The Job Ladder: Inflation vs. Reallocation

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ONLINE APPENDIX

1 Log-linearization

1.1 Equilibrium recap

We begin by reviewing all equilibrium conditions in the extended model, with intensive margins of recruiting and labor supply.

To simplify notation, we define the relative price of an efficiency unit of Service input in terms of final good:

$$\boxed{x_t = \frac{\omega_t}{z_t P_t}}$$

and the marginal utility of consumption, with a preference shock Υ_t

$$\boxed{\mathcal{M}_t = \left(\frac{C_t}{\Upsilon_t}\right)^{-\frac{1}{\sigma}}}$$

We then list all equilibrium conditions:

• Euler equation

$$\boxed{\beta \mathbb{E}_t \left[\frac{\mathcal{M}_{t+1}}{\mathcal{M}_t} \frac{1+R_t}{1+\pi_{t+1}}\right] = 1}$$

Now let $\tilde{y} = y^{1+\Xi}$. The the expected flow match output from an unemployed job applicant equals:

$$\boxed{\mu_0 = \sum_{k=1}^{K} \tilde{y}_k \left(\Gamma_k - \Gamma_{k-1}\right)}$$

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and the expected flow match output from an employed job applicant be $\mu_{1,t-1}$, where:

$$\mu_{1,t} = \sum_{k=1}^{K} \sum_{j=1}^{k} \left(\tilde{y}_k - \tilde{y}_j\right) \frac{L_{j,t} - L_{j-1,t}}{1 - u_t} \left(\Gamma_k - \Gamma_{k-1}\right)$$

Decomposing the sums, we can express $\mathbb{I}_{1,t} = (1 - u_t)\mu_{1,t}$ as follows:

$$\begin{aligned}
\mathbb{I}_{1,t} &= \sum_{k=1}^{K} \sum_{j=1}^{k} \left(\tilde{y}_k - \tilde{y}_j\right) \left(L_{j,t} - L_{j-1,t}\right) \left(\Gamma_k - \Gamma_{k-1}\right) = \sum_{k=1}^{K} \left(\Gamma_k - \Gamma_{k-1}\right) \sum_{j=1}^{k} \left(\tilde{y}_k - \tilde{y}_j\right) \left(L_{j,t} - L_{j-1,t}\right) \\
&= \sum_{k=1}^{K} \Gamma_k \sum_{j=1}^{k} \left(\tilde{y}_k - \tilde{y}_j\right) \left(L_{j,t} - L_{j-1,t}\right) - \sum_{k=1}^{K} \Gamma_{k-1} \sum_{j=1}^{k} \left(\tilde{y}_k - \tilde{y}_j\right) \left(L_{j,t} - L_{j-1,t}\right) \\
&= \sum_{k=1}^{K} \Gamma_k \sum_{j=1}^{k} \left(\tilde{y}_k - \tilde{y}_j\right) \left(L_{j,t} - L_{j-1,t}\right) - \sum_{k=1}^{K-1} \Gamma_k \sum_{j=1}^{k+1} \left(\tilde{y}_{k+1} - \tilde{y}_j\right) \left(L_{j,t} - L_{j-1,t}\right) \\
&= \sum_{j=1}^{K} \left(\tilde{y}_K - \tilde{y}_j\right) \left(L_{j,t} - L_{j-1,t}\right) + \sum_{k=1}^{K-1} \Gamma_k \left[\sum_{j=1}^{k} \left(\tilde{y}_k - \tilde{y}_j\right) \left(L_{j,t} - L_{j-1,t}\right) - \sum_{j=1}^{k+1} \left(\tilde{y}_{k+1} - \tilde{y}_j\right) \left(L_{j,t} - L_{j-1,t}\right)\right] \\
&= \tilde{y}_K L_{K,t} - \sum_{j=1}^{K} \tilde{y}_j (L_{j,t} - L_{j-1,t}) + \\
&+ \sum_{k=1}^{K-1} \Gamma_k \left[\tilde{y}_k L_{k,t} - \sum_{j=1}^{k} \tilde{y}_j (L_{j,t} - L_{j-1,t}) - \tilde{y}_{k+1} L_{k+1,t} + \sum_{j=1}^{k+1} \tilde{y}_j (L_{j,t} - L_{j-1,t})\right] \\
&= \tilde{y}_K L_{K,t} - \sum_{j=1}^{K} \tilde{y}_j (L_{j,t} - L_{j-1,t}) + \sum_{k=1}^{K-1} \Gamma_k \left[\tilde{y}_k L_{k,t} - \tilde{y}_{k+1} L_{k+1,t} + \tilde{y}_{k+1} (L_{k+1,t} - L_{k,t})\right] \\
&= \tilde{y}_K L_{K,t} - \sum_{j=1}^{K} \tilde{y}_j (L_{j,t} - L_{j-1,t}) - \sum_{k=1}^{K-1} \Gamma_k \left(\tilde{y}_{k+1} - \tilde{y}_k\right) L_{k,t} \\
&= \tilde{y}_K L_{K,t} - \tilde{y}_K (L_{K,t} - L_{K-1,t}) - \tilde{y}_{K-1} (L_{K-1,t} - L_{K-2,t}) - ... \tilde{y}_1 L_{1,t} - \sum_{k=1}^{K-1} \Gamma_k \left(\tilde{y}_{k+1} - \tilde{y}_k\right) L_{k,t} \\
&= \sum_{j=1}^{K-1} \left(\tilde{y}_{j+1} - \tilde{y}_j\right) L_{j,t} - \sum_{j=1}^{K-1} \Gamma_j \left(\tilde{y}_{j+1} - \tilde{y}_j\right) L_{j,t}
\end{aligned}$$

so finally

$$\mathbb{I}_{1,t} = \sum_{j=1}^{K-1} (1 - \Gamma_j) \left(\tilde{y}_{j+1} - \tilde{y}_j\right) L_{j,t}$$

and

$$\boxed{\mu_{1,t} = \sum_{k=1}^{K-1} (1 - \Gamma_k) \left(\tilde{y}_{k+1} - \tilde{y}_k\right) \frac{L_{k,t}}{1 - u_t}}$$

- Recruiting efforts.

Apply the additive separable utility specification to the following equations derived in the paper:

$$\mathcal{L}_t = \frac{b}{\mathcal{M}_t[1-\beta(1-\delta)]}$$
$$\mathbb{E}_t\left[\frac{\mathcal{M}_{t+1}}{\mathcal{M}_t}\mathcal{L}_{t+1}\right] = \frac{\beta b}{\mathcal{M}_t[1-\beta(1-\delta)]} = \beta\mathcal{L}_t$$
$$r^*_{i,t} = \left(\frac{P_t\Omega_{i,t}}{\kappa_s\omega_t}\right)^{\frac{1}{\iota}} = \left(\frac{\Omega_{i,t}}{\kappa_s x_t z_t}\right)^{\frac{1}{\iota}}$$
$$\Omega_{0,t} = \mathbb{E}_t\left[\frac{\mathcal{M}_{t+1}}{\mathcal{M}_t}W_{t+1}\right]\mu_0 - \mathbb{E}_t\left[\frac{\mathcal{M}_{t+1}}{\mathcal{M}_t}\mathcal{L}_{t+1}\right]$$
$$\Omega_{1,t} = \mathbb{E}_t\left[\frac{\mathcal{M}_{t+1}}{\mathcal{M}_t}W_{t+1}\right]\mu_{1,t-1}$$

Therefore:

$$\boxed{r^*_{0,t} = \left(\beta\frac{\mu_0\mathbb{E}_t\left[\mathcal{M}_{t+1}W_{t+1}\right] - \frac{b}{1-\beta(1-\delta)}}{\kappa_s x_t z_t\mathcal{M}_t}\right)^{\frac{1}{\iota}}}$$

and

$$\boxed{r^*_{1,t} = \left(\beta\frac{\mu_{1,t-1}\mathbb{E}_t\left[\mathcal{M}_{t+1}W_{t+1}\right]}{\kappa_s x_t z_t\mathcal{M}_t}\right)^{\frac{1}{\iota}}}$$

- Final good market-clearing:

$$\boxed{Q_t = \Upsilon_t\mathcal{M}_t^{-\sigma} + \kappa_v\theta_t\left[u_{t-1} + \delta(1-u_{t-1})s_0 + (1-\delta)(1-u_{t-1})s_1\right]}$$

- Free entry:

$$\boxed{\frac{\kappa_v\theta_t}{\varphi_t\phi(\theta_t)} = \kappa_s x_t z_t\frac{\iota}{1+\iota}\frac{\left[u_{t-1}+\delta(1-u_{t-1})s_0\right]{r^*_{0,t}}^{1+\iota} + (1-\delta)(1-u_{t-1})s_1{r^*_{1,t}}^{1+\iota}}{u_{t-1}+\delta(1-u_{t-1})s_0+(1-\delta)(1-u_{t-1})s_1}}$$

- Employment and unemployment dynamics:

$$\boxed{L_{k,t} = (1-\delta)\left[1 - s_1\varphi_t\phi(\theta_t)r^*_{1,t}(1-\Gamma_k)\right]L_{k,t-1} + \varphi_t\phi(\theta_t)r^*_{0,t}\Gamma_k\left[u_{t-1}+\delta s_0(1-u_{t-1})\right]}$$

$$\boxed{u_t = \left(1-\varphi_t\phi(\theta_t)r^*_{0,t}\right)u_{t-1} + \delta\left(1 - s_0\varphi_t\phi(\theta_t)r^*_{0,t}\right)(1-u_{t-1})}$$

- Recursion for the real MC:

$$\boxed{W_t = \frac{(x_t z_t)^{1+\Xi}}{1+\Xi}\left(\frac{\mathcal{M}_t}{\mathcal{B}}\right)^{\Xi} + (1-\delta)\beta\mathbb{E}_t\left[\frac{\mathcal{M}_{t+1}}{\mathcal{M}_t}W_{t+1}\right]}$$

- Optimal pricing:

$$\boxed{p_t^{\star\frac{\eta}{\zeta}+1-\eta} = \frac{\eta}{\eta-1}\frac{1}{\zeta}\cdot\frac{\sum_{\tau=0}^{+\infty}(1-\nu)^\tau\beta^\tau\mathbb{E}_t\left[\mathcal{M}_{t+\tau}Q_{t+\tau}^{\frac{1}{\zeta}}P_{t+\tau}^{\frac{\eta}{\zeta}}x_{t+\tau}\right]}{\sum_{\tau=0}^{+\infty}(1-\nu)^\tau\beta^\tau\mathbb{E}_t\left[\mathcal{M}_{t+\tau}Q_{t+\tau}P_{t+\tau}^{\eta-1}\right]}}$$

- Market-clearing in the Service market:

$$\boxed{\left(\frac{x_t z_t \mathcal{M}_t}{\mathcal{B}}\right)^{\Xi}\sum_{k=1}^{K} y_k^{1+\Xi}\left(L_{k,t-1}-L_{k-1,t-1}\right) = \frac{Q_t^{\frac{1}{\zeta}}}{z_t}\left(\frac{P_t}{\widetilde{P}_t}\right)^{\frac{\eta}{\zeta}} + \frac{Q_t-\Upsilon_t\mathcal{M}_t^{-\sigma}}{\iota x_t z_t}}$$

Note that the last term on the right, the demand for Service for recruiting activities, is simplified from its original form

$$\frac{\kappa_s\varphi_t\phi(\theta_t)}{1+\iota}\left[\left(u_{t-1}+\delta(1-u_{t-1})s_0\right)r_{0,t}^{*\,1+\iota}+(1-\delta)(1-u_{t-1})s_1 r_{1,t}^{*\,1+\iota}\right].$$

by using the free entry condition.

- Taylor rule:

$$\boxed{\ln(1+R_t) = \rho_R\ln\left(1+R_{t-1}\right)+(1-\rho_R)\left[\psi_\pi\ln\left(1+\pi_t\right)+\psi_u\ln\frac{u_t}{u}-\ln\beta\right]+\ln\varsigma_t}$$

- Structural shocks:
  For $\o \in \{z,\Upsilon,\varphi,\varsigma\}$

$$\boxed{\ln\o_t = \rho_{\o}\ln\o_{t-1}+\sigma_{\o}\varepsilon_t^{\o}}$$

1.2 Steady state

In the absence of shocks, we obtain a steady state equilibrium. Normalize to one the steady price level P (measured in dollars). Solve for steady W and replace this expression in the optimal stationary recruiting efforts, so that W no longer appears. Then, steady state equilibrium solves the following set of algebraic equations:

$$z=\varphi=\Upsilon=1, \qquad R=\frac{1}{\beta}, \qquad P=\widetilde{P}=p^*=1$$

$$\omega = x = \frac{\eta-1}{\eta}\zeta Q^{\frac{\zeta-1}{\zeta}}$$

$$u=\frac{\delta[1-s_0\phi(\theta)r_0^*]}{\delta[1-s_0\phi(\theta)r_0^*]+\phi(\theta)r_0^*}, \qquad L_k=\frac{\phi(\theta)r_0^*\Gamma_k\left[u+\delta(1-u)s_0\right]}{\delta+(1-\delta)s_1\phi(\theta)r_1^*(1-\Gamma_k)}$$

$$\mu_0=\sum_{k=1}^{K}\left(\Gamma_k-\Gamma_{k-1}\right)y_k^{1+\Xi}, \qquad \mu_1=\sum_{k=1}^{K-1}(1-\Gamma_k)\left(y_{k+1}^{1+\Xi}-y_k^{1+\Xi}\right)\frac{L_k}{1-u}$$

$$r_0^* = \left\{ \frac{\beta}{1-\beta(1-\delta)} \left[\frac{\mu_0}{\kappa_s} \frac{1}{1+\Xi} \left(\frac{x\mathcal{M}}{\mathcal{B}} \right)^{\Xi} - \frac{b}{\kappa_s x \mathcal{M}} \right] \right\}^{\frac{1}{\iota}}$$

$$r_1^* = \left[\frac{\beta}{1-\beta(1-\delta)} \frac{\mu_1}{\kappa_s} \frac{1}{1+\Xi} \left(\frac{x\mathcal{M}}{\mathcal{B}} \right)^{\Xi} \right]^{\frac{1}{\iota}}$$

$$\left(\frac{x\mathcal{M}}{\mathcal{B}} \right)^{\Xi} \sum_{k=1}^{K} y_k^{1+\Xi} (L_k - L_{k-1}) = Q^{\frac{1}{\zeta}} + \frac{Q - \mathcal{M}^{-\sigma}}{\iota x}$$

$$Q = \mathcal{M}^{-\sigma} + \kappa_v \theta \left[u + \delta(1-u)s_0 + (1-\delta)(1-u)s_1 \right]$$

Finally, to write the free entry condition in steady state, solve for $\kappa_v\theta$ from the previous equation and replace it:

$$\frac{Q - \mathcal{M}^{-\sigma}}{\phi(\theta)} \frac{1+\iota}{\iota x \kappa_s} = \left[u + \delta s_0 \left(1-u\right) \right] r_0^{*1+\iota} + (1-\delta)(1-u)s_1 r_1^{*1+\iota}$$

1.3 Log-linear approximation and matrix representation of the linearized model

We use hats to denote log deviations from the steady state with zero inflation, such as $\widehat{\theta}_t = \ln\theta_t - \ln\theta$. For inflation, since we cannot take logs of $\pi = 0$, we use a linearization in levels: $\widehat{\pi}_t = \pi_t - \pi = \pi_t = \ln P_t - \ln P_{t-1}$. Moreover, in steady state, from the Euler equation $R = -\ln\beta$ and we define $\widehat{R}_t = R_t + \ln\beta$. We use the first-order approximation rules

$$x_t = a + by_t + cy_tz_t \;\Rightarrow\; x\widehat{x}_t = by\widehat{y}_t + cyz(\widehat{y}_t + \widehat{z}_t) \Rightarrow\; \widehat{x}_t = \frac{by\widehat{y}_t + cyz(\widehat{y}_t + \widehat{z}_t)}{a + by + cyz}$$

$$x_t = \frac{a + by_t + cy_tz_t}{d + ew_t + fw_th_t} \;\Rightarrow\; \widehat{x}_t = \frac{by\widehat{y}_t + cyz(\widehat{y}_t + \widehat{z}_t)}{a + by + cyz} - \frac{ew\widehat{w}_t + fwh(\widehat{w}_t + \widehat{h}_t)}{d + ew + fwh}$$

$$x_t = a + b\mathbb{E}_t[y_{t+1}z_{t+1}] = a + b\mathbb{E}_t[e^{\ln y_{t+1} + \ln z_{t+1}}] = a + byz\mathbb{E}_t[1 + \widehat{y}_{t+1} + \widehat{z}_{t+1}]$$

$$\Rightarrow x_t - x = x\widehat{x}_t = byz\mathbb{E}_t[\widehat{y}_{t+1} + \widehat{z}_{t+1}]$$

$$y_t = f(x_{1,t}, .., x_{n,t}) \Rightarrow \widehat{y}_t = y^{-1} \sum_{i=1}^{n} \frac{\partial f(x_1, ...x_n)}{\partial x_i} x_i \widehat{x}_{i,t}$$

The resulting system of approximated equilibrium conditions comprises $13 + K$ linear stochastic difference equations, boxed and labeled [$\mathcal{M}$,R,W, SC$_0$, SC$_1$, Q, FEC, $\{\text{L}_k\}_{k=1,\cdots,K-1}$, u, PC, MC, z, Υ, φ, MP] in the $13 + K$ variables, stacked in a column vector $\widehat{\chi}_t$, where:

$$\chi_t^\top = \left(\underbrace{r_{0,t}^*, r_{1,t}^*, Q_t, \theta_t, x_t}_{\text{5 static variables } \mathcal{Y}_t^\top} \quad \underbrace{\pi_t, W_t, \mathcal{M}_t}_{\text{3 jump variables } \mathcal{X}_t^\top} \quad \underbrace{z_t, \varphi_t, \Upsilon_t, \varsigma_t,}_{\text{4 exogenous states } \mathcal{Z}_t^\top} \quad \underbrace{R_t, u_t, L_{1,t}, \cdots, L_{K-1,t}}_{\text{1+K endogenous states } \mathcal{S}_t^\top} \right)$$

where $\mathcal{Y}_t, \mathcal{X}_t, \mathcal{Z}_t, \mathcal{S}_t$ are column vectors, recalling that $u_t = 1 - L_{K,t}$ and $K \geq 1$ is the finite cardinality of the support of match quality. Here "static" variables are endogenous variables that appear in the system only dated at time t; "jump" variables are endogenous variables that appear in the system only dated at time t and (in expectation) $t+1$. Static and jump variables do not appear dated at $t-1$, so, unlike states, they have no predetermined values. States can be exogenous or endogenous. Price indices P_t and $\tilde{P}_t$ no longer appear, because only their growth rate π_t is relevant to equilibrium.[1]

The linearized system has the matrix representation:

$$\boxed{\mathsf{A}\widehat{\chi}_t + \mathsf{B}\widehat{\chi}_{t-1} + \mathsf{C}\mathbb{E}_t\widehat{\chi}_{t+1} + \mathsf{D}\varepsilon_t = \mathsf{0}_{(13+K)\times 1}} \tag{1}$$

where $\mathsf{A}, \mathsf{B}, \mathsf{C}$ are $(13+K)\times(13+K)$ coefficient matrices, D is $(13+K)\times 4$. Equations are on the rows of these matrices, and variables are on the columns. Matrices $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ have all zero entries, except the following. Given our labeling of equations and variables, rather than numbering rows and columns, we denote each row with the corresponding equation label, and each column with the corresponding variable.

Consumption Euler Equation $[\mathcal{M}]$

$$\boxed{\mathbb{E}_t\widehat{\mathcal{M}}_{t+1} - \widehat{\mathcal{M}}_t + \widehat{R}_t - \mathbb{E}_t\pi_{t+1} = 0}$$

So

$$\mathsf{A}_{\mathcal{M},\mathcal{M}} = -1$$
$$\mathsf{A}_{\mathcal{M},R} = 1$$

and

$$\mathsf{C}_{\mathcal{M},\mathcal{M}} = 1$$
$$\mathsf{C}_{\mathcal{M},\pi} = -1$$

Present value of Service relative price [W]

$$W\widehat{W}_t = \frac{x^{1+\Xi}}{1+\Xi}\left(\frac{\mathcal{M}}{\mathcal{B}}\right)^{\Xi}\left[(1+\Xi)\left(\widehat{x}_t+\widehat{z}_t\right)+\Xi\widehat{\mathcal{M}}_t\right] + \beta(1-\delta)W\mathbb{E}_t\left[\widehat{\mathcal{M}}_{t+1} - \widehat{\mathcal{M}}_t + \widehat{W}_{t+1}\right]$$

Dividing through by W and using its s.s. expression:

$$\boxed{\widehat{W}_t - [1-\beta(1-\delta)]\left[(1+\Xi)\left(\widehat{x}_t+\widehat{z}_t\right)+\Xi\widehat{\mathcal{M}}_t\right] - \beta(1-\delta)\left[\mathbb{E}_t\widehat{\mathcal{M}}_{t+1} - \widehat{\mathcal{M}}_t + \mathbb{E}_t\widehat{W}_{t+1}\right] = 0}$$

So

$$\begin{aligned}
\mathsf{A}_{W,x} &= -[1-\beta(1-\delta)](1+\Xi)\\
\mathsf{A}_{W,\mathcal{M}} &= \mathsf{A}_{W,x} + 1\\
\mathsf{A}_{W,W} &= 1\\
\mathsf{A}_{W,z} &= \mathsf{A}_{W,x}
\end{aligned}$$

[1]Note that, if we solve the Euler equation [C] for $\mathbb{E}_t\widehat{\mathcal{M}}_{t+1} = \widehat{\mathcal{M}}_t - \widehat{R}_t + \mathbb{E}_t\pi_{t+1}$, and replace everywhere in the other equations, we eliminate the Euler equation and $\mathcal{M}_t$ becomes a static variable, which no longer appears in expectation.

and

$$\begin{aligned}\mathsf{C}_{W,\mathcal{M}} &= -\beta(1-\delta)\\ \mathsf{C}_{W,W} &= \mathsf{C}_{W,\mathcal{M}}\end{aligned}$$

Recruiting intensity (Screening) of unemployed job applicants [SC$_0$] Using

$$r_{0,t}^{*\iota}\kappa_s x_t z_t \mathcal{M}_t = \frac{\mu_0}{\mu_{1,t-1}} r_{1,t}^{*\iota}\kappa_s x_t z_t \mathcal{M}_t - \frac{\beta b}{1-\beta(1-\delta)}$$

and log linearizing

$$r_0^{*\iota}\left(\iota\widehat{r}_{0,t}^* + \widehat{x}_t + \widehat{z}_t + \widehat{\mathcal{M}}_t\right) = \frac{\mu_0}{\mu_1} r_1^{*\iota}\left(\iota\widehat{r}_{1,t}^* + \widehat{x}_t + \widehat{z}_t + \widehat{\mathcal{M}}_t - \widehat{\mu}_{1,t-1}\right)$$

and replacing from above

$$\boxed{\left(\frac{r_0^*}{r_1^*}\right)^{\iota}\left(\iota\widehat{r}_{0,t}^* + \widehat{x}_t + \widehat{z}_t + \widehat{\mathcal{M}}_t\right) - \frac{\mu_0}{\mu_1}\left(\mathbb{E}_t\widehat{W}_{t+1} + \mathbb{E}_t\widehat{\mathcal{M}}_{t+1}\right) = 0}$$

So,

$$\begin{aligned}\mathsf{A}_{\mathrm{SC}_0,x} &= \left(\frac{r_0^*}{r_1^*}\right)^{\iota}\\ \mathsf{A}_{\mathrm{SC}_0,\mathcal{M}} &= \mathsf{A}_{\mathrm{SC}_0,x}\\ \mathsf{A}_{\mathrm{SC}_0,z} &= \mathsf{A}_{\mathrm{SC}_0,x}\\ \mathsf{A}_{\mathrm{SC}_0,r_0^*} &= \mathsf{A}_{\mathrm{SC}_0,x}\iota\end{aligned}$$

and

$$\begin{aligned}\mathsf{C}_{\mathrm{SC}_0,\mathcal{M}} &= -\frac{\mu_0}{\mu_1}\\ \mathsf{C}_{\mathrm{SC}_0,W} &= \mathsf{C}_{\mathrm{SC}_0,\mathcal{M}}\end{aligned}$$

Recruiting intensity (Screening) of employed job applicants [SC$_1$] Let

$$\widehat{\mu}_{1,t-1} = \frac{u}{1-u}\widehat{u}_{t-1} + \sum_{k=1}^{K-1}\frac{(1-\Gamma_k)\left(y_{k+1}^{1+\Xi} - y_k^{1+\Xi}\right)L_k}{\sum_{j=1}^{K-1}(1-\Gamma_j)\left(y_{j+1}^{1+\Xi} - y_j^{1+\Xi}\right)L_j}\widehat{L}_{k,t-1}$$

Then

$$\boxed{\iota\widehat{r}_{1,t}^* + \widehat{x}_t + \widehat{z}_t - \mathbb{E}_t\widehat{W}_{t+1} - \mathbb{E}_t\widehat{\mathcal{M}}_{t+1} + \widehat{\mathcal{M}}_t - \widehat{\mu}_{1,t-1} = 0}$$

So

$$\begin{aligned}\mathsf{A}_{\mathrm{SC}_1,x} &= 1\\ \mathsf{A}_{\mathrm{SC}_1,\mathcal{M}} &= 1\\ \mathsf{A}_{\mathrm{SC}_1,z} &= 1\\ \mathsf{A}_{\mathrm{SC}_1,r_1^*} &= \iota\end{aligned}$$

and

$$
\begin{aligned}
\mathsf{B}_{\mathrm{SC}_1,u} &= -\frac{u}{1-u} \\
\mathsf{B}_{\mathrm{SC}_1,L_k} &= -\frac{(1-\Gamma_k)\left(y_{k+1}^{1+\Xi} - y_k^{1+\Xi}\right) L_k}{\sum_{j=1}^{K-1}(1-\Gamma_j)\left(y_{j+1}^{1+\Xi} - y_j^{1+\Xi}\right) L_j}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathsf{C}_{\mathrm{SC}_1,\mathcal{M}} &= -1 \\
\mathsf{C}_{\mathrm{SC}_1,W} &= -1
\end{aligned}
$$

Final good market-clearing [Q]

$$
\boxed{Q\widehat{Q}_t + \mathcal{M}^{-\sigma}(\sigma\widehat{\mathcal{M}}_t - \widehat{\Upsilon}_t) - \kappa_v\theta\left[u + \delta(1-u)s_0 + (1-\delta)(1-u)s_1\right]\widehat{\theta}_t}
$$
$$
\boxed{-\kappa_v\theta u\left[1 - \delta s_0 - (1-\delta)s_1\right]\widehat{u}_{t-1} = 0}
$$

So

$$
\begin{aligned}
\mathsf{A}_{Q,Q} &= Q \\
\mathsf{A}_{Q,\Upsilon} &= -\mathcal{M}^{-\sigma} \\
\mathsf{A}_{Q,\mathcal{M}} &= -\sigma\mathsf{A}_{Q,\Upsilon} \\
\mathsf{A}_{Q,\theta} &= -\kappa_v\theta\left[u + \delta(1-u)s_0 + (1-\delta)(1-u)s_1\right]
\end{aligned}
$$

and

$$
\mathsf{B}_{Q,u} = -\kappa_v\theta u\left[1 - \delta s_0 - (1-\delta)s_1\right]
$$

Price Indices. As is standard, the law of motion of the final good price:

$$
P_t^{1-\eta} = \nu p_t^{*1-\eta} + (1-\nu)P_{t-1}^{1-\eta}
$$

log-linearizes as:

$$
\widehat{P}_t = \nu\widehat{p_t^{\star}} + (1-\nu)\widehat{P}_{t-1}
$$

where we used the fact that in steady state, $P_t = p_t^* = P_{t-1} = P$.

Similarly, the dynamics of $\tilde{P}_t$ can be written as

$$
\tilde{P}_t^{-\eta} = \nu p_t^{*-\eta} + (1-\nu)\tilde{P}_{t-1}^{-\eta}
$$

log-linearize as:

$$
\widehat{\tilde{P}}_t = \nu\widehat{p_t^*} + (1-\nu)\widehat{\tilde{P}}_{t-1}
$$

independently of the value of η. Combining the two log-linear equations, $\left(\widehat{P}_t - \widehat{\tilde{P}}_t\right) = (1-\nu)\left(\widehat{P}_{t-1} - \widehat{\tilde{P}}_{t-1}\right)$. Thus $\widehat{\tilde{P}}_t - \widehat{P}_t$ converges to zero deterministically. Near steady state, prices are close to their steady-state benchmark, there is little price dispersion. This implies that $\widehat{\tilde{P}}_t$ and $\widehat{P}_t$ are approximately the same:

$$
\widehat{\tilde{P}}_t \simeq \widehat{P}_t
$$

Employment distribution $[\mathbf{L}_k]$ For $k = 1, 2, \cdots, K-1$, $K \geq 2$:

$$\boxed{\widehat{L}_{k,t} - (1-\delta)\left[1 - s_1\phi(\theta)r_1^*(1-\Gamma_k)\right]\widehat{L}_{k,t-1} + (1-\delta)s_1\phi(\theta)r_1^*(1-\Gamma_k)\widehat{r}^*_{1,t} - \frac{u+\delta(1-u)s_0}{L_k}\phi(\theta)r_0^*\Gamma_k\widehat{r}^*_{0,t}}$$

$$\boxed{-\frac{(1-\delta s_0)\phi(\theta)r_0^*\Gamma_k}{L_k}u\widehat{u}_{t-1} + \left[(1-\delta)s_1r_1^*(1-\Gamma_k) - \frac{u+\delta(1-u)s_0}{L_k}r_0^*\Gamma_k\right]\phi(\theta)\left(\alpha\widehat{\theta}_t + \widehat{\varphi}_t\right) = 0}$$

So

$$\begin{aligned}
\mathsf{A}_{L_k,r_0^*} &= -\phi(\theta)r_0^*\frac{u+\delta(1-u)s_0}{L_k}\Gamma_k \\
\mathsf{A}_{L_k,r_1^*} &= (1-\delta)s_1\phi(\theta)r_1^*(1-\Gamma_k) \\
\mathsf{A}_{L_k,\varphi} &= \mathsf{A}_{L_k,r_0^*} + \mathsf{A}_{L_k,r_1^*} \\
\mathsf{A}_{L_k,\theta} &= \alpha\mathsf{A}_{L_k,\theta} \\
\mathsf{A}_{L,L_k} &= 1
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{B}_{L_k,u} &= -\frac{(1-\delta s_0)\phi(\theta)r_0^*\Gamma_k}{L_k}u \\
\mathsf{B}_{L_k,L_k} &= -(1-\delta) + \mathsf{A}_{L_k,r_1^*}
\end{aligned}$$

Note that we stop these equations at $k = K-1$ because the equation at $k = K$ is redundant from the identity $L_{K,t} = 1 - u_t$ and:

Unemployment $[\mathbf{u}]$

$$\boxed{\widehat{u}_t - \left[1 - \delta - (1-\delta s_0)\phi(\theta)r_0^*\right]\widehat{u}_{t-1} + \left(1 - \delta s_0 + \frac{\delta s_0}{u}\right)\phi(\theta)r_0^*\left(\alpha\widehat{\theta}_t + \widehat{\varphi}_t + \widehat{r}^*_{0,t}\right) = 0}$$

So

$$\begin{aligned}
\mathsf{A}_{u,\varphi} &= \frac{u+\delta(1-u)s_0}{u}\phi(\theta)r_0^* \\
\mathsf{A}_{u,\theta} &= \alpha\mathsf{A}_{u,\varphi} \\
\mathsf{A}_{u,r_0^*} &= \mathsf{A}_{u,\varphi} \\
\mathsf{A}_{u,u} &= 1
\end{aligned}$$

and

$$\mathsf{B}_{u,u} = -1 + \delta + (1-\delta s_0)\phi(\theta)r_0^*$$

Optimal reset price and Phillips Curve [PC]. Next, we log-linearize the optimal reset pricing equation. The l.h.s. log-linearizes as:

$$\left(\frac{\eta}{\zeta}+1-\eta\right)\widehat{p}_t^{\star}$$

The numerator in the r.h.s. log-linearizes as:

$$[1-\beta(1-\nu)]\,\mathbb{E}_t\sum_{\tau=0}^{+\infty}(1-\nu)^{\tau}\beta^{\tau}\left(\frac{1}{\zeta}\widehat{Q}_{t+\tau}+\frac{\eta}{\zeta}\widehat{P}_{t+\tau}+\widehat{x}_{t+\tau}+\widehat{\mathcal{M}}_{t+\tau}\right)$$

while the (inverse of the) denominator log-linearizes as:

$$-\,[1-\beta(1-\nu)]\,\mathbb{E}_t\sum_{\tau=0}^{+\infty}(1-\nu)^{\tau}\beta^{\tau}\left[\widehat{Q}_{t+\tau}+(\eta-1)\widehat{P}_{t+\tau}+\widehat{\mathcal{M}}_{t+\tau}\right]$$

Putting everything together:

$$\left(\frac{\eta}{\zeta}+1-\eta\right)\widehat{p}_t^{\star}=[1-\beta(1-\nu)]\,\mathbb{E}_t\sum_{\tau=0}^{+\infty}(1-\nu)^{\tau}\beta^{\tau}\left[\frac{1-\zeta}{\zeta}\widehat{Q}_{t+\tau}+\left(\frac{\eta}{\zeta}+1-\eta\right)\widehat{P}_{t+\tau}+\widehat{x}_{t+\tau}\right]$$

This latter equation can be rewritten in recursive form:

$$\left(\frac{\eta}{\zeta}+1-\eta\right)\widehat{p}_t^{\star}=[1-\beta(1-\nu)]\left[\frac{1-\zeta}{\zeta}\widehat{Q}_t+\left(\frac{\eta}{\zeta}+1-\eta\right)\widehat{P}_t+\widehat{x}_t\right]+\beta(1-\nu)\left(\frac{\eta}{\zeta}+1-\eta\right)\mathbb{E}_t\widehat{p}_{t+1}^{\star}$$

Now remember that $\widehat{p}_t^{\star}$ solves:

$$\widehat{P}_t=\nu\widehat{p}_t^{*}+(1-\nu)\widehat{P}_{t-1}\iff\widehat{p}_t^{*}=\widehat{P}_{t-1}+\frac{1}{\nu}\pi_t$$

Substituting:

$$\begin{aligned}\left(\frac{\eta}{\zeta}+1-\eta\right)\left(\widehat{P}_{t-1}+\frac{1}{\nu}\pi_t\right)=[1-\beta(1-\nu)]&\left[\frac{1-\zeta}{\zeta}\widehat{Q}_t+\left(\frac{\eta}{\zeta}+1-\eta\right)\widehat{P}_t+\widehat{x}_t\right]\\&+\beta(1-\nu)\left(\frac{\eta}{\zeta}+1-\eta\right)\mathbb{E}_t\left(\widehat{P}_t+\frac{1}{\nu}\pi_{t+1}\right)\end{aligned}$$

simplifying terms in $\widehat{P}_t$

$$\begin{aligned}\left(\frac{\eta}{\zeta}+1-\eta\right)\left(\widehat{P}_{t-1}+\frac{1}{\nu}\pi_t\right)=&\left(\frac{\eta}{\zeta}+1-\eta\right)\widehat{P}_t\\&+[1-\beta(1-\nu)]\left[\frac{1-\zeta}{\zeta}\widehat{Q}_t+\widehat{x}_t\right]+\beta\frac{1-\nu}{\nu}\left(\frac{\eta}{\zeta}+1-\eta\right)\mathbb{E}_t\left(\pi_{t+1}\right)\end{aligned}$$

Finally, replace $\widehat{P}_{t-1} = \widehat{P}_t - \pi_t$, collate terms and rearrange:

$$\boxed{\pi_t - \frac{\nu}{1-\nu}\frac{1-\beta(1-\nu)}{1+\frac{\eta}{\zeta}-\eta}\left[\frac{1-\zeta}{\zeta}\widehat{Q}_t + \widehat{x}_t\right] - \beta\mathbb{E}_t\pi_{t+1} = 0}$$

So, let

$$\mathsf{A}_{\mathrm{PC}} = -\frac{\nu}{1-\nu}\frac{1-\beta(1-\nu)}{1+\eta\frac{1-\zeta}{\zeta}}$$

Then

$$\begin{aligned}
\mathsf{A}_{\mathrm{PC},\pi} &= 1 \\
\mathsf{A}_{\mathrm{PC},Q} &= \mathsf{A}_{\mathrm{PC}}\frac{1-\zeta}{\zeta} \\
\mathsf{A}_{\mathrm{PC},x} &= \mathsf{A}_{\mathrm{PC}} \\
\mathsf{A}_{\mathrm{PC},\mathcal{M}} &= \mathsf{A}_{\mathrm{PC}} - \mathsf{A}_{\mathrm{PC},x}
\end{aligned}$$

and

$$\mathsf{C}_{\mathrm{PC},\pi} = -\beta$$

With constant returns to scale, $\zeta = 1$, it reduces to the standard NKPC.

$$\pi_t = \nu\frac{1-\beta(1-\nu)}{1-\nu}\widehat{x}_t + \beta\mathbb{E}_t\pi_{t+1}$$

where $\widehat{x}_t$ is the marginal cost. The more decreasing are returns to scale (smaller ζ), the more sensitive is inflation to final output, given marginal cost.

Free-entry condition [FEC] Rewrite it as:

$$\frac{\kappa_v}{\kappa_s}\frac{1+\iota}{\iota}\frac{1}{z_t x_t}\frac{\theta_t}{\varphi_t\phi(\theta_t)} = \frac{\left[u_{t-1}+\delta(1-u_t)s_0\right]{r^*_{0,t}}^{1+\iota} + (1-\delta)(1-u_{t-1})s_1{r^*_{1,t}}^{1+\iota}}{u_{t-1}+\left[\delta s_0+(1-\delta)s_1\right](1-u_{t-1})}$$

The LHS log-linearizes as $-\widehat{z}_t - \widehat{x}_t - \widehat{\varphi}_t + (1-\alpha)\widehat{\theta}_t$, where α is the elasticity of $\phi(\cdot)$ evaluated at the s.s. θ.

The denominator on the RHS log-linearizes as:

$$-\frac{1-\left[\delta s_0+(1-\delta)\,s_1\right]}{u+\left[\delta s_0+(1-\delta)\,s_1\right](1-u)}u\widehat{u}_{t-1}$$

The numerator on the RHS log-linearizes as follows:

$$(1+\iota)\frac{\left[u+\delta(1-u)s_0\right]{r^*_0}^{1+\iota}\widehat{r}^*_{0,t} + (1-\delta)(1-u)s_1{r^*_1}^{1+\iota}\widehat{r}^*_{1,t}}{\left[u+\delta(1-u)s_0\right]{r^*_0}^{1+\iota} + (1-\delta)(1-u)s_1{r^*_1}^{1+\iota}} \\
+ \frac{(1-\delta s_0){r^*_0}^{1+\iota} - (1-\delta)s_1{r^*_1}^{1+\iota}}{\left[u+\delta(1-u)s_0\right]{r^*_0}^{1+\iota} + (1-\delta)(1-u)s_1{r^*_1}^{1+\iota}}u\widehat{u}_t$$

Collecting all terms:

$$\boxed{(1-\alpha)\widehat{\theta}_t-\widehat{\varphi}_t-\widehat{x}_t-\widehat{z}_t-(1+\iota)\frac{[u+\delta(1-u)s_0]\left(\frac{r_0^*}{r_1^*}\right)^{1+\iota}\widehat{r}^*_{0,t}+(1-\delta)(1-u)s_1\widehat{r}^*_{1,t}}{[u+\delta(1-u)s_0]\left(\frac{r_0^*}{r_1^*}\right)^{1+\iota}+(1-\delta)(1-u)s_1}}$$

$$\boxed{+\left\{\frac{1-\delta s_0-(1-\delta)\,s_1}{u+[\delta s_0+(1-\delta)\,s_1]\,(1-u)}-\frac{(1-\delta s_0)\left(\frac{r_0^*}{r_1^*}\right)^{1+\iota}-(1-\delta)s_1}{[u+\delta(1-u)s_0]\left(\frac{r_0^*}{r_1^*}\right)^{1+\iota}+(1-\delta)(1-u)s_1}\right\}u\widehat{u}_{t-1}=0}$$

So, let

$$Den=[u+\delta(1-u)s_0]\left(\frac{r_0^*}{r_1^*}\right)^{1+\iota}+(1-\delta)(1-u)s_1$$

Then:

$$\begin{aligned}
\mathsf{A}_{\mathrm{FEC},\theta}&=1-\alpha\\
\mathsf{A}_{\mathrm{FEC},x}&=-1\\
\mathsf{A}_{\mathrm{FEC},r_0^*}&=-(1+\iota)\frac{u+\delta(1-u)s_0}{Den}\left(\frac{r_0^*}{r_1^*}\right)^{1+\iota}\\
\mathsf{A}_{\mathrm{FEC},r_1^*}&=-(1+\iota)\frac{(1-\delta)(1-u)s_1}{Den}\\
\mathsf{A}_{\mathrm{FEC},z}&=-1\\
\mathsf{A}_{\mathrm{FEC},\varphi}&=-1
\end{aligned}$$

and

$$\mathsf{B}_{\mathrm{FEC},u}=\left[\frac{1-\delta s_0-(1-\delta)s_1}{u+(\delta s_0+(1-\delta)s_1)(1-u)}-\frac{(1-\delta s_0)\left(\frac{r_0^*}{r_1^*}\right)^{1+\iota}-(1-\delta)s_1}{Den}\right]u$$

Market-clearing in the Service market [MC] First, note that we can sum by parts and use $L_{K,t}=1-u_t$:

$$\sum_{k=1}^{K}y_k^{1+\Xi}\left(L_{k,t-1}-L_{k-1,t-1}\right)=y_K^{1+\Xi}(1-u_{t-1})-\sum_{k=1}^{K-1}\left(y_{k+1}^{1+\Xi}-y_k^{1+\Xi}\right)L_{k,t-1}\tag{2}$$

Therefore:

$$\boxed{\left(\frac{x\mathcal{M}}{\mathcal{B}}\right)^{\Xi}\left[\Xi\sum_{k=1}^{K}y_k^{1+\Xi}\left(L_k-L_{k-1}\right)\left(\widehat{\mathcal{M}}_t+\widehat{x}_t+\widehat{z}_t\right)-y_K^{1+\Xi}u\widehat{u}_{t-1}-\sum_{k=1}^{K-1}\left(y_{k+1}^{1+\Xi}-y_k^{1+\Xi}\right)L_k\widehat{L}_{k,t-1}\right]+}$$

$$\boxed{-Q^{\frac{1}{\zeta}}\left(\frac{1}{\zeta}\widehat{Q}_t-\widehat{z}_t\right)-\frac{Q\widehat{Q}_t+\mathcal{M}^{-\sigma}(\sigma\widehat{\mathcal{M}}_t-\widehat{\Upsilon}_t)}{\iota x}-\frac{Q-\mathcal{M}^{-\sigma}}{\iota x}(\widehat{z}_t+\widehat{x}_t)}$$

So, let

$$\mathsf{A}_{\mathrm{MC},1} = \left(\frac{x\mathcal{M}}{\mathcal{B}}\right)^{\Xi} \Xi \sum_{k=1}^{K} y_k^{1+\Xi} \left(L_k - L_{k-1}\right)$$

Then

$$\begin{aligned}
\mathsf{A}_{\mathrm{MC},\mathcal{M}} &= \mathsf{A}_{\mathrm{MC},1} - \frac{\sigma\mathcal{M}^{-\sigma}}{\iota x} \\
\mathsf{A}_{\mathrm{MC},x} &= \mathsf{A}_{\mathrm{MC},1} + \frac{\mathcal{M}^{-\sigma} - Q}{\iota x} \\
\mathsf{A}_{\mathrm{MC},z} &= \mathsf{A}_{\mathrm{MC},x} + Q^{\frac{1}{\zeta}} \\
\mathsf{A}_{\mathrm{MC},Q} &= -\frac{Q^{\frac{1}{\zeta}}}{\zeta} - \frac{Q}{\iota x} \\
\mathsf{A}_{\mathrm{MC},\Upsilon} &= \frac{\mathcal{M}^{-\sigma}}{\iota x}
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{B}_{\mathrm{MC},u} &= -\left(\frac{x\mathcal{M}}{\mathcal{B}}\right)^{\Xi} y_K^{1+\Xi} u \\
\mathsf{B}_{\mathrm{MC},L_k} &= -\left(\frac{x\mathcal{M}}{\mathcal{B}}\right)^{\Xi} \left(y_{k+1}^{1+\Xi} - y_k^{1+\Xi}\right) L_k
\end{aligned}$$

Monetary Policy nominal interest rate rule [R]

$$\boxed{\widehat{R}_t - \rho_R \widehat{R}_{t-1} - (1-\rho_R)\left(\psi_\pi \pi_t + \psi_u u \widehat{u}_t\right) - \widehat{\varsigma}_t = 0}$$

For the last "exogenous" dynamic equations, which are already log linear and require no approximation, we report matrix coefficients all together:

Structural shock dynamics $[z], [\Upsilon], [\varphi]$**, [MP]** .

For $\o \in \{z, \Upsilon, \varphi, \varsigma\}$

$$\boxed{\widehat{\o}_t - \rho_{\o} \widehat{\o}_{t-1} - \sigma_{\o} \varepsilon_t^{\o} = 0}$$

so,

$$\begin{aligned}
\mathsf{A}_{R,\pi} &= -(1-\rho_R)\psi_\pi \\
\mathsf{A}_{R,R} &= 1 \\
\mathsf{A}_{R,\varsigma} &= -1 \\
\mathsf{A}_{R,u} &= -(1-\rho_R)\psi_u u \\
\mathsf{A}_{\o,\o} &= 1 \\
\mathsf{B}_{\o,\o} &= -\rho_{\o} \\
\mathsf{D}_{\o,\o} &= -\sigma_{\o}
\end{aligned}$$

2 Solution: Rational Expectations Equilibrium

2.1 Elimination of static variables

We can solve out for the static variables $\mathcal{Y}_t$ and reduce the system to equations in dynamic (jump and state) variables only, where, with an abuse of notation, χ now denotes the column vector stacking only these other variables, $\mathcal{X}, \mathcal{Z}, \mathcal{S}$. We can choose any five equations where the five static variables $\mathcal{Y}_t$ appear, for example [SC$_0$], [SC$_1$], [MC], [FEC], [PC], collect them in a block denoted by "ST" (for "static"), and build a square submatrix $\mathsf{A}_{ST,\mathcal{Y}}$ with the $\mathsf{A}_{\bullet,\bullet}$ coefficients on each element of $\mathcal{Y}$ in those equations. The rest of the equations are denoted by "DY" (for "dynamic"): $\mathsf{A}_{DY,\bullet}, \mathsf{B}_{DY,\bullet}, \mathsf{C}_{DY,\bullet}, \mathsf{D}_{DY,\bullet}$.

In this new notation, the original system (1), can be written as

$$\begin{pmatrix} \mathsf{A}_{ST,\mathcal{Y}} & \mathsf{A}_{ST,\chi} \\ \mathsf{A}_{DY,\mathcal{Y}} & \mathsf{A}_{DY,\chi} \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{Y}}_t \\ \widehat{\chi}_t \end{pmatrix} + \begin{pmatrix} \mathsf{B}_{ST,\chi} \\ \mathsf{B}_{DY,\chi} \end{pmatrix} \widehat{\chi}_{t-1} + \begin{pmatrix} \mathsf{C}_{ST,\chi} \\ \mathsf{C}_{DY,\chi} \end{pmatrix} \mathbb{E}_t \widehat{\chi}_{t+1} + \begin{pmatrix} \mathsf{D}_{ST} \\ \mathsf{D}_{DY} \end{pmatrix} \varepsilon_t = 0$$

By inspection, $\mathsf{A}_{ST,\mathcal{Y}}$ is generically non singular for many choices of the equations in ST. So, we can solve the upper block

$$\mathsf{A}_{ST,\mathcal{Y}} \widehat{\mathcal{Y}}_t + \mathsf{A}_{ST,\chi} \widehat{\chi}_t + \mathsf{B}_{ST,\chi} \widehat{\chi}_{t-1} + \mathsf{C}_{ST,\chi} \mathbb{E}_t \widehat{\chi}_{t+1} + \mathsf{D}_{ST} \varepsilon_t = 0$$

for the static variables

$$\widehat{\mathcal{Y}}_t = -\mathsf{A}_{ST,\mathcal{Y}}^{-1} \left(\mathsf{A}_{ST,\chi} \widehat{\chi}_t + \mathsf{B}_{ST,\chi} \widehat{\chi}_{t-1} + \mathsf{C}_{ST,\chi} \mathbb{E}_t \widehat{\chi}_{t+1} + \mathsf{D}_{ST} \varepsilon_t \right) \tag{3}$$

The lower block becomes

$$\begin{aligned} \begin{pmatrix} \mathsf{A}_{DY,\mathcal{Y}} & \mathsf{A}_{DY,\chi} \end{pmatrix} \begin{pmatrix} \mathsf{A}_{ST,\mathcal{Y}}^{-1} \left(\mathsf{A}_{ST,\chi} \widehat{\chi}_t + \mathsf{B}_{ST,\chi} \widehat{\chi}_{t-1} + \mathsf{C}_{ST,\chi} \mathbb{E}_t \widehat{\chi}_{t+1} + \mathsf{D}_{ST} \varepsilon_t \right) \\ \widehat{\chi}_t \end{pmatrix} + \\ \mathsf{B}_{DY,\chi} \widehat{\chi}_{t-1} + \mathsf{C}_{DY,\chi} \mathbb{E}_t \widehat{\chi}_{t+1} + \mathsf{D}_{DY} \varepsilon_t = 0 \end{aligned}$$

After rearranging, we can rewrite this lower block in terms of dynamic variables χ_t only,

$$\begin{aligned} \left(-\mathsf{A}_{DY,\mathcal{Y}} \mathsf{A}_{ST,\mathcal{Y}}^{-1} \mathsf{A}_{ST,\chi} + \mathsf{A}_{DY,\chi} \right) \widehat{\chi}_t + \left(-\mathsf{A}_{DY,\mathcal{Y}} \mathsf{A}_{ST,\mathcal{Y}}^{-1} \mathsf{B}_{ST,\chi} + \mathsf{B}_{DY,\chi} \right) \widehat{\chi}_{t-1} + \\ \left(-\mathsf{A}_{DY,\mathcal{Y}} \mathsf{A}_{ST,\mathcal{Y}}^{-1} \mathsf{C}_{ST,\chi} + \mathsf{C}_{DY,\chi} \right) \mathbb{E}_t \widehat{\chi}_{t+1} + \left(-\mathsf{A}_{DY,\mathcal{Y}} \mathsf{A}_{ST,\mathcal{Y}}^{-1} \mathsf{D}_{ST,} + \mathsf{D}_{DY} \right) \varepsilon_t = 0 \end{aligned}$$

and re-define the $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ matrices appropriately. Once we find a solution in terms of χ_t, we recover $\mathcal{Y}_t$ from (3).

2.2 Fundamental solution

We look for the fundamental solution to the system, where expectation errors are only a function of the stochastic realizations of the structural innovations ε. That is, we rule out endogenous expectations errors that give rise to indeterminacy. After estimating parameters, we can check whether the conditions for such determinacy are satisfied.

Because the system of equations (1), which describes equilibrium whether or not we solve out for static variables, is linear, we guess and verify an AR(1) linear solution

$$\widehat{\chi}_t = \Psi \widehat{\chi}_{t-1} + \Lambda \varepsilon_t \tag{4}$$

Using this guess, which implies $\mathbb{E}_t\widehat{\chi}_{t+1} = \Psi\widehat{\chi}_t = \Psi^2\widehat{\chi}_{t-1} + \Psi\Lambda\varepsilon_t$, (1) becomes:

$$\mathsf{A}\Psi\widehat{\chi}_{t-1} + \mathsf{A}\Lambda\varepsilon_t + \mathsf{B}\widehat{\chi}_{t-1} + \mathsf{C}\Psi^2\widehat{\chi}_{t-1} + \mathsf{C}\Psi\Lambda\varepsilon_t + \mathsf{D}\varepsilon_t = \mathsf{0}$$

Thereforem, Ψ solves the quadratic equation:

$$\mathsf{A}\Psi + \mathsf{B} + \mathsf{C}\Psi^2 = 0$$

Since this quadratic equation may have multiple solutions, but the conjectured REE process has exogenous innovations, we need to select a solution Ψ that guarantees stability of REE, i.e. has eigenvalues within the unit circle.

Note that generically A is invertible, as it contains non zero elements in every row and in every column. Therefore, one possible solution method is iterative, whether or not we solve out for static variables. Guess $\Psi_0 = 0$ and for $n = 1, 2\cdots$ iterate

$$\Psi_n = -\mathsf{A}^{-1}(\mathsf{B} + \mathsf{C}\Psi_{n-1}^2)$$

If this recursion converges, we have a solution. Uhlig (1999) proposes an alternative solution method that picks the stable solution, if it exists.

If at the solution Ψ the square matrix $(\mathsf{A} + \mathsf{C}\Psi)$ is non-singular, we can compute

$$\Lambda = -(\mathsf{A} + \mathsf{C}\Psi)^{-1}\mathsf{D}$$

and further characterize the solution. Since

$$(\mathsf{A} + \mathsf{C}\Psi)\Psi = -\mathsf{B}$$

and, by inspection, the first six columns of B are all zero, then a non-singular $(\mathsf{A} + \mathsf{C}\Psi)$ implies that the first six columns of Ψ are also all zero. We also note that the first six rows of D are all zero. Therefore, the six "jump" variables $\widehat{C}_t, \widehat{\theta}_t, \widehat{\pi}_t, \widehat{x}_t, \widehat{W}_t$ are only a linear, deterministic function of the other, predetermined, state variables lagged, a function that we can think of as a policy function, while the $5+K$ state variables are only a linear function of themselves lagged and structural innovations ε, which immediately yields the first component of the model state-space representation, the "state" or "transition" equations. We now show a method to derive this representation in general, even when the matrix $(\mathsf{A} + \mathsf{C}\Psi)$ is not invertible.

3 State-space representation

We now express the model dynamics in a state-space representation. This representation allows to estimate the model either by Maximum Likelihood, using the Kalman Filter, or by Bayesian methods, or by a method of moments, simulating data and computing moments to be matched to empirical ones. Once we have estimated/calibrated the parameter values, we can use the state-space representation also to simulate the equilibrium dynamics from any initial condition and for any draw of innovations, for example IRFs starting from s.s. and introducing once-and-for-all structural innovations, as well as policy experiments.

Recall that we denoted by $\mathcal{X}, \mathcal{Z}, \mathcal{S}$ the three column vectors of (resp.) non-predetermined (jump) variables, exogenous states and endogenous states, after solving out for the static variables. The state-space representation is a "transition" or "state" equation, a linear map from lagged states and structural innovations to current states, where this map should not include jump endogenous variables $\mathcal{X}_t$, and a "measurement" equation, a linear map from states and noise, which could include fundamental innovations as well as measurement error, to observables Y_t, which can include some of the states themselves (in which case the map is the identity), some of the jump endogenous variables $\mathcal{X}_t$, and other variables that the model generates and are observable in the data.

Given the structure of our model, and the available data that determine observables, the natural state-space representation is the following. We can stack the $5+K$ states $\mathcal{Z}_t, \mathcal{S}_t$ into a vector S_t, and the N_Y observables into a vector Y_t, so that transition and measurement equations read

$$\begin{aligned} S_t &= \underline{\mathsf{Q}} S_{t-1} + \overline{\mathsf{V}} \varepsilon_t \\ Y_t &= \overline{\mathsf{N}} S_t + \underline{\mathsf{N}} S_{t-1} + \mathsf{R} \epsilon_t \end{aligned} \tag{5}$$

where $\underline{\mathsf{Q}}$ is a square matrix of dimension $5+K$, $\overline{\mathsf{V}}$ is $(5+K)\times 5$, each $\overline{\mathsf{N}}, \underline{\mathsf{N}}$ is $N_Y \times (5+K)$, R is a square matrix of dimension N_Y, $\epsilon_t \sim \mathbb{N}(0, \mathsf{I}_{N_Y})$ is a column vector of multivariate Gaussian white noise measurement error, which may be required to make sure the log likelihood of the sample is finite. $\underline{\mathsf{Q}}, \overline{\mathsf{V}}, \underline{\mathsf{N}}, \overline{\mathsf{N}}, \mathsf{R}$ are either known from above or can be written in terms of structural parameters. We now show their expressions.

3.1 Transition (state) equations

We first reformulate the system of equilibrium conditions for each of the three subvectors. To that end, we introduce the following notation:

$$\mathsf{A} = \begin{pmatrix} \mathsf{A}_{\mathcal{X}\mathcal{X}} & \mathsf{A}_{\mathcal{X}\mathcal{Z}} & \mathsf{A}_{\mathcal{X}\mathcal{S}} \\ \mathsf{A}_{\mathcal{Z}\mathcal{X}} & \mathsf{A}_{\mathcal{Z}\mathcal{Z}} & \mathsf{A}_{\mathcal{Z}\mathcal{S}} \\ \mathsf{A}_{\mathcal{S}\mathcal{X}} & \mathsf{A}_{\mathcal{S}\mathcal{Z}} & \mathsf{A}_{\mathcal{S}\mathcal{S}} \end{pmatrix}$$

where the matrix $\mathsf{A}_{\mathcal{X}\mathcal{X}}$ collects the coefficients in the three remaining equations of the three remaining forward-looking (jump) variables $\mathcal{X}_t = (\widehat{C}_t, \widehat{\pi}_t, \widehat{W}_t)$, the matrix $\mathsf{A}_{\mathcal{X}\mathcal{Z}}$ the coefficients in the same three equations of the four exogenous state variables $\mathcal{Z}_t = (\widehat{z}_t, \widehat{\varsigma}_t, \widehat{\Upsilon}_t, \widehat{\varphi}_t)$, and so on. Similarly for B, C, and Ψ In this notation, by inspection (each with the appropriate dimensions):

$$\mathsf{A} = \begin{pmatrix} \mathsf{A}_{\mathcal{X}\mathcal{X}} & \mathsf{A}_{\mathcal{X}\mathcal{Z}} & \mathsf{A}_{\mathcal{X}\mathcal{S}} \\ 0 & \mathsf{I} & 0 \\ \mathsf{A}_{\mathcal{S}\mathcal{X}} & \mathsf{A}_{\mathcal{S}\mathcal{Z}} & \mathsf{A}_{\mathcal{S}\mathcal{S}} \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 0 & 0 & \mathsf{B}_{\mathcal{X}\mathcal{S}} \\ 0 & \mathsf{B}_{\mathcal{Z}\mathcal{Z}} & 0 \\ 0 & 0 & \mathsf{B}_{\mathcal{S}\mathcal{S}} \end{pmatrix}$$

and

$$\mathsf{C} = \begin{pmatrix} \mathsf{C}_{\mathcal{X}\mathcal{X}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \Psi_{\mathcal{X}\mathcal{X}} & \Psi_{\mathcal{X}\mathcal{Z}} & \Psi_{\mathcal{X}\mathcal{S}} \\ 0 & \Psi_{\mathcal{Z}\mathcal{Z}} & 0 \\ \Psi_{\mathcal{S}\mathcal{X}} & \Psi_{\mathcal{S}\mathcal{Z}} & \Psi_{\mathcal{S}\mathcal{S}} \end{pmatrix}$$

Moreover:

$$\Psi_{\mathcal{ZZ}} = -\mathsf{B}_{\mathcal{ZZ}} = \begin{pmatrix} \rho_z & 0 & 0 & 0 \\ 0 & \rho_\varsigma & 0 & 0 \\ 0 & 0 & \rho_\varphi & 0 \\ 0 & 0 & 0 & \rho_\Upsilon \end{pmatrix}$$

We finally define the following "standard deviation" matrix:

$$\mathsf{D} = \begin{pmatrix} -\sigma_z & 0 & 0 & 0 \\ 0 & -\sigma_\varsigma & 0 & 0 \\ 0 & 0 & -\sigma_\varphi & 0 \\ 0 & 0 & 0 & -\sigma_\Upsilon \end{pmatrix}$$

We then start with the simplest set of subvectors, namely exogenous states:

$$\mathcal{Z}_t = \Psi_{\mathcal{ZZ}}\mathcal{Z}_{t-1} - \mathsf{D}\varepsilon_t \tag{6}$$

Next, the dynamics of the endogenous states $\mathcal{S}$ in our model are governed by:

$$\mathsf{A}_{\mathcal{SS}}\mathcal{S}_t = -\mathsf{A}_{\mathcal{SX}}\mathcal{X}_t - \mathsf{A}_{\mathcal{SZ}}\mathcal{Z}_t - \mathsf{B}_{\mathcal{SS}}\mathcal{S}_{t-1} \tag{7}$$

Finally, for the jump variables

$$\mathsf{A}_{\mathcal{XX}}\mathcal{X}_t + \mathsf{A}_{\mathcal{XZ}}\mathcal{Z}_t + \mathsf{A}_{\mathcal{XS}}\mathcal{S}_t + \mathsf{B}_{\mathcal{XS}}\mathcal{S}_{t-1} + \mathsf{C}_{\mathcal{XX}}\mathbb{E}_t\left[\mathcal{X}_{t+1}\right] = 0 \tag{8}$$

To find the transition equation(s), we need to "solve out" the jump variables. Under the fundamental solution, we have:

$$\mathbb{E}_t\left[\mathcal{X}_{t+1}\right] = \Psi_{\mathcal{XX}}\mathcal{X}_t + \Psi_{\mathcal{XZ}}\mathcal{Z}_t + \Psi_{\mathcal{XS}}\mathcal{S}_t$$

Replacing in Eq. (8), we can collect terms and, assuming invertibility of the 5×5 matrix $\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}$, solve for the "policy functions", i.e. for $\mathcal{X}_t$ as a function of the states alone, without stochastic innovations:

$$\mathcal{X}_t = -\left(\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}\right)^{-1}\left[\left(\mathsf{A}_{\mathcal{XZ}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XZ}}\right)\mathcal{Z}_t + \left(\mathsf{A}_{\mathcal{XS}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XS}}\right)\mathcal{S}_t + \mathsf{B}_{\mathcal{XS}}\mathcal{S}_{t-1}\right] \tag{9}$$

Replacing in Eq. (7) and rearranging

$$\begin{aligned}\left[\mathsf{A}_{\mathcal{SS}} - \mathsf{A}_{\mathcal{SX}}\left(\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}\right)^{-1}\left(\mathsf{A}_{\mathcal{XS}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XS}}\right)\right]\mathcal{S}_t \\ = \left[\mathsf{A}_{\mathcal{SX}}\left(\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}\right)^{-1}\left(\mathsf{A}_{\mathcal{XZ}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XZ}}\right) - \mathsf{A}_{\mathcal{SZ}}\right]\mathcal{Z}_t \\ + \left[\mathsf{A}_{\mathcal{SX}}\left(\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}\right)^{-1}\mathsf{B}_{\mathcal{XS}} - \mathsf{B}_{\mathcal{SS}}\right]\mathcal{S}_{t-1}\end{aligned}$$

Further substituting (6):

$$\begin{aligned}\underbrace{\left[\mathsf{A}_{\mathcal{SS}} - \mathsf{A}_{\mathcal{SX}}\left(\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}\right)^{-1}\left(\mathsf{A}_{\mathcal{XS}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XS}}\right)\right]}_{\mathsf{Q}_{\mathcal{S}}}\mathcal{S}_t \\ = \underbrace{\left[\mathsf{A}_{\mathcal{SX}}\left(\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}\right)^{-1}\left(\mathsf{A}_{\mathcal{XZ}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XZ}}\right) - \mathsf{A}_{\mathcal{SZ}}\right]}_{\mathsf{Q}_{\mathcal{SZ}}}\left(\Psi_{\mathcal{ZZ}}\mathcal{Z}_{t-1} - \mathsf{D}\varepsilon_t\right) \\ + \underbrace{\left[\mathsf{A}_{\mathcal{SX}}\left(\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}\right)^{-1}\mathsf{B}_{\mathcal{XS}} - \mathsf{B}_{\mathcal{SS}}\right]}_{\mathsf{Q}_{\mathcal{SS}}}\mathcal{S}_{t-1}\end{aligned}$$

Thus, if the $(K+1)\times(K+1)$ matrix $\mathsf{Q}_{\mathcal{S}}$ is invertible, the model admits a state-space representation as in (5), with:

$$\underline{\mathsf{Q}} = \begin{pmatrix} \mathsf{I} & 0_{4\times(K+1)} \\ \mathsf{Q}_{\mathcal{SZ}} & \mathsf{Q}_{\mathcal{SS}} \end{pmatrix} \begin{pmatrix} \Psi_{\mathcal{ZZ}} \\ \mathsf{I} \end{pmatrix} \qquad \text{and} \qquad \overline{\mathsf{V}} = -\begin{pmatrix} \mathsf{I} \\ \mathsf{Q}_{\mathcal{SZ}} \end{pmatrix} \mathsf{D} \tag{10}$$

where:

$$\begin{aligned} \mathsf{Q}_{\mathcal{SZ}} &= \mathsf{Q}_{\mathcal{S}}^{-1}\left[\mathsf{A}_{\mathcal{SX}}\left(\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}\right)^{-1}\left(\mathsf{A}_{\mathcal{XZ}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XZ}}\right) - \mathsf{A}_{\mathcal{SZ}}\right] \\ \mathsf{Q}_{\mathcal{SS}} &= \mathsf{Q}_{\mathcal{S}}^{-1}\left[\mathsf{A}_{\mathcal{SX}}\left(\mathsf{A}_{\mathcal{XX}} + \mathsf{C}_{\mathcal{XX}}\Psi_{\mathcal{XX}}\right)^{-1}\mathsf{B}_{\mathcal{XS}} - \mathsf{B}_{\mathcal{SS}}\right] \end{aligned}$$

In that representation, $S_t = (\mathcal{Z}_t, \mathcal{S}_t)^\top$ only depend on $S_{t-1} = (\mathcal{Z}_{t-1}, \mathcal{S}_{t-1})^\top$ and orthogonal innovations, with coefficient matrices that we know in terms of structural parameters.

3.2 Measurement equations

We have data on seven statistics that directly correspond to model objects:

1. Personal Consumption Expenditure C_t;
2. monthly PCE inflation π_t, which can be turned into total inflation over the past 12 months $\bar{\pi}_t = \sum_{\tau=0}^{11} \pi_{t-\tau}$ by keeping track of the past 11 lags of monthly PCE inflation, which enters the Taylor rule;
3. Federal Funds Rate R_t;
4. unemployment rate u_t;
5. UE probability $\varphi_t \theta_t^\alpha r_{0,t}^*$, log-linearized $\alpha\widehat{\theta}_t + \widehat{\varphi}_t + \widehat{r}_{0,t}^*$;
6. vacancies $v_t = \theta_t \times [u_{t-1} + [\delta s_0 + (1-\delta)s_1](1-u_{t-1})]$, log-linearized

$$\widehat{v}_t = \widehat{\theta}_t + \frac{1-(1-\delta)s_1 - \delta s_0}{u + [\delta s_0 + (1-\delta)s_1](1-u)} u\widehat{u}_{t-1}$$

7. time-aggregated EE probability

$$\mathrm{EE}_t = \delta s_0 \varphi_t \phi(\theta_t) r_{0,t}^* + (1-\delta)s_1\varphi_t\phi(\theta_t) r_{1,t}^* \sum_{k=1}^{K} \overline{\Gamma}_k \frac{L_{k,t-1} - L_{k-1,t-1}}{1-u_{t-1}}$$

   log-linearized:

$$\widehat{\varphi}_t + \widehat{\alpha}\widehat{\theta}_t + \frac{\delta s_0 \phi(\theta) r_0^*}{\mathrm{EE}} \widehat{r}_{0,t}^* + \left(1 - \frac{\delta s_0\phi(\theta) r_0^*}{\mathrm{EE}}\right)\left(\widehat{r}_{1,t}^* + \frac{u}{1-u}\widehat{u}_{t-1}\right) \\ + \frac{(1-\delta)s_1\phi(\theta)r_1^*}{\mathrm{EE}} \sum_{k=1}^{K} \overline{\Gamma}_k \frac{L_k \widehat{L}_{k,t-1} - L_{k-1}\widehat{L}_{k-1,t-1}}{1-u}$$

8. average hours per employee

$$H_t = \sum_{k=1}^{K} \left(\frac{\mathcal{M}_t}{\mathcal{B}} x_t z_t y_k \right)^{\Xi} \frac{L_{k,t-1} - L_{k-1,t-1}}{1 - u_{t-1}} = \left(\frac{\mathcal{M}_t}{\mathcal{B}} x_t z_t \right)^{\Xi} \frac{1}{1 - u_{t-1}} \sum_{k=1}^{K} y_k^{\Xi} \left(L_{k,t-1} - L_{k-1,t-1} \right)$$

This is the same expression as the supply of Service good, in units of the numeraire, except that y is raised to the power Ξ rather than $1+\Xi$. Therefore, following the same steps, and summing by parts as in (2), we obtain:

$$\widehat{H}_t \quad = \Xi \left(\widehat{\mathcal{M}}_t + \widehat{x}_t + \widehat{z}_t \right) + \tfrac{u}{1-u} \widehat{u}_{t-1} - \tfrac{y_K^{\Xi} u \widehat{u}_{t-1} + \sum_{k=1}^{K-1} \left(y_{k+1}^{\Xi} - y_k^{\Xi} \right) L_k \widehat{L}_{k,t-1}}{y_K^{\Xi} (1-u) + \sum_{k=1}^{K-1} \left(y_{k+1}^{\Xi} - y_k^{\Xi} \right) L_k}$$

It is harder to map total output Q_t of Final goods into data, because in the model Q_t only includes private consumption and vacancy costs (the only form of investment), while empirical GDP includes other forms of investment, as well as Government spending and exports.

The model generates log deviations from steady state. Accordingly, in the data, we consider HP-filtered log time series.

This leaves us with seven variables independently generated by the model, with direct empirical counterparts, so that the dimension of the measurement vector Y_t is $N_Y = 7$. All eight of them can be written as *deterministic* functions of state variables. To derive $\overline{\mathsf{N}}, \underline{\mathsf{N}}$, we can just use the policy functions and state equations found before.

3.3 State-space representation in canonical form

In the canonical state-space representation, the lagged state does not appear in the measurement equation. To convert our system to the traditional formulation, it is customary to extend the state space, namely stack the state and its lag into an (extended) state

$$\mathsf{S}_t = \begin{pmatrix} S_t \\ S_{t-1} \end{pmatrix}$$

so that

$$\mathsf{S}_t = \underbrace{\begin{pmatrix} \overline{\mathsf{Q}} & 0 \\ \mathsf{I} & 0 \end{pmatrix}}_{\mathsf{Q}} \mathsf{S}_{t-1} + \underbrace{\begin{pmatrix} \overline{\mathsf{V}} \\ 0 \end{pmatrix}}_{\mathsf{V}} \varepsilon_t$$

the extended state S_t has $2 \cdot (5 + K) = 10 + 2K$ elements, and

$$Y_t = \underbrace{\begin{pmatrix} \overline{\mathsf{N}} & \underline{\mathsf{N}} \end{pmatrix}}_{\mathsf{N}} \mathsf{S}_t + \mathsf{R} \epsilon_t$$

so that the state space representation (5) is in canonical form:

$$\begin{aligned} \mathsf{S}_t &= \mathsf{Q} \mathsf{S}_{t-1} + \mathsf{V} \varepsilon_t \\ Y_t &= \mathsf{N} \mathsf{S}_t + \mathsf{R} \epsilon_t \end{aligned} \tag{11}$$

4 Estimation

We illustrate the estimations steps, and then provide details. We write the Rational Expectations Equilibrium in State-Space canonical form, with 11 lags of inflation as part of the (endogenous component of the) state vector, to keep track of annual inflation. The scale of vacancies is not identified, so we set $\theta = 1$. Then:

1. We pre-calibrate values of β and η.

2. We estimate by GMM the parameters of the Taylor rule $\rho_R, \psi_\pi, \psi_u, \rho_\varsigma, \sigma_\varsigma$.

3. Given parameters of the match distribution Γ_k, a truncated Pareto on $\{y_1, ...y_K\}$ (namely, λ and the upper bound y_K, because the lower bound of the support can be normalized to $y_1 = 1$ WLOG), we use steady state equation and a few moment conditions to estimate δ, s_0.

4. Given values of the following parameters: $\lambda, y_K, \iota, \sigma, \alpha, \zeta, s_1, \Xi, \mathcal{B}$, we can compute all steady state endogenous values and the parameters b, κ_v, κ_s. An optional step and new empirical moment allows to estimate also s_1 in steady state.

5. We estimate $\lambda, y_K, \iota, \sigma, \alpha, \zeta, s_1, \nu, \Xi, \mathcal{B}$, and the six parameters of the other shock processes $\rho_\varnothing, \sigma_\varnothing$ for $\varnothing \in \{z, \Upsilon, \varphi\}$, by a Simulated Method of Moments. Using their values, we go through the previous steps to compute the other parameters and steady state, thus the State-Space representation. We then simulate a time series of the Rational Expectations Equilibrium and finally use the seven measurement equations to estimate the variances, correlations, and first-order autocorrelation of the seven model-generated time series illustrated above. This is a total of 35 moments. We seek values of the 16-17 (depending on the model) parameters which minimize the squared distance between log deviations of model-generated and empirical moments.

We now provide details on the main steps.

4.1 Steady State

4.1.1 Moment conditions

As shown in Appendix B to the paper, by imposing steady state moment conditions, we can estimate ("calibrate") the turnover parameters δ, s_0, and possibly s_1. With a discrete distribution of match quality, the argument goes as follows. The probability of acceptance equals

$$\sum_{k=1}^{K}(1-\Gamma_k)\frac{L_k - L_{k-1}}{1-u} = \frac{L_K}{1-u} - \sum_{k=1}^{K}\Gamma_k\frac{L_k - L_{k-1}}{1-u}$$
$$= 1 - \sum_{k=1}^{K-1} L_k\frac{\Gamma_k - \Gamma_{k+1}}{1-u} - \frac{L_K}{1-u} = \sum_{k=1}^{K-1} L_k\frac{\Gamma_{k+1} - \Gamma_k}{1-u}$$

Using the expression for L_k

$$= \sum_{k=1}^{K-1} \frac{u+\delta(1-u)s_0}{\delta+(1-\delta)s_1 r_1^* \phi(\theta)(1-\Gamma_k)} \phi(\theta) r_0^* \frac{\Gamma_k}{1-u} \left(\Gamma_{k+1}-\Gamma_k\right)$$
$$= \frac{u+\delta(1-u)s_0}{1-u} \phi(\theta) r_0^* \sum_{k=1}^{K-1} \frac{\Gamma_k}{\delta+(1-\delta)s_1 r_1^* \phi(\theta)(1-\Gamma_k)} \left(\Gamma_{k+1}-\Gamma_k\right)$$
$$= \delta \sum_{k=1}^{K-1} \frac{\Gamma_k \left(\Gamma_{k+1}-\Gamma_k\right)}{\delta+(1-\delta)s_1 r_1^* \phi(\theta)(1-\Gamma_k)}$$

Recall:

$$(1-\delta)s_1\phi(\theta)r_1^* = \frac{\mathsf{EE}-\delta+\mathsf{EU}}{\mathsf{AC}} = \frac{\mathsf{EE}-\delta s_0 \mathsf{UE}}{\mathsf{AC}} \tag{12}$$

Then we can define

$$\mathcal{A}(\delta) = \delta \sum_{k=1}^{K-1} \frac{\Gamma_k \left(\Gamma_{k+1}-\Gamma_k\right)}{\delta+(1-\delta)s_1 r_1^* \phi(\theta)(1-\Gamma_k)}$$

Replace $u = \mathsf{UR}$ and the expression for $(1-\delta)s_1\phi(\theta)r_1^*$ from (12). Then

$$\mathcal{A}(\delta) = \delta \mathsf{AC} \sum_{k=1}^{K-1} \frac{\Gamma_k \left(\Gamma_{k+1}-\Gamma_k\right)}{\delta \mathsf{AC} + \left(\mathsf{EE}-\delta+\mathsf{EU}\right)\left(1-\Gamma_k\right)} \tag{13}$$

which can be used to estimate δ. Although this expression appears to depend on the sampling distribution Γ_k, we know from the general case in Appendix B that it does not. For example, we can fix $\Gamma_k = k/K$ to be uniform quantiles and choose the discrete grid $\{y_k\}$ so that the discrete distribution approximates any desired one. Because $\mathcal{A}(\delta)$ does not depend on the grid $\{y_k\}$, it can be evaluated independently of the underlying distribution.

The parameters $\mathcal{B}, b$ enter the steady state equations and the coefficients of the log linearized system only through a "modified" MRS between leisure and consumption on the Hours (intensive) and (un)employment (extensive) margins:

$$\overline{\mathcal{B}} = \frac{1}{1+\Xi}\left(\frac{x\mathcal{M}}{\mathcal{B}}\right)^{\Xi} \qquad \text{and} \qquad \overline{b} = \frac{b}{x\mathcal{M}}$$

Then

$$\kappa_s r_0^{*\iota} = \frac{\beta}{1-\beta(1-\delta)}\left(\overline{\mathcal{B}}\mu_0 - \overline{b}\right) \tag{14}$$

and

$$\kappa_s r_1^{*\iota} = \frac{\beta}{1-\beta(1-\delta)}\overline{\mathcal{B}}\mu_1 \tag{15}$$

Dividing, solve for the ratio of recruiting efforts:

$$r^* := \frac{r_0^*}{r_1^*} = \left(\frac{\overline{\mathcal{B}}\mu_0 - \overline{b}}{\overline{\mathcal{B}}\mu_1}\right)^{\frac{1}{\iota}}$$

Then using Eq. (12):

$$r^* := \frac{r_0^*}{r_1^*} = \frac{(1-\delta)\phi(\theta)r_0^*}{(1-\delta)\phi(\theta)r_1^*} = \frac{s_1\mathsf{AC}(1-\delta)\mathsf{UE}}{\mathsf{EE}-\delta+\mathsf{EU}} \tag{16}$$

Equating the last two expressions for r^* allows to find another expression for the surplus

$$\overline{\mathcal{B}}\mu_0 - \overline{b} = \overline{\mathcal{B}}\mu_1 \left[\frac{s_1\mathsf{AC}(1-\delta)\mathsf{UE}}{\mathsf{EE}-\delta+\mathsf{EU}}\right]^\iota \tag{17}$$

Using again (14) and the last expression, and then (15)

$$\begin{aligned}
\phi\kappa_s r_0^{*1+\iota} &= \phi r_0^*\kappa_s r_0^{*\iota} = \mathsf{UE}\frac{\beta}{1-\beta(1-\delta)}\left(\overline{\mathcal{B}}\mu_0 - \overline{b}\right) = \mathsf{UE}\frac{\beta}{1-\beta(1-\delta)}\left[\frac{s_1\mathsf{AC}(1-\delta)\mathsf{UE}}{\mathsf{EE}-\delta+\mathsf{EU}}\right]^\iota \overline{\mathcal{B}}\mu_1 \\
\phi\kappa_s(1-\delta)s_1 r_1^{*1+\iota} &= (1-\delta)s_1\phi r_1^*\kappa_s r_1^{*\iota} = \frac{\mathsf{EE}-\delta s_0\mathsf{UE}}{\mathsf{AC}}\frac{\beta}{1-\beta(1-\delta)}\overline{\mathcal{B}}\mu_1
\end{aligned}$$

The combined free entry and market-clearing condition can then be written as:

$$\begin{aligned}
Q - \mathcal{M}^{-\sigma} &= \phi\frac{\iota x\kappa_s}{1+\iota}\left\{[\mathsf{UR}+\delta s_0(1-\mathsf{UR})]\, r_0^{*1+\iota} + (1-\delta)(1-\mathsf{UR})s_1 r_1^{*1+\iota}\right\} \qquad (18)\\
&= \frac{\beta}{1-\beta(1-\delta)}\frac{\iota x}{1+\iota}\overline{\mathcal{B}}\mu_1\left\{[\mathsf{UR}+\delta s_0(1-\mathsf{UR})]\,\mathsf{UE}\left[\frac{s_1\mathsf{AC}(1-\delta)\mathsf{UE}}{\mathsf{EE}-\delta+\mathsf{EU}}\right]^\iota + (1-\mathsf{UR})\frac{\mathsf{EE}-\delta s_0\mathsf{UE}}{\mathsf{AC}}\right\}
\end{aligned}$$

The final equation we can rewrite in this new notation is Service market-clearing:

$$\overline{\mathcal{B}}(1+\Xi)\sum_{k=1}^{K} y_k^{1+\Xi}(L_k - L_{k-1}) = Q^{\frac{1}{\zeta}} + \frac{Q-\mathcal{M}^{-\sigma}}{\iota x} \tag{19}$$

4.1.2 Moment estimation

Armed with these expressions, proceed as follows:

1. find the root of $\mathcal{A}(\delta) = \mathsf{AC}$ in (13) to estimate the value of δ.
2. Compute the value of $s_0 = (1-\mathsf{EU}/\delta)/\mathsf{UE}$.
3. OPTIONAL: this step exploits an empirical observation of a reallocation shock incidence GF to estimate the value s_1 once and for all from $\mathsf{GF} = \delta s_0/[\delta s_0 + (1-\delta)s_1]$;
4. From (16), compute the value of r^*.
5. Using the fact that $\phi(\theta)r_0^*\,[u+\delta(1-u)s_0]$ is the total flow of hires from unemployment, which must equal separations $\delta(1-u)$ in steady state, and replacing the expression for $(1-\delta)s_1\phi(\theta)r_1^*$ from Eq. (12), compute the values of the employment distribution:

$$L_k = \frac{\phi(\theta)r_0^*\Gamma_k\,[u+\delta(1-u)s_0]}{\delta + (1-\delta)s_1\phi(\theta)r_1^*(1-\Gamma_k)} = \frac{\delta(1-\mathsf{UR})\Gamma_k}{\delta + \frac{\mathsf{EE}-\delta s_0\mathsf{UE}}{\mathsf{AC}}(1-\Gamma_k)}$$

6. Compute the values of:

$$\mu_0 = \sum_{k=1}^{K} (\Gamma_k - \Gamma_{k-1})\, y_k^{1+\Xi}$$

$$\mu_1 = \sum_{k=1}^{K-1} (1 - \Gamma_k) \left(y_{k+1}^{1+\Xi} - y_k^{1+\Xi} \right) \frac{L_k}{1 - \mathsf{UR}}$$

7. We now have three equations, (18), (19) and the steady state equation

$$x = \frac{\eta - 1}{\eta} \zeta Q^{\frac{\zeta-1}{\zeta}}$$

   where $\bar{b}$ and κ_s no longer appear. Given parameter values $\beta, \delta, s_0, \mu_0, \mu_1, L_k, \Xi, \mathcal{B}, \iota, \zeta$, and possibly s_1 calculated before, we can solve these three equations for the values of $x, Q, \mathcal{M}$. Recall that $\overline{\mathcal{B}}$ also contains x and $\mathcal{M}$.

8. From (17), compute the value of $\bar{b}$ and then of b.

9. Using again (14) and (15), compute the values of $\kappa_s r_0^{*\iota}$ and $\kappa_s r_1^{*\iota}$.

10. From free entry, compute the "composite parameter"

$$\kappa_v \theta = \frac{Q - \mathcal{M}^{-\sigma}}{\mathsf{UR} + [\delta s_0 + (1 - \delta) s_1](1 - \mathsf{UR})}$$

To summarize: given values of the parameters λ, y_K of Γ_k, as well as $\iota, \sigma, \alpha, \zeta$ and, if we skip Step 3, s_1, we can compute all steady state values and the parameter $b, \kappa_v \theta$. The parameters ν, κ_s do not enter steady state equations, while κ_v and θ are not separately identified. Similarly, the s.s. contact rate $\phi(\theta)$ does not appear in these equations.

4.2 Dynamics

Using $\phi(\theta) r_0^* = \mathsf{UE}$, $u = \mathsf{UR}$, $(1 - \delta) s_1 \phi(\theta) r_1^*$ from Eq. (12), the values of $\kappa_v \theta$, $r^* = r_0^* / r_1^*$ estimated above, we can compute the coefficients of the linearized system.

We start with the "labor market block", namely free entry conditions and dynamics of the employment distribution (including unemployment). These coefficients only depend on values of: Γ_k, which allow to estimate δ and then s_0; of s_1, which allows to estimate L_k and r^*; and ι, α.

$$\begin{aligned}
\mathsf{A}_{\mathrm{FEC},\theta} &= 1 - \alpha \\
\mathsf{A}_{\mathrm{FEC},r_0^*} &= -(1 + \iota) \frac{\mathsf{UR} + \delta(1 - \mathsf{UR}) s_0}{[\mathsf{UR} + \delta(1 - \mathsf{UR}) s_0]\, r^{*1+\iota} + (1 - \delta)(1 - \mathsf{UR}) s_1} r^{*1+\iota} \\
\mathsf{A}_{\mathrm{FEC},r_1^*} &= -(1 + \iota) \frac{(1 - \delta)(1 - \mathsf{UR}) s_1}{[\mathsf{UR} + \delta(1 - \mathsf{UR}) s_0]\, r^{*1+\iota} + (1 - \delta)(1 - \mathsf{UR}) s_1} \\
\mathsf{B}_{\mathrm{FEC},u} &= \left[\frac{1 - \delta s_0 - (1 - \delta) s_1}{\mathsf{UR} + (\delta s_0 + (1 - \delta) s_1)(1 - \mathsf{UR})} - \frac{(1 - \delta s_0) r^{*1+\iota} - (1 - \delta) s_1}{[\mathsf{UR} + \delta(1 - \mathsf{UR}) s_0]\, r^{*1+\iota} + (1 - \delta)(1 - \mathsf{UR}) s_1} \right] \mathsf{UR}
\end{aligned}$$

$$
\begin{aligned}
\mathsf{A}_{L_k,r_0^*} &= -\mathsf{UE}\frac{\mathsf{UR}+\delta(1-\mathsf{UR})s_0}{L_k}\Gamma_k \\
\mathsf{A}_{L_k,r_1^*} &= \frac{\mathsf{EE}-\delta+\mathsf{EU}}{\mathsf{AC}} = \frac{\mathsf{EE}-\delta s_0\mathsf{UE}}{\mathsf{AC}}(1-\Gamma_k) \\
\mathsf{B}_{L_k,u} &= -\frac{\mathsf{UE}(1-\delta s_0)\Gamma_k}{L_k}\mathsf{UR} \\
\mathsf{B}_{L_k,L_k} &= -(1-\delta)+\mathsf{A}_{L_k,r_1^*} \\
\mathsf{A}_{u,\varphi} &= \frac{\mathsf{UR}+\delta(1-\mathsf{UR})s_0}{\mathsf{UR}}\mathsf{UE} \\
\mathsf{A}_{u,\theta} &= \alpha\mathsf{A}_{u,\varphi} \\
\mathsf{B}_{u,u} &= -1+\delta+(1-\delta s_0)\mathsf{UE}
\end{aligned}
$$

The remaining equations also depend on the rest of the parameters: pre-calibrated β, η, then $\Xi, \sigma, \nu, \zeta, \mathcal{B}, \kappa_v\theta$:

$$
\begin{aligned}
\mathsf{A}_{W,x} &= -[1-\beta(1-\delta)](1+\Xi) \\
\mathsf{C}_{W,\mathcal{M}} &= -\beta(1-\delta) \\
\mathsf{A}_{\mathrm{SC}_0,x} &= r^{*\iota} \\
\mathsf{A}_{\mathrm{SC}_0,r_0^*} &= \mathsf{A}_{\mathrm{SC}_0,x}\iota
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{C}_{\mathrm{SC}_0,\mathcal{M}} &= -\frac{\mu_0}{\mu_1} \\
\mathsf{C}_{\mathrm{SC}_0,W} &= \mathsf{C}_{\mathrm{SC}_0,\mathcal{M}} \\
\mathsf{A}_{\mathrm{SC}_1,r_1^*} &= \iota \\
\mathsf{B}_{\mathrm{SC}_1,u} &= -\frac{\mathsf{UR}}{1-\mathsf{UR}} \\
\mathsf{B}_{\mathrm{SC}_1,L_k} &= -\frac{(1-\Gamma_k)\left(y_{k+1}^{1+\Xi}-y_k^{1+\Xi}\right)L_k}{\sum_{j=1}^{K-1}(1-\Gamma_j)\left(y_{j+1}^{1+\Xi}-y_j^{1+\Xi}\right)L_j}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{A}_{Q,Q} &= Q \\
\mathsf{A}_{Q,\mathcal{M}} &= \sigma\mathcal{M}^{-\sigma} \\
\mathsf{A}_{Q,\Upsilon} &= -\mathcal{M}^{-\sigma} \\
\mathsf{A}_{Q,\theta} &= -Q+\mathcal{M}^{-\sigma} \\
\mathsf{B}_{Q,u} &= \mathsf{A}_{Q,\theta}\frac{1-\delta s_0-(1-\delta)s_1}{\mathsf{UR}+\delta(1-\mathsf{UR})s_0+(1-\delta)(1-\mathsf{UR})s_1}
\end{aligned}
$$

$$\mathsf{A}_{\mathrm{PC}} = -\frac{\nu}{1-\nu}\frac{1-\beta(1-\nu)}{1+\eta\frac{1-\zeta}{\zeta}}$$
$$\mathsf{A}_{\mathrm{PC},Q} = \mathsf{A}_{\mathrm{PC}}\frac{1-\zeta}{\zeta}$$
$$\mathsf{A}_{\mathrm{PC},x} = \mathsf{A}_{\mathrm{PC}}$$

$$\mathsf{A}_{\mathrm{MC},\mathcal{M}} = \left(\frac{x\mathcal{M}}{\mathcal{B}}\right)^{\Xi} \Xi \sum_{k=1}^{K} y_k^{1+\Xi}\left(L_k - L_{k-1}\right) - \frac{\sigma\mathcal{M}^{-\sigma}}{\iota x}$$
$$\mathsf{A}_{\mathrm{MC},x} = \left(\frac{x\mathcal{M}}{\mathcal{B}}\right)^{\Xi} \Xi \sum_{k=1}^{K} y_k^{1+\Xi}\left(L_k - L_{k-1}\right) + \frac{\mathcal{M}^{-\sigma} - Q}{\iota x}$$
$$\mathsf{A}_{\mathrm{MC},z} = \mathsf{A}_{\mathrm{MC},x} + Q^{\frac{1}{\zeta}}$$
$$\mathsf{A}_{\mathrm{MC},Q} = -\frac{Q^{\frac{1}{\zeta}}}{\zeta} - \frac{Q}{\iota x}$$
$$\mathsf{A}_{\mathrm{MC},\Upsilon} = \frac{\mathcal{M}^{-\sigma}}{\iota x}$$

and

$$\mathsf{B}_{\mathrm{MC},u} = -\left(\frac{x\mathcal{M}}{\mathcal{B}}\right)^{\Xi} y_K^{1+\Xi} u$$
$$\mathsf{B}_{\mathrm{MC},L_k} = -\left(\frac{x\mathcal{M}}{\mathcal{B}}\right)^{\Xi} \left(y_{k+1}^{1+\Xi} - y_k^{1+\Xi}\right) L_k$$

$$\mathsf{A}_{\mathcal{M},\mathcal{M}} = -1$$
$$\mathsf{A}_{\mathcal{M},R} = 1$$

$$\mathsf{C}_{\mathcal{M},\mathcal{M}} = 1$$
$$\mathsf{C}_{\mathcal{M},\pi} = -1$$

4.3 Identification

By inspection, the coefficients of the dynamical system, reported above (excluding the exogenous shock processes and the Taylor rule), depend on r_0^*, r_1^* only through their ratio r^*. Furthermore, $\kappa_s, \phi, \kappa_v\theta$ do not appear anywhere in these coefficients and, as we saw, $\kappa_v\theta$ enters only as a composite parameter in steady state. Therefore, steady state equations and dynamic moments do not identify the scale of hiring costs κ_s, κ_v separately from the scale of vacancies, thus of θ. WLOG, we can set $\theta = 1$, thus $\phi = 1$, compute in steady state κ_v instead of $\kappa_v\theta$ and κ_s from Step 7 and $r_0^* = \mathsf{UE}$.

To summarize: given some empirical moments, we can estimate δ, s_0; given also values of the parameters λ, y_K of Γ_k, as well as $\iota, \sigma, \alpha, \zeta, \Xi$ and, if we skip Step 3, s_1, we can compute all steady state values and estimate the parameters b, κ_v, κ_s. Dynamic equations identify the remaining parameters.