Appendix to "Difference-in-Differences in the Marketplace"

Factor demand equations in differential form

In levels, the factor demand equations are:

$$K = \frac{\partial C(w,r,Y;A,B)}{\partial r} \quad and \quad L = \frac{\partial C(w,r,Y;A,B)}{\partial w}$$

where the cost function C is the minimum factor cost of achieving output Y:

$$C(w,r,Y;A,B) = \min_{L,K} wL + rK \quad s.t. \quad \frac{A}{B}F(L,KB) = Y$$

As usual, C is homogeneous of degree one in prices and of degree one in output. The special case $C(w,1,1;1,1)$ is of general interest because:

$$C(w,r,Y;A,B) = C\left(w,\frac{r}{B},Y\frac{B}{A};1,1\right) = \frac{rY}{A}C\left(B\frac{w}{r},1,1;1,1\right)$$

where the final term uses the two homogeneity conditions. One interpretation of the middle cost function is that the choice variables of L and K have been replaced with choice variables L and KB and we recognize that the price of BK differs from the price of K by a factor of B. In this notation, L demand is:

$$L = \frac{\partial C(w,r,Y;A,B)}{\partial w} = \frac{\partial}{\partial w}\frac{rY}{A}C\left(B\frac{w}{r},1,1;1,1\right) = Y\frac{B}{A}\frac{\partial C\left(B\frac{w}{r},1,1;1,1\right)}{\partial\left(\frac{Bw}{r}\right)} \tag{1}$$

Recall that the definition of the elasticity of substitution in F, which we denote $\sigma > 0$, is the cross-price derivative of $C(w,r,1;1;1)$ times $C(w,r,1;1;1)/[(1-s_L)s_L]$. That makes the cross-price elasticity of either conditional factor demand equal to the product of σ and the other factor's share. By homogeneity, its own price elasticity is the negation of its cross-price elasticity. From (1), the log-derivative form of L demand is therefore:

$$\Delta L = \Delta\left(Y\frac{B}{A}\right) - (1-s_L)\sigma\Delta\left(B\frac{w}{r}\right) = \Delta\left(\frac{Y}{A}\right) - (1-s_L)\sigma\Delta\left(\frac{w}{r}\right) + [1-\sigma+s_L\sigma]\Delta B \tag{2}$$

Using price homogeneity, the levels form of K demand is (3):

$$K = \frac{\partial}{\partial r}\frac{rY}{A}C\left(B\frac{w}{r},1,1;1,1\right) = Bw\frac{Y}{A}\frac{\partial}{\partial r}C\left(1,\frac{r}{Bw},1;1,1\right) = \frac{Y}{A}\frac{\partial C\left(1,\frac{r}{wB},1;1,1\right)}{\partial\left(\frac{r}{Bw}\right)} \tag{3}$$

Differentiating (3) with respect to $r/(Bw)$ has just an own-price term, whose elasticity is $-s_L\sigma$. Therefore, the log-differential form of K demand is:

$$\Delta K = \Delta\left(\frac{Y}{A}\right) + s_L\sigma\Delta\left(B\frac{w}{r}\right) = \Delta\left(\frac{Y}{A}\right) + s_L\sigma\Delta\left(\frac{w}{r}\right) + s_L\sigma\Delta B \tag{4}$$