Internet Appendix for "Misallocation and Asset Prices"

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I. Cross-Sectional Evidence

Our theory's main implication is that the volatile and persistent time-series variation in misallocation captures the low-frequency component of the time-series variation in aggregate growth. Although our model does not analyze cross-sectional implications, we provide cross-sectional evidence to further support the theoretical mechanism. In Subsection I.A, we estimate the market price of risk for the misallocation factor and study its cross-sectional asset pricing implications. In Subsection I.B, we show that firms with higher book-to-market ratios are more negatively exposed to the misallocation factor, which provides further support that the slow-moving misallocation captures low-frequency growth fluctuations.

A. Misallocation as A Macroeconomic Risk Factor

Our model implies that the misallocation M_t plays a significant role in determining the SDF of representative agent through its effects on aggregate consumption growth. To examine the empirical relevance of this mechanism, we test whether the empirical misallocation measure M_t is a risk factor significantly priced in the cross section of assets.

We consider standard test assets, including 25 size-sorted and book-to-market-sorted portfolios, 10 momentum-sorted portfolios, and 6 maturity-sorted Treasury bond portfolios. For each asset i, we estimate the factor loadings using the following time-series regression:

$$R^e_{i,t} = c_i + \sum_k \beta_{i,k} f_{k,t} + \varepsilon_{i,t}, \tag{IA.1}$$

where $R^e_{i,t} = R_{i,t} - r_{f,t}$ is the excess return of asset i over the risk-free rate and $f_{k,t}$ represents risk factor k. We then estimate the cross-sectional price of risk associated with the factors $f_{k,t}$ by running a cross-sectional regression of time-series average excess returns, $\mathbb{E}[R^e_{i,t}]$, on risk factor exposures estimated in equation (IA.1) as follows,

$$\mathbb{E}[R^e_{i,t}] = \alpha + \sum_k \widehat{\beta}_{i,k} \lambda_k + \epsilon_i, \tag{IA.2}$$

where the estimated $\widehat{\lambda}_k$ is the price of risk for factor k and $\widehat{\alpha}$ is the average cross-sectional pricing error or zero-beta rate.

The above estimation procedure is implemented using different linear factor models. The results are presented in Table IA.I and visualized in Figures IA.1 and IA.2. As a benchmark, column (1)

Table IA.I: Portfolio returns and model fit.

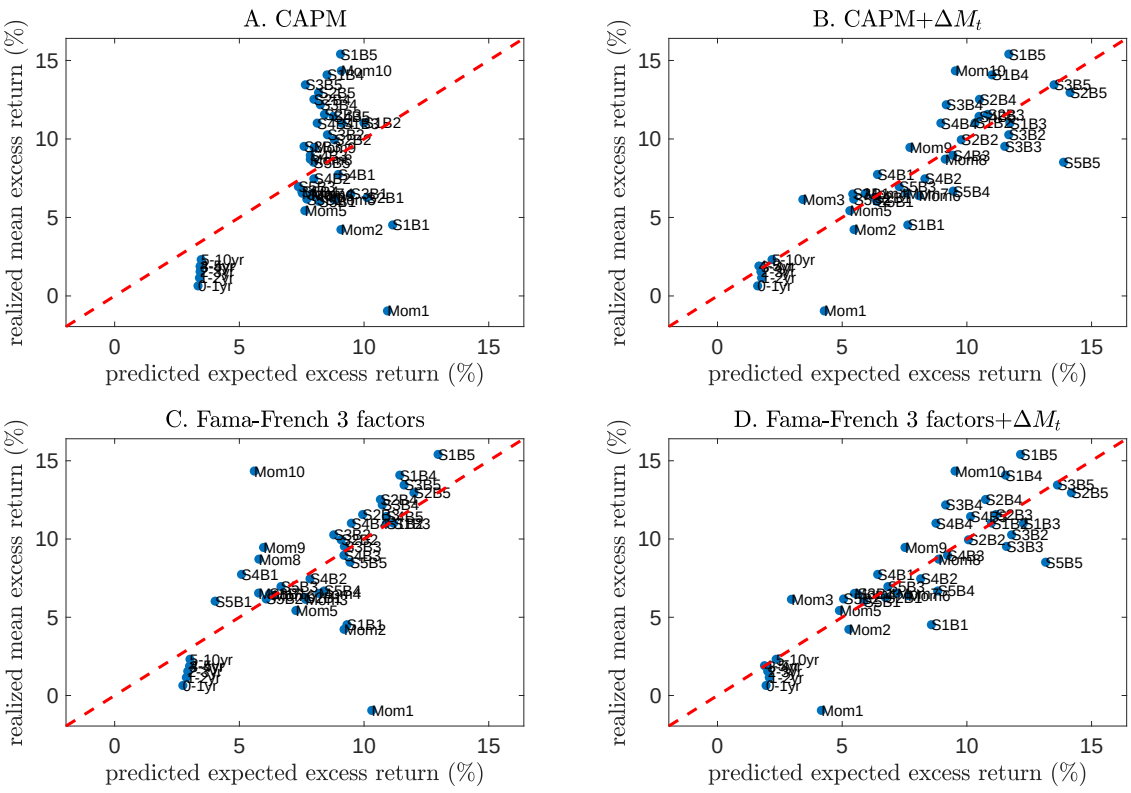
	(1) Mkt	(2) $Mkt, \Delta M$	(3) FF	(4) $FF, \Delta M$	(5) Mkt, CG	(6) $Mkt, \Delta M, CG$	(7) FF, CG	(8) $FF, \Delta M, CG$
Panel A: Prices of risk								
Intercept	3.336	1.425	2.665	1.781	2.438	1.412	2.336	1.464
	(0.962)	(1.702)	(0.730)	(1.177)	(1.385)	(1.594)	(0.882)	(1.263)
Mkt	4.919	6.076	3.658	5.353	4.610	5.828	4.533	6.196
	(3.851)	(6.237)	(3.742)	(5.589)	(4.874)	(5.750)	(4.211)	(5.903)
ΔM		−0.079		−0.078		−0.070		−0.079
		(0.031)		(0.030)		(0.025)		(0.032)
SMB			3.006	2.182			2.641	1.831
			(2.874)	(4.307)			(3.232)	(4.555)
HML			4.355	4.804			4.396	4.842
			(2.975)	(4.458)			(3.354)	(4.724)
CG					0.023	0.011	0.019	0.019
					(0.011)	(0.013)	(0.007)	(0.011)
Panel B: Test diagnostics								
Total MAPE	2.764	1.465	1.902	1.450	2.029	1.393	1.955	1.469
Size and B/M 25	2.768	1.527	1.304	1.472	1.493	1.401	1.377	1.576
Momentum 10	3.304	1.964	3.721	1.975	3.617	1.970	3.713	1.753
Bond 6	1.847	0.375	1.361	0.483	1.613	0.398	1.433	0.550
Adjusted R-squared	0.303	0.446	0.619	0.705	0.453	0.531	0.629	0.718

Note: This table presents pricing results for 41 test assets, including 25 size-sorted and book-to-market-sorted portfolios, 10 momentum-sorted portfolios, and 6 maturity-sorted Treasury bond portfolios. Each model is estimated using equation (IA.2). Mkt is the market's excess return over the risk-free rate. ΔM is the misallocation factor, which is the year-on-year changes in the empirical misallocation measure M_t. SMB and HML are the two factors in the FF3 model, capturing the excess returns of small caps over big caps and of value stocks over growth stocks, respectively. Panel A reports the prices of risk. Shanken standard errors are reported in brackets. Panel B reports test diagnostics, including MAPE and the adjusted R-squared. The sample is yearly and spans the period from 1965 to 2016.

of Table IA.I reports the results of CAPM, which includes market excess returns as the single risk factor. It clearly shows that the exposure to market risk cannot explain the spread in average returns across portfolios. The cross-sectional intercept is statistically significant and the factor price of risk is statistically insignificant. The pricing errors are large, with a high total mean absolute pricing error (MAPE) of 2.764% and a low adjusted R-squared of 0.303. Column (2) of Table IA.I presents the results based on a two-factor model that includes the year-on-year changes in the empirical misallocation measure, ΔM_t, as an additional risk factor. The price of risk for ΔM_t is -0.079, which is negative and statistically significant as implied by our model.[1] Relative to CAPM, the adjusted R-squared increases significantly to 0.446 and the total MAPE declines significantly to 1.465%. The test assets are lined up very close to the 45-degree line in the two factor model (Panel B of Figure IA.1), which is in sharp contrast to the prediction of CAPM (Panel A of Figure IA.1).

As another benchmark, column (3) of Table IA.I presents the results of the Fama-French three-factor (FF3) model. Comparing columns (2) and (3) of Table IA.I, the FF3 model achieves a higher

[1]The magnitude of the price of risk for ΔM_t does not represent the risk premium of ΔM_t because the misallocation factor ΔM_t does not lie in the space of excess returns.

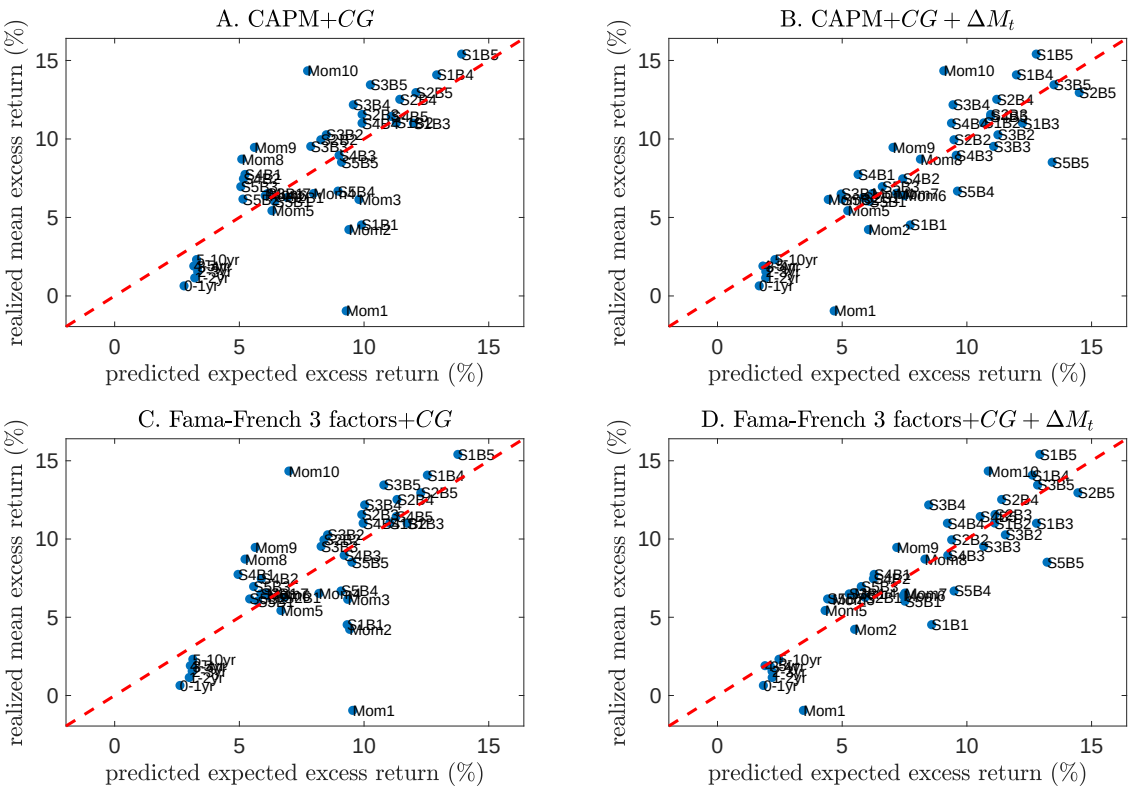


Note: This figure plots the realized mean excess returns of portfolios against the expected excess returns predicted by various linear factor asset pricing models. The sample is yearly and spans the period from 1965 to 2016.

Figure IA.1: Realized versus predicted mean excess returns in factor models with M_t.

adjusted R-squared of 0.619. However, the two-factor model with market returns and the misallocation factor ΔM_t has a lower total MAPE. The two-factor model outperforms the FF3 model especially for the 10 momentum-sorted portfolios (3.721% compared to 1.964%). It is well known that the FF3 model has a poor explanatory power for momentum-sorted portfolio returns. The cross-sectional fit is clearly displayed in Panels B and C of Figure IA.1, which shows that the two-factor model outperforms the FF3 model mainly due to the improved fit for momentum-sorted portfolios. In column (4) of Table IA.I, we further include the misallocation factor ΔM_t to the FF3 model to construct a four-factor model. Compared with the FF3 model, the cross-sectional fit further improves as shown by the lower total MAPE and higher adjusted R-squared in the four-factor model. The improvement is mainly due to improved explanatory power for momentum-sorted portfolio.

Our model suggests that the low-frequency component of aggregate consumption growth is generated by the slow-moving misallocation. If this mechanism is empirically relevant, we expect the long-run expected consumption growth to have little explanatory power for portfolio returns after including the misallocation factor ΔM_t in linear factor models. Following Parker and Julliard (2005), we use accumulated future consumption growth to approximate long-run expected consumption growth. Column (5) of Table IA.I and Panel A of Figure IA.2 show that the two-factor model with market returns and accumulated future consumption growth can fit the returns of our test portfolios well, with an adjusted R-squared of 0.453. In column (6) of Table IA.I and Panel B of Figure IA.2,



Note: This figure plots the realized mean excess returns of portfolios against the expected excess returns predicted by various linear factor asset pricing models. The sample is yearly and spans the period from 1965 to 2016.

Figure IA.2: Realized versus predicted mean excess returns in factor models with M_t and accumulated future consumption growth.

we augment this two-factor model with the misallocation factor ΔM_t to construct a three-factor model. We find that the relation between realized mean excess returns and predicted mean excess returns across our test portfolios stays almost unchanged, implying that expected consumption growth and misallocation are indeed similarly priced in the cross section of test assets. However, the coefficient on accumulated future consumption growth becomes statistically insignificant after including ΔM_t as a factor whereas the coefficient on ΔM_t is statistically significant. Similar patterns are shown in columns (7) and (8) of Table IA.I and Panels C and D of Figure IA.2, when we include the misallocation factor ΔM_t in a four-factor model that contains the Fama-French three factors and accumulated future consumption growth.

The strong pricing power of misallocation factor, as a (macro) nontradable asset pricing factor, is an important, nontrivial empirical finding. As emphasized by Cochrane (2017), it is the sole job of macro-finance to understand what are the primitive sources of systematic risk, by suggesting (macro) nontradable factors, and explain why they earn a premium.[2] However, not many studies find that (macro) nontradable factors motivated by macro-finance models empirically outperform or drive out (ad hoc) tradable factors such as Fama-French factors in explaining the cross section of expected asset

[2]Other recent reviews on macro-finance models also highlight this point (e.g., Brunnermeier, Eisenbach and Sannikov, 2012; Dou et al., 2020).

Table IA.II: Exposure to misallocation M_t across firms sorted on the book-to-market ratio.

$BE_{i,t-1}/ME_{i,t-1}$	Q1 (low)	Q2	Q3	Q4	Q5 (high)	Q5−Q1
β_1^p	0.026	−0.169	−0.331	−0.421	−0.510	−0.536
	(0.143)	(0.089)	(0.107)	(0.173)	(0.183)	(0.196)

Note: In each year t, we sort firms into quintiles on their book-to-market ratios $BE_{i,t-1}/ME_{i,t-1}$ in year $t-1$. For each quintile portfolio, we estimate β_1^p according to specification (IA.3). The sample spans the period from 1965 to 2016. Robust standard errors are reported in brackets.

returns,[3] partly because the measurement error in nontradable factors causes attenuation bias in the estimates of factor exposures.

B. Cash Flow Exposure to the Misallocation Factor

To support the key theoretical mechanism that the slow-moving misallocation drives low-frequency growth fluctuations, we provide further cross-sectional evidence on firms' cash flow exposure to the misallocation factor. Our starting point is the robust evidence found in the asset pricing literature (Bansal, Dittmar and Lundblad, 2005; Parker and Julliard, 2005; Hansen, Heaton and Li, 2008; Santos and Veronesi, 2010): the cash flows of value firms load more positively on accumulated consumption growth than those of growth firms. Given that a higher misallocation predicts a lower consumption growth over long horizons in both the data and model (Panel B of Table 3 of the main text), if the theoretical mechanism has empirical relevance, we should find the cash flows of value firms load more negatively on misallocation in the data.

To test this prediction, we follow the empirical strategy of Santos and Veronesi (2010). In each year t, we sort firms into quintiles based on their book-to-market ratios $BE_{i,t-1}/ME_{i,t-1}$ in year $t-1$, where $BE_{i,t-1}$ is the book equity from Compustat and $ME_{i,t-1}$ is the market equity from CRSP. For each quintile portfolio, we compute the value-weighted return on equity (ROE) across all firms within the portfolio, where a firm's ROE is its income before extraordinary items divided by its common equity. Let $ROE^p_{t+j,j+1}$ denote the value-weighted ROE at year $t+j$ of the portfolio p, which was formed $j+1$ years earlier, i.e., in year $t-1$. We run a regression similar to the specification adopted by Santos and Veronesi (2010), except for including accumulated misallocation shocks as an additional independent variable:

$$\sum_{j=0}^{4}\rho^j ROE^p_{t+j,j+1} = \beta_0^p + \beta_1^p \sum_{j=0}^{4}\rho^j \Delta M_{t+j} + \beta_2^p \sum_{j=0}^{4}\rho^j ROE^{Mkt}_{t+j} + \varepsilon_t, \tag{IA.3}$$

where $\rho = 0.95$ is a constant as in Santos and Veronesi (2010). The variable ΔM_t is the year-on-year changes in M_t and the variable ROE^{Mkt}_t is the ROE of the market portfolio. The coefficient of interest is β_1^p, which captures the loadings of accumulated ROE on accumulated misallocation shocks.

Table IA.II presents the results. The accumulated ROE of firms with high book-to-market ratios (i.e., value firms in the quintile group 5 labeled as Q5) is significantly more negatively exposed to

[3] A few exceptions include durable consumption growth (Yogo, 2006; Gomes, Kogan and Yogo, 2009), expenditure shares of housing (Piazzesi, Schneider and Tuzel, 2007), market liquidity (Pástor and Stambaugh, 2003), intermediary leverages (Adrian, Etula and Muir, 2014; He, Kelly and Manela, 2017), and common fund flows (Dou, Kogan and Wu, 2023), among others.

accumulated year-on-year changes in misallocation than that of firms with low book-to-market ratios (i.e., growth firms in the quintile group 1 labeled as Q1). The loadings monotonically decrease from 0.026 to -0.510 as the book-to-market ratio increases from Q1 to Q5. The difference in the loadings between Q1 and Q5 (Q5$-$Q1) is -0.536, which is statistically significant.

II. Proofs

A. *Proof of Proposition 1*

To ensure that shareholders do not intervene, the manager pays a dividend flow of $\zeta a_{i,t}\mathrm{d}t$ to shareholders over $[t, t+\mathrm{d}t)$ where ζ is to be determined. Thus, firm i's total dividend payment is $(\tau+\zeta)a_{i,t}\mathrm{d}t$ over $[t, t+\mathrm{d}t)$, which includes the rents paid to the manager and the dividends paid to shareholders. If the manager follows this payout policy consistently, shareholders will be willing to defer intervention continuously. Payouts and rents evolve in lockstep.

We now derive the relation between ζ, ρ, and τ. Per our discussion in Subsection II.B, the manager's value is proportional to the firm's capital, given by $\xi_{i,t}a_{i,t}$, where $\xi_{i,t}$ depends on the firm's idiosyncratic productivity $z_{i,t}$ and the aggregate state of the economy. If shareholders do not intervene, they receive a dividend payment that is a fraction ζ/τ of the manager's private benefits. Thus, shareholders' value is $(\zeta/\tau)\xi_{i,t}a_{i,t}$. If shareholders intervene, the firm's value will drop to $(1-\tau/\rho)\xi_{i,t}a_{i,t}$ due to the loss of capital. However, because shareholders now are also managers, they will have the claim to all dividends, which generate a value of $(1+\zeta/\tau)(1-\tau/\rho)\xi_{i,t}a_{i,t}$. Thus, the manager chooses the intensity ζ of dividends to shareholders such that shareholders are indifferent about having an intervention or not:

$$\frac{\zeta}{\tau}\xi_{i,t}a_{i,t} = (1+\zeta/\tau)(1-\tau/\rho)\xi_{i,t}a_{i,t}, \tag{IA.4}$$

which implies $\zeta = (1-\tau/\rho)\tau/[1-(1-\tau/\rho)]$. The firm's total dividend payout ratio is

$$\tau + \zeta = \rho. \tag{IA.5}$$

B. *Proof of Lemma 1*

Given capital $k_{i,t} = a_{i,t} + \widehat{a}_{i,t}$, utilization intensity $u_{i,t}$, and intermediate composite $x_{i,t}$, firm i solves a static maximization problem when choosing $\ell_{i,t}$ and $x_{i,j,t}$. Taking the first-order condition with respect to $\ell_{i,t}$ on the right-hand side of equation (17) of the main text, we obtain the optimal labor demand:

$$\ell_{i,t} = \left[\frac{w_t}{(1-\alpha)(1-\varepsilon)(z_{i,t}u_{i,t}k_{i,t})^{\alpha(1-\varepsilon)}x_{i,t}^{\varepsilon}}\right]^{\frac{1}{(1-\alpha)(1-\varepsilon)-1}}. \tag{IA.6}$$

Substituting equations (14) and (17) of the main text and equation (IA.6) into equation (2) of the

main text:

$$\mathrm{d}a_{i,t} = -\int_0^{N_t} p_{j,t}x_{i,j,t}\mathrm{d}j\mathrm{d}t - u_{i,t}k_{i,t}\left(\delta_k\mathrm{d}t + \sigma_k\mathrm{d}W_t\right) + a_{i,t}(-\delta_a\mathrm{d}t + \sigma_{a,t}\mathrm{d}W_t) - r_{f,t}\widehat{a}_{i,t}\mathrm{d}t - \rho a_{i,t}\mathrm{d}t$$
$$+ [1-(1-\alpha)(1-\varepsilon)]\left[\frac{w_t}{(1-\alpha)(1-\varepsilon)}\right]^{\frac{(1-\alpha)(1-\varepsilon)}{(1-\alpha)(1-\varepsilon)-1}}(z_{i,t}u_{i,t}k_{i,t})^{\frac{\alpha(1-\varepsilon)}{1-(1-\alpha)(1-\varepsilon)}}x_{i,t}^{\frac{\varepsilon}{1-(1-\alpha)(1-\varepsilon)}}\mathrm{d}t. \quad \text{(IA.7)}$$

Taking the first-order condition with respect to $x_{i,j,t}$ on the right-hand side of equation (IA.7), we derive firm i's optimal demand for intermediate good $j \in [0, N_t]$:

$$x_{i,j,t} = \left[\frac{\varepsilon}{p_{j,t}}\left[\frac{w_t}{(1-\alpha)(1-\varepsilon)}\right]^{\frac{(1-\alpha)(1-\varepsilon)}{(1-\alpha)(1-\varepsilon)-1}}(z_{i,t}u_{i,t}k_{i,t})^{\frac{\alpha(1-\varepsilon)}{1-(1-\alpha)(1-\varepsilon)}}x_{i,t}^{1-\nu-\frac{\alpha(1-\varepsilon)}{1-(1-\alpha)(1-\varepsilon)}}\right]^{\frac{1}{1-\nu}}. \quad \text{(IA.8)}$$

Substituting into equation (3) of the main text, we derive $x_{i,t}$:

$$x_{i,t} = \left(\frac{\varepsilon}{p_t}\right)^{\frac{1-(1-\alpha)(1-\varepsilon)}{\alpha(1-\varepsilon)}}\left[\frac{(1-\alpha)(1-\varepsilon)}{w_t}\right]^{\frac{1-\alpha}{\alpha}}z_{i,t}u_{i,t}k_{i,t}, \quad \text{(IA.9)}$$

where the price index p_t is given by

$$p_t = \left(\int_0^{N_t} p_{j,t}^{\frac{\nu}{\nu-1}}\mathrm{d}j\right)^{\frac{\nu-1}{\nu}}. \quad \text{(IA.10)}$$

Substituting equation (IA.9) into (IA.6), we obtain equation (22) of the main text. Substituting equation (IA.9) into (IA.8), we obtain equation (23) of the main text. Substituting equation (IA.9) and $\int_0^{N_t} p_{j,t}x_{i,j,t}\mathrm{d}j = p_t x_{i,t}$ into (IA.7), we obtain

$$\mathrm{d}a_{i,t} = -u_{i,t}k_{i,t}\left(\delta_k\mathrm{d}t + \sigma_k\mathrm{d}W_t\right) + a_{i,t}(-\delta_a\mathrm{d}t + \sigma_{a,t}\mathrm{d}W_t) - r_{f,t}\widehat{a}_{i,t}\mathrm{d}t - \rho a_{i,t}\mathrm{d}t$$
$$+ \alpha(1-\varepsilon)\left(\frac{\varepsilon}{p_t}\right)^{\frac{\varepsilon}{\alpha(1-\varepsilon)}}\left[\frac{(1-\alpha)(1-\varepsilon)}{w_t}\right]^{\frac{1-\alpha}{\alpha}}z_{i,t}u_{i,t}k_{i,t}\mathrm{d}t. \quad \text{(IA.11)}$$

Thus, the manager's problem defined in equation (16) of the main text can be simplified and characterized recursively as follows:

$$0 = \max_{\widehat{a}_{i,t},u_{i,t}} \tau a_{i,t}\mathrm{d}t + \mathbb{E}_t\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t}J_{i,t} + \mathrm{d}J_{i,t} + \frac{\mathrm{d}\Lambda_t}{\Lambda_t}\mathrm{d}J_{i,t}\right]. \quad \text{(IA.12)}$$

subject to the budget constraint (IA.11). Because the technology, budget constraint, and collateral constraint are all linear in $a_{i,t}$, the value $J_{i,t}$ is also linear in $a_{i,t}$ with the following form:

$$J_{i,t} \equiv J_t(a_{i,t}, z_{i,t}) = \xi_t(z_{i,t})a_{i,t}, \quad \text{(IA.13)}$$

where $\xi_{i,t} \equiv \xi_t(z_{i,t})$ captures the marginal value of capital to firm i's manager, which depends on the firm's idiosyncratic productivity $z_{i,t}$ and the aggregate state of the economy. Substituting equation

(18) of the main text and equation (IA.13) into (IA.12), we obtain

$$0 = \max_{\widehat{a}_{i,t}, u_{i,t}} \quad \tau a_{i,t} \mathrm{d}t + \mathbb{E}_t \left[(-r_{f,t} \mathrm{d}t - \eta_t \mathrm{d}W_t) \xi_{i,t} a_{i,t} \right] + \mathbb{E}_t \left[(1 - r_{f,t} \mathrm{d}t - \eta_t \mathrm{d}W_t)(\mathrm{d}\xi_{i,t} a_{i,t} + \xi_{i,t} \mathrm{d}a_{i,t} + \mathrm{d}\xi_{i,t} \mathrm{d}a_{i,t}) \right]. \tag{IA.14}$$

The variable $\xi_{i,t}$ evolves as follows:

$$\mathrm{d}\xi_{i,t} / \xi_{i,t} = \mu_{\xi,i,t} \mathrm{d}t + \sigma_{\xi,i,t} \mathrm{d}W_t + \sigma_{w,i,t} \mathrm{d}W_{i,t}, \tag{IA.15}$$

where $\mu_{\xi,i,t} \equiv \mu_{\xi,t}(z_{i,t})$, $\sigma_{\xi,i,t} \equiv \sigma_{\xi,t}(z_{i,t})$, and $\sigma_{w,i,t} \equiv \sigma_{w,t}(z_{i,t})$ are endogenously determined in equilibrium. Using equations (IA.11) and (IA.15), and the properties that $(\mathrm{d}W_t)^2 = \mathrm{d}t$, $\mathbb{E}_t[\mathrm{d}W_{i,t}] = \mathbb{E}_t[\mathrm{d}W_t] = \mathbb{E}_t[\mathrm{d}W_t \mathrm{d}W_{i,t}] = 0$, we rewrite (IA.14) as follows, after omitting higher-order terms:

$$\begin{aligned} 0 = \max_{\widehat{a}_{i,t}, u_{i,t}} \quad & \tau a_{i,t} \mathrm{d}t - r_{f,t} a_{i,t} \xi_{i,t} \mathrm{d}t + \mu_{\xi,i,t} a_{i,t} \xi_{i,t} \mathrm{d}t - \eta_t \sigma_{\xi,i,t} a_{i,t} \xi_{i,t} \mathrm{d}t \\ & + [\sigma_k(\eta_t - \sigma_{\xi,i,t}) - \delta_k] u_{i,t} k_{i,t} \xi_{i,t} \mathrm{d}t - \sigma_{a,t}(\eta_t - \sigma_{\xi,i,t}) \xi_{i,t} a_{i,t} \mathrm{d}t - r_{f,t} \widehat{a}_{i,t} \xi_{i,t} \mathrm{d}t - (\rho + \delta_a) a_{i,t} \xi_{i,t} \mathrm{d}t \\ & + \alpha(1-\varepsilon) \left(\frac{\varepsilon}{p_t} \right)^{\frac{\varepsilon}{\alpha(1-\varepsilon)}} \left[\frac{(1-\alpha)(1-\varepsilon)}{w_t} \right]^{\frac{1-\alpha}{\alpha}} z_{i,t} u_{i,t} k_{i,t} \xi_{i,t} \mathrm{d}t. \end{aligned} \tag{IA.16}$$

Using $k_{i,t} = a_{i,t} + \widehat{a}_{i,t}$, we can see that maximizing (IA.16) is essentially the same as maximizing

$$\begin{aligned} 0 = \max_{\widehat{a}_{i,t}, u_{i,t}} \quad & [\sigma_k(\eta_t - \sigma_{\xi,i,t}) - \delta_k] u_{i,t} \widehat{a}_{i,t} \xi_{i,t} \mathrm{d}t - r_{f,t} \widehat{a}_{i,t} \xi_{i,t} \mathrm{d}t \\ & + \alpha(1-\varepsilon) \left(\frac{\varepsilon}{p_t} \right)^{\frac{\varepsilon}{\alpha(1-\varepsilon)}} \left[\frac{(1-\alpha)(1-\varepsilon)}{w_t} \right]^{\frac{1-\alpha}{\alpha}} z_{i,t} u_{i,t} \widehat{a}_{i,t} \xi_{i,t} \mathrm{d}t. \end{aligned} \tag{IA.17}$$

Because a positive shock ($\mathrm{d}W_t > 0$) increases misallocation through higher capital depreciation of productive firms, we have $\eta_t < 0$ in equilibrium. Moreover, because $\xi_{i,t}$ is not affected by the manager's choice of $\widehat{a}_{i,t}$. The objective function (IA.17) is linear in both $\widehat{a}_{i,t}$ and $u_{i,t}$. Thus, conditional on $u_{i,t} = 1$, we can characterize the productivity cutoff $\underline{z}_t$ that makes the manager indifferent about leasing capital as follows:

$$\underline{z}_t \kappa_t = r_{f,t} + \delta_k + \sigma_k(\sigma_{\xi,t}(\underline{z}_t) - \eta_t), \tag{IA.18}$$

where

$$\kappa_t = \alpha(1-\varepsilon) \left(\frac{\varepsilon}{p_t} \right)^{\frac{\varepsilon}{\alpha(1-\varepsilon)}} \left[\frac{(1-\alpha)(1-\varepsilon)}{w_t} \right]^{\frac{1-\alpha}{\alpha}}. \tag{IA.19}$$

Because $r_{f,t} > 0$ in equilibrium, it is clear that firms will optimally choose $u_{i,t} = 1$ for $z_{i,t} \geq \underline{z}_t$.[4] The optimal decisions follow bang-bang solutions:

$$\widehat{a}_t(a,z) = \begin{cases} \lambda a, & z \geq \underline{z}_t \\ -a & z < \underline{z}_t \end{cases}, \quad k_t(a,z) = \begin{cases} (1+\lambda)a, & z \geq \underline{z}_t \\ 0 & z < \underline{z}_t \end{cases}, \quad u_t(z) = \begin{cases} 1, & z \geq \underline{z}_t \\ 0 & z < \underline{z}_t \end{cases}. \tag{IA.20}$$

[4] In other words, the (latent) cutoff productivity for $u_{i,t}$ is lower than the cutoff productivity $\underline{z}_t$ for $\widehat{a}_{i,t}$ when $r_{f,t} > 0$.

C. *Proof of Proposition 2*

Define the productivity H_t of the final goods sector as

$$H_t = \left[\frac{1}{K_t}\int_{\underline{z}_t}^{\infty}\int_0^{\infty} z u_t(z) k_t(a,z)\varphi_t(a,z)\mathrm{d}a\mathrm{d}z\right]^{\alpha}. \tag{IA.21}$$

Using (21), (27), and (32) of the main text and $k_t(a,z) = a + \widehat{a}_t(a,z)$, H_t can be written as

$$H_t = \left[(1+\lambda)\frac{A_t}{K_t}\int_{\underline{z}_t}^{\infty} z\omega_t(z)\mathrm{d}z\right]^{\alpha}. \tag{IA.22}$$

Substituting (IA.20) into the capital market-clearing condition (31) of the main text, we obtain

$$(1+\lambda)\int_{\underline{z}_t}^{\infty}\int_0^{\infty} a\varphi_t(a,z)\mathrm{d}a\mathrm{d}z = K_t. \tag{IA.23}$$

Given the definition of capital share in (32) of the main text, the left-hand side of equation (IA.23) can be simplified as

$$(1+\lambda)\int_{\underline{z}_t}^{\infty}\int_0^{\infty} a\varphi_t(a,z)\mathrm{d}a\mathrm{d}z = (1+\lambda)A_t\int_{\underline{z}_t}^{\infty}\omega_t(z)\mathrm{d}z = (1+\lambda)A_t(1-\Omega_t(\underline{z}_t)). \tag{IA.24}$$

Thus, we have the following equation

$$(1+\lambda)[1-\Omega_t(\underline{z}_t)] = K_t/A_t, \tag{IA.25}$$

which determines the equilibrium K_t/A_t. Substituting equation (IA.25) into (IA.22), we obtain

$$H_t = \left[\frac{\int_{\underline{z}_t}^{\infty} z\omega_t(z)\mathrm{d}z}{1-\Omega_t(\underline{z}_t)}\right]^{\alpha}. \tag{IA.26}$$

Substituting equation (6) of the main text into equation (5) of the main text, we obtain

$$\pi_{j,t} = \max_{p_{j,t}} \ (p_{j,t}-1)\left(p_{j,t}/p_t\right)^{\frac{1}{\nu-1}} X_t. \tag{IA.27}$$

Taking the first-order condition, we obtain

$$p_{j,t} = 1/\nu \ \text{ for all } j. \tag{IA.28}$$

Substituting equation (IA.28) into (IA.10), we obtain

$$p_t = N_t^{\frac{\nu-1}{\nu}} p_{j,t} = N_t^{\frac{\nu-1}{\nu}}/\nu. \tag{IA.29}$$

Substituting equation (22) of the main text into (29) of the main text and using (IA.21), we obtain

$$L_t = \left(\frac{\varepsilon}{p_t}\right)^{\frac{\varepsilon}{\alpha(1-\varepsilon)}}\left[\frac{(1-\alpha)(1-\varepsilon)}{w_t}\right]^{\frac{1}{\alpha}} H_t^{\frac{1}{\alpha}} K_t. \tag{IA.30}$$

Substituting equation (IA.29) into (IA.30), we derive the equilibrium wage w_t:

$$w_t = (1-\alpha)(1-\varepsilon)(\varepsilon\nu)^{\frac{\varepsilon}{1-\varepsilon}} N_t^{\frac{(1-\nu)\varepsilon}{\nu(1-\varepsilon)}} H_t (K_t/L_t)^{\alpha}. \tag{IA.31}$$

By definition, the aggregate output Y_t is

$$Y_t = \int_{\underline{z}_t}^{\infty} \int_0^{\infty} \left[(z u_t(z) k_t(a,z))^{\alpha} (\ell_t(a,z))^{1-\alpha} \right]^{1-\varepsilon} x_t(a,z)^{\varepsilon} \varphi_t(a,z) \mathrm{d}a\mathrm{d}z. \tag{IA.32}$$

Substituting equation (22) of the main text and (IA.9) into (IA.32), we obtain

$$Y_t = \left(\frac{\varepsilon}{p_t}\right)^{\frac{\varepsilon}{\alpha(1-\varepsilon)}} \left[\frac{(1-\alpha)(1-\varepsilon)}{w_t}\right]^{\frac{1-\alpha}{\alpha}} \int_{\underline{z}_t}^{\infty} \int_0^{\infty} z u_t(z) k_t(a,z) \varphi_t(a,z) \mathrm{d}a\mathrm{d}z. \tag{IA.33}$$

Further, substituting equations (IA.21), (IA.29) and (IA.31) into the above equation, we obtain

$$Y_t = (\varepsilon\nu)^{\frac{\varepsilon}{1-\varepsilon}} H_t N_t^{\frac{(1-\nu)\varepsilon}{\nu(1-\varepsilon)}} K_t^{\alpha} L_t^{1-\alpha} = (\varepsilon\nu)^{\frac{\varepsilon}{1-\varepsilon}} H_t N_t^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha}. \tag{IA.34}$$

Using equation (IA.34), the equilibrium wage w_t in (IA.31) can be simplified as

$$w_t = (1-\alpha)(1-\varepsilon)\frac{Y_t}{L_t}. \tag{IA.35}$$

Equation (IA.19) can be simplified by substituting out w_t and p_t using (IA.29) and (IA.35):

$$\kappa_t = \alpha(1-\varepsilon)(\varepsilon\nu)^{\frac{\varepsilon}{1-\varepsilon}} H_t^{\frac{\alpha-1}{\alpha}} N_t^{\frac{(1-\nu)\varepsilon}{\nu(1-\varepsilon)}} K_t^{\alpha-1} L_t^{1-\alpha} = \alpha(1-\varepsilon) H_t^{-\frac{1}{\alpha}} \frac{Y_t}{K_t}. \tag{IA.36}$$

Substituting equations (IA.9), (IA.28) and (IA.29) into (IA.27) and using (IA.21), we obtain

$$\pi_t = \frac{1-\nu}{\nu} (\varepsilon\nu)^{\frac{1-(1-\alpha)(1-\varepsilon)}{\alpha(1-\varepsilon)}} H_t^{\frac{1}{\alpha}} N_t^{\frac{1-\nu}{\nu}\left[\frac{1-(1-\alpha)(1-\varepsilon)}{\alpha(1-\varepsilon)} - \frac{1}{1-\nu}\right]} \left[\frac{(1-\alpha)(1-\varepsilon)}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_t. \tag{IA.37}$$

Further, substituting equation (IA.31) into the above equation and using (IA.34), we obtain

$$\pi_t = \frac{1-\nu}{\nu} (\varepsilon q)^{\frac{1}{1-\varepsilon}} H_t N_t^{\frac{\varepsilon-\nu}{\nu(1-\varepsilon)}} K_t^{\alpha} L_t^{1-\alpha} = (1-\nu)\varepsilon \frac{Y_t}{N_t}. \tag{IA.38}$$

Thus, we have

$$\int_{j=0}^{N_t} \pi_t \mathrm{d}j = N_t \pi_t = (1-\nu)\varepsilon Y_t. \tag{IA.39}$$

Substituting equation (9) of the main text into $\vartheta_t = \chi (N_t/S_t)^h$, we obtain

$$S_t = (\chi q_t)^{\frac{1}{h}} N_t. \tag{IA.40}$$

D. *Proof of Proposition 3*

Let $\psi_t(\widetilde{a},\widetilde{z})$ be the joint distribution of $\widetilde{a}$ and $\widetilde{z}$. Define $\Gamma_t \equiv \text{Cov}(\widetilde{a}_{i,t},\widetilde{z}_{i,t})$. Under our approximation, as discussed in Section 3.1 of the main text, $\widetilde{a}$ and $\widetilde{z}$ follow a joint normal distribution. Thus, $\psi_t(\widetilde{a},\widetilde{z})$ is the PDF of a joint normal distribution, with the covariance between $\widetilde{a}$ and $\widetilde{z}$ being Γ_t.

The PDF $\varphi_t(a,z)$ is related to $\psi_t(\widetilde{a},\widetilde{z})$ through the Jacobian matrix, as follows:

$$\varphi_t(a,z) = \left| \begin{array}{cc} \partial\widetilde{a}/\partial a & \partial\widetilde{a}/\partial z \\ \partial\widetilde{z}/\partial a & \partial\widetilde{z}/\partial z \end{array} \right| \psi_t(\widetilde{a},\widetilde{z}) = \frac{1}{az}\psi_t(\widetilde{a},\widetilde{z}). \tag{IA.41}$$

Let $f(\widetilde{z})$ be the PDF of $\widetilde{z}$, which follows a normal distribution, $N(0,\sigma^2/2)$, in the stationary equilibrium. Using equation (IA.41), we have

$$\int_0^\infty a\varphi_t(a,z)\mathrm{d}a = \int_{-\infty}^\infty \frac{a}{z}\psi_t(\widetilde{a},\widetilde{z})\mathrm{d}\widetilde{a} = \int_{-\infty}^\infty \frac{a}{z}\psi_t(\widetilde{a}|\widetilde{z})f(\widetilde{z})\mathrm{d}\widetilde{a} = \mathbb{E}\left[\exp(\widetilde{a}_{i,t})|\widetilde{z}\right]\frac{f(\widetilde{z})}{z}. \tag{IA.42}$$

Using equation (IA.41), the variable A_t defined in (27) of the main text can be written as

$$A_t = \int_{-\infty}^\infty \int_{-\infty}^\infty a\frac{1}{az}\psi_t(\widetilde{a},\widetilde{z})az\mathrm{d}\widetilde{a}\mathrm{d}\widetilde{z} = \mathbb{E}\left[\exp(\widetilde{a}_{i,t})\right]. \tag{IA.43}$$

Substituting equations (IA.42) and (IA.43) into equation (32) of the main text, we obtain

$$\omega_t(z) = \frac{\mathbb{E}[\exp(\widetilde{a}_{i,t})|\widetilde{z}]}{\mathbb{E}[\exp(\widetilde{a}_{i,t})]}\frac{f(\widetilde{z})}{z}. \tag{IA.44}$$

Because $\widetilde{a}_{i,t}$ and $\widetilde{z}_{i,t}$ follow a joint normal distribution with covariance Γ_t, we have

$$\mathbb{E}[\exp(\widetilde{a}_{i,t})|\widetilde{z}] = \exp\left(\mathbb{E}[\widetilde{a}_{i,t}|\widetilde{z}] + \frac{1}{2}\text{var}(\widetilde{a}_{i,t}|\widetilde{z})\right) \text{ and } \mathbb{E}[\exp(\widetilde{a}_{i,t})] = \exp\left(\mathbb{E}[\widetilde{a}_{i,t}] + \frac{1}{2}\text{var}(\widetilde{a}_{i,t})\right), \tag{IA.45}$$

where

$$\mathbb{E}[\widetilde{a}_{i,t}|\widetilde{z}] = \mathbb{E}[\widetilde{a}_{i,t}] + 2\widetilde{z}\Gamma_t/\sigma^2 \text{ and } \text{var}(\widetilde{a}_{i,t}|\widetilde{z}) = \text{var}(\widetilde{a}_{i,t}) - 2\Gamma_t^2/\sigma^2. \tag{IA.46}$$

Substituting equations (IA.45) and (IA.46) into (IA.44), we obtain

$$\omega_t(z) = \frac{f(\widetilde{z})}{z}\exp\left(\frac{2\widetilde{z}\Gamma_t}{\sigma^2}\right)\exp\left(-\frac{\Gamma_t^2}{\sigma^2}\right) = \frac{1}{z\sigma\sqrt{\pi}}\exp\left(-\frac{(\ln z - \Gamma_t)^2}{\sigma^2}\right). \tag{IA.47}$$

This formula turns out to be the same as equation (29) of Moll (2014). Substituting out $\Gamma_t = -M_t\text{var}(\widetilde{z}_{i,t}) = -\sigma^2 M_t/2$, we obtain equation (38) in the main text.

E. *Proof of Proposition 4*

Substituting equation (38) of the main text into (IA.25), we obtain the equation that determines the productivity cutoff $\underline{z}_t$ under our approximation of $\omega_t(z)$:

$$\frac{1}{1+\lambda}\frac{K_t}{A_t} = 1 - \Omega_t(\underline{z}_t) = \int_{\widetilde{\underline{z}}_t}^\infty \frac{1}{\sigma\sqrt{\pi}}\exp\left(-\frac{(\widetilde{z}+\sigma^2 M_t/2)^2}{\sigma^2}\right)\mathrm{d}\widetilde{z} = \Phi\left(\frac{-\sigma^2 M_t/2 - \widetilde{\underline{z}}_t}{\sigma/\sqrt{2}}\right). \tag{IA.48}$$

Rearranging the above equation, we obtain $\underline{z}_t$:

$$\underline{z}_t = \exp\left[-\frac{\sigma^2}{2}M_t - \Phi^{-1}\left(\frac{1}{1+\lambda}\frac{K_t}{A_t}\right)\frac{\sigma}{\sqrt{2}}\right]. \tag{IA.49}$$

The term $\int_{\underline{z}_t}^{\infty} z\omega_t(z)\mathrm{d}z$ can be simplified using (38) of the main text, as follows

$$\int_{\underline{z}_t}^{\infty} z\omega_t(z)\mathrm{d}z = \exp\left(-\frac{\sigma^2}{2}M_t + \frac{\sigma^2}{4}\right)\Phi\left(\frac{-\sigma^2 M_t/2 + \sigma^2/2 - \widetilde{\underline{z}}_t}{\sigma/\sqrt{2}}\right). \tag{IA.50}$$

Substituting (IA.49) and (IA.50) into (35) of the main text, we obtain

$$H_t = \left[(1+\lambda)\frac{A_t}{K_t}\exp\left(-\frac{\sigma^2}{2}M_t + \frac{\sigma^2}{4}\right)\Phi\left(\Phi^{-1}\left(\frac{1}{1+\lambda}\frac{K_t}{A_t}\right) + \frac{\sigma}{\sqrt{2}}\right)\right]^{\alpha}. \tag{IA.51}$$

Substituting into equation (35) of the main text, we obtain equation (39) of the main text,

$$Z_t = (\varepsilon\nu)^{\frac{\varepsilon}{1-\varepsilon}} N_t^{1-\alpha}\left[(1+\lambda)\frac{A_t}{K_t}\exp\left(-\frac{\sigma^2}{2}M_t + \frac{\sigma^2}{4}\right)\Phi\left(\Phi^{-1}\left(\frac{1}{1+\lambda}\frac{K_t}{A_t}\right) + \frac{\sigma}{\sqrt{2}}\right)\right]^{\alpha}. \tag{IA.52}$$

F. *Proof of Proposition 5*

Equation (27) of the main text implies

$$A_{t+\mathrm{d}t} - A_t = \int_0^{\infty}\int_0^{\infty} \mathrm{d}a_t(a,z)\varphi_t(a,z)\mathrm{d}a\mathrm{d}z. \tag{IA.53}$$

Substituting equations (2) and (25) of the main text into the above equation, we obtain

$$\begin{aligned} A_{t+\mathrm{d}t} - A_t = {} & (1+\lambda)\kappa_t\int_{\underline{z}_t}^{\infty}\int_0^{\infty} za\mathrm{d}t\varphi_t(a,z)\mathrm{d}a\mathrm{d}z - (1+\lambda)r_{f,t}\int_{\underline{z}_t}^{\infty}\int_0^{\infty} a\mathrm{d}t\varphi_t(a,z)\mathrm{d}a\mathrm{d}z \\ & - (1+\lambda)\left(\delta_k\mathrm{d}t + \sigma_k\mathrm{d}W_t\right)\int_{\underline{z}_t}^{\infty}\int_0^{\infty} a\varphi_t(a,z)\mathrm{d}a\mathrm{d}z + \sigma_{a,t}A_t\mathrm{d}W_t + (r_{f,t} - \rho - \delta_a)A_t\mathrm{d}t. \end{aligned} \tag{IA.54}$$

Using equations (IA.21), (IA.23), and (IA.36), the above equation can be simplified as

$$\mathrm{d}A_t = \left[\alpha(1-\varepsilon)Y_t - \delta_k K_t - \delta_a A_t - r_{f,t}B_t - \rho A_t\right]\mathrm{d}t - (\sigma_k K_t - \sigma_{a,t}A_t)\mathrm{d}W_t. \tag{IA.55}$$

Substituting $\vartheta_t = \chi\left(N_t/S_t\right)^h$ and (37) of the main text into (8) of the main text, we obtain

$$\frac{\mathrm{d}N_t}{N_t} = \chi\left(\chi q_t\right)^{\frac{1-h}{h}}\mathrm{d}t - \delta_b\mathrm{d}t. \tag{IA.56}$$

Define $\Gamma_t \equiv \mathrm{Cov}(\widetilde{a}_{i,t}, \widetilde{z}_{i,t}) = -M_t\mathrm{var}(\widetilde{z}_{i,t}) = -\sigma^2 M_t/2$. Next, we derive the evolution of Γ_t. By definition, $\Gamma_{t+\mathrm{d}t} \equiv \mathrm{Cov}(\widetilde{a}_{i,t+\mathrm{d}t}, \widetilde{z}_{i,t+\mathrm{d}t})$. According to equation (4) of the main text, we have

$$\widetilde{z}_{i,t+\mathrm{d}t} = \widetilde{z}_{i,t} - \theta\widetilde{z}_{i,t}\mathrm{d}t + \sigma\sqrt{\theta}\mathrm{d}W_{i,t}. \tag{IA.57}$$

Thus,

$$\begin{aligned}\mathrm{d}\Gamma_t =&\mathrm{Cov}\left(\widetilde{a}_{i,t}+\mathrm{d}\widetilde{a}_{i,t},\widetilde{z}_{i,t}-\theta\widetilde{z}_{i,t}\mathrm{d}t+\sigma\sqrt{\theta}\mathrm{d}W_{i,t}\right)-\Gamma_t\\ =&(1-\theta\mathrm{d}t)\mathrm{Cov}(\widetilde{a}_{i,t}+\mathrm{d}\widetilde{a}_{i,t},\widetilde{z}_{i,t})-\Gamma_t=-\theta\Gamma_t\mathrm{d}t+(1-\theta\mathrm{d}t)\mathrm{Cov}(\widetilde{z}_{i,t},\mathrm{d}\widetilde{a}_{i,t}).\end{aligned}\tag{IA.58}$$

Omitting the higher-order term $\mathrm{Cov}(\widetilde{z}_{i,t},\mathrm{d}\widetilde{a}_{i,t})\mathrm{d}t$, we obtain

$$\mathrm{d}\Gamma_t=-\theta\Gamma_t\mathrm{d}t+\mathrm{Cov}(\widetilde{z}_{i,t},\mathrm{d}\widetilde{a}_{i,t}).\tag{IA.59}$$

Substituting out $\Gamma_t=-\sigma^2M_t/2$, we obtain equation (44) of the main text.

We now derive the expression for $\mathrm{Cov}(\widetilde{z}_{i,t},\mathrm{d}\widetilde{a}_{i,t})$ under the approximation that $\widetilde{a}_{i,t}$ follows a normal distribution. Using Ito's lemma, we have

$$\mathrm{d}\widetilde{a}_{i,t}=1/a_{i,t}\mathrm{d}a_{i,t}-1/(2a_{i,t}^2)(\mathrm{d}a_{i,t})^2.\tag{IA.60}$$

Substituting equations (2) and (25) of the main text into the above equation, we obtain the evolution of $\widetilde{a}_{i,t}$. In particular, for $z_{i,t}<\underline{z}_t$, we have

$$\mathrm{d}\widetilde{a}_{i,t}=\left(r_{f,t}-\rho-\delta_a-1/2\sigma_{a,t}^2\right)\mathrm{d}t+\sigma_{a,t}\mathrm{d}W_t.\tag{IA.61}$$

For $z_{i,t}\geq\underline{z}_t$, we have

$$\begin{aligned}\mathrm{d}\widetilde{a}_{i,t}=&(1+\lambda)\left[\kappa_t z_{i,t}\mathrm{d}t-(\delta_k\mathrm{d}t+\sigma_k\mathrm{d}W_t)-r_{f,t}\mathrm{d}t\right]+\left(r_{f,t}-\rho-\delta_a-1/2\sigma_{a,t}^2\right)\mathrm{d}t\\ &-1/2(1+\lambda)^2\sigma_k^2\mathrm{d}t+(1+\lambda)\sigma_k\sigma_{a,t}\mathrm{d}t+\sigma_{a,t}\mathrm{d}W_t.\end{aligned}\tag{IA.62}$$

Because $\mathbb{E}[\widetilde{z}_{i,t}]=0$, we have

$$\mathrm{Cov}(\widetilde{z}_{i,t},\mathrm{d}\widetilde{a}_{i,t})=\mathbb{E}[\widetilde{z}_{i,t}\mathrm{d}\widetilde{a}_{i,t}].\tag{IA.63}$$

Substituting equations (IA.61) and (IA.62) into (IA.63), we obtain

$$\begin{aligned}\mathrm{Cov}(\widetilde{z}_{i,t},\mathrm{d}\widetilde{a}_{i,t})=&(1+\lambda)\kappa_t\mathrm{d}t\int_{\widetilde{\underline{z}}_t}^{\infty}\widetilde{z}zf(\widetilde{z})\mathrm{d}\widetilde{z}\\ &-(1+\lambda)\left[\left(r_{f,t}+\delta_k+\frac{1}{2}(1+\lambda)\sigma_k^2-\sigma_k\sigma_{a,t}\right)\mathrm{d}t+\sigma_k\mathrm{d}W_t\right]\int_{\widetilde{\underline{z}}_t}^{\infty}\widetilde{z}f(\widetilde{z})\mathrm{d}\widetilde{z},\end{aligned}\tag{IA.64}$$

where $f(\widetilde{z})$ is the PDF of $\widetilde{z}$, which follows $N(0,\sigma^2/2)$ in the stationary equilibrium.

Substituting out $f(\widetilde{z})$, the term $\int_{\widetilde{\underline{z}}_t}^{\infty}\widetilde{z}f(\widetilde{z})\mathrm{d}\widetilde{z}$ in equation (IA.64) can be simplified as follows:

$$\int_{\widetilde{\underline{z}}_t}^{\infty}\widetilde{z}f(\widetilde{z})\mathrm{d}\widetilde{z}=\int_{\widetilde{\underline{z}}_t}^{\infty}\widetilde{z}\frac{1}{\sigma\sqrt{\pi}}\exp\left(-\frac{\widetilde{z}^2}{\sigma^2}\right)\mathrm{d}\widetilde{z}=-\frac{\sigma}{2\sqrt{\pi}}\int_{\widetilde{\underline{z}}_t}^{\infty}\mathrm{d}\exp\left(-\frac{\widetilde{z}^2}{\sigma^2}\right)=\frac{\sigma}{2\sqrt{\pi}}\exp\left(-\frac{\widetilde{\underline{z}}_t^2}{\sigma^2}\right).\tag{IA.65}$$

The term $\int_{\widetilde{z}_t}^{\infty} \widetilde{z} z f(\widetilde{z}) \mathrm{d}\widetilde{z}$ in equation (IA.64) can be simplified as follows:

$$\begin{aligned}
&\int_{\widetilde{z}_t}^{\infty} \widetilde{z} z f(\widetilde{z}) \mathrm{d}\widetilde{z} = \frac{1}{\sigma\sqrt{\pi}} \int_{\widetilde{z}_t}^{\infty} \widetilde{z} \exp\left(\widetilde{z} - \frac{\widetilde{z}^2}{\sigma^2}\right) \mathrm{d}\widetilde{z} = \frac{1}{\sigma\sqrt{\pi}} \exp\left(\frac{\sigma^2}{4}\right) \int_{\widetilde{z}_t}^{\infty} \widetilde{z} \exp\left(-\frac{1}{\sigma^2}\left(\widetilde{z} - \frac{\sigma^2}{2}\right)^2\right) \mathrm{d}\widetilde{z} \\
&= \frac{1}{\sigma\sqrt{\pi}} \exp\left(\frac{\sigma^2}{4}\right) \left[\int_{\widetilde{z}_t}^{\infty} \left(\widetilde{z} - \frac{\sigma^2}{2}\right) \exp\left(-\frac{1}{\sigma^2}\left(\widetilde{z} - \frac{\sigma^2}{2}\right)^2\right) \mathrm{d}\widetilde{z} + \int_{\widetilde{z}_t}^{\infty} \frac{\sigma^2}{2} \exp\left(-\frac{1}{\sigma^2}\left(\widetilde{z} - \frac{\sigma^2}{2}\right)^2\right) \mathrm{d}\widetilde{z}\right] \\
&= \frac{\sigma}{2} \exp\left(\frac{\sigma^2}{4}\right) \left[\frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{\sigma^2}\left(\widetilde{z}_t - \frac{\sigma^2}{2}\right)^2\right) + \sigma\Phi\left(\frac{\sigma^2/2 - \widetilde{z}_t}{\sigma/\sqrt{2}}\right)\right]. \qquad \text{(IA.66)}
\end{aligned}$$

Substituting equations (IA.65) and (IA.66) into (IA.64), we obtain

$$\begin{aligned}
\mathrm{Cov}(\widetilde{z}_{i,t}, \mathrm{d}\widetilde{a}_{i,t}) =& \frac{(1+\lambda)\sigma^2\kappa_t}{2} \exp\left(\frac{\sigma^2}{4}\right) \Phi\left(\frac{\sigma^2/2 - \widetilde{z}_t}{\sigma/\sqrt{2}}\right) \mathrm{d}t \qquad \text{(IA.67)} \\
&+ \frac{(1+\lambda)\sigma}{2\sqrt{\pi}} [(\underline{z}_t\kappa_t - r_{f,t} - \delta_k - 0.5(1+\lambda)\sigma_k^2 + \sigma_k\sigma_{a,t})\mathrm{d}t - \sigma_k \mathrm{d}W_t] \exp\left(-\frac{\widetilde{z}_t^2}{\sigma^2}\right).
\end{aligned}$$

G. *Proof of Proposition 6*

In the absence of aggregate shocks, the evolution of aggregate capital A_t (equation (42) of the main text) becomes

$$\frac{\mathrm{d}A_t}{A_t} = \alpha(1-\varepsilon)\frac{Y_t}{A_t}\mathrm{d}t - (r_{f,t} + \delta_k)\frac{K_t}{A_t}\mathrm{d}t - (\rho + \delta_a - r_{f,t})\mathrm{d}t. \qquad \text{(IA.68)}$$

In the deterministic balanced growth path, aggregate output Y_t, consumption C_t, capital A_t, and knowledge stock N_t all grow at a constant rate g:

$$\frac{\mathrm{d}Y_t}{Y_t} = \frac{\mathrm{d}C_t}{C_t} = \frac{\mathrm{d}A_t}{A_t} = \frac{\mathrm{d}N_t}{N_t} = g\mathrm{d}t. \qquad \text{(IA.69)}$$

The variables E_t, K_t/A_t, Y_t/A_t, M_t, H_t, $\underline{z}_t$, κ_t, $r_{f,t}$, and q_t are all constant. Thus, we omit the subscript t and use the notations E, $\overline{K/A}$, $\overline{Y/A}$, M, H, $\underline{z}$, κ, r_f, and q to represent these variables in the deterministic balanced growth path. The risk-free rate r_f is determined by the representative agent's first-order condition

$$\frac{\mathrm{d}C_t}{C_t} = \psi(r_f - \delta)\mathrm{d}t. \qquad \text{(IA.70)}$$

Substituting equation (IA.69) into (IA.70), (43) of the main text, and (IA.68), respectively, we obtain

$$r_f = g/\psi + \delta, \qquad \text{(IA.71)}$$

$$g = \chi(\chi q)^{\frac{1-h}{h}} - \delta_b, \qquad \text{(IA.72)}$$

$$\overline{Y/A} = \frac{g + \rho + \delta_a - r_f + (r_f + \delta_k)\overline{K/A}}{\alpha(1-\varepsilon)}. \qquad \text{(IA.73)}$$

Dividing both sides of equation (34) of the main text by A_t and using $L_t \equiv 1$, we obtain

$$E = \left[\frac{1}{(\varepsilon\nu)^{\frac{\varepsilon}{1-\varepsilon}} H} \frac{\overline{Y/A}}{\overline{K/A}^{\alpha}} \right]^{\frac{1}{1-\alpha}}. \tag{IA.74}$$

The flow profit π to each intermediate-goods producer is a constant and given by equation (37) of the main text,

$$\pi = \frac{(1-\nu)\varepsilon\overline{Y/A}}{E}. \tag{IA.75}$$

Substituting equation (18) of the main text into equation (7) of the main text, we obtain the value of blueprint q

$$q = \frac{\pi}{r_f + \delta_b}. \tag{IA.76}$$

Substituting equations (IA.71), (IA.74), and (IA.75) into (IA.76), we obtain q

$$q = \frac{(1-\nu)\varepsilon(\varepsilon\nu)^{\frac{\varepsilon}{(1-\varepsilon)(1-\alpha)}}}{g/\psi + \delta + \delta_b} \left(\frac{\overline{K/A}}{\overline{Y/A}} \right)^{\frac{\alpha}{1-\alpha}} H^{\frac{1}{1-\alpha}}. \tag{IA.77}$$

Substituting equation (39) of the main text into (35) of the main text, we obtain H as follows:

$$H = \left[\frac{1+\lambda}{\overline{K/A}} \exp\left(-\frac{\sigma^2}{2}M + \frac{\sigma^2}{4} \right) \Phi\left(\Phi^{-1}\left(\frac{\overline{K/A}}{1+\lambda} \right) + \frac{\sigma}{\sqrt{2}} \right) \right]^{\alpha}, \tag{IA.78}$$

where the misallocation M in the deterministic balanced growth path is obtained by setting $\mathrm{d}M_t = 0$ in equation (44) of the main text:

$$M = -\frac{2\mathrm{Cov}(\widetilde{z}_{i,t}, \mathrm{d}\widetilde{a}_{i,t})}{\theta\sigma^2 \mathrm{d}t}. \tag{IA.79}$$

Substituting (IA.67) into (IA.79) and using $\sigma_k = 0$ in the deterministic balanced growth path, we obtain:

$$M = -\frac{(1+\lambda)\kappa}{\theta} \exp\left(\frac{\sigma^2}{4} \right) \Phi\left(\frac{\sigma^2/2 - \ln \underline{z}}{\sigma/\sqrt{2}} \right) - \frac{1+\lambda}{\theta\sigma\sqrt{\pi}} (\underline{z}\kappa - r_f - \delta_k) \exp\left(-\frac{\widetilde{z}^2}{\sigma^2} \right). \tag{IA.80}$$

When solving above equations, we need to know $\overline{K/A}$, $\underline{z}$, and κ. They are given by equation (IA.49), (24) of the main text (setting $\sigma_k = 0$), and (IA.36), respectively, as follows

$$\underline{z} = \exp\left(-\frac{\sigma^2}{2}M - \Phi^{-1}\left(\frac{\overline{K/A}}{1+\lambda} \right) \frac{\sigma}{\sqrt{2}} \right), \tag{IA.81}$$

$$\underline{z}\kappa = r_f + \delta_k, \tag{IA.82}$$

$$\kappa = \alpha(1-\varepsilon) H^{-\frac{1}{\alpha}} \frac{\overline{Y/A}}{\overline{K/A}}. \tag{IA.83}$$

H. *Proof of Proposition 7*

Substituting equation (IA.75) into (IA.76), we obtain

$$q = \frac{(1-\nu)\varepsilon}{r_f + \delta_b}\overline{Y/N}. \tag{IA.84}$$

Substituting equations (34) and (39) of the main text into (IA.84), and using $L_t \equiv 1$, we obtain

$$\begin{aligned}\ln q = &-\frac{\alpha\sigma^2}{2}M + \frac{\alpha\sigma^2}{4} + \ln\left(\frac{(1-\nu)\varepsilon(\varepsilon\nu)^{\frac{\varepsilon}{1-\varepsilon}}}{r_f+\delta_b}\right) + \alpha\ln\left(\Phi\left(\Phi^{-1}\left(\frac{\overline{K/A}}{1+\lambda}\right) + \frac{\sigma}{\sqrt{2}}\right)\right) \\ &+ \alpha\ln(1+\lambda) - \alpha\ln E.\end{aligned} \tag{IA.85}$$

III. Assessment of Our Parametric Approximation Method

We evaluate the accuracy of our parametric approximation method. In Subsection III.A, we provide a heuristic discussion that the actual distribution of $\widetilde{a}_{i,t} \equiv \ln a_{i,t}$ is approximately a normal distribution. In Subsection III.B, we review the existing numerical approximation methods and point out their differences from our parametric approximation method in the end. In Subsection III.C, we exactly solve the model in the absence of aggregate shocks and show that the implied variables and distributions in the deterministic balanced growth path are similar to the solutions obtained using our parametric approximation method.

In the presence of aggregate shocks, it is difficult to accurately solve the model using existing numerical methods. This is because agents need to base their optimal decisions today on the evolution of future prices that are consistent with the economy's law of motion. Because of the heterogeneity across agents, future prices will, through market-clearing conditions, depend on the future distribution of agents, which is infinite dimensional. Thus, the cross-sectional distribution of agents becomes part of the aggregate state space, substantially increasing the computational complexity. In principle, it is impossible to "exactly solve" dynamic stochastic general equilibrium models with heterogeneous agents that simultaneously have four features: heterogeneous agents, aggregate shocks, endogenous equilibrium prices, and persistent state variables.[5] Despite the difficulties, in Subsection III.D, we take an off-the-shelf higher-degree numerical approximation method to solve the model. We show that this method yields results similar to those obtained based on our parametric approximation method.

[5]It is the combination of all four features that makes it impossible to solve the model exactly. For example, without heterogeneous agents, the aggregate state space has a finite dimension. The model can be solved using standard iteration or projection methods based on discretized grids of the aggregate state space. Moreover, without aggregate shocks, there exists a deterministic steady state with stationary cross-sectional distributions, which can be solved using a standard shooting algorithm (e.g., Buera and Shin, 2013; Dabla-Norris et al., 2021; Dou and Ji, 2021; Ji, Teng and Townsend, 2023). Finally, without endogenous equilibrium prices (i.e., a model in partial equilibrium), agents make decisions based on the exogenously specified evolution of aggregate prices, and the future cross-sectional distribution is irrelevant from agents' perspective. Thus, the distribution of agents is not part of the aggregate state space when we solve for the optimal policy functions, making standard iteration and projection methods tractable.

A. *Discussions for the Distribution of* $\ln a_{i,t}$

We provide a heuristic discussion that the actual distribution of $\widetilde{a}_{i,t} \equiv \ln a_{i,t}$ is approximately a normal distribution. In the absence of aggregate shocks, consider the deterministic balanced growth path. Thus, all equilibrium prices are constant as shown in the proof of Proposition 6. The productivity cutoff $\underline{z}$ determined by equation (IA.18) becomes:

$$\underline{z}\kappa = r_f + \delta_k. \tag{IA.86}$$

Rewrite equations (2) and (25) of the main text using (IA.86) as follows:

$$\frac{\mathrm{d}a_{i,t}}{\mathrm{d}t} = s(z_{i,t})a_{i,t}, \tag{IA.87}$$

where

$$s(z) = (1+\lambda)\kappa \max\{z - \underline{z}, 0\} + r_f - \rho - \delta_a, \tag{IA.88}$$

and κ is given by equation (IA.83).

To better illustrate intuitions, we rewrite (IA.87) in discrete time with a time interval $\Delta t \approx 0$:

$$a_{i,t+\Delta t} = [1 + s(z_{i,t})\Delta t]\, a_{i,t}. \tag{IA.89}$$

We denote $a_{i,n} \equiv a_{i,n\Delta t}$ and $z_{i,n} \equiv z_{i,n\Delta t}$ for $n = 1, 2, ...$ Then, it follows that

$$a_{i,n+1} = [1 + s(z_{i,n})\Delta t]\, a_{i,n}. \tag{IA.90}$$

Define $\xi_{i,n} \equiv \ln(1 + s(z_{i,n})\Delta t) - \overline{\xi}$ with $\overline{\xi} \equiv \mathbb{E}\left[\ln(1 + s(z_{i,n})\Delta t)\right]$, and thus (IA.90) can be written as

$$\ln a_{i,n+1} = \ln a_{i,n} + \overline{\xi} + \xi_{i,n}. \tag{IA.91}$$

For a large $T > 0$, suppose we set $N_T = T/\Delta t$ (without loss of generality, we assume that N_T is an integer), then equation (IA.91) implies

$$\ln a_{i,T} = \ln a_{i,1} + (N_T - 1)\overline{\xi} + \sum_{n=1}^{N_T - 1} \xi_{i,n}. \tag{IA.92}$$

In the deterministic balanced growth path, $z_{i,n}$ follows a stationary process evolving according to equation (4) of the main text. Thus, the process $\xi_{i,n}$ is also stationary.

The evolution of $\ln z_{i,n}$ can be directly obtained from equation (4) of the main text, as follows:

$$\ln z_{i,n+1} = e^{-\theta\Delta t} \ln z_{i,n} + \sigma\sqrt{\frac{1 - e^{-2\theta\Delta t}}{2}}\varepsilon_{i,n+1}, \tag{IA.93}$$

where $\varepsilon_{i,n+1}$ is a standard normal variable.

According to Andrews (1983), the process $z_{i,n}$ is strong mixing with mixing coefficients dominated

by an exponentially declining sequence. Let

$$\sigma_{N_T}^2 = \mathbb{E}\left[\xi_{i,1}^2\right] + 2\sum_{n=1}^{N_T-1}\left(1-\frac{n}{N_T}\right)\mathbb{E}\left[\xi_{i,1}\xi_{i,n}\right]. \tag{IA.94}$$

Using the Berry-Esseen bound developed by Tikhomirov (1980) and Bentkus, Gotze and Tikhomoirov (1997), we obtain

$$\sup_x \left|\mathbb{P}\left\{\sum_{n=1}^{N_T-1}\xi_{i,n} \leq \sigma_{N_T}x\right\} - \Phi(x)\right| \leq AN_T^{-1/2}\ln^2 N_T, \tag{IA.95}$$

where $\Phi(x)$ is the CDF of a standard normal random variable, and A is a constant that depends on model parameters.

B. Discussions of Existing Numerical Approximation Methods

The literature has proposed various approximation algorithms since the influential work of Krusell and Smith (1998). Generally speaking, there are three methods classified by how we approximate the cross-sectional distribution and solve for aggregate dynamics. The first method is proposed by Algan, Allais and Den Haan (2008), who approximate the cross-sectional distribution using a flexible parametric family, which reduces the infinite-dimensional distribution to a finite number of parameters. They then solve for the dynamics of these parameters using a globally accurate projection technique. The second method is proposed by Reiter (2009), who approximates the distribution with a fine histogram, which again reduces the infinite dimensional distribution to a finite number of parameters representing the probability mass of each discretized agent type. Because it requires a large number of discretized bins to approximate the distribution, Reiter (2009) needs to solve the model using locally accurate approximations with respect to the aggregate state vector. The third method is recently proposed by Winberry (2018), who approximates the cross-sectional distribution using the parametric family proposed by Algan, Allais and Den Haan (2008). Instead of solving the model using a global projection method, Winberry (2018) solves the model using a locally accurate perturbation method, similar to Reiter (2009).

Our proposed parametric approximation method in Section 3.1 of the main text is related to the numerical approximation methods discussed above with one crucial difference. In our method, the cross-sectional distribution of agents is assumed to be joint log normal (in $z_{i,t}$ and $a_{i,t}$) at any point in time, which allows us to derive "analytical formulas" for the evolution of the parameters (i.e., the misallocation measure M_t) that characterize the distribution. By contrast, in the numerical approximation methods of Algan, Allais and Den Haan (2008) and Winberry (2018), the assumed parametric function is used to "numerically fit" the cross-sectional distribution of agents, and as a result, analytical formulas are not derived.

Broadly speaking, our idea of approximating the joint distribution of firms using the covariance between log productivity and log capital is related to several seminal papers in the macroeconomics literature. In the model of Krusell and Smith (1998), agents face idiosyncratic labor productivity shocks (employed or unemployed). Because the covariance between individual productivity and

individual wealth does not affect aggregate output, Krusell and Smith (1998) find that the behavior of macroeconomic aggregates can be almost perfectly described by the mean of the wealth distribution. By contrast, in our model, the covariance between firm productivity and firm capital determines the degree of misallocation, the aggregate TFP, and the growth rate. Thus, both the mean of the capital distribution and the covariance have to be included to describe the behavior of macroeconomic aggregates. Angeletos (2007) considers idiosyncratic investment risk and endogenous wealth accumulation. In his model, the covariance between individual productivity and individual wealth affects the aggregate output. However, by assuming that productivity shocks are i.i.d. over time, the covariance is no longer a state variable; and the mean of wealth serves as a single state variable that sufficiently captures the economy's dynamics. Our model is closest to that of Moll (2014), who considers persistent idiosyncratic productivity shocks and endogenous wealth accumulation in a general equilibrium model with heterogeneous agents. Moll (2014) solves the model without aggregate shocks using the finite difference method for partial differential equations (PDE). Building on Moll (2014)'s characterization of wealth shares, we propose an analytically tractable approximation for the infinite-dimensional joint distribution using the covariance between log productivity and log capital across firms. Thus, misallocation naturally emerges as a crucial endogenous state variable. Our approximation method remains tractable even in the presence of aggregate shocks.

C. *Deterministic Steady States without Aggregate Shocks*

We begin by evaluating the accuracy of our parametric approximation method in the deterministic balanced growth path. Because there are no aggregate shocks, the capital share distribution, $\omega(z)$, is stationary in the deterministic balanced growth path. We are able to solve it accurately by approximating $\omega(z)$ nonparametrically using a fine histogram, as in Buera and Shin (2013) and Moll (2014). To ensure accuracy, we choose 251 equal-spaced grids for idiosyncratic productivity z over the interval $[z_{min}, z_{max}]$, with $z_{min} = 0$ and $z_{max} = \exp(\Phi^{-1}(0.99)\sigma/\sqrt{2})$. The choice of z_{max} corresponds to the 99th percentile of the steady-state distribution of $z_{i,t}$. We discretize the time horizon using a short time period, $\Delta t = 1/200$, and verify that the solution does not change when finer grid points are chosen.

Table IA.III compares the solution of our parametric approximation method and the solution based on the histogram method for various key endogenous aggregate variables. When the yearly autocorrelation in idiosyncratic productivity is 0.5 (i.e., $\exp(-\theta) = 0.5$), our parametric approximation method yields solutions almost identical to those of the histogram method. When idiosyncratic productivity becomes more persistent, the accuracy of our method becomes slightly worse. Importantly, in the benchmark calibration with $\exp(-\theta) = 0.85$, our parametric approximation method still yields results very similar to the histogram solution. Not only for these aggregate variables, our baseline approximation method also well captures the endogenous capital share distribution. Figure IA.3 compares the capital share distribution $\omega(z)$ solved by our parametric approximation method (blue solid line) and that solved by the histogram method (black dashed line). Panels A and B show that when the yearly autocorrelation of idiosyncratic productivity $z_{i,t}$ is 0.5 or 0.7, the two curves are almost overlapping with each other, indicating that our parametric approximation method provides extremely high accuracy. Panel C shows that when the yearly autocorrelation is 0.85, as in

Table IA.III: Accuracy of our parametric approximation in the deterministic balanced growth path.

Variables	$\exp(-\theta) = 0.5$		$\exp(-\theta) = 0.7$		$\exp(-\theta) = 0.85$	
	baseline	histogram	baseline	histogram	baseline	histogram
Firm profitability, κ	0.035	0.035	0.032	0.032	0.027	0.028
Productivity cutoff, $\underline{z}$	1.357	1.357	1.481	1.475	1.729	1.709
Value of blueprints, q	0.420	0.420	0.439	0.439	0.473	0.473
Productivity, H	1.494	1.465	1.537	1.498	1.618	1.557
Flow profit of innovators, π	0.071	0.071	0.074	0.074	0.080	0.080
Wage-capital ratio, w/A	0.214	0.224	0.214	0.225	0.214	0.231
Dividend-capital ratio, D/A	0.038	0.038	0.038	0.039	0.038	0.039
R&D-capital ratio, S/A	0.127	0.133	0.127	0.133	0.127	0.137
Knowledge stock-capital ratio, N/A	1.802	1.881	1.726	1.817	1.600	1.729
Growth rate, g (%)	1.752	1.766	1.754	1.727	1.750	1.709
Risk-free rate, r_f (%)	1.947	1.954	1.948	1.934	1.946	1.924

Note: This table evaluates the accuracy of our parametric approximation method in the deterministic balanced growth path (i.e., no aggregate shocks) for different persistence of idiosyncratic productivity (θ). The columns labeled "baseline" present the steady-state values of corresponding variables solved by our parametric approximation method. The columns labeled "histogram" present the steady-state values solved by the histogram method with 251 equal-spaced grids for productivity $z_{i,t}$. All the parameter values are taken from our benchmark calibration in Table 1 of the main text except for θ (we consider three values, $\exp(-\theta) = 0.5, 0.7, 0.85$) and ψ (we set its value to match a growth rate of 1.75% for corresponding θ). All other parameter values are taken from our benchmark calibration in Table 1 of the main text.

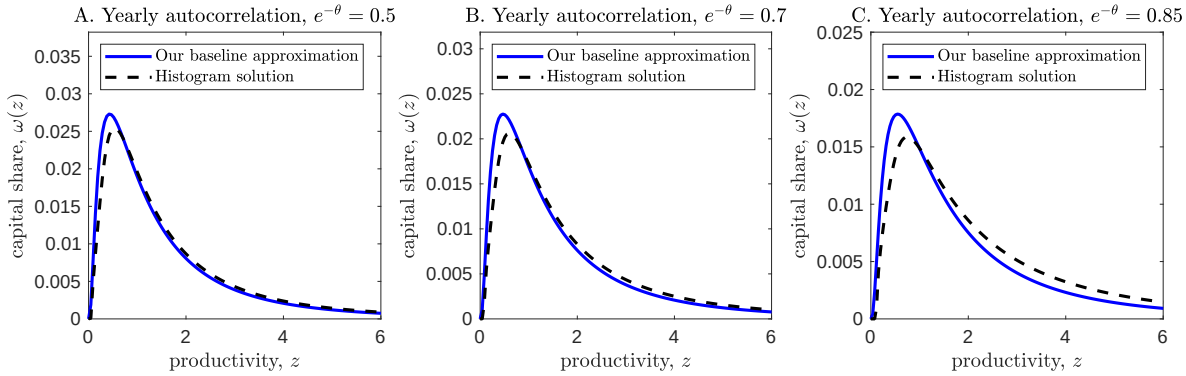
our benchmark calibration, the endogenous capital share distributions solved by the two solution methods remain very similar.

If idiosyncratic productivity becomes more persistent, the capital share distribution solved by our parametric approximation method will further diverge from that solved by the histogram method. Intuitively, an extremely high persistence of idiosyncratic productivity will endogenously generate a fat right tail for the capital share distribution because productive firms will accumulate significant amounts of capital in the deterministic balanced growth path. This fat right tail cannot be well approximated by a log-normal distribution specified in equation (38) of the main text, resulting in relatively large numerical errors from our parametric approximation method. Despite the potential bias at extremely low values of θ (i.e., high values of persistence, $\exp(-\theta)$), Figure IA.3 shows that our parametric approximation method is sufficiently accurate for a wide range of empirically relevant θ estimated by Asker, Collard-Wexler and Loecker (2014) based on the U.S. data.

D. Stochastic Steady States with Aggregate Shocks

Per our discussions in Subsection III.B, there are three numerical approximation methods that we can benchmark to when assessing the accuracy of our parametric approximation method. Among these three methods, we choose the most recent method proposed by Winberry (2018) for its numerical tractability and stability.[6] The main advantages of this method are the fast computation speed and

[6]Because of our model has large nonlinearities introduced by the endogenous SDF, the projection method of Algan, Allais and Den Haan (2008) is difficult to implement at higher-order approximations. Moreover, our model features kinked decision rules introduced by the endogenous time-varying productivity cutoff $\underline{z}_t$. Thus, the method of Reiter (2009) is difficult to be implemented accurately as the fixed grids of histograms generate large numerical errors around the cutoff $\underline{z}_t$ when using local perturbation approaches to approximate the evolution of aggregate state space.



Note: This figure compares the capital share $\omega(z)$ in the deterministic balanced growth path (i.e., no aggregate shocks) solved by our parametric approximation method (see equation (38) of the main text) and that solved by the histogram method with 251 equal-spaced grids for productivity $z_{i,t}$. The blue solid line in each panel represents our parametric approximation method and the black dashed line represents the histogram method. All the parameter values are taken from our benchmark calibration in Table 1 of the main text except for θ (consider three values, $\exp(-\theta) = 0.5, 0.7, 0.85$) and ψ (set its value to match a growth rate of 1.75% for corresponding θ). All other parameter values are taken from our benchmark calibration in Table 1 of the main text.

Figure IA.3: Accuracy of capital share distributions in the deterministic balanced growth path.

flexibility in the degree of approximation for the cross-sectional distribution.

In particular, we use a flexible parametric family to approximate the capital share $\omega_t(z)$ at any point in time t. With a 4th-order approximation, this boils down to using 4 parameters to match the mean, variance, skewness, and kurtosis of $\omega_t(z)$. The evolution of these 4 parameters can be computed using numerical integration methods (Gauss-Hermite and Gauss Legendre quadratures) based on Kolmogorov forward equations. The implementation details of this method are presented in Section 3.1 of the online note on additional materials. Similar to the results of Winberry (2018), we also find that a 4th-order approximation for the distribution can sufficiently capture complicated shapes, and further increasing the degree of approximation will not change the model-implied dynamics much.

In Table IA.IV, we compare the key variables solved by our baseline parametric approximation method and those solved by the higher-degree numerical approximation methods. To ensure that the results obtained from the two solution methods are comparable, we implement both methods using the perturbation approach around the corresponding deterministic steady state. Table IA.IV presents the results of our baseline parametric approximation method and the 2nd-order, 3rd-order, and 4th-order numerical approximation methods. The results of our parametric approximation method are very similar to the results of the 2nd-order numerical approximation method, because our parametric approximation method essentially keeps track of the first and second moments of $\omega_t(z)$.[7] However, the results of our parametric approximation method do not exactly match the results of the 2nd-order numerical approximation method because there are numerical errors in the latter method. A discussion is provided in footnote 2 in Section 3.1 of the online note on additional materials. Table IA.IV also shows that the differences between our baseline parametric approximation method and the 4th-order numerical approximation method are generally within 10% for most variables. These findings are consistent with the numerical results of Winberry (2018). Specifically,

[7]The first moment is $m_{1,t} = -M_t\sigma^2/2$ and the second moment is $m_{2,t} = \sigma^2/2$, which is a constant under our parametric approximation method.

Table IA.IV: Accuracy of our parametric approximation in stochastic steady states.

Variables	Baseline	2nd-order	3rd-order	4th-order
Firm profitability, $\mathbb{E}[\kappa_t]$	0.027	0.027	0.027	0.027
Productivity cutoff, $\mathbb{E}[\log(\underline{z}_t)]$	1.526	1.514	1.393	1.308
Value of blueprints, $\mathbb{E}[q_t]$	0.483	0.483	0.483	0.483
Productivity, $\mathbb{E}[\ln H_t]$	0.664	0.679	0.660	0.645
Flow profit of innovators, $\mathbb{E}[\pi_t]$	0.081	0.081	0.081	0.081
Wage-capital ratio, $\mathbb{E}[\ln(w_t/A_t)]$	−1.900	−1.901	−1.861	−1.834
Dividend-capital ratio, $\mathbb{E}[\ln(D_t/A_t)]$	−3.337	−3.334	−3.314	−3.303
Consumption-capital ratio, $\mathbb{E}[\ln(C_t/A_t)]$	−1.696	−1.697	−1.659	−1.634
R&D-capital ratio, $\mathbb{E}[\ln(S_t/A_t)]$	−2.412	−2.413	−2.376	−2.351
Knowledge stock-capital ratio, $\mathbb{E}[\ln(N_t/A_t)]$	0.104	0.103	0.140	0.165
Capital growth, $\mathbb{E}[\Delta \ln A_t]$ (%)	1.754	1.750	1.743	1.730
Consumption growth, $\mathbb{E}[\Delta \ln C_t]$ (%)	1.753	1.750	1.743	1.730
Volatility of consumption growth, $\text{var}[\Delta \ln C_t]$ (%)	1.664	1.671	1.292	1.358
Risk-free rate, $\mathbb{E}[r_{f,t}]$ (%)	1.580	1.590	1.697	1.745
Consumption claim's return, $\mathbb{E}[\ln R_{m,t}]$ (%)	2.124	2.112	2.074	2.040

Note: This table evaluates the accuracy of our parametric approximation method in the full stochastic steady states with aggregate shocks. The column labeled "baseline" presents the steady-state values of corresponding variables solved by our parametric approximation method. The columns labeled "2nd-order", "3rd-order", and "4th-order" present the results corresponding to the second-, third-, and fourth-order numerical approximation methods for the capital share distribution $\omega_t(z)$ based on equation (77) in the online note on additional materials. All the parameter values are taken from our benchmark calibration in Table 1 of the main text.

Winberry (2018) approximates the distribution of heterogeneous firms in a DSGE model using a flexible parametric family and shows that the implied dynamics of aggregate variables, such as consumption, output, investment, SDF, and the autocorrelations in the covariance between log capital and log productivity, based on the log-normal approximation are very close to the results based on higher-order approximations, in the stochastic steady state.

IV. TFP Formula

Our TFP Z_t in equation (35) of the main text depends on the final goods sector's productivity H_t (can be thought of as the TFP of the final goods sector) and the economy's aggregate knowledge stock N_t. In this appendix section, we show that our formula for H_t is consistent with the TFP formula of Hsieh and Klenow (2009) when goods are homogeneous and the industry is not distorted by wedges.

According to equation (21) of the main text, $u_t(z) = 1$ for $z \geq \underline{z}_t$ and $u_t(z) = k_t(a,z) = 0$ for $z < \underline{z}_t$. Thus, the final goods sector's productivity H_t given by equation (IA.21) can be written as

$$H_t = \left[\frac{1}{K_t}\int_{\underline{z}_t}^{\infty}\int_0^{\infty} z k_t(a,z)\varphi_t(a,z)\mathrm{d}a\mathrm{d}z\right]^{\alpha} = \left[\frac{1}{K_t}\int_0^{\infty}\int_0^{\infty} z k_t(a,z)\varphi_t(a,z)\mathrm{d}a\mathrm{d}z\right]^{\alpha}. \tag{IA.96}$$

Without loss of generality, we rewrite equation (IA.96) to focus on a countable number of firms,

$$H_t = \left(\frac{1}{K_t}\sum_i z_i k_i\right)^{\alpha}. \tag{IA.97}$$

Because the firm-level productivity in equation (1) of the main text is z_i^{α} not z_i, we do a change of variables by replacing z_i with $z_i^{1/\alpha}$, so that z_i represents firm i's productivity, as in Hsieh and Klenow (2009). Thus, equation (IA.97) becomes

$$H_t = \left(\frac{1}{K_t} \sum_i z_i^{1/\alpha} k_i \right)^{\alpha}. \tag{IA.98}$$

Next, we show that equation (IA.98) is consistent with the industry-level TFP formula used by Hsieh and Klenow (2009) when goods are homogeneous and the industry is not distorted by wedges. In the model of Hsieh and Klenow (2009), there are s industries and each industry has M_s firms. They define a single industry's TFP as

$$TFP_s = \frac{Y_s}{K_s^{\alpha_s} L_s^{1-\alpha_s}}. \tag{IA.99}$$

To be consistent with our model setup, we focus on deriving TFP_s in one single industry in the model of Hsieh and Klenow (2009), which corresponds to our final goods sector. Moreover, without loss of generality, we also normalize the aggregate labor in industry s to one, i.e., $L_s = 1$, as in our model. We derive the formula of TFP_s using the original notations of Hsieh and Klenow (2009).

Substituting $L_s = 1$ and equations (3) and (4) of Hsieh and Klenow (2009) into (IA.99),

$$TFP_s = \frac{1}{K_s^{\alpha_s}} \left(\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = \frac{1}{K_s^{\alpha_s}} \left[\sum_{i=1}^{M_s} \left(A_{si} K_{si}^{\alpha_s} L_{si}^{1-\alpha_s} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}. \tag{IA.100}$$

Using the first-order condition, labor L_{si} in the model of Hsieh and Klenow (2009) is as follows

$$L_{si} = \left[\frac{(1-\tau_{Ysi}) P_{si} A_{si} K_{si}^{\alpha_s} (1-\alpha_s)}{w} \right]^{\frac{1}{\alpha_s}}. \tag{IA.101}$$

Substituting equation (IA.101) into (IA.100), we obtain

$$TFP_s = \frac{1}{K_s^{\alpha_s}} \left(\frac{1-\alpha_s}{w} \right)^{\frac{1-\alpha_s}{\alpha_s}} \left[\sum_{i=1}^{M_s} \left[A_{si}^{1/\alpha_s} K_{si} \left[(1-\tau_{Ysi}) P_{si} \right]^{\frac{1-\alpha_s}{\alpha_s}} \right]^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}. \tag{IA.102}$$

The labor market clearing condition in the model of Hsieh and Klenow (2009) implies

$$\sum_{i=1}^{M_s} \left[\frac{(1-\tau_{Ysi}) P_{si} A_{si} K_{si}^{\alpha_s} (1-\alpha_s)}{w} \right]^{\frac{1}{\alpha_s}} = 1. \tag{IA.103}$$

Substituting (IA.103) into (IA.102),

$$TFP_s = \frac{1}{K_s^{\alpha_s}} \frac{\left[\sum_{i=1}^{M_s} \left[A_{si}^{1/\alpha_s} K_{si} \left[(1-\tau_{Ysi}) P_{si} \right]^{\frac{1-\alpha_s}{\alpha_s}} \right]^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}}{\left[\sum_{i=1}^{M_s} A_{si}^{1/\alpha_s} K_{si} \left[(1-\tau_{Ysi}) P_{si} \right]^{1/\alpha_s} \right]^{1-\alpha_s}}. \tag{IA.104}$$

If goods are homogeneous and the industry is not distorted by wedges, i.e., $\sigma \to \infty$ and $\tau_{Ysi} = 0$, then P_{si} is equalized across all i, i.e., $P_{si} \equiv P_s$. This assumption allows us to simplify (IA.104) as follows,

$$TFP_s = \frac{1}{K_s^{\alpha_s}} \frac{\sum_{i=1}^{M_s} A_{si}^{1/\alpha_s} K_{si}}{\left(\sum_{i=1}^{M_s} A_{si}^{1/\alpha_s} K_{si}\right)^{1-\alpha_s}} = \left[\frac{1}{K_s} \sum_{i=1}^{M_s} A_{si}^{1/\alpha_s} K_{si}\right]^{\alpha_s}. \tag{IA.105}$$

Except for notational differences, the formula (IA.105) is identical to (IA.98).

V. Output Gains from Capital Reallocation

In both the data and model, capital reallocation is procyclical as the aggregate reallocation rate is higher in booms than recessions. Eisfeldt and Shi (2018) propose a method to quantify the cost of misallocation fluctuations over business cycles. The key idea of this method is to measure the potential output gain if the amount of capital reallocation observed in booms could be achieved in recessions. This method's main advantage is that it incorporates flow data on capital reallocation to help measure the cost of increased misallocation during recessions. The quantity data on flows are presumably more precisely measured than MRPK, which depends on particular model specifications.

In the model, productive firms ($z_{i,t} \geq \underline{z}_t$) lease capital from unproductive firms ($z_{i,t} < \underline{z}_t$) as shown in Lemma 1. Thus, the aggregate reallocation rate over $[t, t + \mathrm{d}t)$ is given by

$$Realloc_t = \frac{1}{A_t} \int_{\underline{z}_t}^{\infty} \int_0^{\infty} \lambda a \varphi_t(a, z) \mathrm{d}a \mathrm{d}z = \lambda(1 - \Omega_t(\underline{z}_t)), \tag{IA.106}$$

where $1 - \Omega_t(\underline{z}_t)$ captures the share of aggregate capital in the final goods sector held by productive firms ($z_{i,t} \geq \underline{z}_t$). Under our calibration, $\underline{z}_t$ is countercyclical, with $\mathrm{corr}(\ln \underline{z}_t, \Delta \ln C_t) \approx -0.35$. Thus, the aggregate reallocation rate $Realloc_t$ is procyclical even with a constant λ.

Using the method proposed by Eisfeldt and Shi (2018), we quantify the potential output gain in recessions if the reallocation rate of capital among firms during recessions is assumed to be as high as that during booms.[8] We apply this method to both the actual data in our 1965-2016 sample and the simulated data of our model. The estimated potential gains in recessions from capital reallocation are 3.58% and 3.09%, respectively, indicating that misallocation fluctuations have large effects on output fluctuations. The similarity in the two estimates provides a further validation of the model.

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[8] The implementation details are in Section 2.4 of the online note on additional materials.

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