar{x}(t)} m(z,t)dz \\ &= 1 - \left[M_0(\bar{x}(t))e^{-\nu t} + F(\bar{x}(t))\left(1 - e^{-\nu t}\right)\right] \end{aligned}$$

where the second line uses that $\bar{x}(t)$ is decreasing in time. Taking the derivative of $N(t)$ with respect to time:

$$N_t(t) = \left[m_0(\bar{x}(t))e^{-\nu t} + f(\bar{x}(t))\left(1 - e^{-\nu t}\right)\right]\bar{x}_t(t) - \nu e^{-\nu t}\left[F(\bar{x}(t)) - M_0(\bar{x}(t))\right]$$

□

Proof. (of Lemma 8) The proof has two parts. The first part establishes that $\bar{x}$ is decreasing. The second one uses that property to obtain the law of motion of M. Differentiating the definition of N

$$N(t) = 1 - \int_0^{\bar{x}(t)} m(x,t)dx$$

with respect to t, and using that $m(\cdot,t)$ is zero for $x > \bar{x}(t)$ is in general strictly positive at the left limit $m(\bar{x}(t)^-,t)$, we have:

$$N_t(t) = \bar{x}_t(t)1_{\{\bar{x}_t(t)\leq 0\}}m(\bar{x}(t)^-,t) - \int_0^{\bar{x}(t)} m_t(x,t)dx$$

Using the law of motion of m we have:

$$\begin{aligned}
N_t(t) &= \bar{x}_t(t)1_{\{\bar{x}_t(t)\leq 0\}}m(\bar{x}(t)^-,t) + \nu\int_0^{\bar{x}(t)}\left(m_(x,t) - f(x)\right)dx \\
&= \bar{x}_t(t)1_{\{\bar{x}_t(t)\leq 0\}}m(\bar{x}(t)^-,t) + \nu\int_0^{\bar{x}(t)}\left(m_(x,t) - f(x)\right)dx \\
&= \bar{x}_t(t)1_{\{\bar{x}_t(t)\leq 0\}}m(\bar{x}(t)^-,t) + \nu\left(M(x,t) - F(x,t)\right)
\end{aligned}$$

But since, $M(x,t) \leq F(x,t)$ for all x, then if $N_t(t) \geq 0$, then $\bar{x}_t(t) \leq 0$.

Now, let use that $\bar{x}(t)$ is decreasing. In this case if $x < \bar{x}(t)$ then it must be that $\bar{s} \leq \bar{x}(s)$ for all $s \leq t$. Then for such x we have:

$$m_t(x,s) = -\nu\left(m(x,s) - f(x)\right) \text{ for all } s \leq t$$

We can solve this o.d.e. for each x, using the boundary $m(x,s) = m_0(x)$. This gives

$$m(x,t) = \begin{cases} (1 - e^{-\nu t})f(x) + e^{-\nu t}m_0(x) & \text{for all } x \leq \bar{x}(t) \\ 0 & \text{for all } x > \bar{x}(t) \end{cases}$$

Integrating this density we get the desired result of its CDF $M(x,t)$. □