Online Appendix for Why Global and Local Solutions of Open-Economy Models with Incomplete Markets Differ and Why it Matters*

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July 2023

Abstract

This appendix provides a review of the closure methods used in the academic literature and policy circles, details on the global and local solution methods used to solve the endowment and Sudden Stops models in the main paper. The appendix also includes some mathematical derivations of expressions and results referred to in the paper, and additional quantitative results for each model.

JEL Classification: *C63, F41, E44*

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Contents

A Review of the solution methods used in the literature **2**

B Endowment economy **5**
B.1 Global methods . . . 6
B.1.1 FiPIt method . . . 6
B.1.2 VFI method . . . 7
B.2 Local methods . . . 8
B.2.1 Perturbation methods . . . 9
B.2.2 Risk-adjusted steady state method . . . 11
B.2.3 Linear-with-occasionally-binding-constraint methods . . . 13
B.3 Calibration & comparison of quantitative results . . . 15
B.3.1 Calibration . . . 15
B.3.2 NFA decision rule and net exports . . . 15
B.3.3 Long-run moments & impulse response functions . . . 19
B.3.4 Spectral densities . . . 20
B.3.5 Comparison of DEIR vs. Endogenous Discounting . . . 23
B.3.6 Linear with occasionally binding constraint method: DynareOBC . . . 26
B.3.7 Analytical solution under savings-under-uncertainty framework . . . 29

C The Sudden Stops model **33**
C.1 *FiPIt* Global Solution Method . . . 36
C.2 Additional quantitative results of the Sudden Stops model . . . 41
C.2.1 Spectral analysis . . . 41
C.2.2 *FiPIt* v. DynareOBC speed comparisons . . . 41
C.2.3 Financial premia & quintile analysis for Sudden Stops model . . . 43

A Review of the solution methods used in the literature

Global and local methods have been widely used in research and policy applications. Tables 1 and 2 document the usage of these methods for solving open-economy models with incomplete markets in research articles and policy institutions. Table 1 includes 101 papers. To construct the table, we took the 100 most cited items in *Google Scholar* that cite Schmitt-Grohé and Uribe (2003), and then excluded textbooks and review articles. It also includes all quantitative papers in the references of this paper and several well-known papers going back to the early 1990s when the first numerical solutions of open-economy models with incomplete markets were produced. Table 2 lists the models used in eight policy institutions, using information obtained from publicly available documents.

Table 1: Methods Used to Solve Open-Economy Incomplete Markets Models

Authors	Year	Publication	Type of model	Solution method	Stationarity assumption	ψ	
Adolfson et al.	2007	JIE	SOE	1OA	DEIR	0.145	e
Aguiar and Gopinath	2007	JPE	SOE	1OA	DEIR	0.001	s
Anand et al.	2015	JME	SOE	1OA	DEIR	0.0007	s
Angeloni and Ehrmann	2007	BEJ Ma	$N = 12$	1OA	DEIR	0.1	c
Arellano	2008	AER	SOE	GLB			
Arellano and Mendoza	2002	NBER	SOE	GLB			
Arezki et al.	2017	QJE	SOE	1OA	DEIR	0.0001	s
Baxter and Crucini	1995	IER	$N = 2$	1OA	AHC		
Bems and de Carvalho	2011	JIE	SOE	GLB			
Benes et al.	2015	OER	SOE	RSS	NSIA		
Bengui et al.	2012	JME	$N = 2$	GLB			
Benigno and Thoenissen	2008	JIMF	$N = 2$	1OA	AHC		
Benigno et al.	2016	JME	SOE	GLB			
Beraja et al.	2019	JES	SOE	1OA	ED		
Bergin	2006	JIMF	$N = 2$	1OA	DEIR	0.00384	e
Bianchi	2016	AER	SOE	GLB			
Bianchi	2011	AER	SOE	GLB			
Bianchi and Mendoza	2018	JPE	SOE	GLB			
Bianchi and Mendoza	2021	JEDC	SOE	GLB			
Bianchi and Mendoza	2018	JPE	SOE	GLB			
Bianchi et al.	2012	IMFER	SOE	GLB			
Bianchi et al.	2016	JIE	SOE	GLB			
Bodenstein	2011	JIE	$N = 2$	1OA	ED		
Bodenstein et al.	2017	CJE	SOE	1OA	DEIR	0.001	c
Bodenstein et al.	2011	JIE	$N = 2$	1OA	DEIR	0.0001	s
Bom et al.	2014	JEDC	SOE	1OA	OLG		
Bouakez and Eyquem	2015	JIMF	SOE	1OA	DEIR	0.0017	e
Boz and Mendoza	2014	JME	SOE	GLB			
Boz et al.	2015	JMCB	SOE	GLB			
Boz et al.	2011	JME	SOE	1OA	DEIR	0.001	s
Buch et al.	2005	JIMF	$N = 2$	1OA	AHC		
Cavallo and Ghironi	2002	JME	$N = 2$	1OA	OLG		
Chang and Fernández	2013	IER	SOE	1OA	DEIR	0.001	s
Choudri et al.	2005	JIE	SOE	1OA	DEIR	0.001	s
Coeurdacier et al.	2020	JME	$N = 2$	GLB			
Coeurdacier et al.	2011	AER PP	SOE	RSS	NSIA		
Correia et al.	1995	EER	SOE	1OA	AHC		
Corsetti et al.	2014	JME	$N = 2$	1OA	DEIR	—	c
Corsetti et al.	2008	REStud	$N = 2$	1OA	ED		
Cuadra and Sapriza	2008	JIE	SOE	GLB			
Davis and Presno	2017	JME	SOE	1OA	DEIR	0.015	c
Devereux and Sutherl and	2010	JMCB	$N = 2$	1OA	ED		
Devereux and Sutherl and	2011	JEEA	$N = 2$	1OA	ED		
Devereux et al.	2006	EJ	SOE	1OA	AHC		
Drechsel and Tenreyro	2018	JIE	SOE	1OA	DEIR	0.199	e
Durdu and Mendoza	2006	JIE	SOE	GLB			
Durdu et al.	2013	JIE	SOE	GLB			
Durdu et al.	2009	JDE	SOE	GLB			
Eichenbaum et al.	2021	REStud	N=3	1OA	CM		
Enders et al.	2011	JIE	$N = 2$	1OA	ED		
Engel and Wang	2011	JIE	$N = 2$	1OA	AHC		
Fern and ez and Chang	2013	IER	SOE	1OA	DEIR	0.001	s
Fern and ez-Villaverde et al.	2011	AER	SOE	3OA	AHC		
Fernández et al.	2015	RED	SOE	1OA	DEIR	0.00001	s

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Table 1 – continued from previous page

Authors	Year	Journal	Type of model	Solution method	Stationarity assumption	ψ	
Fernández et al.	2018	JIE	SOE	1OA	DEIR	0.001	s
Ferrero	2015	JMCB	$N = 2$	1OA	CM		
Fogli and Perri	2006	NBER	$N = 2$	GLB			
Gabaix and Maggiori	2017	QJE	$N = 2$	1OA	CM		
Gadatsch et al.	2016	EM	$N = 2$	1OA	DEIR	0.01	e
Garcia-Cicco et al.	2010	AER	SOE	1OA	DEIR	0.001,2.8	s,e
Gertler et al.	2007	JMCB	SOE	1OA	DEIR	0.0001	s
Ghironi	2006	JIE	$N = 2$	1OA	OLG		
Ghironi and Melitz	2005	QJE	$N = 2$	1OA	AHC		
Glocker and Towbin	2012	WP	SOE	1OA	AHC		
Gourinchas et al.	2017	NBER MA	SOE	1OA	DEIR	0.0001	s
Gourio et al.	2013	JIE	$N = 2$	GLB			
Hatchondo and Martinez	2009	JIE	SOE	GLB			
Heathcote and Perri	2016	IMF ER	SOE	3OA	NSIA		
Heathcote and Perri	2002	JME	$N = 2$	1OA	AHC		
Heathcote and Perri	2013	JPE	$N = 2$	3OA	NSIA		
Itskhoki and Mukhin	2021	JPE	N=2	1OA	DEIR	0.001	s
Jaimovich and Rebelo	2008	JMCB	SOE	1OA	DEIR	0.00001	s
Justiniano and Preston	2010	JIE	SOE	1OA	DEIR	0.01	c
Kollmann	2013	JMCB	$N = 2$	1OA	ED		
Korinek and Mendoza	2014	ARE	SOE	GLB			
Liu and Spiegel	2015	IMF ER	SOE	1OA	AHC		
Lubik and Schorfheide	2005	NBER MA	$N = 2$	1OA	CM		
Martín and Gulan	2015	AEJ Ma	SOE	1OA	DEIR	0.108	e
Martin and Philippon	2017	AER	SOE	1OA	DEIR	0.001	s
Melina et al.	2016	EM	SOE	GLB			
Mendoza	1991	AER	SOE	GLB			
Mendoza	1992	IMF WP	SOE	GLB			
Mendoza	1995	IER	SOE	GLB			
Mendoza	2010	AER	SOE	GLB			
Mendoza and Smith	2006	JIE	SOE	GLB			
Mendoza and Yue	2012	QJE	SOE	GLB			
Mendoza et al.	2009	JPE	$N = 2, 3$	GLB			
Monacelli	2005	JMCB	SOE	1OA	CM		
Nason and Rogers	2006	JIE	SOE	1OA	DEIR	0.00014,0.007	e
Neumeyer and Perri	2005	JME	SOE	1OA	AHC		
Punzi	2013	RIE	$N = 2$	1OA	ED		
Rabanal and Tuesta	2010	JEDC	$N = 2$	1OA	AHC		
Raffo	2008	JIE	$N = 2$	1OA	AHC		
Rebelo and Vegh	1995	REStud	SOE	1OA	AHC		
Schmitt-Gorhé and Uribe	2016	JPE	SOE	GLB			
Schmitt-Gorhé and Uribe	2018	IER	SOE	1OA	DEIR	0.001–79.0	e
Shapiro and Mandelman	2016	JIE	SOE	1OA	DEIR	0.001	c
Smets and Wouters	2002	JME	SOE	1OA	OLG		
Stahler and Thomas	2012	EM	N=2	GLB	DEIR	0.01	s
Unsal	2011	WP	N=2	2OA	DEIR	0.0075	
Uribe and Yue	2006	JIE	SOE	1OA	AHC		

Note: SOE denotes a small open economy model. $N =$ denotes a multicountry model with N countries. 1OA, 2OA and 3OA are the first-, second- and third-order approximation methods respectively, RSS is the risky steady state method, and GA indicates models solved with global methods (including models with standard preferences and $\beta R < 1$, endogenous discounting, or overlapping generations). The approaches used to induce stationarity when using local methods are the debt-elastic interest rate (DEIR), asset holding costs (AHC), endogenous discounting (ED), overlapping generations (OLG) and complete markets (CM). NSIA denotes no-stationarity inducing assumption. For cases using DEIR, s, c, and e denote whether the debt-elasticity parameter ψ was chosen to be small, estimated, or calibrated respectively.

Table 2: Solution Methods Used in Policy Models

Institution	Model name	Type of model	Solution method	Stationarity assumption	ψ
Bank of Canada	GEM	$N = 5$	1OA	DEIR	n.d.
Bank of England	COMPASS	SOE	1OA	ED	
ECB	NAWM	$N = 2$ and SOE	1OA	DEIR	.01(s)
European Commission	QUEST	SOE	1OA	DEIR	.02(e)
Federal Reserve Board	SIGMA	$N = 2$	1OA	PAC	
IMF	GIMF	$N \geq 2$	1OA	OLG	
Norges Bank	NEMO	SOE	1OA	DEIR	n.d.
Riksbank	RAMSES	SOE	1OA	DEIR	.01(c)

Note: See Table 1 note for details on abbreviations. n.d. denotes there is no public document disclosing the value used.

When local methods are used, 1OA is the most common in research and is used in all eight policy models, and from the assumptions to induce stationarity, DEIR is the most common followed by NFA adjustment costs and ED preferences. From all DEIR solutions, the ψ values range from 0.00001 to 0.01, and the most common is 0.001, which is the value proposed by Schmitt-Grohé and Uribe (2003).[1] In other cases, the value of ψ is set by calibration or obtained via estimation. In calibrated cases (three research papers and three policy models), ψ ranges from 0.01 to 0.1, and in estimated cases (four research papers and one policy model), the point estimates or the medians of posterior distributions in Bayesian estimation range from 0.00014 to 2.8.[2] See the main draft for more discussion on these tables.

Global (GA) methods are also widely used. Their use dates back to the Mendoza (1991) RBC model of a small open economy, and there are many applications in quantitative studies of sovereign default (e.g., Aguiar and Gopinath, 2007; Arellano, 2008), emerging markets business cycles (e.g., Mendoza, 1995), global imbalances (e.g., Mendoza et al., 2009), Sudden Stops (e.g., Durdu et al., 2009; Mendoza, 2010; Mendoza and Smith, 2006), and financial regulation (e.g., Benigno et al., 2016; Bianchi, 2011; Bianchi and Mendoza, 2018; Schmitt-Grohé and Uribe, 2017)

B Endowment economy

As described in the main draft, we use a small-open-economy model with stochastic endowment income to derive analytical results and characterize NFA dynamics under incomplete markets. The

[1] Note, however, that the DEIR functional forms are not always the same, so ψ values are not directly comparable. When relevant for our quantitative analysis, we control for this by making comparisons in terms of the elasticity of the interest rate with respect to percent deviations of NFA from steady state.

[2] A country specific estimate of 79 was also reported in one case but this estimate appears to be an outlier case.

economy is inhabited by a representative agent with preferences given by:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}, \quad u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \tag{1}$$

where $\beta \in (0,1)$ is the subjective discount factor, c_t is consumption and σ is the CRRA coefficient.

The economy's resource constraint is given by

$$c_t = e^{z_t}\bar{y} - A + b_t - qb_{t+1}, \tag{2}$$

where $e^{z_t}\bar{y}$ is stochastic income that fluctuates around a mean $\bar{y}$ with shocks z_t of exponential term e^{z_t}, b_t denotes the NFA position in one-period, non-state-contingent discount bonds traded in a frictionless global credit market at exogenous price $q = \frac{1}{1+r}$, where r is the world real interest rate, and A is a constant that represents investment and government expenditures (which is introduced so the model can be calibrated to observed average consumption-GDP ratios). Income shocks follow an AR(1) process: $z_t = \rho_z z_{t-1} + \sigma_z \varepsilon_t^z$ where ε_t^z is i.i.d.

For further details of the model and the definition of competitive equilibrium, see the main draft. Now we turn to the description of the quantitative methods we used to solve this model.

B.1 Global methods

GA methods solve the model in recursive form over a discrete state space of (b, z) pairs. Since the competitive equilibrium is efficient, it can be represented as a dynamic programming problem:

$$V(b,z) = \max_{c,b'} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta \sum_{z'} \pi(z',z) V\left(b',z'\right) \right\}, \tag{3}$$

$$s.t. \qquad c = e^z\bar{y} - A + b - qb', \quad b' \geq \varphi.$$

The AR(1) process of income is approximated as a discrete Markov chain with transition probability matrix $\pi(z', z)$. The solution to the Bellman equation is characterized by a decision rule $b'(b, z)$ and the associated value function $V(b, z)$. This decision rule and the Markov process of the shocks induce a joint ergodic (unconditional) distribution of NFA and income $\lambda(b, z)$.

We use two alternative global solution methods: *FiPIt* method and Value Function Iteration (VFI) method.

B.1.1 FiPIt method

While using the *FiPIt* method, we solve for $b'(b, z)$, by solving for the recursive equilibrium conditions using a fixed-point iteration algorithm (see Mendoza and Villalvazo, 2020, for details).[3] In

[3]This method solves Euler equations directly instead of using a non-linear solver. It is fast and easy to implement in Matlab. Mendoza and Villalvazo (2020) show that it performs better than standard time iteration and endogenous grids methods, particularly for models with two endogenous state variables and occasionally binding constraints, because the time iteration method requires solving nonlinear equations and the endogenous grids method requires complex interpolation techniques because the endogenous grids are irregular.

this case, the *FiPIt* method iterates on the following representation of the Euler equation:

$$c_{j+1}(b,z) = \left\{ \beta\,(1+r) \sum_{z'} \pi(z',z) \left[\left(c_j(\hat{b}'_j(b,z),z') \right)^{-\sigma} \right] \right\}^{-\frac{1}{\sigma}}. \tag{4}$$

Given a conjecture of the decision rule $\hat{b}'_j(b,z)$ in iteration j, the associated consumption function is $c_j(b,z) = e^z\bar{y} - A + b - q\hat{b}'_j(b,z)$. This consumption function is interpolated over its first argument in order to determine $c_j(\hat{b}'_j(b,z),z')$, so that the above equation solves directly for a new consumption function $c_{j+1}(b,z)$. Using the resource constraint, the new consumption function yields a new decision rule for bonds $b'_{j+1}(b,z)$, which is re-set to $b'_{j+1}(b,z) = \varphi$ if $b'_{j+1}(b,z) \leq \varphi$. Then the decision rule conjecture is updated to $\hat{b}'_{j+1}(b,z)$ as a convex combination of $\hat{b}'_j(b,z)$ and $b'_{j+1}(b,z)$, and the process is repeated until $b'_{j+1}(b,z) = \hat{b}'_j(b,z)$ up to a convergence criterion.

B.1.2 VFI method

The VFI algorithm is implemented as follows: Given the endogenous state variable b and exogenous state $z = \varepsilon^z$, the equilibrium of the endowment model is represented as the solution to this dynamic programming problem:

$$V(b,z) = \max_{b',c} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta E\left[V(b',z')\right] \right\} \tag{5}$$

s.t.

$$c = e^z\bar{y} - A - q^b b' + b \tag{6}$$

where $V(b,z)$ is the value function and $q^b = \frac{1}{1+r}$. The VFI algorithm solves this problem in the following steps:

1. Define grids for b and z with nb nodes for the bonds grid and ne nodes for the income shocks, and set also a convergence criterion for the value function ϵ.

2. Define grids of bonds $\{b_1, b_2, b_3, ..., b_{nb}\}$ and shocks $\{z_1, z_2, z_3, ..., z_{ne}\}$. The bonds grid is defined by a set of nodes with a lower bound equal to the natural or ad-hoc debt limits, and an upper bound high enough to have zero mass in the ergodic distribution of bonds. The grid of shocks is defined by a Markov process with ne nodes, usually constructed to approximate an AR(1) process of an actual income process in the data using Tauchen's quadrature method.

3. For each state (b_i, z_h), set a value for the initial value function (at iteration $j = 0$, where j is the iteration counter) so as to construct $V^0(b,z)$. A trivial initial condition is $V^0(b,z) = 0$.

4. Maximization step: Update the value function to obtain $V^{j+1}(b,z)$ as follows:

   a For each state (b_i, z_h), evaluate the value of consumption if the decision rule for bonds b_{ii} were to be assigned to each of the nb possible values it can take on the bonds grid (i.e., for $ii = 1, .., nb$):

$$c(b_i, z_h; b_{ii}) = e^{z_h}\bar{y} - A - q^b b_{ii} + b_i$$

b Compute the value of choosing b_{ii} if the state of nature is (b_i, z_h), denoted $W(b_i, z_h; b_{ii})$, by evaluating the Bellman equation as follows:

$$W^j(b_i, z_h; b_{ii}) = \frac{c(b_i, z_h; b_{ii})^{1-\sigma}}{1-\sigma} + \beta E_{hh}\left[V^j(b_{ii}, z_{hh})\right]$$

c For each (b_i, z_h), maximize $W^j(b_i, z_h; b_{ii})$ over b_{ii}. The decision rule for bonds at iteration j is given by $b'^j(b_i, z_h) = argmax_{b_{ii}}\left\{W^j(b_i, z_h; b_{ii})\right\}$ and the "new" value function is defined by $V^{j+1}(b_i, z_h) = W^j(b_i, z_h; b'^j(b_i, z_h))$

5. Compare V^j and V^{j+1} and compute the convergence criterion d as

$$d^j = \max_{i\in\{1,\ldots,nb\}, h\in\{1,\ldots,ne\}} |V^j(b_i, z_h) - V^{j+1}(b_i, z_h)|$$

6. If $d > \epsilon$, convergence has not been attained Set $V^j(b, z) = V^{j+1}(b, z)$ and go back to step 4.

7. If $d \leq \epsilon$, the value funtion has converged to $V(b, z) = V^{j+1}(b, z)$ with associated decision rules given by $b'(b, z) = b'^j(b_i, z_h)$.

The VFI method is slow in models with more than one endogenous state variable and cannot be used when equilibria are inefficient, but does not require differentiability or convexity of optimization problems. Compared to VFI, *FiPIt* is a fast, simple method that applies fixed-point iteration to solve Euler equations. It can be used when equilibria are inefficient but requires differentiability (see Mendoza and Villalvazo, 2020, for details).

B.2 Local methods

This section provides a textbook treatment of the local solution methods used for the endowment economy model: first- and second-order perturbation (1OA and 2OA, respectively); risk-adjusted steady state (RSS); and linear-with-occasionally-binding-constraint (DynareOBC). We keep the notation generic so that our description also applies to the RBC model for the first three solution method, and the solution of the Sudden Stops model with DynareOBC. Note that all four solutions are local in that they construct an approximation in the neighborhood of a steady state. However, the aim of all four methods is to construct good numerical approximations of *non-local* features.

Notation Let the set of equilibrium conditions for the dynamic general equilibrium model we wish to study be written as

$$0 = \mathbb{E}_t f\left(\mathbb{y}_{t+1}, \mathbb{y}_t, \mathbb{y}_{t-1}, \varepsilon_t\right), \tag{7}$$

where $\mathbb{E}_t$ denotes the expectations operator. The vector $\mathbb{y}_t$ of variables is of size n_y and the vector ε_t of shocks is of size n_ε. The function f maps $\mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_\varepsilon}$ into $\mathbb{R}^{n_y}$. The vector ε_t is assumed to have bounded support and be independently and identically distributed with mean zero and with variance-covariance matrix I, the identity matrix.[4] The unknown solution (or decision rule) to the model given in (7) that we wish to approximate is of the form

$$\mathbb{y}_t = g\left(\mathbb{y}_{t-1}, \varepsilon_t\right), \tag{8}$$

where g maps $\mathbb{R}^{n_y} \times \mathbb{R}^{n_\varepsilon}$ into $\mathbb{R}^{n_y}$.

[4] This is without loss of generality since the var-cov structure of the fundamental shocks can be specified in f.

B.2.1 Perturbation methods

Although linearizing around a steady state is a special version of perturbation methods and has been used to solve dynamic equilibrium models since the early 1980s, perturbation methods were only formerly introduced into Economics by Judd and Judd (1998). In Judd, the methods were based upon a value function formulation of the problem. In this paper, we follow Schmitt-Grohé and Uribe (2004) in constructing a second-order perturbation method using the equilibrium conditions directly rather than a value function formulation. Toolkits for applying perturbation methods abound. `Dynare`, in particular, is an efficient, user friendly, and freely available toolkit.

The basic idea of perturbation methods is to find a particular case of the model that has a known solution and use that particular case and its solution as a starting point for computing approximate solutions to nearby problems. This approach relies on the implicit function theorem and Taylor series expansion. A special version of perturbation methods is linearizing around a steady state.

In particular, with perturbation methods we approximate the decision rule of the model near the steady state of the model's deterministic counterpart. The method then uses local information to calculate linear and higher-order approximations of the solution near the steady state.

The result of perturbation methods is a polynomial which approximates the true solution in the neighborhood of the deterministic steady state. To construct this approximation, we add an auxiliary *perturbation* parameter, $\sigma \geq 0$, to the unknown decision rule in (8) as follows

$$\mathrm{y}_t = g\left(\mathrm{y}_{t-1}, \varepsilon_t, \sigma\right) \quad \text{and} \quad \mathrm{y}_{t+\tau} = g\left(\mathrm{y}_{t+\tau-1}, \sigma\varepsilon_{t+\tau}, \sigma\right) \quad \text{for} \quad \tau > 0, \tag{9}$$

When $\sigma = 0$ we have the perfect-foresight analog of that model—i.e., the model without future uncertainty. Since this parameter scales the variance-covariance matrix of the shocks, it is typically normalized to 1.

The perturbation methods begins by finding a known solution to a nearby problem. In a dynamic stochastic model, this known solution is the deterministic steady state, when $\sigma = 0$, $\varepsilon_t = 0$, and y solves

$$0 = f\left(\mathrm{y}, \mathrm{y}, \mathrm{y}, 0\right). \tag{10}$$

Notice that when $\sigma = 0$ the expectations operator disappears because there is no future uncertainty. Armed with the deterministic steady state, the perturbation method proceeds as follows. Substituting (8) into (7) gives

$$0 = \mathbb{E}_t f\left(g\left(g\left(\mathrm{y}_{t-1}, \varepsilon_t, \sigma\right), \sigma\varepsilon_{t+1}, \sigma\right), g\left(\mathrm{y}_{t-1}, \varepsilon_t, \sigma\right), \mathrm{y}_{t-1}, \varepsilon_t\right). \tag{11}$$

Equation (11) can be written more concisely as

$$0 = \mathbb{E}_t F\left(\mathrm{y}_{t-1}, \varepsilon_t, \sigma\right). \tag{12}$$

Since (12) is equal to zero, all derivatives of (12) must also be equal to zero. These derivatives provide identifying restrictions for determining the partial derivatives of g.

First-order Differentiating (12) with respect to y_{t-1} and ε_t gives

$$\begin{aligned} [F_{\mathrm{y}}]^i_j &= [f_1]^i_k [g_{\mathrm{y}}]^k_l [g_{\mathrm{y}}]^l_j + [f_2]^i_k [g_{\mathrm{y}}]^k_j + [f_3]^i_j = 0; \quad i,j,k,l = 1,\ldots,n_{\mathrm{y}}, & (13)\\ [F_{\varepsilon}]^i_m &= [f_1]^i_k [g_{\mathrm{y}}]^k_l [g_{\varepsilon}]^l_m + [f_2]^i_k [g_{\varepsilon}]^k_m + [f_4]^i_m = 0; \quad m = 1,\ldots,n_{\varepsilon}, & (14)\end{aligned}$$

respectively, where all the derivatives of f have been evaluated at the deterministic steady state: $(\mathrm{y}_{t-1}, \varepsilon_t, \sigma) = (\mathrm{y}, 0, 0)$, and are therefore known.

A word on the notation is necessary here. The term $[f_3]^i_j$, for example, is the (i,j) element of the Hessian of f with respect its third argument, $\mathbb{y}_{t-1}$. Also, $[f_2]^i_k [g_\mathbb{y}]^k_j$, for example, is short-hand for

$$\sum_{k=1}^{n} \frac{\partial f^i}{\partial \mathbb{y}^k_t} \frac{\partial g^k}{\partial \mathbb{y}^j_{t-1}}. \tag{15}$$

The derivatives of f evaluated at $(\mathbb{y}, \mathbb{y}, \mathbb{y}, 0)$ are known. The first expression above represents a system of $n_\mathbb{y} \times n_\mathbb{y}$ quadratic equations in $n_\mathbb{y} \times n_\mathbb{y}$ unknowns given by the elements of $g_\mathbb{y}$. Similarly, with $g_\mathbb{y}$ known, the second expression above represents a system of $n_\mathbb{y} \times n_\varepsilon$ (linear) equations in $n_\mathbb{y} \times n_\varepsilon$ unknowns given by the elements of g_ε.

Thus, using standard results on the solution of quadratic matrix equations, $g_\mathbb{y}$ and g_ε can be found as solutions. In general, in the neighborhood of the deterministic steady state, up to first-order, we have that $g_\sigma = 0$. This result holds because the system of equations $[F_\sigma]^i = 0$ are linear and homogeneous in g_σ. We do not prove the result here but direct the interested reader to Schmitt-Grohé and Uribe (2004). This result is important though, because it says that a first-order approximation in the neighborhood of the deterministic steady state is always certainty equivalent.

The first-order approximation of the solution in the neighborhood of the deterministic steady state is therefore given by

$$g(\mathbb{x}_t, \sigma) = g(\mathbb{x}, 0) + g_\mathbb{x}(\mathbb{x}, 0)(\mathbb{x}_t - \mathbb{x}), \tag{16}$$

where, for simplicity, we write $\mathbb{x}_t \equiv \left[\mathbb{y}'_{t-1}, \varepsilon'_t\right]'$. It follows that the deterministic steady state, $\mathbb{x}$, is simply $\mathbb{x} = (\mathbb{y}, 0)$.

Second-order Given the deterministic steady state, the first-derivatives of (12) provided identifying restrictions to solve for the $g_\mathbb{x}$ terms of the unknown decision rule. Similarly, we will use second-derivatives of (12) combined with the deterministic steady state and the known values for $g_\mathbb{x}$ to derive identifying restrictions for $g_{\mathbb{xx}}$ and $g_{\sigma\sigma}$. Specifically

$$\begin{aligned}
[F_{\mathbb{yy}}]^i_{jn} &= \left([f_{11}]^i_{ko}[g_\mathbb{y}]^o_p[g_\mathbb{y}]^p_n + 2[f_{12}]^i_{ko}[g_\mathbb{y}]^o_n + 2[f_{13}]^i_n\right)[g_\mathbb{y}]^k_l[g_\mathbb{y}]^l_j \\
&\quad + \left([f_{22}]^i_{ko}[g_\mathbb{y}]^o_n + 2[f_{23}]^i_n\right)[g_\mathbb{y}]^k_j + [f_{33}]^i_n \\
&\quad + [f_1]^i_k\left([g_{\mathbb{yy}}]^k_{lo}[g_\mathbb{y}]^o_n[g_\mathbb{y}]^l_j + [g_\mathbb{y}]^k_l[g_{\mathbb{yy}}]^l_{jn}\right) + [f_2]^i_k[g_{\mathbb{yy}}]^k_{jn} = 0, \qquad (17)\\
[F_{\mathbb{y}\varepsilon}]^i_{jm} &= \left([f_{11}]^i_{ko}[g_\mathbb{y}]^o_p[g_\varepsilon]^p_m + 2[f_{12}]^i_{ko}[g_\varepsilon]^o_m + 2[f_{14}]^i_m\right)[g_\mathbb{y}]^k_l[g_\mathbb{y}]^l_j \\
&\quad + \left([f_{22}]^i_{ko}[g_\varepsilon]^o_m + [f_{24}]^i_m\right)[g_\mathbb{y}]^k_j + \left([f_{31}]^i_{ko}[g_\mathbb{y}]^o_p[g_\varepsilon]^p_m + [f_{32}]^i_{ko}[g_\varepsilon]^o_m + [f_{34}]^i_m\right) \\
&\quad + [f_1]^i_k\left([g_{\mathbb{yy}}]^k_{lo}[g_\varepsilon]^o_m[g_\mathbb{y}]^l_j + [g_\mathbb{y}]^k_l[g_{\mathbb{y}\varepsilon}]^l_{jm}\right) + [f_2]^i_k[g_{\mathbb{y}\varepsilon}]^k_{jm} = 0, \qquad (18)\\
[F_{\varepsilon\varepsilon}]^i_{mq} &= \left([f_{11}]^i_{ko}[g_\mathbb{y}]^o_p[g_\varepsilon]^p_q + [f_{12}]^i_{ko}[g_\varepsilon]^o_q + [f_{14}]^i_q\right)[g_\mathbb{y}]^k_l[g_\varepsilon]^l_m \\
&\quad + \left([f_{21}]^i_{ko}[g_\mathbb{y}]^o_p[g_\varepsilon]^p_q + [f_{22}]^i_{ko}[g_\varepsilon]^o_q + [f_{24}]^i_q\right)[g_\varepsilon]^k_m + [f_{44}]^i_q \\
&\quad + [f_1]^i_k\left([g_{\mathbb{yy}}]^k_{lo}[g_\varepsilon]^o_1[g_\varepsilon]^l_m + [g_\mathbb{y}]^k_l[g_{\mathbb{y}\varepsilon}]^l_{jq}\right) + [f_2]^i_k[g_{\varepsilon\varepsilon}]^k_{mq} = 0; \qquad (19)\\
n, o, p &= 1, \ldots, n_\mathbb{y}; \quad q = 1, \ldots, n_\varepsilon
\end{aligned}$$

The first expression above represents a system of $n_\mathbb{y} \times n_\mathbb{y} \times n_\mathbb{y}$ linear equations in $n_\mathbb{y} \times n_\mathbb{y} \times n_\mathbb{y}$ unknowns given by the elements of $g_{\mathbb{yy}}$. With $g_{\mathbb{yy}}$ known, the second expression above represents a system of $n_\mathbb{y} \times n_\mathbb{y} \times n_\varepsilon$ linear equations in $n_\mathbb{y} \times n_\mathbb{y} \times n_\varepsilon$ unknowns given by the elements of

$g_{\mathbb{y}\varepsilon}$. And, with $g_{\mathbb{y}\varepsilon}$ known, the third expression above represents a system of $n_{\mathbb{y}} \times n_{\varepsilon} \times n_{\varepsilon}$ linear equations in $n_{\mathbb{y}} \times n_{\varepsilon} \times n_{\varepsilon}$ unknowns given by the elements of $g_{\varepsilon\varepsilon}$.

In general, in the neighborhood of the deterministic steady state, $g_{\mathbb{y}\sigma} = g_{\varepsilon\sigma} = 0$. This results because the system of equations $[F_{\mathbb{y}\sigma}]^i_j = 0$ and $[F_{\varepsilon\sigma}]^i_j = 0$ are homogenous in $g_{\mathbb{y}\sigma}$ and $g_{\varepsilon\sigma}$, respectively. Finally, the $n_{\mathbb{y}}$ unknowns given by the elements of $g_{\sigma\sigma}$ can be obtained by solving the $n_{\mathbb{y}} \times 1$ linear equations

$$[F_{\sigma\sigma}]^i = [f_{11}]^i_{jk} [g_{\varepsilon}]^j_m [g_{\varepsilon}]^k_q [I]^q_m + [f_2]^i_j [g_{\sigma\sigma}]^j + [f_1]^i_j \left([g_{\mathbb{y}}]^j_k [g_{\sigma\sigma}]^k + [g_{\varepsilon\varepsilon}]^j_{mq} [I]^m_q + [g_{\sigma\sigma}]^j \right) = 0. \quad (20)$$

The second-order approximation of the decision rule in the neighborhood of the deterministic steady state of a stochastic model differs from that of its non-stochastic counterpart only in a constant term, $g_{\sigma\sigma}$. The second-order approximation of the solution in the neighborhood of the deterministic steady state is therefore given by

$$\begin{aligned} &[g(\mathbb{x}_t, \sigma)]^i = [g(x, 0)]^i + [g_{\mathbb{x}}(x, 0)]^i_r [\mathbb{x}_t - \mathbb{x}]^r \\ &+ \frac{1}{2} [g_{\mathbb{x}\mathbb{x}}(x, 0)]^i_{rs} [\mathbb{x}_t - \mathbb{x}]^r [\mathbb{x}_t - \mathbb{x}]^s + \frac{1}{2} [g_{\sigma\sigma}(x, 0)]^i, \quad r, s = 1, \ldots, n_{\mathbb{y}} + n_{\varepsilon}. \end{aligned} \quad (21)$$

An increase in uncertainty—increasing the standard deviation of the shock innovations—will change $g_{\sigma\sigma}$ and shift the mean of the endogenous variables away from the deterministic steady state.

$\mathfrak{m}^{th}$*-order* In principle, one can solve a polynomial approximation of g up to any order, $\mathfrak{m}$, using perturbation methods. In each step, the $(\mathfrak{m}-1)^{th}$ approximation of g is known and the $\mathfrak{m}^{th}$ derivatives of (12) will provide identifying restrictions for the $\mathfrak{m}^{th}$ coefficients of g. In each case, the system of equations are linear. However, the number of unknowns increase rapidly in $\mathfrak{m}$. Thus, even for a medium-size DSGE model, orders of approximation above 3 are computationally demanding.

B.2.2 Risk-adjusted steady state method

The risk-adjusted steady state method differs from standard perturbation methods in constructing an approximate solution, not in the neighborhood of the deterministic steady state, but rather in the neighborhood of an approximation of the full stochastic model's steady state. In principle, just like standard perturbation methods, an approximate decision rule can be constructed up to any order. However, we focus attention on a first-order approximation in the neighborhood of a risk-adjusted steady state. Several approaches to constructing such approximations have been proposed in the literature, beginning with Collard and Juillard (2001), and with more recent contributions by Coeurdacier et al. (2011), Juillard (2011), de Groot (2013), and Meyer-Gohde (2015). The method proposed by Coeurdacier et al. (2011) is a focus of this section since they apply their *risky* steady state method to solving a small open economy model as we do.

Unlike (9), the solution is not parameterized explicitly by the perturbation parameter. Rather

$$\mathbb{y}_t = g(\mathbb{y}_{t-1}, \varepsilon_t) \quad \text{and} \quad \mathbb{y}_{t+\tau} = g(\mathbb{y}_{t+\tau-1}, \sigma\varepsilon_{t+\tau}) \quad \text{for} \quad \tau > 0. \quad (22)$$

Substituting (9) into (7) gives

$$0 = \mathbb{E}_t f(g(g(\mathbb{y}_{t-1}, \varepsilon_t), \sigma\varepsilon_{t+1}), g(\mathbb{y}_{t-1}, \varepsilon_t), \mathbb{y}_{t-1}, \varepsilon_t). \quad (23)$$

We can define the *exact* risk-adjusted steady state: When $\sigma = 1$ and $\varepsilon = 0$, $\mathbb{y}^{er}$ is the exact risk-

adjusted steady state if it solves the problem

$$0 = \mathbb{E}_t f\left(g\left(\mathbb{y}^{er}, \varepsilon_{t+1}\right), \mathbb{y}^{er}, \mathbb{y}^{er}, 0\right). \tag{24}$$

However, this object is not very useful since the definition relies on g in order to solve for $\mathbb{y}^{er}$. Instead, it is possible to solve jointly for an approximation of the risk-adjusted steady state and linear dynamics.[5] Equation (23) can be written more concisely as

$$0 = F\left(\mathbb{x}_t, \sigma\right). \tag{25}$$

Consider the Taylor expansion of this function up to second-order in the neighborhood of an approximate risk-adjusted steady state $(\mathbb{y}^r)$, given by

$$\left[F\left(\mathbb{y}_{t-1}, \varepsilon_t, \sigma\right)\right]^i = \left[F\left(\mathbb{y}^r, 0, 0\right)\right]^i + \frac{1}{2}\left[F_{\sigma\sigma}\right]^i + \left[F_{\mathbb{y}}\right]^i_j \left[\mathbb{x}_t - \mathbb{x}\right]^j + \frac{1}{2}\left[F_{\mathbb{y}\mathbb{y}}\right]^i_{jk}\left[\mathbb{x}_t - \mathbb{x}\right]^j\left[\mathbb{x}_t - \mathbb{x}\right]^k. \tag{26}$$

All the derivatives of F are evaluated at $(\mathbb{y}^r, 0, 0)$, although for notational simplicity this information has been removed. So far we haven't made much progress since even the derivatives of F are unknown. However, for this equation to hold for any $\mathbb{x}_t$ then it must be the case that

$$\left[F\right]^i + \frac{1}{2}\left[F_{\sigma\sigma}\right]^i = 0, \quad \left[F_{\mathbb{y}}\right]^i_j = 0, \quad \text{and} \quad \left[F_{\mathbb{y}\mathbb{y}}\right]^i_{jk} = 0. \tag{27}$$

All of these terms above have exactly the same structure as those given for the perturbation methods except $\left[F_{\sigma\sigma}\right]^i$, which is given by

$$\left[F_{\sigma\sigma}\right]^i = \left[f_{11}\right]^i_{jk}\left[g_\varepsilon\right]^j_m\left[g_\varepsilon\right]^k_q\left[I\right]^q_m + \left[f_1\right]^i_j\left[g_{\varepsilon\varepsilon}\right]^j_{mq}\left[I\right]^m_q = 0. \tag{28}$$

This shows that for a second-order approximation in σ, we get second-order terms, $g_{\varepsilon\varepsilon}$. Thus, to construct a linear (first-order approximation), we follow Couerdacier et al. (2011) and assume that the second-order terms, $g_{\sigma\sigma}$, are equal to zero, and ignore the second-derivative conditions, $F_{\mathbb{y}\mathbb{y}}$.

Bringing everything together, we have the following set of equations: The steady state equations

$$\mathbb{F}\left(\mathbb{y}^r, g_{\mathbb{x}}\right) \equiv \left[f\left(\mathbb{y}^r, \mathbb{y}^r, \mathbb{y}^r, 0\right)\right]^i + \left[f_{11}\right]^i_{jk}\left[g_\varepsilon\right]^j_m\left[g_\varepsilon\right]^k_q\left[I\right]^q_m = 0, \tag{29}$$

and the first-derivative conditions

$$\left[F_{\mathbb{y}}\right]^i_j = \left[f_1\right]^i_k\left[g_{\mathbb{y}}\right]^k_l\left[g_{\mathbb{y}}\right]^l_j + \left[f_2\right]^i_k\left[g_{\mathbb{y}}\right]^k_j + \left[f_3\right]^i_j = 0; \quad i, j, k, l = 1, \ldots, n_{\mathbb{y}}, \tag{30}$$

$$\left[F_\varepsilon\right]^i_m = \left[f_1\right]^i_k\left[g_{\mathbb{y}}\right]^k_l\left[g_\varepsilon\right]^l_m + \left[f_2\right]^i_k\left[g_\varepsilon\right]^k_m + \left[f_4\right]^i_m = 0; \quad m = 1, \ldots, n_\varepsilon. \tag{31}$$

Since both sets of equations depend on both the steady state $\mathbb{y}^r$ and on the linearized dynamics, $g_{\mathbb{y}}$ and g_ε, these equations need to be solved simultaneously. We use the following algorithm:

1. Use the deterministic steady state as an initial guess, $\mathbb{y}^0$, of the risk-adjusted steady state and calculate the linearized dynamics, $g^0_{\mathbb{x}}$.
2. Evaluate (29) using $\left(\mathbb{y}^0, g^0_{\mathbb{x}}\right)$.

[5] The exact risk-adjusted steady state is not the mean of the ergodic distribution of y. The ergodic mean requires an unconditional expectations operator, while in (24) the expectations operator is conditional on time t information.

if $\left|\mathbb{F}\left(\mathbb{y}^0, g_{\mathbb{x}}^0\right)\right| > \epsilon$ (where ϵ is a tolerance parameter) then update the steady state guess to $\mathbb{y}^1$ and proceed to step 3.

else STOP. A solution to the fixed-point problem has been found.

3. Use $\mathbb{y}^1$ to calculate the updated linearized dynamics $g_{\mathbb{x}}^1$. Return to step 2 using $\left(\mathbb{y}^1, g_{\mathbb{x}}^1\right)$ in the place of $\left(\mathbb{y}^0, g_{\mathbb{x}}^0\right)$.

The linear approximation of the solution in the neighborhood of the risk-adjusted steady state is therefore given by

$$g\left(\mathbb{y}_{t-1}, \varepsilon_t, \sigma\right) = \mathbb{y}^r + g_{\mathbb{y}}\left(\mathbb{y}^r, 0\right)\left(\mathbb{y}_{t-1} - \mathbb{y}^r\right) + g_{\varepsilon}\left(\mathbb{y}^r, 0\right)\varepsilon_t. \tag{32}$$

Notice that, unlike its standard perturbation counterpart, this linear approximation of the decision rule is not certainty equivalent. Changes in the variance-covariance of shocks will alter both the steady state and the linearized dynamics around that steady state.

Alternative risk-adjusted approximations are possible. Juillard (2011) solves the system of equations in (27) and thus constructs a second-order approximation in the neighborhood of the deterministic steady state. In contrast, de Groot (2013) derives the second-order approximation in the neighborhood of the deterministic steady state and takes a first-order Taylor expansion of this function to construct a linear approximation in the neighborhood of the risk-adjusted steady state. This methods benefits from not requiring a fixed-point problem be solved.

B.2.3 Linear-with-occasionally-binding-constraint methods

In this section, we describe how perturbation solutions can handle the model with an occassionally binding collateral constraint. Since the zero lower bound became a binding constraint for monetary policy during the 2007-08 financial crisis, the literature has developed extensions to local approximation methods to solve models with occasionally binding constraints. There are two main computational toolkits publicly available. For the model in this paper, both approaches delivered the same candidate solution paths for endogenous variables.[6] The first, `OccBin` (Guerrieri and Iacoviello (2015), exploits a piecewise linear solution technique. The second, `DynareOBC` (Holden (2016)), exploits the use of "news" shocks to construct a solution.[7]

Holden (2016): We first present the set up without the constraint. The linearized version of (7) can be written as

$$0 = (C + B + A)\mathbb{x} + C\mathbb{x}_{t+1} + B\mathbb{x}_t + A\mathbb{x}_{t-1}, \tag{33}$$

where $\mathbb{x}_t \equiv \left[y_t', \varepsilon_{t+1}'\right]'$ and $\mathbb{x}$ is the deterministic steady state. Assume the first element of the steady state vector, $\mathbb{x}_1$, is positive. We will return to this assumption below. The matrices are given by

$$C \equiv \begin{bmatrix} f_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B \equiv \begin{bmatrix} f_2 & 0 \\ 0 & I \end{bmatrix}, \quad \text{and} \quad A \equiv \begin{bmatrix} f_3 & f_4 \\ 0 & 0 \end{bmatrix}. \tag{34}$$

where, for example, f_1 is the Hessian of f with respect to the first argument, evaluated at the deterministic steady state. The solution to this problem is the linearized decision rule

$$\mathbb{x}_t = \left(I - g_{\mathbb{x}}\right)\mathbb{x} + g_{\mathbb{x}}\mathbb{x}_{t-1} \quad \text{where} \quad g_{\mathbb{x}} = -\left(B + Cg_{\mathbb{x}}\right)^{-1} A. \tag{35}$$

[6] Other local approximation methods may also be amenable to solving models with occasionally binding constraints. For extended path algorithms, see Adjemian and Juillard (2013). For penalty based methods see McGrattan (1996).

[7] Antecedents of this approach include Laséen and Svensson (2011) and Holden and Paetz (2012).

The occasionally binding constraint is introduced by augmenting (33) as follows

$$\mathbb{x}_{1,t} = \max\left(0, (C_1 + B_1 + A_1)\mathbb{x} + C_1\mathbb{x}_{t+1} + (B_1 + I_1)\,\mathbb{x}_t + A_1\mathbb{x}_{t-1}\right), \tag{36}$$

$$0 = (C_{-1} + B_{-1} + A_{-1})x + C_{-1}\mathbb{x}_{t+1} + B_{-1}\mathbb{x}_t + A_{-1}\mathbb{x}_{t-1}, \tag{37}$$

where C_1, for example, denotes the first row of C and C_{-1} denotes C excluding the first row. The occasionally binding constraint has been introduced using a $\max$ operator in the first equation. The fact that the constraint has been introduced as a positivity constraint is without loss of generality since it is always possible to rewrite a constraint in this form.

The problem that needs to be solved can be formulated as follows: Given a vector of initial conditions, $\mathbb{x}_0$, find a sequence $\{\mathbb{x}_t\}_{t=0}^{\infty}$ in $\mathbb{R}^n$ such that (36) and (37) hold and that $\mathbb{x}_t \to \mathbb{x}$ as $t \to \infty$. Above we assume that $\mathbb{x}_1 > 0$. This means i) that the constraint does not bind in the steady state and ii) we consider only solutions in which there is some $T \in \mathbb{N}$ such that for $t > T$ the constraint no longer binds.

The problem described above is solved by solving a complementary problem using "news" shocks. Consider the following system

$$0 = (C + B + A)\mathbb{x} + C\mathbb{x}_{t+1} + B\mathbb{x}_t + A\mathbb{x}_{t-1} + I_{c1}y_{1,t-1} \tag{38}$$

In time 0 there is a $T \times 1$ vector, y_0, of news shocks, where $y_{t,0}$ is the shock t periods ahead. Thus, $y_{i+1,t-1} = y_{i,t}$ and, for $t > T$, $y_{1,t-1} = 0$. I_{c1} denotes the first column of the identity matrix. The algorithm then searches for $y_{1,t-1}$ such that the `max` operator in (36) is adhered to in all periods.

Guerrieri and Iacoviello (2015): In contrast to using news shocks to enforce the constraint, this method concatenate decision rules to construct a piecewise-linear solution. They begin by defining two regimes, the "unconstrained" and "constrained" regimes

$$0 = (C + B + A)\mathbb{x} + C\mathbb{x}_{t+1} + B\mathbb{x}_t + A\mathbb{x}_{t-1}, \tag{39}$$

$$0 = (C^* + B^* + A^*)\mathbb{x} + C^*\mathbb{x}_{t+1} + B^*\mathbb{x}_t + A^*\mathbb{x}_{t-1}, \tag{40}$$

where the latter is the constrained regime (i.e., the first term in the `max` operator above). The coefficients of this second (constrained) set of linearized equilibrium conditions are evaluated at the same deterministic steady state as the first (unconstrained) set of linearized equilibrium conditions. A solution for the model is then a function that maps $\mathbb{x}_{t-1}$ into $\mathbb{x}_t$ such that the conditions under the unconstrained or constrained set of equilibrium conditions hold, depending on the evaluation of the constraint in (36). The algorithm used to solve for this time-varying decision rule employs a guess-and-verify approach. In effect, for any initial state configuration, $\mathbb{x}_0$, the algorithm guesses the periods $t \geq 0$ for which the economy is in the constrained and unconstrained regimes. This guess is characterized by a vector of 0s and 1s. Again, let T denote the last period in which the constraint binds. For $t \geq T$, the decision rule will be (35). For period $T - 1$, the solution for $\mathbb{x}_{T-1}$ given $\mathbb{x}_{T-2}$ can then be derived by solving the following equation

$$0 = (C^* + B^* + A^*)\mathbb{x} + C^*\left((I - g_{\mathbb{x}})\,\mathbb{x} + g_{\mathbb{x}}\mathbb{x}_{t-1}\right) + B^*\mathbb{x}_{t-1} + A^*\mathbb{x}_{t-2}. \tag{41}$$

It is possible to iterate back in this fashion, using either (35) or (41), depending on the regime, until $\mathbb{x}_0$ is reached, giving a candidate equilibrium path. Next, the `max` function is used to check the vector that summarizes the initial guess of regimes against the actual outturn in the candidate path. If the sequence of regime guesses is verified, then the solution is complete. Otherwise, the

guess for when regimes apply is updated and the process is repeated.

Next, we describe the calibration used for the endowment model and present quantitative results that highlight differences and similarities implied by the local and global solutions.

B.3 Calibration & comparison of quantitative results

B.3.1 Calibration

The baseline calibration is based on the calibration constructed by Mendoza (2010) using quarterly Mexican data at annualized rates for the 1993-2005 period. We use the same values of $\sigma = 2$ and $R = 1.086$. For the income process, we set $\sigma_z = 0.0272$ and $\rho_z = 0.749$ to match Mendoza's estimates of the standard deviation and first-order autocorrelation of the cyclical component of Mexico's GDP. The implied standard deviation of income innovations is $\sigma_\varepsilon = 0.01802$. The local methods use this income process directly. In the GA solution, we approximate it as a five-point Markov chain using the improved Tauchen and Hussey (1991) quadrature method developed by Flodén (2008).[8] The mean NFA-GDP and consumption-GDP ratios in the Mexican data are also taken from Mendoza's calibration ($E(b/y) = -0.363$, $E(c/y) = 0.65$). Given the value of R and the resource constraint, the value of the autonomous spending share that captures investment and government absorption is $A = 0.32$. For further details of the calibration see the main draft Section 2.4.

B.3.2 NFA decision rule and net exports

This section details the derivations of the decision rules reported in Section 2.4 of the main paper.

Analytical solution

Assume that NFA, b_t, follows a first-order autoregressive process, where ρ and σ^2 denote the autocorrelation and variance of NFA, respectively.[9] Net exports are given by $nx_t = \frac{b_t}{R} - b_{t-1}$. The autocorrelation of nx implied by the NFA autoregressive process is derived as follows

$$\begin{aligned}
E\left(nx_t nx_{t-1}\right) &= E\left(\frac{b_t}{R} - b_{t-1}\right)\left(\frac{b_{t-1}}{R} - b_{t-2}\right) \\
&= E\left(\frac{b_t b_{t-1}}{R^2} - \frac{b_{t-1}^2}{R} - \frac{b_t b_{t-2}}{R} + b_{t-1}b_{t-2}\right) \\
&= \sigma^2\left(\frac{\rho}{R^2} - \frac{1}{R} - \frac{\rho^2}{R} + \rho\right) = \frac{\sigma^2}{R^2}\left(\rho - R - \rho^2 R + R^2\rho\right). \\
E\left(nx_t^2\right) &= E\left(\frac{b_t^2}{R^2} - 2\frac{b_t b_{t-1}}{R} + b_{t-1}^2\right) = \frac{\sigma^2}{R^2}\left(1 - 2\rho R + R^2\right).
\end{aligned}$$

Hence

$$\text{ac}\left(nx_t\right) \equiv \frac{E\left(nx_t nx_{t-1}\right)}{E\left(nx_t^2\right)} = \frac{\rho\left(R^2+1\right) - R\left(\rho^2+1\right)}{R^2 - 2\rho R + 1}. \tag{42}$$

[8] The Markov process is discrete with bounded support whereas the AR(1) is normally-distributed with unbounded support. However, Flodén (2008) showed that for other than highly-persistent shocks, Markov processes using quadrature methods match closely the unconditional moments of the AR(1) even with few nodes.

[9] We set the mean of NFA to zero without loss of generality.

Furthermore

$$\frac{d\text{ac}\left(nx_t\right)}{d\rho} = 1 + \frac{2\left(R-\rho\right)R\left(\rho R-1\right)}{\left(R^2-2\rho R+1\right)^2} > 0, \tag{43}$$

$$\frac{d^2\text{ac}\left(nx_t\right)}{d\rho^2} = \frac{2R\left(R^2-1\right)^2}{\left(R^2-2\rho R+1\right)^3} > 0. \tag{44}$$

This implies that $\text{ac}\left(nx_t\right) > 0$ when $\rho R > 1$. Furthermore, since the denominator is always positive, $\text{ac}\left(nx_t\right)$ is increasing and convex in ρ. Finally,

$$\lim_{\rho\to 1}\frac{d\text{ac}\left(nx_t\right)}{d\rho} = \frac{R^2+1}{\left(R-1\right)^2}. \tag{45}$$

Since the denominator of this limit is close to zero, the sensitivity of the autocorrelation of net exports to the autocorrelation of NFA becomes extremely large when NFA is close to a unit root.[10]

This nonlinear relationship is key to understanding the differences between the local and global solutions that we observe.

Local solution

To derive decision rules using the local solutions, we use the following equilibrium conditions (with $\bar{y} = 1$ and $A = 0$, for simplicity)

$$\begin{aligned}
c_t &= e^{z_t} + b_t - \frac{b_{t+1}}{1+r_t},\\
1 &= \beta\mathbb{E}_t\left(c_t/c_{t+1}\right)^{\sigma}\left(1+r_t\right),\\
r_t &= r + \psi\left(e^{b^*-b_{t+1}}-1\right),\\
z_t &= \rho_z z_{t-1} + \sigma_z\varepsilon_t^z.
\end{aligned}$$

Written in the form of (7), we have

$$\mathbb{E}_t f\left(\mathbb{y}_{t+1},\mathbb{y}_t,\mathbb{y}_{t-1},\varepsilon_t\right) = \mathbb{E}_t\begin{bmatrix} -\mathbb{y}_t^1 + \exp\left(\mathbb{y}_t^3\right) + \mathbb{y}_{t-1}^2 - \frac{\mathbb{y}_t^2}{1+r+\psi\left(\exp\left(b-\mathbb{y}_t^2\right)-1\right)} \\ -1+\beta\left(\mathbb{y}_t^1/\mathbb{y}_{t+1}^1\right)^{\sigma}\left(1+r+\psi\left(\exp\left(b^*-\mathbb{y}_t^2\right)-1\right)\right) \\ -\mathbb{y}_t^3 + \rho_z\mathbb{y}_{t-1}^3 + \sigma_z\varepsilon_t^z \end{bmatrix}, \tag{46}$$

where $\mathbb{y}_t \equiv \left[c_t, b_{t+1}, z_t\right]'$. The deterministic steady states of NFA, b^{dss}, is pinned down by the parameter b^*. The endowment process, z_t, has a steady state of 0. The deterministic steady state of consumption is given by $c^{dss} = 1 - b^{dss}r/(1+r)$. We can derive numerical values for the policy functions. Using the baseline calibration, $\{r = 0.08571, b^* = -0.51, \sigma = 2, \psi = 0.001, \rho_z = 0.597, \sigma_z = 0.0327\}$, the 1OA decision rule is given by

$$\begin{bmatrix} c^{dss} \\ b^{dss} \end{bmatrix} = \begin{bmatrix} .960 \\ -.510 \end{bmatrix} \quad \text{and} \quad g_{\mathbb{x}} = \begin{bmatrix} .084 & .186 \\ .995 & .884 \end{bmatrix}, \tag{47}$$

where the rows are the endogenous variables, $[c_t, b_{t+1}]$, and the columns are the state variables,

[10] To give a numerical sense of the nonlinearity, consider the following values: When $\rho = 0$, the $\text{ac}\left(nx_t\right) = -R/\left(R^2-1\right) \approx -0.5$ and $\frac{d\text{ac}(nx_t)}{d\rho} = 1 - 2R/\left(R^2-1\right)^2 \approx 0.5$. When $\rho = 1/R$, the $\text{ac}\left(nx_t\right) = 0$ and $\frac{d\text{ac}(nx_t)}{d\rho} = 1$. Finally, when $\rho = 1$, the $\text{ac}\left(nx_t\right) = 1$ and $\frac{d\text{ac}(nx_t)}{d\rho} = \frac{R^2+1}{(R-1)^2} \approx 220$ when $R = 1.1$.

$[b_t, z_t]'$. The additional 2OA decision rule terms are given by

$$g^c_{\mathbb{x}\mathbb{x}} = \begin{bmatrix} -.005 & -.005 \\ -.005 & .104 \end{bmatrix}, \quad g^b_{\mathbb{x}\mathbb{x}} = \begin{bmatrix} .004 & .005 \\ .005 & .972 \end{bmatrix}, \quad \text{and} \quad g_{\sigma\sigma} = \begin{bmatrix} -4.8 \times 10^{-5} \\ 5.2 \times 10^{-5} \end{bmatrix}. \tag{48}$$

For the RSS decision rules, the steady state conditions are given by

$$0 = -c^{rss} + 1 + b^{rss} - \frac{b^{rss}}{1 + r + \psi\left(e^{b^* - b^{rss}} - 1\right)}, \tag{49}$$

$$0 = -1 + \beta\left(1 + r + \psi\left(e^{b^* - b^{rss}} - 1\right)\right)\left(1 + \frac{\gamma(\gamma - 1)}{2}\mathbb{E}_t\left(c_{t+1} - c^{rss}\right)^2\right), \tag{50}$$

where $\mathbb{E}_t\left(c_{t+1} - c^{rss}\right)^2 = g^c_z \sigma^2_z$. Solving jointly for the risk-adjusted steady state, $[c^{rss}, b^{rss}]'$, and the first-order coefficients, $g^{rss}_{\mathbb{x}}$, gives

$$\begin{bmatrix} c_t \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} c^{rss} \\ b^{rss} \end{bmatrix} + g^{rss}_{\mathbb{x}} \begin{bmatrix} b_t - b^{rss} \\ z_t \end{bmatrix} \tag{51}$$

where

$$\begin{bmatrix} c^{rss} \\ b^{rss} \end{bmatrix}' = \begin{bmatrix} .970 \\ -.376 \end{bmatrix}, \quad g^{rss}_{\mathbb{x}} = \begin{bmatrix} .083 & .184 \\ .995 & .886 \end{bmatrix}. \tag{52}$$

Since the DynareOBC method generates time-and-state dependent decision rules, it is not possible to write down decision rules as we have for the perturbation and RSS methods.

Analytical results using local solutions for a simplified endowment model with log-utility and i.i.d. shocks

It is possible derive analytical expressions for the 2OA and RSS decision rules in a simplified endowment economy. In particular, we set $b^* = 0$ so the deterministic steady state is $[c, b] = [1, 0]$; $\rho_z = 0$ so endowment fluctuations are i.i.d.; and $\sigma = 1$ so we have log-preferences. Finally, we set $r = 0$ and $\beta = 1$. The 2OA decision rules are given by

$$\tilde{c}_t = g\left(b_t, z_t, \sigma\right) = g_b b_t + g_z z_t + \frac{1}{2}\left(g_{bb} b_t^2 + g_{zz} z_t^2 + g_{\sigma\sigma}\right) + g_{bz} b_t z_t,$$

$$b_{t+1} = h\left(b_t, z_t, \sigma\right) = h_b b_t + h_y z_t + \frac{1}{2}\left(h_{bb} b_t^2 + h_{zz} z_t^2 + h_{\sigma\sigma}\right) + h_{bz} b_t z_t,$$

where $\tilde{c}_t \equiv \log\left(c_t\right)$. Substituting into the Euler equation and budget constraint gives

$$1 = \mathbb{E}_t \exp\left(g\left(b_t, z_t, \sigma_z\right) - g\left(h\left(b_t, z_t, \sigma_z\right), \sigma_z \sigma \varepsilon_{t+1}, \sigma\right)\right)\left(1 + \psi\left(\exp\left(-h\left(b_t, z_t, \sigma_z\right)\right) - 1\right)\right),$$

$$\exp\left(g\left(b_t, z_t, \sigma_z\right)\right) = \exp\left(z_t\right) + b_t - h\left(b_t, z_t, \sigma_z\right)\left(1 + \psi\left(\exp\left(-h\left(b_t, z_t, \sigma_z\right)\right) - 1\right)\right)^{-1}.$$

2OA solution

Solving for h_b gives

$$\begin{aligned} h_b\left(\psi\right) &= 1 - \frac{1}{2}\left(\sqrt{\psi^2 + 4\psi} - \psi\right), \\ &= 1 - 2\frac{\psi}{\psi + \sqrt{\psi^2 + 4\psi}} < 1, \end{aligned}$$

for $\psi > 0$. It also tells us that $h_b\left(0\right) = 1$, $h'_b\left(\psi\right) < 0$, $h''_b\left(\psi\right) > 0$, $\lim_{\psi \to \infty} h_b\left(\psi\right) = 0$, and

$\lim_{\psi\to 0} h_b'(\psi) = -\infty$. We also have that

$$0 = g_z - h_z(g_b + \psi),$$
$$0 = 1 - g_z - h_z.$$

As a result, it is straightforward to show that $h_z = h_b$ and $g_b = g_z = 1 - h_b$.The 2OA coefficients for NFA are given by

$$h_{bb} = h_{bz} = -\psi^3 + \sqrt{\psi+4}\psi^{5/2} - 4\psi^2 + 2\sqrt{\psi+4}\psi^{3/2} - \frac{5\psi}{2} + \frac{(\psi+1)\sqrt{\psi+4}\sqrt{\psi}}{2(\psi+3)},$$
$$h_{zz} = -\psi^3 + \sqrt{\psi+4}\psi^{5/2} - 4\psi^2 + 2\sqrt{\psi+4}\psi^{3/2} - 2\psi - \frac{\sqrt{\psi+4}\sqrt{\psi}}{\psi+3} + 1,$$
$$h_{\sigma\sigma} = \frac{\left(-2\psi^2 + 2\sqrt{\psi+4}\psi^{3/2} - 7\psi + 5\sqrt{\psi+4}\sqrt{\psi} - 4\right)}{(\psi+3)\left(\psi + \sqrt{\psi+4}\sqrt{\psi} + 2\right)}.$$

Clearly, the coefficients quickly become complicated functions of ψ. However, $h_{bb}(0) = h_{bz}(0) = 0$, $h_{zz}(0) = 1$, and $h_{\sigma\sigma}(0) = -2/3$.

RSS solution
In the simplified endowment economy with i.i.d. endowment risk and a simplified DEIR function of the form $1 + r_t = (1+r)\, e^{\psi(b^* - b_{t+1})}$, the "full RSS" solution is identical to the "partial RSS" solution described in the main paper. We begin by substituting into the Euler equation, giving

$$1 = \beta(1+r)\,\mathbb{E}_t \exp\left(\begin{array}{c} -\gamma\left(\tilde{c}^{rss} + g_b b_{t+1} + g_z\left(\rho_y z_t + \sigma_z \varepsilon_{t+1}\right)\right) \\ +\gamma\left(\tilde{c}^{rss} + g_b\left(b_t - b^{rss}\right) + g_z z_t\right) + \psi\left(b^* - b_{t+1}\right) \end{array}\right).$$

The steady state equation is given by

$$\begin{aligned} 1 &= \beta(1+r)\,\mathbb{E}_t \exp\left(-\gamma g_z \sigma_z \varepsilon_{t+1} - \psi\left(b^{rss} - b^*\right)\right) \\ &= \beta(1+r)\exp\left(\frac{1}{2}\gamma^2 g_z^2 \sigma_z^2 - \psi\left(b^{rss} - b^*\right)\right). \end{aligned}$$

The second line is either derived from the properties of the log-normal distribution, or by taking a second-order expansion around $\sigma_z = 0$ and then applying $\log(1+x) \approx x$ for small x. Rearranging gives

$$b^{rss} = b^* + \frac{\gamma^2 g_z^2 \sigma_z^2}{2\psi}.$$

When $b^* = 0$ and $\gamma = 1$ (log-preferences), this simplifies to $b^{rss} = \frac{g_z^2\sigma_z^2}{2\psi}$. After setting $\beta = 1$ and $r = 0$, the 1OA term, h_b, is given by

$$h_b = \frac{1}{2}\left(e^{-b\psi} + (1 - b\psi) - \sqrt{\left(e^{-b\psi} - (1 - b\psi)\right)^2 + 4e^{-b\psi}\psi}\right). \tag{53}$$

where, for clarity, b denotes b^{rss}. The only difference between h_b under RSS and 2OA is that the

expression is evaluated at b^{rss} rather than $b^{dss} = 0$. The remaining 1OA terms are given by

$$h_z = \frac{1}{1 - e^{b\psi}(h_b + b\psi - 1)}, \quad g_b = \frac{e^{b\psi}(h_b + b\psi) - 1}{(e^{b\psi} - 1)\, b - 1}, \quad g_z = \frac{1 - e^{b\psi}(h_b + b\psi)}{(1 - (e^{b\psi} - 1)\, b)\,(1 - e^{b\psi}(h_b + b\psi - 1))}.$$

These coefficients clarify that the first-order dynamics of the RSS solution differ from the dynamics of the 1OA solution.

B.3.3 Long-run moments & impulse response functions

In this subsection, we provide pertinent details for the results related to the long-run moments and impulse response functions in the main draft.

The result showing that precautionary savings nearly vanish from 2OA and partial RSS solutions as ψ rises implies that the terms driving the deviation of the unconditional average of b from b^{dss} in their decision rules are vanishing too. To shed light on why this happens, we use again the decision rules for log utility and i.i.d. shocks (together with the quantitative result that the quadratic and interaction terms of the 2OA solutions are negligible) to obtain these expressions:[11]

$$E[b]^{2OA} = b^{dss} + \sigma_z^2 \frac{h_{\sigma\sigma}}{2(1 - h_b)}, \qquad E[b]^{RSS} = b^{dss} + \sigma_z^2 \frac{g_y^2}{2\psi}, \tag{54}$$

where g_y is the coefficient of the consumption decision rule on income. Since h_b is decreasing in ψ for $\psi < 0.5$ (recall Figure 1 in the main draft), the denominators in the right-hand-side of the above expressions rise with ψ, which brings the unconditional means closer to b^{dss}. The coefficients $h_{\sigma_z\sigma_z}$ and g_y also depend on ψ.

Second-order terms & moments

The main text of the paper argues that the quadratic and interaction terms of the 2OA solutions are negligible. This section addresses this point in more detail. Figure 1 plots three rows of unconditional moments (mean, standard deviation, and persistence of NFA) from the endowment model using Andreasen et al. (2018) pruning. Each of the three columns varies one structural parameter at a time (ψ, σ, and ρ_z, respectively) while holding the rest at the baseline calibration. We compare the true SOA solution with a "poor man's" analog in which we set $h_{bb} = h_{bz} = h_{zz} = 0$ (i.e., leaving only the $h_{\sigma\sigma}$ term from the true SOA decision rule).

The top-left panel shows that the mean-NFA under the true SOA and under the poor-man's SOA only diverge for small value of ψ when NFA is highly persistent. The difference is significant for the baseline calibration ($\psi = 0.001$) but not for the targetted calibration ($\psi = 0.047$). Rows 2 and 3 show that first-order moments such as the standard deviation and persistence of NFA are unaffected by the existence of second-order terms, except in the case when the calibration is extreme and far from the baseline (e.g., when risk averse or the persistence of the endowment shocks is very high).

To see why the unconditional mean-NFA is affected by second-order terms, it is helpful to write out its analytical expression. The pruned SOA decision rule is given by

$$b_{t+1} = b_{1t+1} + b_{2t+1}, \tag{55}$$

$$b_{1t+1} = h_b b_{1t} + h_z z_t, \tag{56}$$

[11] When ψ is small and h_b is close to 1 (e.g., when $\psi = 0.001$), the other quadratic and interaction terms do matter for the value of $E[b]$. However, they become negligible again if $\psi > 0.01$. See Online Appendix C.3.3 for details.

$$b_{2t+1} = h_b b_{2t} + \frac{1}{2}\left(h_{bb} b_{1t}^2 + h_{zz} z_t^2 + h_{\sigma\sigma}\right) + h_{bz} b_{1t} z_t. \tag{57}$$

Taking unconditional expectations gives

$$\mathbb{E}\left(b_{t+1}\right) = \frac{1}{1-h_b}\frac{1}{2}\left(h_{bb}\mathbb{E}\left(b_{1t}^2\right) + 2h_{bz}\mathbb{E}\left(b_{1t}z_t\right) + h_{zz}\mathbb{E}\left(z_t^2\right) + h_{\sigma\sigma}\right), \tag{58}$$

and since

$$\mathbb{E}\left(b_{1t+1}^2\right) = \frac{1}{1-h_b^2} h_z \frac{\sigma_z^2}{1-\rho^2}, \quad \mathbb{E}\left(b_{1t}z_t\right) = \frac{\sigma_z^2}{1-\rho_z^2}, \quad \mathbb{E}\left(z_t^2\right) = \rho_z \frac{\sigma_z^2}{1-\rho_z^2}, \tag{59}$$

we can combine these terms to give

$$\mathbb{E}\left(b_{t+1}\right) = \frac{1}{1-h_b}\frac{1}{2}\left(\left(h_{bb}\frac{1}{1-h_b^2}h_z + 2h_{bz}\rho + h_{zz}\right)\frac{\sigma_z^2}{1-\rho_z^2} + h_{\sigma\sigma}\right) \tag{60}$$

The poor man's [pm] SOA ignores all second-order terms except for $h_{\sigma\sigma}$. In this case:

$$b_{t+1}^{pm} = h_b b_t^{pm} + h_z z_t + \frac{1}{2}h_{\sigma\sigma}, \tag{61}$$

and

$$\mathbb{E}\left(b_{t+1}^{pm}\right) = \frac{1}{1-h_b}\frac{1}{2}h_{\sigma\sigma}. \tag{62}$$

Thus, the discrepancy, err, is given by

$$\text{err} = \frac{1}{1-h_b}\frac{1}{2}\left(\left(h_{bb}\frac{1}{1-h_b^2}h_z + 2h_{bz}\rho_z + h_{zz}\right)\frac{\sigma_z^2}{1-\rho_z^2}\right). \tag{63}$$

This final expression clarifies that when h_b is close to 1, that all interaction terms, but especially h_{bb}, gets magnified, and becomes important for accurately calculating mean-NFA.

B.3.4 Spectral densities

In this section, we examine nonparametric sample periodograms of simulated data produced by the various solution methods. A summary of our discussion here is also presented in the main draft. The goal is to determine whether they yield different predictions about the relevance of fluctuations at different frequencies for overall variability. Figure 2 shows periodograms of b, c and nx for a multivariate spectrum from long time-series simulations with 4500 periods and a Bartlett window with the smoothing parameter set to 100.[12] The y-axis shows the population spectrum, the x-axis shows the frequency in years, and the vertical lines isolate the business cycle frequency. The panels on the left (right) are for the baseline (targeted) calibrations. As in the previous charts, the plots for the GA solution are identical in both sets of plots, because the global solution has a single calibration. In addition, as with the time-series results, the spectral density functions are nearly identical for 2OA and RSS methods, because the local decision rules have similar h_b terms and the quadratic and interaction terms of the 2OA solutions are irrelevant.

[12] We follow Hamilton (1994) in setting the value of the smoothing parameter. The results for parametric estimates of the spectral densities are generally smoother but show similar patterns as those of the nonparametric estimates.

Figure 1: Second-order terms & moments

Note: Moments calculated analytically using the Andreasen et al. (2018) pruning algorithm. The Poor man's SOA sets $h_{bb} = h_{zz} = h_{bz} = 0$.

All the periodograms are generally downward sloping because the equilibrium stochastic processes are similar to AR(1) processes. Hence, the contribution of lower frequencies to the variances of the variables exceeds that of business cycle and lower frequencies. The results show, however, that the local methods under the baseline calibration overestimate the contribution of low frequency movements to the total variance of all three series, which is consistent with their slower mean-reversion and higher values of ρ_b relative to the GA solution. Moreover, while the contribution of fluctuations at the business cycle frequency or higher for the variability of b is slightly higher with the local solutions than in the GA solution, for nx the local methods overestimate it and for c they underestimate it. In particular, the local methods underestimate significantly the fraction of consumption fluctuations explained by business cycle and higher frequencies and under-predict significantly the contribution of low frequencies.

For targeted calibrations, the periodograms of b for local solutions are more similar to those of global solution, but the periodograms of c and nx for the local solutions still differ sharply from the global ones. They underestimate significantly the contribution of consumption fluctuations at business cycle and higher frequencies to overall consumption variance.

The frequency analysis shows that the results of GA and local methods differ at most frequen-

Figure 2: Spectral Density Functions in the Endowment Economy Model

Note: These graphs show parametric estimates of spectral density functions. GA, 2OA and RSS denote the global, second-order and risky-steady state solution, respectively.

cies, and not just in business cycle moments and long-run averages. For the endowment economy with the baseline calibrations, local methods overestimate significantly the contribution of low-frequency movements to the variability of NFA and net exports (nx), in line with the result indicating that they overestimate the autocorrelation of NFA. The local methods also overestimate (underestimate) the contribution of low-frequency (high-frequency) movements to consumption fluctuations. The targeted calibrations perform better at approximating the spectral density of NFA and nx, but for consumption they still underestimate the contribution of high-frequency fluctuations by a large margin. Similar results are obtained for NFA, nx and consumption in the RBC model.

B.3.5 Comparison of DEIR vs. Endogenous Discounting

In this section, we examine the implications of endogenizing the discount factor as an alternative approach to induce stationarity in open-economy, incomplete-markets models. Although the most-commonly used method in the literature is the use of DEIR function, as surveyed in Tables 1 and 2, the endogenous discount factor (ED) is also widely used. We compare below local (2OA) ED solutions with GA solutions and 2OA solutions with DEIR for the same endowment economy model of Section 2 in the paper.

Analytical comparison of DEIR and Endogenous Discounting (ED)

Section 8.2 of Schmitt-Grohé and Uribe (2003) presents a simplified endowment economy closed with an endogenous discount factor without internalization (henceforth, ED) and finds that $h_b = 1 - \theta c^*$, where θ and c^* are given by the functional form

$$\beta\left(c_t\right) = \frac{\left(1 + c_t - c^*\right)^{-\theta}}{1 + r}. \tag{64}$$

As with the DEIR model, we assume $\rho_z = 0$, $\sigma = 1$, and $1 = \beta\left(1 + r\right)$. The 1OA decision rule for the ED specification (with $r \neq 0$ and $c^* = 1$) is given by

$$h_b = h_z = 1 - \theta \quad \text{and} \quad g_b = g_z = \frac{r + \theta}{r + 1}, \tag{65}$$

and with $r = 0$, $g_b = g_z = \theta$. Thus, the h_b coefficient of the DEIR and ED models are identical if $r = 0$, $c^* = 1$ and if

$$\frac{\theta}{2} = \frac{\psi}{\psi + \sqrt{\psi^2 + 4\psi}}. \tag{66}$$

Thus, while there is a mapping from ψ to θ when matching h_b, the relationship is nonlinear.

It is not possible to solve a SOA of the UE model when $r = 0$ because $h_{\sigma\sigma}$ is undefined. Instead, solving with $r > 0$, the 2OA coefficients for the ED specification are given by

$$h_{bb} = h_{bz} = \frac{(\theta - 1)\theta(\theta + r)^2}{(r + 1)(r - (\theta - 2)\theta)}, \tag{67}$$

$$h_{zz} = \frac{(\theta - 1)^2\left(r^2 + 3\theta r + r + \theta(\theta + 2)\right)}{(r + 1)(r - (\theta - 2)\theta)}, \tag{68}$$

$$h_{\sigma\sigma} = -\frac{(\theta + r)\left(-r^2 - 3\theta r + r + \theta((\theta - 4)\theta + 2)\right)}{r(r + 1)(r - (\theta - 2)\theta)}. \tag{69}$$

When $r = 0$, we have $h_b = h_y = 1 - \theta$, $g_b = g_y = \theta$, $h_{bb} = h_{by} = -(1 - \theta)\theta^2/(2 - \theta)$, and $h_{yy} = (2 - 3\theta + \theta^3)/(2 - \theta)$ with $h_{\sigma\sigma}$ undefined. When $\theta = 0$, then $h_{bb} = h_{by} = 0$, $h_{yy} = 1$ and $h_{\sigma\sigma} = (r - 1)/(r + 1)$.

Figure 3 shows the coefficients of the policy function for DEIR vs. ED under various calibrations. The x-axis varies ψ (and θ for the ED model that it delivers the same h_b). It shows that the DEIR decision rule at second-order has more non-linearity arising from h_{bb}, whereas the ED decision rule has a more significant shift term arising from $h_{\sigma\sigma}$. Moreover, $h_{\sigma\sigma}$ is sensitive to small values of r for the ED decision rule. Figure 4 highlights the differences for the calibration of $r = 0.005$ and $\psi = 0.001$. Notice is that the shift term ($h_{\sigma\sigma}$) is very important for UE model, whereas the DEIR model has more curvature coming from h_{bb}.

Figure 3: Coefficients of the SOA: DEIR vs. ED

Note: The x-axis is always ψ. For the ED model, θ has been chosen to match h_b. Under ED, $h_{\sigma\sigma}$ is sensitive to r and is undefined ($h_{\sigma\sigma} \to -\infty$) for $r = 0$.

Comparison of DEIR and Endogenous Discounting (ED) with local and global solutions

When ED is used, period utility of date $t+1$ is discounted by a factor θ_{t+1} such that $\theta_{t+1} = \beta(c_{t+1})\theta_t$ where:

$$\beta(c) = (1+c)^{-\psi^{ED}},$$

and ψ^{ED} denotes the elasticity of the period discount factor with respect to consumption. Preferences with this endogenous discount factor are known as Uzawa-Epstein (UE) preferences. They feature a rate of time preference for date t that is an increasing function of the history of consumption up to date t, and an "impatience effect" by which agents take into account that increasing date-t consumption reduces the discount factor that applies to all utility flows from date-t forward. In closing the small open economy with ED for the local solutions, Schmitt-Grohé and Uribe (2003) proposed an alternative formulation in which the impatience effect is removed by assuming that $\beta(\cdot)$ depends on "aggregate consumption" so that agents do not internalize the dependency of the discount factor on consumption. We denote this formulation as SGU-ED. With this assumption, the Euler Equation for NFA becomes:

$$u_c(c_t) = (1+r)\beta(\tilde{c}_t)E_t[u_c(c_{t+1})],$$

Figure 4: SOA decision rules: DEIR vs. ED

Note: For $\sigma_z = 0.03$, $r = 0.008$ and $\psi = 0.001$ and a θ that delivers the same h_b. Decision rules are plotted for $z = \{-0.03, 0, 0.03\}$. The close-to-linear decision rules are the ED specification whereas the convex decision rules are the DEIR specification.

where $\tilde{c}_t$ denotes aggregate consumption, which household takes as given but equals c_t at equilibrium. Schmitt-Grohé and Uribe (2003) found that local solutions using UE or SGU-ED are nearly identical, but SGU-ED has a much simpler representation since it removes the impatience effect. Hence our choice to use SGU-ED for the analysis conducted here. The rest of the endowment model structure remains the same as described in the paper.

To explore how using SGU-ED affects long-run moments and precautionary savings we compare local solutions to two alternative variants of the GA solution, one with $\beta R < 1$ preferences (GA-$\beta R < 1$), which is the case analyzed in the main draft, and the other with Uzawa-Epstein (UE) preferences (GA-UE). The results are shown in Table 3. For the local solutions, we present 2OA results for the baseline DEIR case shown in the paper and SGU-ED setup.

For the calibration of GA-UE and 2OA-ED, we calibrate the respective ψ^{ED} parameter for each solution to match an average NFA-GDP ratio of -0.363, estimated in data for the Mexican economy (see the main draft calibration for details). This calibration approach yields $\psi^{ED} = 0.164$ both for GA-UE and 2OA-ED solutions.

Table summarizes the main results. The table highlights that first order moments are similar across all solution methods. But this is largely by construction and not a general result. It reflects the calibration strategy that all methods target an average NFA-GDP ratio (b/y) in the Mexican data (-36.3%). Higher moments with 2OA-ED closer to GA-UE, suggesting with UE preferences, local solution provides a reasonably good match to the global solution.

To provide further evidence of the above finding, we show in Figure 5 how the mean NFA position changes as the variability of income increases. The plot makes it clear that the 2OA-ED solution is similar to the GA-UE solution, but still significantly differs from GLB-$\beta R < 1$ solution. Both GA-UE and 2OA-ED underestimate the mean of NFA and precautionary savings. As Durdu et al. (2009) showed, the UE setup yields significantly smaller precautionary savings than the $\beta R < 1$ model as the variability of the income shocks rises. In turn, as explained in Section 2.4 of the paper, precautionary savings incentives are weaker with ED because the marginal benefit of

Table 3: Long-run Moments: Comparison of GA vs SGU-ED in the Endowment Economy Model

	Data	GA		2OA	
		$\beta R < 1$	UE	DEIR	UE
Selected parameters					
ψ^{ED}	.	.	.	.	0.164
Mean					
c	0.650	0.650	0.651	0.650	0.651
nx/y	-0.014	0.028	0.028	0.028	0.028
b/y	-0.363	-0.363	-0.363	-0.363	-0.363
Standard deviation relative to GDP					
c	1.247	1.352	1.391	3.326	1.416
nx/y	0.775	0.540	0.815	2.168	0.839
b/y	3.128	9.766	18.656	71.159	19.695
Correlation with GDP					
c	0.895	0.834	0.625	0.213	0.607
nx/y	-0.688	0.431	0.496	0.223	0.489
b/y	0.842	0.531	0.423	0.150	0.412
First-order autocorrelation					
c	0.701	0.849	0.963	0.997	0.968
nx/y	0.797	0.768	0.833	0.972	0.844
b/y	0.669	0.971	0.993	0.999	0.993

Note: GA refers to the global solution, UE refers to the global solution with Uzawa-Epstein preferences, SGU-ED denotes local solution with endogenous discounting. $\sigma(\cdot)$ denotes the coefficient of variation for variables in levels and the standard deviation for variables in ratios (nx/y and b/y).

savings falls as β_t falls when agents borrow more. The graph shows that the mean NFA predicted by the local solution and GA-UE is similar, and both underestimate significantly the comparable GA-$\beta R < 1$ solution.

B.3.6 Linear with occasionally binding constraint method: DynareOBC

In this section, we provide details on the DynareOBC approach applied to the endowment economy model with an occasionally binding constraint (the ad-hoc debt limit) that we considered in the main draft and also presented in Section B. We consider alternative parameterizations in which the constraint in the reference regime (i.e., at the deterministic steady-state) is binding or not. We use this example to illustrate how DynareOBC constructs solutions using perfect foresight paths along a time-series simulation. This example demonstrates that unconditional mean values for endogenous variables differ from the deterministic steady state even with a first-order approximation and without the presence of precautionary savings.

For tractability, we rewrite the model here. Assume that the problem of the representative agent

Figure 5: Average NFA in the endowment economy as the variability output rise

Note: GA-UE refers to the global solution with Uzawa-Epstein preferences, 2OA-ED denotes local solution with endogenous discounting.

is given by

$$\max_{c_t} \sum_{t=0}^{\infty} \frac{c_t^{1-\sigma}}{1-\sigma} \quad \text{s.t.} \quad c_t + q_t b_{t+1} = e^{z_t} + b_t, \quad b_{t+1} \geq \varphi, \tag{70}$$

where b_{t+1} is bonds, φ is the ad-hoc debt limit, and the endowment process is $z_{t+1} = \rho z_t + \sigma_z \varepsilon_{t+1}$. Denote the Lagrange multiplier on the constraint by $q_t c_t^{-\sigma} \lambda_t$. The FOC is given by

$$1 - \lambda_t = \beta \mathbb{E}_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} r_t, \tag{71}$$

where

$$r_t = 1/q_t = 1/\beta + \psi \left(e^{b^* - b_{t+1}} - 1 \right). \tag{72}$$

b^* denotes the bond value in the DEIR specification. Typically, this value is set to the deterministic steady state that the model calibration targets. In this example, however, the deterministic steady state b^{dss} would be equal to the greater of b^* and ad-hoc debt limit φ. Put differently, the steady state (or the reference regime) is unconstrained if $b^* > \varphi$ and constrained if $b^* \leq \varphi$.

- Unconstrained SS: $b = b^*$, $r = 1/\beta$, $c = 1 + (1 - 1/r)\, b^*$, $\lambda = 0$.
- Constrained SS: $b = \varphi$, $r = 1/\beta + \psi \left(e^{b^* - \varphi} - 1 \right) < 1/\beta$, $c = 1 + (1 - 1/r)\, \varphi$, $\lambda = 1 - \beta r > 0$.

In `DynareOBC` the equilibrium conditions are written as follows for Regime 1:

$$1 - \lambda_t = \beta \mathbb{E}_t \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} r_t, \tag{73}$$

$$c_t + q_t b_{t+1} = e^{z_t} + b_t, \tag{74}$$

$$r_t = 1/q_t = 1/\beta + \psi \left(e^{b^* - b_{t+1}} - 1 \right), \tag{75}$$

$$0 = \max \left(\varphi, b^* - \log \left(\frac{1}{\psi \beta} \left(\frac{c_{t+1}}{c_t} \right)^{\sigma} + 1 \right) \right). \tag{76}$$

For illustration, Figure 6 displays a stochastic simulation of the model.[13] The black line plots the realized path for consumption and bonds, respectively. At each date, for a given state vector, x_t, DynareOBC solves for a perfect foresight path, presented by the red-dash lines. These represent the equilibrium paths for the economy at each point in time conditional on no future shocks such that the economy converges back to the deterministic steady-state in T periods. The red-dashed line starting, for example, in period t is used to generate the realized economic outcome in period $t+1$. For the purposes of the stochastic simulation, the rest of that perfect foresight path is discarded. In $t+1$ the economy is hit by a news shock and the state vector is given by x_{t+1}. DynareOBC then generates a new perfect foresight path, and so on.

Figure 6: Perfect foresight paths for DynareOBC

The bottom row of Figure 6 focuses on a single perfect foresight path from period 141 in the stochastic simulation. In solving for this perfect foresight path, the algorithm needs to determine the periods in which the constraint is binding. This problem is not trivial. The naive (unconstrained) perfect foresight path given by the black-dashed line suggests that the constraint might bind immediately and continue to bind past period 180. In fact, the equilbrium perfect foresight path with the constraint is one in which the constraint is expected to start binding in period 144 (3 periods after the shock) and remain so until only period 163. Of course, subsequent shocks meant

[13] Note that φ is set to -0.1 in this example, hence the constraint is nonbinding in the deterministic steady state.

that the constraint became unbinding even earlier in period 153. The implication of this analysis is that when the NFA process is highly persistent, the method requires long extended perfect foresight paths (i.e., T needs to be sufficiently high) to find a solution. Additionally, the number of time periods used for the simulations also need to be sufficiently long to achieve convergence of long-run moments.[14]

We next illustrate how alternative values for the ad-hoc debt limit, in relation to the b^* value in the DEIR specification, affects the deterministic steady state and long-run value of NFA. Figure 7 shows the results. Some useful insights can be derived from this figure that can guide intuition for our results in the main draft. The reference regime is constrained (unconstrained) when φ is higher (lower) than the b^* value of 0. Regardless of whether the reference regime is constrained or unconstrained the unconditional mean of NFA is always higher than the deterministic steady state value of NFA as long as the constraint binds with a positive probability. This implied higher value of NFA is mainly due to the model's asymmetric (or nonlinear) response to shocks in the presence of an occasionally binding constraint. A shock that causes the constraint to bind causes households to lower bond holdings slower than an increase the same shock with an opposite sign would lead to. As a result, the realized upward movements in bonds is greater than the downward movements, causing the economy to endogenously move away from the constraint. This move away from the constraint would be realized even without a precautionary savings incentive.

Figure 7: Comparative statics

The shaded area illustrates the constrained region. As the constraint becomes tighter the probability of the constraint being binding increases monotonically as shown in the left panel. When the φ is less than 0, the deterministic steady state value of NFA (shown in orange) equals b^*. When φ is higher than 0, the deterministic steady state value of NFA equals φ. The unconditional mean of NFA increases with φ and it is always higher than the deterministic steady state of NFA as long as the probability of the constraint being binding is non-zero.

B.3.7 Analytical solution under savings-under-uncertainty framework

The models we considered for the endowment model both in the main draft and in the appendix do not feature analytical solutions under CRRA preferences. So, it is not possible to assess how

[14]For the solution of the Sudden Stops model, we set the `TimetoEscapeBounds` option equal to 60. That is, the algorithm searches for a path for the endogenous variables for which the economy returns back to the reference regime within 60 quarters from the current period. This is necessary because, with near-unit root debt dynamics, it can take a long period of time for the economy to revert back to the reference regime.

GA solution compares to an analytical or true solution, although our premise is that GA is nearly identical to the true solution. To understand this issue better, we analyze how global and local solutions compare to the true solution using Levhari and Srinivasan (1969) savings-under-uncertainty framework, which features an analytical solution. We start with describing the model.

At each period t, a household either consumes his entire wealth b_t or invests part of it using a risky asset. Denoting consumption as c_t, investment will then be $b_t - c_t$. The random return on investment from the risky asset is given by an exogenous interest rate R_t. The objective of the household then is to maximize

$$E\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}\right] \quad \text{s.t.} \tag{77}$$

$$b_{t+1} = (b_t - c_t)R_{t+1}, \tag{78}$$

where b_0 is given and $b_t \geq c_t \geq 0$. The optimality conditions for this model is given as

$$c_t^{-\sigma} = \beta \mathbb{E}_t \left(c_{t+1}^{-\sigma} R_{t+1}\right), \tag{79}$$

$$b_{t+1} = (b_t - c_t) R_{t+1}, \tag{80}$$

where $\log(R_{t+1}) = \mu + \sigma_\varepsilon \varepsilon_{t+1}$, $\varepsilon_{t+1} \sim (0,1)$. This model is different from the other models we considered in the main draft in that the true solution is a random walk, hence long-run averages for variables in levels (consumption or investment) are not well-defined. However, we can solve this model by detrending. In particular, we can normalize the variables by b_t to give

$$x_t^{-\sigma} = \beta \mathbb{E}_t \left(x_{t+1}^{-\sigma} g_{t+1}^{-\sigma} R_{t+1}\right), \tag{81}$$

$$g_{t+1} = (1 - x_t) R_{t+1}, \tag{82}$$

where $x_t \equiv c_t/b_t$ and $g_t = b_t/b_{t-1}$. Substituting the constraint into the Euler equation gives

$$x_t^{-\sigma} = \beta \mathbb{E}_t \left(x_{t+1}^{-\sigma} (1 - x_t)^{-\sigma} R_{t+1}^{1-\sigma}\right). \tag{83}$$

The model contains neither endogenous nor exogenous state variables. Thus, the decision rule is a constant: $x_t = \lambda$, and the exact analytical solution is given by

$$\lambda^{EXA} = 1 - \beta^{1/\sigma} \exp\left(\frac{1-\sigma}{\sigma}\mu\right) \exp\left(\frac{(1-\sigma)^2}{\sigma}\frac{\sigma_\varepsilon^2}{2}\right). \tag{84}$$

For the solution of the model, note that variance matters but persistence does not, and as we mentioned above, there is no long-run average of the model in levels. The ones in detrended form are well defined. Now we discuss how we can derive the global and local approximations for this model.

Global solution The global solution to this problem can easily be derived by implementing FiPIt for the Euler Equation (83). The solution will start with a guess of x_j in iteration j to be plugged in to the right hand side of the Euler equation to derive an updated guess of x_{j+1}. If $||x_{j+1} - x_j|| < \epsilon$ then the solution is achieved, otherwise we update our guess with x_{j+1} and iterate to convergence. We found that for all parameterizations, the GA solution converges to the exact analytical solution.

First-order approximation and risk-adjusted steady state solution To derive the first-order approximation (1OA) and RSS solution we follow the following steps.

Step 1: Substitute the decision rule into the equilibrium condition to give

$$F(\sigma_\varepsilon) \equiv \beta \mathbb{E}_t \left(\exp\left((1-\sigma)(\mu + \sigma_\varepsilon \varepsilon_{t+1})\right)\right) - (1-\lambda)^\sigma = 0. \tag{85}$$

Step 2: For RSS, take a second-order approximation with respect to σ_ε to give

$$0 = F(0) + \frac{\sigma_\varepsilon^2}{2} F''(0), \tag{86}$$

$$= \beta \exp\left((1-\sigma)\mu\right)\left(1 + (1-\sigma)^2 \frac{\sigma_\varepsilon^2}{2}\right) - (1-\lambda)^\sigma. \tag{87}$$

Rearranging gives the following 1OA and analytical RSS solution:

$$\lambda^{1OA} = 1 - \beta^{1/\sigma} \exp\left(\frac{1-\sigma}{\sigma}\mu\right), \tag{88}$$

$$\lambda^{RSS} = 1 - \beta^{1/\sigma} \exp\left(\frac{1-\sigma}{\sigma}\mu\right)\left(1 + (1-\sigma)^2 \frac{\sigma_\varepsilon^2}{2}\right)^{1/\sigma}. \tag{89}$$

Higher-order perturbation approximation Define the decision rule as $x_t = f(\sigma_\varepsilon)$. The second-order approximation (2OA) around the deterministic steady state is given by $\lambda^{2OA} = f(0) + f''(0)\frac{\sigma_\varepsilon^2}{2}$ (since odd-derivatives are generically zero). The methodology proceeds as follows:
Step 1: Substitute the decision rule into the equilibrium condition to give

$$F(\sigma_\varepsilon) \equiv \beta \mathbb{E}_t \exp\left((1-\sigma)(\mu + \sigma_\varepsilon \varepsilon)\right) - (1 - f(\sigma_\varepsilon))^\sigma = 0, \tag{90}$$

Step 2: $F(0) = 0$ gives

$$\beta \exp\left((1-\sigma)\mu\right) - (1 - f(0))^\sigma = 0, \tag{91}$$

$$f(0) = 1 - \beta^{1/\sigma} \exp\left(\frac{1-\sigma}{\sigma}\mu\right) \tag{92}$$

Step 3: $F''(0) = 0$ gives

$$0 = (1-\sigma)^2 + \sigma (1 - f(0))^{-1} f''(0), \tag{93}$$

$$f''(0) = -\frac{(1-\sigma)^2}{\sigma}(1 - f(0)). \tag{94}$$

Combining these results gives

$$\lambda^{2OA} = 1 - \beta^{1/\sigma} \exp\left(\frac{1-\sigma}{\sigma}\mu\right)\left(1 + \frac{(1-\sigma)^2}{\sigma}\frac{\sigma_\varepsilon^2}{2}\right). \tag{95}$$

In fact, perturbation methods produce a Taylor expansion of the exact decision rule. Thus, the 4OA is given by

$$\lambda^{4OA} = 1 - \beta^{1/\sigma} \exp\left(\frac{1-\sigma}{\sigma}\mu\right)\left(1 + \theta\sigma_\varepsilon^2 + \frac{\theta^2}{2}\sigma_\varepsilon^4\right), \tag{96}$$

where $\theta \equiv (1-\sigma)^2/(2\sigma)$.

To illustrate how local solutions differ from the analytical (or GA) solution, we plot the savings rate implied by each solution as we vary the variance parameter σ_ε (see panel (a) in Figure 8).[15] We also plot the ratio of the savings rate implied by the GA solution to each of the local solution (see panel (b)). In the graphs, the vertical black line corresponds to the level of σ_ε above which the implied GA savings rate becomes infeasible. This is the point at which the GA savings rate crosses the horizontal red line in panel (a). At this level of σ_ε implied savings rate reaches 1, hence consumption equals 0. Further increases in σ_ε would imply negative consumption. As can be seen from both panels, 1OA, 2OA or RSS solutions all imply significantly different savings rate relative to the GA solution for a wide range of σ_ε values. 4OA solution is quite close to the GA solution for virtually all σ_ε values.

Figure 8: Implied savings rate

Note: These graphs show the implied savings rate $1 - \lambda$ in panel a and the ratio of the GA savings rate to the local solution savings rate in panel b. The vertical line in each chart shows the level of σ_ε above which the savings rate becomes infeasible for the analytical solution. This point is reached when the GA line crosses the red horizontal line in panel a.

[15] We set the other parameters as follows. β is 0.94, mean interest rate is 1.7. As we increase σ_ε, we recompute $\mu = 1.7 - \sigma_\varepsilon^2/2$, so that we implement a mean preserving spread increase in interest rate.

To further illustrate the discrepancy implied by local solutions relative to the GA, we next plot the simulated series for the logarithm of bond decisions and logarithm of consumption for σ_ε value set to 0.4601. With this parameterization, the implied savings rate for each solution is as follows: GA=0.99577, 1OA=0.70314, RSS=0.86118, 2OA=0.94825, and 4OA=0.99045. Figure 9 show the bond decisions on the left and consumption on the right.[16] Rearranging the budget constraint and using the optimality decision rule for consumption, we can show that the bond decision behaves as a random walk with drift

$$\log(b_{t+1}) = \log(1-\lambda) + \log(b_t) + \log(R_t). \tag{97}$$

With the exception of 4OA, the drift component $(\log(1-\lambda))$ yields large differences for local solutions relative to the GA.

The analysis in this section shows that GA is identical to the true analytical solution, but all the local solutions with the exception of 4OA imply large differences relative to the analytical solution. However, the accuracy of 4OA we see for this model does not extend to the models in the main draft, because in those models, the center of approximation is not known with local approximations, hence they need to be induced by imposing DEIR.

Figure 9: Time-series simulation of bond decisions and consumption

Note: This figure shows the simulated series of bond decisions and consumption in log terms for all solution methods.

C The Sudden Stops model

This section presents a complete description of the Sudden Stops model used for local solutions. The RBC model has the same set of optimality conditions but shuts down the occasionally binding borrowing constraint. The equations are written consistent with using a `Dynare` compatible toolbox. There are 26 variables

$$b_{t+1}, ca_t, c_t, g_t, i_t, l_t, \lambda_t, r_t^k, R_t, tb_{t+1}, \nu_t, w_t, y_t^{gdp}, y_t, s_t, q_t, k_{t+1}, b_{t+1}$$
$$\varepsilon_t^A, \varepsilon_t^P, \varepsilon_t^r, \mu_t, eb_{t+1}, x_{1,t}, x_{2,t}, lev_t,$$

[16] For each simulation, the time interval is annual and we start with the same initial conditions and burn the first 200 years.

and 26 equations, (98)—(122), given below. Equation (98) is the occasionally binding constraint, written using a max operator. The first term inside the max operator shows the value of capital consistent with the borrowing constraint binding. The second term shows the optimal value of capital derived from the capital Euler equation (equation 101) and the capital demand equation (equation 106) when the Lagrange multiplier on the borrowing constraint, $\mu_t = 0$.
Borrowing constraint:

$$k_{t+1} = \max\left(\frac{\phi R_t\left(w_t l_t + e^{\varepsilon_t^P} p\nu_t\right) - \frac{b_{t+1}}{R_t}}{\kappa q_t}, \left(\frac{x_{2,t}}{x_{1,t}}\right)^{\frac{1}{\alpha-1}}\right) \tag{98}$$

Aggregate resource constraint:

$$c_t + i_t + g_t = b_t - b_{t+1}/R_t + y_t - e^{\varepsilon_t^P} p\nu_t - (R_t - 1)\,\phi\left(w_t l_t + e^{\varepsilon_t^P} p\nu_t\right) \tag{99}$$

Bond Euler equation:

$$1 = \beta R_t \mathbb{E}_t \lambda_{t+1}/\lambda_t + \mu_t/\lambda_t \tag{100}$$

Capital Euler equation:

$$1 + a\left(\frac{k_t - k_{t-1}}{k_{t-1}}\right) = \beta\mathbb{E}_t\frac{\lambda_{t+1}}{\lambda_t}\left(r_{t+1}^k + 1 - \delta + a\left(\frac{k_{t+1} - k_t}{k_t}\right) + \frac{a}{2}\left(\frac{k_{t+1} - k_t}{k_t}\right)^2\right) + \frac{\mu_t}{\lambda_t}\kappa q_t \tag{101}$$

Marginal utility of consumption and labour supply:

$$\lambda_t\,(1+g) = \left(c_t - \theta\frac{l_t^\omega}{\omega}\right)^{-\gamma}, \tag{102}$$

$$\lambda_t w_t = \theta l_t^{\omega-1}\left(c_t - \theta l_t^\omega/\omega\right) \tag{103}$$

Capital accumulation:

$$i_t = \delta k_{t-1} + (k_{t+1} - k_t)\left(1 + \frac{a}{2}\left(\frac{k_{t+1} - k_t}{k_t}\right)\right) \tag{104}$$

Labour demand:

$$w_t = \eta A e^{\varepsilon_t^A} k_t^\alpha l_t^{\eta-1}\nu_t^{1-\alpha-\eta} - (R_t - 1)\,\phi w_t - R_t\phi w_t\mu_t/\lambda_t \tag{105}$$

Capital demand:

$$r_t^k = \alpha A e^{\varepsilon_t^A} k_t^{\alpha-1} l_t^\eta \nu_t^{1-\alpha-\eta} \tag{106}$$

Intermediate input demand:

$$e^{\varepsilon_t^P} p = (1-\alpha-\eta)\,A e^{\varepsilon_t^A} k_t^\alpha l_t^\eta \nu_t^{-\alpha-\eta} - (R_t - 1)\,\phi e^{\varepsilon_t^P} p - R_t\phi e^{\varepsilon_t^P} p\mu_t/\lambda_t \tag{107}$$

Debt-elastic interest rate:

$$R_t = e^{\varepsilon_t^R}(1+r) + \psi\left(e^{\left(\frac{b^*}{y^{gdp}} - \frac{b_{t+1}}{y^{gdp}}\right)} - 1\right) \tag{108}$$

Auxiliary equations:

$$x_{1,t} = \beta \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} \alpha A e^{\varepsilon_t^A} l_{t+1}^{\eta} \nu_{t+1}^{1-\alpha-\eta}, \tag{109}$$

$$x_{2,t} = 1 + a\left(\frac{k_t - k_{t-1}}{k_t}\right) - \beta \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t}\left(1 - \delta + a\left(\frac{k_{t+1} - k_t}{k_t}\right) + \frac{a}{2}\left(\frac{k_{t+1} - k_t}{k_t}\right)^2\right) \tag{110}$$

Excess borrowing:

$$eb_t = b_{t+1}/R_t - R_t\phi\left(w_t l_t + e^{\varepsilon_t^P} p\nu_t\right) + \kappa q_t k_{t+1} \tag{111}$$

Current account, net exports and domestic savings:

$$ca_t = -b_{t+1} + b_t, \tag{112}$$

$$nx_t = y_t - e^{\varepsilon_t^P} p\nu_t - c_t - i_t - g_t, \tag{113}$$

$$s_t = (-b_t + b_{t+1}) + i_t \tag{114}$$

GDP and production function:

$$y_t^{gdp} = y_t - e^{\varepsilon_t^P} p\nu_t \tag{115}$$

$$y_t = A e^{\varepsilon_t^A} k_t^{\alpha} l_t^{\eta} \nu_t^{1-\alpha-\eta} \tag{116}$$

Price of capital and government spending:

$$q_t = 1 + a\left(\frac{k_t - k_{t-1}}{k_{t-1}}\right), \tag{117}$$

$$g_t = g \times c_t \tag{118}$$

Leverage:

$$lev_t = \frac{\frac{b_{t+1}}{R_t} - R_t\phi\left(w_t l_t + e^{\varepsilon_t^P} p\nu_t\right)}{q_t k_{t+1}} \tag{119}$$

Shocks:

$$\varepsilon_t^A = \rho_a \varepsilon_{t-1}^A + \sigma_a \epsilon_t^1 + \sigma_{ar}\epsilon_t^3 \tag{120}$$

$$\varepsilon_t^P = \rho_p \varepsilon_{t-1}^P + \sigma_p \epsilon_t^2 \tag{121}$$

$$\varepsilon_t^r = \rho_r \varepsilon_{t-1}^r + \sigma_{ar}\epsilon_t^1 + \sigma_r \epsilon_t^3, \tag{122}$$

where $\left(\epsilon_t^1, \epsilon_t^2, \epsilon_t^3\right)$ are mean zero i.i.d. random variables with variance-covariance matrix I_3.

C.1 *FiPIt* Global Solution Method

This section provides a short description of the fixed-point iteration method we used to solve the RBC and Sudden Stops models. This method is developed by Mendoza and Villalvazo (2020) (see article for full details, a users guide and Matlab codes). *FiPIt* has two major advantages over other global methods that are used for solving models with more than one endogenous state variable: It solves Euler equations without requiring a non-linear solver (unlike standard time iteration methods that need to solve them as a non-linear system) and it uses simple multi-linear interpolation (unlike the endogenous grids method that needs interpolation techniques for irregular grids such as Delaunay interpolation). We summarize the algorithm focusing on the Sudden Stops model. The RBC algorithm is the same but with the κ parameter set high enough so that the collateral constraint never binds (in which case the algorithm never enters step 5 in the description provided below).

The *FiPIt* method is in the class of "Euler equation" methods that solve for recursive equilibria using a model's optimality conditions in recursive form. For the RBC and SS models we solved, the state space consists of two endogenous states, $[b, k]$, and three exogenous states $\varepsilon = \left[\varepsilon^A, \varepsilon^R, \varepsilon^P\right]$.[17] The optimality conditions in recursive form are the following:

$$\left(c(b,k,\varepsilon) - \frac{L(b,k,\varepsilon)}{\omega}\right)^{-\sigma} = \lambda(b,k,\varepsilon)(1+\tau) \tag{123}$$

$$\alpha A k^{\gamma} L(b,k,\varepsilon)^{\alpha-1} \upsilon(b,k,\varepsilon)^{\eta} = w(b,k,\varepsilon)\left(1 + \phi(R-1) + \frac{\mu(b,k,\varepsilon)}{\lambda(b,k,\varepsilon)}\phi R\right) \tag{124}$$

$$\alpha A k^{\gamma} L(b,k,\varepsilon)^{\alpha} \upsilon(b,k,\varepsilon)^{\eta-1} = p\left(1 + \phi(R-1) + \frac{\mu(b,k,\varepsilon)}{\lambda(b,k,\varepsilon)}\phi R\right) \tag{125}$$

$$\lambda(b,k,\varepsilon) = R\beta E[\lambda b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon')] + \mu(b,k,\varepsilon) \tag{126}$$

$$\begin{aligned}\lambda(b,k,\varepsilon) = &\frac{1}{q(b,k,\varepsilon)}\beta E[\lambda[b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon')(d(b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon') \\ &+ q'(b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon'))] + \mu(b,k,\varepsilon)\kappa\end{aligned} \tag{127}$$

$$d(b,k,\varepsilon) = \gamma A k^{\gamma-1} L(b,k,\varepsilon)^{\alpha} \upsilon(b,k,\varepsilon) - \delta\frac{a}{2}\frac{(k'(b,k,\varepsilon)-k)^2}{k^2} \tag{128}$$

$$q(b,k,\varepsilon) = 1 + a\left(\frac{k'(b,k,\varepsilon)-k}{k}\right) \tag{129}$$

$$w(b,k,\varepsilon) = L(b,k,\varepsilon)^{\omega-1}(1+\tau) \tag{130}$$

$$\begin{aligned}(1+\tau)c(b,k,\varepsilon) + i(b,k,\varepsilon) = &e^{\varepsilon^A}F(k,L,\upsilon) - e^{\varepsilon^P}p\upsilon - \phi(R-1)(wL(b,k,\varepsilon) \\ &+ e^{\varepsilon^P}p\upsilon(b,k,\varepsilon)) - q^b b'(b,k,\varepsilon) + b\end{aligned} \tag{131}$$

where $i(b,k,\varepsilon) = \delta k + (k'(b,k,\varepsilon) - k)\left[1 + \frac{a}{2}\left(\frac{k'(b,k,\varepsilon)-k}{k}\right)\right]$, $F(k,L,\upsilon) = Ak^{\gamma}L^{\alpha}\upsilon^{\eta}$ with $0 \leq \alpha, \gamma, \eta \leq 1$, $\alpha + \gamma + \eta = 1$, and $A > 0$. Finally, $1/q^b \equiv e^{\varepsilon^R}R$, and q denotes Tobin's q. The algorithm proceeds in the following eight steps:

1. **Define the discrete state space**: We use a grid of bonds with M nodes and a grid of capital with N nodes. The size of the grids for the shocks depends on the approach used to construct

[17] We used the same method to solve the endowment model, in which case bonds are the only endogenous state variable and endowment income is the only exogenous shock. The model reduces to a single recursive equilibrium condition, which is the bonds Euler equation (see Mendoza and Villalvazo (2020) for details).

the Markov processes of the shocks. In the paper, we used the same Markov processes as in Mendoza (2010), which feature two realizations for each of the three shocks. Hence, the state space has $M \times N \times 8$ elements.

2. **Initial conjectures for iteration** j: For each iteration j, define conjectured functions for the price of capital $\hat{q}_j(b,k,\varepsilon)$, the decision rule for bonds $\hat{B}_j(b,k,\varepsilon)$,and the multiplier ratio $\hat{\tilde{\mu}}_j(b,k,\varepsilon)$ which is the ratio of the multiplier on the borrowing constraint, μ, to the multiplier on the budget constraint, λ. For the first iteration, the conjectures are: $\hat{\tilde{\mu}}_j(b,k,\varepsilon) = 0$, $\hat{q}_j(b,k,\varepsilon) = 1$ and $B_j(b,k,\varepsilon) = b$ (i.e., the credit constraint does not bind, Tobin's q is at the value consistent with a capital decision rule equal to the current state, and the bonds decision rule is also set equal to the current state).

3. **Compute iteration-j implied decision rules.** The conjectures set above and the recursive equilibrium conditions imply the following decision rules:

$$K_j(b,k,\varepsilon) = \frac{k}{a}\left[\hat{q}_j(b,k,\varepsilon) - 1 + a\right] \tag{132}$$

$$i_j(b,k,\varepsilon) = (K_j(b,k,\varepsilon) - k)\left[1 + \frac{a}{2}\left(\frac{K_j(b,k,\varepsilon) - k}{k}\right)\right] - \delta k \tag{133}$$

$$v_j(b,k,\varepsilon) = \left\{\frac{e^{\varepsilon^A} A k^\beta \eta^{\frac{\omega-\alpha}{\omega}} \frac{\alpha}{1+\tau}^{\frac{\alpha}{\omega}}}{p^{\frac{\omega-\alpha}{\omega}}\left[1 + \phi(R-1) + \hat{\tilde{\mu}}_j(b,k,\varepsilon)\phi R\right]}\right\}^{\frac{\omega}{\omega(1-\eta)-\alpha}}, \tag{134}$$

$$L_j(b,k,\varepsilon) = \left\{\frac{\alpha}{\eta(1+\tau)} p v_j((b,k,\varepsilon)\right\}^{\frac{1}{\omega}}, \tag{135}$$

$$y_j(b,k,\varepsilon) = e^{\varepsilon^A} A k^\beta L_j(b,k,\varepsilon)^\alpha v_j(b,k,\varepsilon)^\eta, \tag{136}$$

$$\begin{aligned}(1+\tau)c_j(b,k,\varepsilon) = {} & y_j(b,k,\varepsilon) - p v_j(b,k,\varepsilon) - \phi(R-1)\left[(1+\tau)L_j(b,k,\varepsilon)^\omega + p v_j(b,k,\varepsilon)\right] \\ & - i_j(b,k,\varepsilon) - \frac{\hat{B}_j(b,k,\varepsilon)}{R} + b.\end{aligned} \tag{137}$$

4. **Solve for "new" (i.e., $j+1$) decision rules assuming the credit constraint does not bind.** Assume $\hat{\tilde{\mu}}_{j+1}(b,k,\varepsilon) = 0$, and proceed as follows:

   - The optimality conditions for factor allocations and the production function can be solved separately from the rest of the solution to obtain:

$$v_{j+1}(b,k,\varepsilon) = \left\{\frac{e^{\varepsilon^A} k^\beta \eta^{\frac{\omega-\alpha}{\omega}} \frac{\alpha}{1+\tau}^{\frac{\alpha}{\omega}}}{p^{\frac{\omega-\alpha}{\omega}}\left[1 + \phi(R-1)\right]}\right\}^{\frac{\omega}{\omega(1-\eta)-\alpha}} \tag{138}$$

$$L_{j+1}(b,k,\varepsilon) = \left\{\frac{\alpha}{\eta(1+\tau)} p v_{j+1}(b,k,\varepsilon)\right\}^{\frac{1}{\omega}} \tag{139}$$

$$y_{j+1}(b,k,\varepsilon) = e^{\varepsilon^A} A k^\beta L_{j+1}(b,k,\varepsilon)^\alpha v_{j+1}(b,k,\varepsilon)^\eta \tag{140}$$

   - c_{j+1} is solved directly from the bonds Euler equation (this step uses fixed-point itera-

tion):

$$c_{j+1}(b,k,\varepsilon) = \left\{\beta R E\left[\left(c_j(\hat{B}_j(b,k,\varepsilon),K_j(b,k,\varepsilon),\varepsilon') - \frac{L_j(\hat{B}_j(b,k,\varepsilon),K_j(b,k,\varepsilon),\varepsilon')^\omega}{\omega}\right)^{-\sigma}\right]\right\}^{-\frac{1}{\sigma}} + \frac{L_{j+1}(b,k,\varepsilon)^\omega}{\omega} \tag{141}$$

Bi-linear interpolation of the labor and consumption functions is used to determine the values of $c_j(\hat{B}_j(b,k,\varepsilon),K_j(b,k,\varepsilon),\varepsilon')$ and $L_j(\hat{B}_j(b,k,\varepsilon),K_j(b,k,\varepsilon),\varepsilon')$.

- Solve for $B_{j+1}(b,k,\varepsilon)$ using the resource constraint:

$$B_{j+1}(b,k,\varepsilon)/R = y_{j+1}(b,k,\varepsilon) - pv_{j+1}(b,k,\varepsilon) - \phi(R-1)\left[(1+\tau)L_{j+1}(b,k,\varepsilon)^\omega + pv_{j+1}(b,k,\varepsilon)\right] - i_j(b,k,\varepsilon) - (1+\tau)c_{j+1}(b,k,\varepsilon) + b \tag{142}$$

- Evaluate if the collateral constraint binds. If

$$\frac{B_{j+1}(b,k,\varepsilon)}{R} - \phi R\left[(1+\tau)L_{j+1}(b,k,\varepsilon)^\omega + pv_{j+1}(b,k,\varepsilon)\right] + \kappa\hat{q}_j(b,k,\varepsilon)K_j(b,k,\varepsilon) \geq \varepsilon^b \tag{143}$$

for small ε^b, the constraint does not bind at the point (b,k,ε), the functions with $j+1$ subscripts are saved, and the algorithm jumps to step 6. Otherwise, the constraint binds, the $j+1$ functions are discarded and we move to Step 5.

5. **Solve for $j+1$ decision rules when the credit constraint binds.** Now we impose that the collateral constraint holds with equality for the coordinate (b,k,ε) at which the constraint binds in the previous step. Since $\hat{q}_j(\cdot)$ has not changed, we use the same j-indexed functions $K'_j(\cdot)$ and $\tilde{i}_j(\cdot)$ as before. This step requires a non-linear solver to solve for the values of $L_{j+1}(\cdot), v_{j+1}(\cdot), c_{j+1}(\cdot), B_{j+1}(\cdot), \tilde{\mu}_{j+1}(\cdot)$ by solving the following equation system formed by the recursive optimality conditions, expressed as a function of $\tilde{\mu}$ for simplicity:[18]

$$v(\tilde{\mu}) = \left\{\frac{e^{\varepsilon^A}Ak^\beta\eta^{\frac{\omega-\alpha}{\omega}}\frac{\alpha}{1+\tau}^{\frac{\alpha}{\omega}}}{p^{\frac{\omega-\alpha}{\omega}}\left[1+\phi(R-1)+\tilde{\mu}\phi R\right]}\right\}^{\frac{\omega}{\omega(1-\eta)-\alpha}} \tag{144}$$

$$L(\tilde{\mu}) = \left\{\frac{\alpha}{\eta(1+\tau)}pv(\tilde{\mu})\right\}^{\frac{1}{\omega}} \tag{145}$$

$$\frac{B(\tilde{\mu})}{R} = -\kappa\hat{q}_jK_j + \phi Rpv(\tilde{\mu})\left[1+\frac{\alpha}{\eta}\right] \tag{146}$$

$$(1+\tau)c(\tilde{\mu}) = e^{\varepsilon^A}Ak^\beta L(\tilde{\mu})^\alpha v(\tilde{\mu})^\eta - pv(\tilde{\mu}) - \phi(R-1)pv(\tilde{\mu})\left[1+\frac{\alpha}{\eta}\right] - i_j - \frac{B(\tilde{\mu})}{R} + b \tag{147}$$

In addition to the above four equations, the non-linear system also includes the following

[18]The non-linear solver is needed only because when the constraint binds the value of $\tilde{\mu}$ needs to be solved together with the allocations and prices. If the multiplier could be solved separately, as it is the case for a large class of credit constraints, FiPIt does not need a non-linear solver anywhere (see Mendoza and Villalvazo (2020) for details).

condition for $\tilde{\mu}_{j+1}(b,k,\varepsilon)$ that represents the Euler equation for bonds:

$$\tilde{\mu}_{j+1}(b,k,\varepsilon) = 1 - \frac{\beta R E\left[\left(c_j(\hat{B}_j(b,k,\varepsilon), K_j(b,k,\varepsilon), \varepsilon') - \frac{L_j(\hat{B}_j(b,k,\varepsilon), K_j(b,k,\varepsilon), \varepsilon')^{\omega}}{\omega}\right)^{-\sigma}\right]}{\left(c(\tilde{\mu}_{j+1}(b,k,\varepsilon)) - \frac{L(\tilde{\mu}_{j+1}(b,k,\varepsilon))^{\omega}}{\omega}\right)^{-\sigma}} \tag{148}$$

The numerator in the right-hand-side of this expression is using again fixed-point iteration because it uses j-indexed functions to compute the entire expected marginal utility term. The values of the functions $c_j(\hat{B}_j(b,k,\varepsilon), K_j(b,k,\varepsilon), \varepsilon')$ and $L_j(\hat{B}_j(b,k,\varepsilon), K_j(b,k,\varepsilon), \varepsilon')$ are again determined by bi-linear interpolation.

Algebraic manipulation of the five equations in the nonlinear system reduces the system to a single nonlinear equation in $\tilde{\mu}_{j+1}(.)$ for each coordinate (b,k,ε) where the constraint binds:

$$\begin{aligned}
&(1-\tilde{\mu}_{j+1}(\cdot))\left\{C_1^{\frac{\omega}{(1-\eta)\omega-\alpha}}\left[\frac{\alpha}{1+\phi(R-1)+\tilde{\mu}_{j+1}(\cdot)\phi R}\right]^{\frac{\eta\omega+\alpha}{(1-\eta)\omega-\alpha}}\right. \\
&\quad -\left[\frac{\alpha C_1}{1+\phi(R-1)+\tilde{\mu}_{j+1}(\cdot)\phi R}\right]^{\frac{\omega}{(1-\eta)\omega-\alpha}} C_2 \\
&\quad \left.-\left(\frac{i_j(\cdot)-\kappa\hat{q}_j(\cdot)K_j(\cdot)-b}{1+\tau}\right)\right\}^{-\sigma} \\
&\quad = \beta R E\left[\left(c_j(\hat{B}_j(\cdot), K_j(\cdot), \varepsilon') - \frac{L_j(\hat{B}_j(\cdot), K_j(\cdot), \varepsilon')^{\omega}}{\omega}\right)^{-\sigma}\right]
\end{aligned} \tag{149}$$

where:

$$C_1 \equiv \left(\frac{1}{1+\tau}\right)^{1-\eta} e^{\varepsilon^A} A k^{\beta}\left(\frac{\eta}{\alpha p}\right)^{\eta} \quad \text{and} \quad C_2 \equiv \frac{1}{\omega} + \frac{\eta}{\alpha} + \phi\left(1+\frac{\eta}{\alpha}\right)(2R-1). \tag{150}$$

Once $\tilde{\mu}_{j+1}(b,k,\varepsilon)$ is solved, the rest of the functions can be updated to their j+1 forms using the previous equations as follows:

$$v_{j+1}(b,k,\varepsilon) = \left\{\frac{e^{\varepsilon^A} A k^{\beta} \eta^{\frac{\omega-\alpha}{\omega}} \frac{\alpha}{1+\tau}^{\frac{\alpha}{\omega}}}{p^{\frac{\omega-\alpha}{\omega}}\left[1+\phi(R-1)+\tilde{\mu}_{j+1}(b,k,\varepsilon)\phi R\right]}\right\}^{\frac{\omega}{\omega(1-\eta)-\alpha}} \tag{151}$$

$$L_{j+1}(b,k,\varepsilon) = \left\{\frac{\alpha}{\eta(1+\tau)} p v_{j+1}(b,k,\varepsilon)\right\}^{\frac{1}{\omega}} \tag{152}$$

$$B_{j+1}(b,k,\varepsilon) = R\left\{-\kappa\hat{q}_j(b,k,\varepsilon)K_j(b,k,\varepsilon) + \phi R p v_{j+1}(b,k,\varepsilon)\left[1+\frac{\alpha}{\eta}\right]\right\} \tag{153}$$

$$\begin{aligned}
(1+\tau)c_{j+1}(b,k,\varepsilon) = {} & e^{\varepsilon^A} A k^{\beta} L_{j+1}(b,k,\varepsilon)^{\alpha} v_{j+1}(b,k,\varepsilon)^{\eta} \\
& - p v_{j+1}(b,k,\varepsilon) - \phi(R-1) p v_{j+1}(b,k,\varepsilon)\left[1+\frac{\alpha}{\eta}\right] \\
& - i_j(b,k,\varepsilon) - \frac{B_{j+1}(b,k,\varepsilon)}{R} + b
\end{aligned} \tag{154}$$

The functions with $j+1$ subscripts are saved, and we move to Step 5.

6. **Complete iteration-j solutions for the entire state space:** Return to Step 3 and repeat for all triples (b, k, ε) in the state space.

7. **Compute the *new* capital pricing function:** The new pricing function is solved for by applying the new decision rules for $c_{j+1}(\cdot), L_{j+1}(\cdot), b'_{j+1}(\cdot), \tilde{\mu}_{j+1}(\cdot)$ to the Euler equation for capital and solving it so as to obtain the following analytical solution for $q_{j+1}(b, k, \varepsilon)$:

$$q_{j+1}(b,k,\varepsilon) = \frac{\beta E_t\left[\left(c_{j+1}\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right) - \frac{L_{j+1}\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right)^{\omega}}{\omega}\right)^{-\sigma}\left[d'(\cdot) + \hat{q}_j\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right)\right]\right]}{\left(c_{j+1}(\cdot) - \frac{L_{j+1}(\cdot)^{\omega}}{\omega}\right)^{-\sigma}(1-\kappa\tilde{\mu}_{j+1}(\cdot))} \tag{155}$$

where

$$d'\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right) = \gamma A' k'_j(\cdot)^{\gamma-1} L_{j+1}\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right)^{\alpha} v_{j+1}\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right)^{\eta} - \delta + \frac{a}{2}\frac{(k'_j(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon') - k'_j(\cdot))^2}{k'_j(\cdot)^2}$$

This step also uses fixed-point iteration, because the capital price used in the right-hand-side of (155) is the conjecture set in step 2, and since all the functions in the right-hand-side are known, the equation solves directly for $q_{j+1}(b, k, \varepsilon)$. The values of $c_{j+1}\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right))$, $L_{j+1}\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right)$ and $\hat{q}_j\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right)$ are determined by bi-linear interpolation. The value of $d'(\cdot)$ is obtained by applying bi-linear interpolation to evaluate $L_{j+1}\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right)$, $v_{j+1}\left(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon'\right)$ and $k'_j(b'_{j+1}(\cdot),k'_j(\cdot),\varepsilon')$. Notice that the decision rule setting the value of b_{t+1} at which all these functions are interpolated is a j+1-indexed function but over the capital dimension we are still using the j-indexed decision rule.

8. **Check convergence:** Evaluate the convergence of the relevant functions using the following criteria, for small ε^f:

$$||q_{j+1}(b,k,\varepsilon) - \hat{q}_j(b,k,\varepsilon)|| \leq \varepsilon^f \tag{156}$$
$$||B_{j+1}(b,k,\varepsilon) - \hat{B}_j(b,k,\varepsilon)|| \leq \varepsilon^f \tag{157}$$
$$||\tilde{\mu}_{j+1}(b,k,\varepsilon) - \hat{\tilde{\mu}}_j(b,k,\varepsilon)|| \leq \varepsilon^f \tag{158}$$

If all of these conditions hold, the recursive equilibrium of the Sudden Stops model has been solved. If convergence fails, update the conjectured functions using a convex combination of the last conjectures and the new functions to dampen possible overshooting using:

$$\hat{x}_j^{new}(b,k,\varepsilon) = \rho\hat{x}_j(b,k,\varepsilon) + (1-\rho)x_{j+1}(b,k,\varepsilon) \tag{159}$$

for $x = q, B, \tilde{\mu}$ and some $-1 \leq \rho \leq 1$. Set $\hat{x}_j(b,k,\varepsilon)$=$\hat{x}_j^{new}(b,k,\varepsilon)$ for all functions, return to step 2 and repeat until convergence is attained.

C.2 Additional quantitative results of the Sudden Stops model

The main results for the Sudden Stops model is presented in the main draft. Here we discuss additional quantitative results.

C.2.1 Spectral analysis

Figure 10 shows the nonparametric periodograms for the DynareOBC and GA solutions. As in the endowment and RBC models, since all of the variables follow AR(1)-like processes, the periodograms are generally downward sloping, indicating that low frequencies account for a larger fraction of the variance of the variables than business cycle and higher frequencies. Moreover, the periodograms for the GA solution are very similar to those for the RBC GA solution, in line with the finding that the GA impulse responses of the RBC and SS models are similar because the credit constraint binds infrequently. The periodograms of the DynareOBC solution differ from those of the 2OA and RSS solutions of the RBC model, so the local methods fail to match the property of the GA solutions that spectral densities of the RBC and SS models are similar.

The DynareOBC periodograms for NFA, consumption, net exports, investment and labor differ sharply from the GA results. Consumption has the highest variance in the GA solution (121.4), followed by DynareOBC-DEIR (119.2) and DynareOBC-$\beta R < 1$ with a much lower variance (65.8). In contrast, the autocorrelation of consumption is highest in DynareOBC-DEIR (0.91) and about the same in GA and DynareOBC-$\beta R < 1$ (0.83). As a result, the consumption periodograms for the latter two have the same intercept but the one for GA is uniformly higher otherwise, while the periodogram for DynareOBC-DEIR has the highest intercept but is generally below the periodogram for GA. The DynareOBC solutions assign significantly less consumption variability to business cycle and lower frequencies than the GA solution. Net exports also show higher persistence in the DynareOBC-DEIR solution while DynareOBC-$\beta R < 1$ and GA have similar persistence, and opposite from what we observe for consumption, the GA solution has less overall variance and less variability at all frequencies. Investment has higher variance and persistence in the GA than in the local solutions, and it has uniformly higher variability at all frequencies.

C.2.2 *FiPIt* v. DynareOBC speed comparisons

Table 4 provides additional comparisons of execution times for alternative specifications of the DynareOBC and *FiPIt* solutions. As explained in the paper, on one hand, *FiPIt* suffers from the curse of dimensionality typical of GA methods related to the number of state variables, and more so if the model specification requires using a root-finder when the constraint binds. But once the decision rules are solved for, generating stochastic time-series simulations is very fast. On the other hand, the number of state variables is much less of an issue for DynareOBC, but execution time rises with the required length of extended perfect-foresight paths for each date-t solution, the length of the full time-series simulation needed for convergence of long-run moments, and the iterations required to compute the news-shocks sequences that implement the constraint.

The above tradeoffs between the two algorithms is illustrated by the results shown in Table 4. First-order DynareOBC is much slower than *FiPIt* if the simulation length rises to 150,000 periods (350 seconds v. 268 seconds) or with fewer exogenous shocks so that the curse of dimensionality is less severe for the global solution (with TFP shocks only, *FiPIt* solves in 42 seconds v. 230 seconds with DynareOBC). Increasing borrowing capacity by setting κ to 0.3 also results in *FiPIt* solving much faster than DynareOBC (137 v. 228 seconds).

Figure 10: Spectral Density Functions for the Sudden Stops Model

Table 4: Execution Times: Sudden Stops model

	Execution time
Baseline	
GA	268.0
DynareOBC	243.5
150,000 period simulations	
GA	268.0
DynareOBC	350.9
κ 0.3	
GA	137.3
DynareOBC	228.4
TFP shock only	
GA	41.7
DynareOBC	229.9

C.2.3 Financial premia & quintile analysis for Sudden Stops model

The financial premia for the *FiPIt* solution are computed using their recursive representations. For each coordinate in the state space (b, k, ε), we used the recursive representation of equations (20) and (21) of the paper and the recursive optimal decision rules to obtain the following expressions for the shadow interest premium (SIP), the equity premium (EP), the risk component of the equity premium (RP) and the covariance between marginal utility and equity returns (COV):

$$SIP(b,k,\varepsilon) = \frac{R\mu(b,k,\varepsilon)(1+\tau)}{u'(c(b,k,\varepsilon)) - \mu(b,k,\varepsilon)(1+\tau)}, \tag{160}$$

$$EP_{\varepsilon'}(b,k,\varepsilon)(b,k,\varepsilon) = E_{\varepsilon'}[R^q(b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon')] - R, \tag{161}$$

$$\begin{aligned} COV_{\varepsilon'}(b,k,\varepsilon) = {} & E_{\varepsilon'}\left\{\left[u'(c(b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon') - E_{\varepsilon'}\left(u'(c(b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon')\right)\right]\right. \\ & \times \left.\left[R^q(b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon') - E'_{\varepsilon}\left(R^q(b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon')\right)\right]\right\} \end{aligned} \tag{162}$$

$$RP_{\varepsilon'}(b,k,\varepsilon) \equiv -\frac{COV_{\varepsilon'}(b,k,\varepsilon)}{E_{\varepsilon'}[u'(c(b'(b,k,\varepsilon), k'(b,k,\varepsilon), \varepsilon'))]}. \tag{163}$$

The conditional expectations and covariance terms in the above expressions are calculated taking expectations over ε' using the Markov transition probabilities $\pi(\varepsilon, \varepsilon')$:

The means of the financial premia conditional on the quintile distribution of $\mu > 0$ reported in the paper are constructed as follows: First, we construct the quintile distribution of $\mu(b,k,\varepsilon) > 0$ using the ergodic distribution $P(b,k,\varepsilon)$ produced by the *FiPIt* algorithm. To compute this quintile distribution, we start by sorting the values of $\mu(b,k,\varepsilon)$ by size removing all those that are zero. The ergodic distribution of $\mu(b,k,\varepsilon) > 0$ is given by $P(b,k,\varepsilon)/\Sigma_{\{(b,k,\varepsilon):\mu(b,k,\varepsilon)>0\}}P(b,k,\varepsilon)$. The associated cumulative ergodic distribution of $\mu(b,k,\varepsilon) > 0$ is denoted $CumP_{\mu>0}(b,k,\varepsilon)$. Then we assign the values of $\mu(b,k,\varepsilon) > 0$ into quintiles by identifying the values of μ associated with the quintile boundaries at which $CumP_{\mu>0}(b,k,\varepsilon) = 0.2, 0.4, 0.6, 0.8, 1$ and then allocating each value

of $\mu(b, k, \varepsilon) > 0$ into its corresponding quintile using these boundaries. Each value of $\mu(b, k, \varepsilon) > 0$ has associated with it a value for each of the financial premia indicators shown above, since both μ and the financial premia are recursive functions of the state variables. Hence, once the quintile distribution of $\mu(b, k, \varepsilon) > 0$ is formed, the means of the financial premia for each quintile are computed by taking averages using the probability distribution of (b, k, ε) within each quintile (i.e., ergodic distributions scaled by the cumulative probability of being in each quintile) and we can also compute means across the entire quintile distribution, which are the same as means conditional on $\mu(b, k, \varepsilon) > 0$.

For DynareOBC, we use time-series simulations to construct the financial premia indicators. It is not feasible to construct conditional expectations, so we use realized values over the time-series simulations. To construct the quintile distribution of $\mu > 0$, we find the quintile boundaries for the subset of the full time-series simulation solution for which $\mu_t > 0$, and associate with each value μ_t allocated to each quintile the corresponding values of SIP_t, EP_t, COV_t, RP_t. Then we compute within-quintile means of these financial premia as the averages of the values allocated to each quintile, and we can also compute averages across all five quintiles which are equivalent to averages conditional on $\mu_t > 0$.

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