

A Theory of Labor Markets with Inefficient Turnover

Online Appendix—Not for Publication

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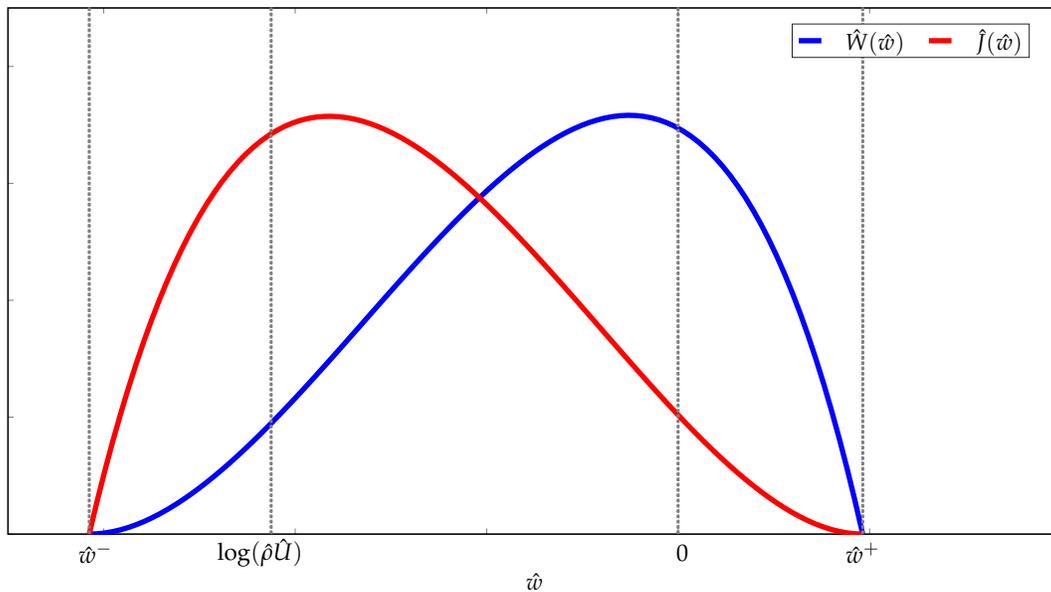
I Additional Results for Section 2: A Model of Labor Markets with Inefficient Turnover

This section presents additional results extending the analysis in Section 2.

I.1 Equilibrium Value Functions

The following figure shows the equilibrium values and continuation sets of the worker (blue solid line) and the firm (red solid line) in our baseline model.

FIGURE I1. EQUILIBRIUM VALUE FUNCTIONS AND CONTINUATION SETS



Notes: The figure plots the equilibrium value functions of the firm (i.e., $\hat{f}(\hat{w})$) and the employed worker (i.e., $\hat{W}(\hat{w})$) as a function of $\hat{w} = w - z$. The blue and red solid lines show the value function of the employed worker and the value function of the firm, respectively. The dotted vertical lines mark the boundaries of the firms' continuation set $(-\infty, \hat{w}^+)$ and the worker's continuation set (\hat{w}^-, ∞) .

I.2 Sequential and Recursive Formulation of the Model

Here, we present the sequential formulation of the problem and show the one-to-one equivalence to our recursive formulation.

Environment. The environment—i.e., preferences, technology, shocks, and frictions—is the same as in Section 2. To focus on the novel component of the paper and to simplify the notation, we assume that a recursive representation holds across employment and unemployment spells.

An unemployed worker's choice of submarket $(z; w)$ is associated with a job-finding rate $f(z; w)$, which induces a stochastic job offer arrival time τ^u . The value of an unemployed worker with productivity z is

$$U(z) = \max_{\{w_t\}_{t=0}^{\tau^u}} \mathbb{E}_0 \left[\int_0^{\tau^u} e^{-\rho t} \tilde{B} e^{z t} dt + e^{-\rho \tau^u} H(z_{\tau^u}; w_{\tau^u}; \bar{\tau}^m(w_{\tau^u}, z_{\tau^u})) \right]. \quad (\text{I.1})$$

That is, an unemployed worker searches for a job in submarket $(z_t; w_t)$ at time $t \leq \tau^u$ until becoming employed at wage w_{τ^u} and receiving the value of employment $H(z_{\tau^u}; w_{\tau^u}; \bar{\tau}^m(w_{\tau^u}, z_{\tau^u}))$ at time τ^u .

Given the (fixed) wage w and current productivity z , a matched worker chooses when to quit, which induces a stopping time τ^h . Based on the same (w, z) pair, a matched firm chooses when to lay off the worker, which induces a stopping time τ^j . Given the choices by workers and firms in addition to the exogenous stopping time τ^δ , the actual match duration is the minimum stopping time in the vector $\bar{\tau}^m = (\tau^h, \tau^j, \tau^\delta)$, denoted $\tau^m = \min\{\tau^h, \tau^j, \tau^\delta\}$. Given a vector of stopping times $\bar{\tau}^m$, the value of a worker employed at wage w with productivity z is

$$H(z; w, \bar{\tau}^m) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} e^w dt + e^{-\rho \tau^m} U(z_{\tau^m}) \right]. \quad (\text{I.2})$$

That is, an employed worker consumes a constant wage w until time τ^m when she either endogenously or exogenously transitions to unemployment. Similarly, given a vector of stopping times $\bar{\tau}^m$, the value of a firm matched with a worker with wage w and productivity z is

$$J(z; w, \bar{\tau}^m) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} [e^{z t} - e^w] dt \right]. \quad (\text{I.3})$$

That is, the match produces $e^{z t}$, of which e^w is paid to the worker until it gets dissolved at time τ^m .

Free Entry. In choosing the number of vacancies to post in each submarket, firms trade off the expected benefit—i.e., the product of the filling rate $q(\theta(z; w))$ and the value of a filled job $J(z; w, \bar{\tau}^m(z; w))$ —with the flow cost $\tilde{K}e^z$ of posting a vacancy. In each submarket, firms post vacancies up to the point at which the marginal vacancy posting cost exceeds its expected benefits. Thus, free entry requires that

$$\min \{ \tilde{K}e^z - q(\theta(z; w))J(z; w, \bar{\tau}^m(z; w)), \theta(z; w) \} = 0, \quad (\text{I.4})$$

for all $(z; w)$.

Equilibrium Definition. We are now ready to define an equilibrium. Let \mathcal{T} be the set of all stopping times for a given match. Given the state $(z; w)$, staying in the match is a *weakly dominant strategy* for the worker if there exists a stopping time $\tau^{h*}(z; w) \in \mathcal{T}$ such that $\Pr(\tau^{h*}(z; w) > 0) = 1$ and

$$H(z; w, \tau^{h*}(z; w), \tau^j, \tau^\delta) \geq H(z; w, \tau^h, \tau^j, \tau^\delta), \quad \forall \tau^h, \tau^j \in \mathcal{T},$$

with strict inequality for some τ^j . Similarly, given $(z; w)$, staying in the match is a weakly dominant strategy for the firm if there exists a stopping time $\tau^{j*}(z; w) \in \mathcal{T}$ such that $\Pr(\tau^{j*}(z; w) > 0) = 1$ and

$$J(z; w, \tau^h, \tau^{j*}(z; w), \tau^\delta) \geq J(z; w, \tau^h, \tau^j, \tau^\delta), \quad \forall \tau^h, \tau^j \in \mathcal{T},$$

with strict inequality for some τ^h .

Definition I.2. An equilibrium consists of a set of value functions $\{H(z; w, \bar{\tau}^m), J(z; w, \bar{\tau}^m), U(z)\}$, a market tightness function $\theta(z; w)$, and policy functions $\{\tau^{h*}(z; w), \tau^{j*}(z; w), w^*(z_t)\}$, such that:

1. Given $H(z; w, \bar{\tau}^{m*}(z; w))$, $U(z)$, and $\theta(z; w)$, the search strategy $\{w^*(z_t)\}_{t=0}^{\tau^{u*}}$ solves equation (I.1).
2. Given $J(z; w, \bar{\tau}^{m*}(z; w))$, market tightness $\theta(z; w)$ solves the free-entry condition (I.4).
3. Given $U(z)$, $(\tau^{h*}(z; w), \tau^{j*}(z; w))$ is a nontrivial Nash equilibrium with stopping times (τ^h, τ^j) that satisfy

$$H(z; w, \tau^{h*}(z; w), \tau^{j*}(z; w), \tau^\delta) \geq H(z; w, \tau^h, \tau^{j*}(z; w), \tau^\delta), \quad \forall (z; w)$$

$$J(z; w, \tau^{h*}(z; w), \tau^{j*}(z; w), \tau^\delta) \geq J(z; w, \tau^{h*}(z; w), \tau^j, \tau^\delta), \quad \forall (z; w)$$

and $\Pr(\tau^{h*}(z; w) > 0) = 1$ (resp. $\Pr(\tau^{j*}(z; w) > 0) = 1$) whenever staying in the match is a weakly dominant strategy for the worker (resp. the firm) given the state $(z; w)$.

Recursive Equilibrium Conditions. Define the recursive equilibrium conditions:

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 u(z)}{\partial z^2} + \max_w f(\theta(z; w)) [h(z; w) - u(z)], \quad \forall z \in \mathbb{R} \quad (\text{I.5})$$

$$0 = \min \{ \tilde{K}e^z - q(\theta(z; w))j(z; w), \theta(z; w) \}, \quad \forall (z; w) \in \mathbb{R}^2$$

$$z \in (\mathcal{Z}^j(w))^c \Rightarrow h(z; w) = u(z), \quad (\text{I.6})$$

$$z \in (\mathcal{Z}^h(w))^c \Rightarrow j(z; w) = 0, \quad (\text{I.7})$$

$$0 = \max \{ u(z) - h(z; w), \mathcal{A}^h h(z; w) + e^w \}, \quad \forall z \in \mathcal{Z}^j(w), h(\cdot; w) \in \mathbf{C}^1(\mathcal{Z}^j(w)) \cap \mathbf{C}(\mathbb{R}), (\text{I.8})$$

$$0 = \max\{-j(z; w), \mathcal{A}^j j(z; w) + e^z - e^w\}, \quad \forall z \in \mathcal{Z}^h(w), j(\cdot; w) \in \mathbf{C}^1(\mathcal{Z}^h(w)) \cap \mathbf{C}(\mathbb{R}), \quad (\text{I.9})$$

$$\mathcal{Z}^h(w) := \text{int} \left\{ z \in \mathbb{R} : h(z; w) > u(z) \text{ or } \mathcal{A}^h u(z) + e^w > 0 \right\}, \quad (\text{I.10})$$

$$\mathcal{Z}^j(w) := \text{int} \left\{ z \in \mathbb{R} : j(z; w) > 0 \text{ or } e^z - e^w > 0 \right\}, \quad (\text{I.11})$$

where we define the characteristic operator for any function $v(z)$ for the firm and the worker as

$$\begin{aligned} \mathcal{A}^h(v(z)) &:= -\rho v + \delta(u(z) - v(z)) + \gamma \frac{\partial v(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 v(z)}{\partial z^2} \\ \mathcal{A}^j(v(z)) &:= -\rho v + \delta(0 - v(z)) + \gamma \frac{\partial v(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 v(z)}{\partial z^2}. \end{aligned}$$

Lemma I.1. *The policy functions $\{\tau^{h*}, \tau^{j*}, w^*(z)\}$ and the value functions $\{U(z), H(z; w, \bar{\tau}^m), J(z; w, \bar{\tau}^m)\}$ given by (I.1), (I.2) and (I.3) and the market tightness function $\theta(z; w)$ form a BRE iff. $\{u(z), h(z; w), j(z; w)\}$ satisfy equations (I.5)–(I.11) and*

$$\begin{aligned} u(z) &= U(z), \\ h(z; w) &= H(z; w, \tau^{h*}(z; w), \tau^{j*}(z; w), \tau^\delta), \\ j(z; w) &= J(z; w, \tau^{h*}(z; w), \tau^{j*}(z; w), \tau^\delta). \end{aligned}$$

To simplify the exposition, we divide the proof into a sequence of steps.

Proposition I.1. *Let $x := (z; w)$. If there exist two functions $h(z; w)$ and $j(z; w)$ satisfying (I.6), (I.7), (I.8) and (I.9) given the continuation sets (I.10) and (I.11), then*

$$\begin{aligned} \tau^{h*}(x) &= \inf \left\{ t \geq 0 : z_t \notin \mathcal{Z}^h(w) \right\}, \\ \tau^{j*}(x) &= \inf \left\{ t \geq 0 : z_t \notin \mathcal{Z}^j(w) \right\} \end{aligned}$$

form a nontrivial Nash equilibrium and

$$h(z; w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta), \quad j(z; w) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta).$$

Moreover, if $(\tau^{h}(x), \tau^{j*}(x))$ is a nontrivial Nash equilibrium, then $h(z; w)$ and $j(z; w)$ satisfy (I.6) to (I.9).*

Proof. **Variational inequalities as sufficient conditions for Nash Equilibrium.** First, we prove that

if $h(z; w)$ and $j(z; w)$ satisfy (I.6) to (I.9), then

$$h(z; w) = H(x, \tau^{h^*}(x), \tau^{j^*}(x), \tau^\delta) \geq H(x, \tau^h(x), \tau^{j^*}(x), \tau^\delta)$$

for any $\tau^h \in \mathcal{T}$. The proof of the statement

$$j(z; w) = J(x, \tau^{h^*}(x), \tau^{j^*}(x), \tau^\delta) \geq J(x, \tau^{h^*}(x), \tau^j(x), \tau^\delta),$$

for any $\tau^j \in \mathcal{T}$, follows the same arguments.

Step 1: Here, we show that $h(z; w) \geq H(x, \tau^h(x), \tau^{j^*}(x), \tau^\delta)$. Let τ^h be any stopping time (not necessarily the optimal). Without loss of generality, we restrict the attention to $\tau^h \leq \tau_{(-\infty, a)}$, where $\tau_{(-\infty, a)} = \inf\{t > 0 : z_t \notin (-\infty, a)\}$. Intuitively, it is never optimal for the worker to stay in the job at wage w when productivity is sufficiently large. Let $U_k \subset \mathbb{R}$ be an increasing sequence of bounded sets s.t. $\cup_{k=1}^\infty U_k = \mathbb{R}$. Let $\tau_k = \inf\{t > 0 : z_t \notin U_k\}$. Since each U_k is bounded, we do not need to assume compact support of the function to apply Proposition A.1. Applying Dynkin's Lemma to the stopping time $\tau_k^h = \tau^h \wedge \tau^{j^*} \wedge \tau^\delta \wedge \tau_k$,

$$\mathbb{E}[e^{-\rho\tau_k^h} h(z_{\tau_k^h}) | z_0 = z] = h(z; w) + \mathbb{E} \left[\int_0^{\tau_k^h} \mathcal{A}^h h(z_t; w) dt | z_0 = z \right].$$

Using condition (I.8), we have that $h(z; w) \geq u(z)$ for all $z \in \mathcal{Z}^j(w)$. Moreover, $h(z; w) = u(z)$ for all $z \in (\mathcal{Z}^j(w))^c$. Therefore, $h(z; w) \geq u(z)$ for all $z \in \mathbb{R}$. Thus, we have that $\mathbb{E}[e^{-\rho\tau_k^h} h(z_{\tau_k^h}; w) | z_0 = z] \geq \mathbb{E}[e^{-\rho\tau_k^h} u(z_{\tau_k^h}) | z_0 = z]$. Thus,

$$\mathbb{E}[e^{-\rho\tau_k^h} u(z_{\tau_k^h}) | z_0 = z] - \mathbb{E} \left[\int_0^{\tau_k^h} \mathcal{A}^h h(z_t; w) dt | z_0 = z \right] \leq h(z; w).$$

From condition (I.8), we have $\mathcal{A}^h h(z; w) + e^w \leq 0$ for all z . Thus,

$$\mathbb{E} \left[\int_0^{\tau_k^h} e^{-\rho t} e^w dt | z_0 = z \right] \leq -\mathbb{E} \left[\int_0^{\tau_k^h} \mathcal{A}^h h(z; w) dt | z_0 = z \right].$$

Using this result

$$\mathbb{E} \left[e^{-\rho\tau_k^h} u(z_{\tau_k^h}) + \int_0^{\tau_k^h} e^{-\rho t} e^w dt | z_0 = z \right] \leq h(z; w)$$

Now, we take the limit $k \rightarrow \infty$. It is easy to see that $\int_0^{\tau^h \wedge \tau^{j^*} \wedge \tau^\delta \wedge \tau_k} e^{-\rho t + w} dt \leq \frac{1}{\rho} e^w$ a.e., so using the

dominated convergence theorem $\lim_{k \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau^h \wedge \tau^{j^*} \wedge \tau^\delta \wedge \tau_k} e^{-\rho t + w} dt | z_0 = z \right] = \mathbb{E} \left[\int_0^{\tau^h \wedge \tau^{j^*} \wedge \tau^\delta} e^{-\rho t + w} dt | z_0 = z \right]$.

As we show below, $u(z) \propto e^z$ and since $e^{zt} \leq e^a$ for all $t \leq \tau^h \leq \tau_{(-\infty, a)}$, we have that $0 \leq e^{-\rho t} u(z_t) \leq e^a$. Applying the monotone convergence theorem, we have that

$$\lim_{k \rightarrow \infty} \mathbb{E} \left[e^{-\rho(\tau^h \wedge \tau^{j^*} \wedge \tau^\delta \wedge \tau_k)} u(z_{\tau^h \wedge \tau^{j^*} \wedge \tau^\delta \wedge \tau_k}) | z_0 = z \right] = \mathbb{E} \left[e^{-\rho(\tau^h \wedge \tau^{j^*} \wedge \tau^\delta)} u(z_{\tau^h \wedge \tau^{j^*} \wedge \tau^\delta}) | z_0 = z \right].$$

Therefore, taking the limit $k \rightarrow \infty$, we finally obtain

$$h(z; w) \geq H(x, \tau^h(x), \tau^{j^*}(x), \tau^\delta).$$

Step 2: Now, we show that $h(z; w) = H(x, \tau^{h^*}(x), \tau^{j^*}(x), \tau^\delta)$. Applying Proposition A.1 to the stopping time $\tau_k^{h^*} = \tau^{h^*} \wedge \tau^{j^*} \wedge \tau_k \wedge \tau^\delta$ we obtain

$$\mathbb{E} [e^{-\rho \tau_k^{h^*}} h(z_{\tau_k^{h^*}}; w) | z_0 = z] = h(z; w) + \mathbb{E} \left[\int_0^{\tau_k^{h^*}} \mathcal{A}^h h(z_t; w) dt | z_0 = z \right].$$

For all $t < \tau_k^{h^*}$, we have that $u(z) < h(z; w)$. Therefore, by (I.8), $\mathcal{A}^h h(z; w) + e^w = 0$ for all z . Thus,

$$\mathbb{E} \left[e^{-\rho \tau_k^{h^*}} h(z_{\tau_k^{h^*}}; w) + \int_0^{\tau_k^{h^*}} e^{-\rho t} e^w dt | z_0 = z \right] = h(z; w).$$

Taking the limit $k \rightarrow \infty$ and following similar arguments as above, we obtain

$$\mathbb{E} \left[e^{-\rho(\tau^{h^*} \wedge \tau^{j^*} \wedge \tau^\delta)} h(z_{\tau^{h^*} \wedge \tau^{j^*} \wedge \tau^\delta}; w) + \int_0^{\tau^{h^*} \wedge \tau^{j^*} \wedge \tau^\delta} e^{-\rho t} e^w dt | z_0 = z \right] = h(z; w).$$

which, given Proposition A.2, is equivalent to

$$\mathbb{E} \left[e^{-(\rho+\delta)(\tau^{h^*} \wedge \tau^{j^*})} h(z_{\tau^{h^*} \wedge \tau^{j^*}}; w) + \int_0^{\tau^{h^*} \wedge \tau^{j^*}} e^{-(\rho+\delta)t} (\delta u(z_t) + e^w) dt | z_0 = z \right] = h(z; w).$$

Since $z_{\tau^{h^*} \wedge \tau^{j^*}} \in \partial(\mathcal{Z}^h(w^*(z)) \cap \mathcal{Z}^j(w^*(z)))$ and $h(\cdot; w)$ is continuous, we have that

$$\mathbb{E} \left[e^{-(\rho+\delta)(\tau^{h^*} \wedge \tau^{j^*})} u(z_{\tau^{h^*} \wedge \tau^{j^*}}; w) + \int_0^{\tau^{h^*} \wedge \tau^{j^*}} e^{-(\rho+\delta)t} (\delta u(z_t) + e^w) dt | z_0 = z \right] = h(z; w).$$

and $h(z; w) = H(x, \tau^{h^*}(x), \tau^{j^*}(x), \tau^\delta)$.

Variational inequalities as sufficient conditions for Nontrivial Nash Equilibrium. This part of

the proof is constructive. Define $\text{WD}^h(w) = \{z \in \mathbb{R} : 0 < e^w + \mathcal{A}^h u(z)\}$ and $\tau^*(z; w) = \inf\{t \geq 0 : z_t \notin \text{WD}^h(w), z_0 = z\}$. Now, we check that this set is where continuation is a weakly dominating strategy. Applying Dynkin's Lemma (and using similar arguments as before), for any stopping time τ we obtain

$$\mathbb{E} \left[e^{-\rho\tau} u(z_\tau) | z_0 = z \right] = u(z) + \mathbb{E} \left[\int_0^{\tau(z;w)} \mathcal{A}^h u(z_t) dt | z_0 = z \right].$$

Using $\tau(z; w) = \min \{ \tau^*(z; w), \tau^j(z; w), \tau^\delta(z; w) \}$,

$$\begin{aligned} u(z) &= \mathbb{E} \left[e^{-\rho\tau(z;w)} u(z_{\tau(z;w)}) | z_0 = z \right] - \mathbb{E} \left[\int_0^{\tau(z;w)} \mathcal{A}^h u(z_t) dt | z_0 = z \right] \\ &\leq \mathbb{E} \left[e^{-\rho\tau(z;w)} u(z_{\tau(z;w)}) | z_0 = z \right] + \mathbb{E} \left[\int_0^{\tau(z;w)} e^{-\rho t + w} dt | z_0 = z \right]. \end{aligned}$$

with strict inequality, if $Pr(\tau(z; w) > 0) = 1$. Thus, staying in the match weakly dominates dissolving the match.

Variational inequalities as necessary conditions. Now, we prove that if $\tau^{h^*}(x)$ and $\tau^{j^*}(x)$ is a nontrivial Nash equilibrium, then $h(z; w)$, $j(z; w)$ satisfy (I.6) to (I.11). Notice that under the assumption that τ^j and τ^h are characterized by continuation sets, we can focus on these sets to prove conditions (I.6) to (I.11). By definition, we have that

$$h(z; w) = \max_{\tau^h} \mathbb{E} \left[\int_0^{\tau^h \wedge \tau^{j^*} \wedge \tau^\delta} e^{-\rho t + w} dt + e^{-\rho(\tau^h \wedge \tau^{j^*} \wedge \tau^\delta)} u(z_{\tau^h \wedge \tau^{j^*} \wedge \tau^\delta}; w) dt | z_0 = z \right]. \quad (\text{I.12})$$

- Condition (I.6): If $z \in (\mathcal{Z}^j(w))^c$, then $\tau^{j^*}(x) = 0$ and $Pr[\min\{\tau^{h^*}(x), \tau^{j^*}(x), \tau^\delta(x)\} \leq \tau^{j^*}(x)] = 1$, and $h(z; w) = u(z)$. A similar argument holds for the firm.
- Condition (I.8): Observe that this condition is the best response of the worker, given that the firm continues. See Øksendal (2007) and Brekke and Øksendal (1990) for a discussion of the necessity of the smooth pasting condition.
- Condition (I.10): For this part, we will assume that u is C^2 and the set of productivities for which $e^w + \mathcal{A}^h u(z) = 0$ has measure zero (we show this property in Lemma 1). To show this, we need to characterize the continuation set in the Nash equilibrium that survives the iterated elimination of weakly dominated strategies. First, from the problem (I.12), if

$\Pr(\tau^{j^*}(x) > 0) = 1$, then $\Pr(\tau^{h^*}(x) > 0) = 1$ iff.

$$z \in \text{int} \{z \in \mathbb{R} : h(z; w) > u(z)\}.$$

Next, we proceed by contradiction. Assume that in the state $(z; w)$ staying in the match weakly dominates leaving and

$$0 < e^w + \mathcal{A}^h u(z). \quad (\text{I.13})$$

Notice that here we are ignoring the case $e^w + \mathcal{A}^h u(z) = 0$ since it has measure 0. If $u(z) \in \mathbb{C}^2$, define an open set U , containing the chosen $(z; w)$, where $e^w + \mathcal{A}^h u(z) > 0$ and take any stopping time τ_U . Then, applying Dynkin's Lemma (and using similar arguments as in Step 1), we obtain

$$\mathbb{E} [e^{-\rho\tau_U} u(z_{\tau_U}) | z_0 = z] = u(z) + \mathbb{E} \left[\int_0^{\tau_U} \mathcal{A}u(z_t) dt | z_0 = z \right].$$

Using the inequality in (I.13),

$$\begin{aligned} u(z) &= \mathbb{E} [e^{-\rho\tau_U} u(z_{\tau_U}) | z_0 = z] - \mathbb{E} \left[\int_0^{\tau_U} \mathcal{A}u(z_t) dt | z_0 = z \right] \\ &> \mathbb{E} [e^{-\rho\tau_U} u(z_{\tau_U}) | z_0 = z] + \mathbb{E} \left[\int_0^{\tau_U} e^{-\rho t + w} dt | z_0 = z \right]. \end{aligned}$$

Thus, staying in the match is dominated for τ^U , arriving at a contradiction. □

Proposition I.2. *Define*

$$w^*(z) = \arg \max_w f(\theta(z; w))(h(z; w) - u(z)).$$

and $\tau^{u^*} = \inf\{t \geq 0 : \Delta N_t^{f(\theta(z_t; w^*(z_t)))} = 1\}$ where $N_t^{f(\theta(z_t; w^*(z_t)))}$ is a Poisson counter with arrival rate $f(\theta(z_t; w^*(z_t)))$. The function $u(z)$ satisfies $u(z) \in \mathbb{C}^2(\mathbb{R})$ and (I.5) iff.

$$u(z) = \max_{\{w_i\}_{i=0}^{\tau^u}} \mathbb{E} \left[\int_0^{\tau^u} e^{-\rho t} B(z_t) dt + e^{-\rho\tau^u} h(z_{\tau^u}; w) \right].$$

Proof. The proof is the standard optimality conditions in the HJB (see Øksendal, 2007). □

Lemma I.1. Assume $u(z)$, $h(z; w)$, $j(z; w)$, $\theta(z; w)$ satisfy (I.5)—(I.9) given the continuation sets (I.10) and (I.11). Then $\{\tau^{h^*}, \tau^{j^*}, \{w_t^*\}_{t=0}^{\tau^u}\}$ constructed with

$$\begin{aligned}\tau^{h^*}(x) &= \inf \left\{ t \geq 0 : z_t \notin \mathcal{Z}^h(w) \right\} \\ \tau^{j^*}(x) &= \inf \left\{ t \geq 0 : z_t \notin \mathcal{Z}^j(w) \right\} \\ w^*(z) &= \arg \max_w f(\theta(z; w))(h(z; w) - u(z)).\end{aligned}$$

is a BRE with

$$\begin{aligned}h(z; w) &= H(x, \tau^{h^*}(x), \tau^{j^*}(x), \tau^\delta), \\ j(z; w) &= J(x, \tau^{h^*}(x), \tau^{j^*}(x), \tau^\delta), \\ u(z) &= U(z).\end{aligned}$$

If $\{H(z; w, \bar{\tau}^m), J(z; w, \bar{\tau}^m), U(z)\}$, market tightness $\theta(z; w)$, and policy functions $\{\tau^{h^*}(z; w), \tau^{j^*}(z; w), w^*(z_t)\}$ is a BRE with

$$\begin{aligned}h(z; w) &= H(x, \tau^{h^*}(x), \tau^{j^*}(x), \tau^\delta), \\ j(z; w) &= J(x, \tau^{h^*}(x), \tau^{j^*}(x), \tau^\delta), \\ u(z) &= U(z).\end{aligned}$$

then $u(z)$, $h(z; w)$, $j(z; w)$, $\theta(z; w)$ satisfy (I.5)—(I.9) given the continuation sets (I.10) and (I.11).

Proof. The proof is a combination of Propositions I.1 and I.2. □

I.3 Derivation of Recursive Equilibrium from Discrete Time

This section presents the discrete-time counterpart of the model described in Section 2 in time intervals Δt ; i.e., $t = 0, \Delta t, 2\Delta t, \dots$. We use the equilibrium concept of a Markov perfect equilibrium. We follow [Stokey \(2008\)](#) to construct a discrete-time approximation of the worker's idiosyncratic productivity:

$$z'_\Delta = \begin{cases} z_\Delta + \sigma\sqrt{\Delta t} & \text{with probability } \frac{1 + \frac{\gamma}{\sigma}\sqrt{\Delta t}}{2} \\ z_\Delta - \sigma\sqrt{\Delta t} & \text{with probability } \frac{1 - \frac{\gamma}{\sigma}\sqrt{\Delta t}}{2} \end{cases}. \quad (\text{I.14})$$

Observe that the process is locally consistent with $dz_t = \gamma dt + \sigma dW_t^z$ (see [Kushner and Dupuis, 2001](#)).

Given the discrete-time nature of the problem, the timing within the period is as follows. At the beginning of the period t , workers' idiosyncratic productivity shocks are realized. Then, the labor market opens: exogenous and endogenous separations and new matches are realized. Finally, production takes place, and agents receive their payoffs. We define all the value functions after the realization of the idiosyncratic shocks and before the labor market opens.

Value functions. The value of an unemployed worker $u_{\Delta t}(z)$ is

$$u_{\Delta t}(z) = \max_w \left\{ e^{-f(\theta_{\Delta t}(z;w))\Delta t} \left[\tilde{B}e^z \Delta t + e^{-\rho\Delta t} \mathbb{E}_{z'} [u_{\Delta t}(z') | z] \right] + \left[1 - e^{-f(\theta_{\Delta t}(z;w))\Delta t} \right] \left[e^w \Delta t + e^{-\rho\Delta t} \mathbb{E}_{z'} [h_{\Delta t}(z'; w) | z] \right] \right\}. \quad (\text{I.15})$$

Here, $\tilde{B}e^z \Delta t$ is the flow income from unemployment, $1 - e^{-\Delta t f(\theta(z;w))}$ is the probability of finding a job with flow income $e^w \Delta t$ and continuation value $\mathbb{E}_{z'} [h_{\Delta t}(z'; w) | z]$, $e^{-\rho\Delta t}$ is the discount factor and z' is a random variable with law of motion (I.14). We use the notation $w_{\Delta t}^*(z)$ to denote the optimal search policy of an unemployed worker.

The vacancy cost for a period Δt is $\tilde{K}e^z \Delta t$ and the expected return is $[1 - e^{-q(\theta(z;w))\Delta t}] j_{\Delta t}(z; w)$. The free entry condition is given by

$$\min\{\tilde{K}e^z \Delta t - [1 - e^{-q(\theta_{\Delta t}(z;w))\Delta t}] j_{\Delta t}(z; w), \theta_{\Delta t}(z; w)\} = 0. \quad (\text{I.16})$$

Thus, if the cost of posting vacancies is larger than the expected value of finding a worker—i.e., $\tilde{K}e^z \Delta t - [1 - e^{-q(\theta_{\Delta t}(z;w))\Delta t}] j_{\Delta t}(z; w) > 0$ —then $\theta_{\Delta t}(z; w) = 0$. Similarly, if the submarket $(z; w)$ is open, then the free entry condition holds with equality $\tilde{K}e^z \Delta t = [1 - e^{-q(\theta_{\Delta t}(z;w))\Delta t}] j_{\Delta t}(z; w)$.

Let $\mathbb{I}_{\Delta t}^i(z; w) \in \{0, 1\}$ be an indicator variable equal to one when the worker chooses to continue in the match and 0 if the worker chooses to quit. Similarly, based on the $(z; w)$ pair, a matched firm chooses to lay a worker off when $\mathbb{I}_{\Delta t}^j(z; w) = 0$ and to continue in the match when $\mathbb{I}^j(z; w) = 1$.

Given firm policy $\mathbb{I}^j(z; w)$, the value function of a worker with productivity z employed at wage

w is

$$h_{\Delta t}^j(z; w) = \begin{cases} \max \left\{ e^{-\delta \Delta t} \left[e^w \Delta t + e^{-\rho \Delta t} \mathbb{E}_{z'} \left[h_{\Delta t}^j(z'; w) | z \right] \right] + (1 - e^{-\delta \Delta t}) u_{\Delta t}(z), u_{\Delta t}(z) \right\}, & \mathbb{I}_{\Delta t}^j(z; w) = 1 \\ u_{\Delta t}(z), & \mathbb{I}_{\Delta t}^j(z; w) = 0. \end{cases} \quad (\text{I.17})$$

If the firm chooses not to lay the worker off, then the employed worker chooses between quitting her job or not while consuming the constant wage w . The notation makes it clear that the fixed point in (I.17) depends on the firm's policy function. We define $h_{\Delta t}(z; w) := h_{\Delta t}^{\mathbb{I}_{\Delta t}^{j*}}(z; w)$, where $\mathbb{I}_{\Delta t}^{j*}$ denotes the firm's *optimal* policy function.

Similarly, given a worker's policy $\mathbb{I}_{\Delta t}^h(z; w)$, the value of a firm matched with a worker with wage w and productivity z is

$$j_{\Delta t}^h(z; w) = \begin{cases} \max \left\{ e^{-\delta \Delta t} \left[(e^z - e^w) \Delta t + e^{-\rho \Delta t} \mathbb{E}_{z'} \left[j_{\Delta t}^h(z'; w) | z \right] \right], 0 \right\} & \text{if } \mathbb{I}_{\Delta t}^h(z; w) = 1 \\ 0 & \text{if } \mathbb{I}_{\Delta t}^h(z; w) = 0 \end{cases} \quad (\text{I.18})$$

We define $j_{\Delta t}(z; w) := j_{\Delta t}^{\mathbb{I}_{\Delta t}^{h*}}(z; w)$, where $\mathbb{I}_{\Delta t}^{h*}$ denotes the worker's *optimal* policy function.

We are ready to define a Markov Perfect equilibrium with the additional refinement that continuation in the match needs to be a *weakly* dominant strategy.

Definition I.3. A Markov Perfect equilibrium is a set $\{h_{\Delta t}(z; w), j_{\Delta t}(z; w), u_{\Delta t}(z), \theta_{\Delta t}(z; w)\}$ of value functions and market tightness together with policy functions $\{\mathbb{I}_{\Delta t}^{h*}, \mathbb{I}_{\Delta t}^{j*}, w_{\Delta t}^*(z)\}$ such that:

- (i) Given $h_{\Delta t}(z; w)$ and $\theta_{\Delta t}(z; w)$, $u_{\Delta t}(z)$ satisfies the value function (I.15) with optimal policy function $w_{\Delta t}^*(z)$.
- (ii) Given $j_{\Delta t}(z; w)$, the market tightness $\theta_{\Delta t}(z; w)$ satisfies (I.16).
- (iii) Given $u(z)$ and $\mathbb{I}_{\Delta t}^{j*}$, $h_{\Delta t}(z; w) = h_{\Delta t}^{\mathbb{I}_{\Delta t}^{j*}}(z; w)$ satisfies the value function (I.17) with optimal policy $\mathbb{I}_{\Delta t}^{h*}(z; w)$. Moreover, if for any function $\mathbb{I}_{\Delta t}^j$, the value function in (I.17) given by $h_{\Delta t}^{\mathbb{I}_{\Delta t}^j}(z; w)$ satisfies

$$e^{-\delta \Delta t} \left[e^w \Delta t + e^{-\rho \Delta t} \mathbb{E}_{z'} \left[h_{\Delta t}^j(z'; w) | z \right] \right] + (1 - e^{-\delta \Delta t}) u_{\Delta t}(z) \geq u_{\Delta t}(z)$$

with strict inequality for some $\mathbb{I}_{\Delta t}^j$, then $\mathbb{I}_{\Delta t}^{h*}(z; w) = 1$.

- (iv) Given $\mathbb{I}_{\Delta t}^{h*}$, $j_{\Delta t}(z; w) = j_{\Delta t}^{\mathbb{I}_{\Delta t}^{h*}}(z; w)$ satisfies the value function (I.18) with optimal policy $\mathbb{I}_{\Delta t}^{j*}(z; w)$.

Moreover, if for any function $\mathbb{I}_{\Delta t}^h$, the value function in (I.17) given by $j_{\Delta t}^h(z; w)$ satisfies

$$e^{-\delta\Delta t} \left[(e^z - e^w) \Delta t + e^{-\rho\Delta t} \mathbb{E}_{z'} \left[j_{\Delta t}^h(z'; w) | z \right] \right] \geq 0$$

with strict inequality for some $\mathbb{I}_{\Delta t}^h$, then $\mathbb{I}_{\Delta t}^{j*}(z; w) = 1$.

A comparison with the main text's recursive equilibrium is helpful. First, in the main text, we use the *optimal* continuation set of each agent to define the equilibrium's best response. This is the reason why the value functions were not indexed by the continuation set of the other agent. Second, unmatched workers and firms internalize the outcome of the nontrivial Nash Equilibrium through $h_{\Delta t}(z; w)$ and $j_{\Delta t}(z; w)$, respectively. Third, the Nash equilibrium part of the definition imposes that the worker's optimal quit strategy is the best response to the firm's layoff policy and vice versa. Fourth, the refinement based on weakly dominating continuation strategies is applied in two steps. In the first step, we solve the decision problem of an agent for a given continuation policy of the other agent. In the second step, we verify that continuing in the match weakly dominates leaving it for all continuation policies of the other agent—not necessarily the optimal one.

We now proceed to derive the equilibrium conditions when $\Delta t \downarrow 0$. Define the following limits

$$\begin{aligned} u(z) &= \lim_{\Delta t \downarrow 0} u_{\Delta t}(z), \quad h(z; w) = \lim_{\Delta t \downarrow 0} h_{\Delta t}(z; w), \\ j(z; w) &= \lim_{\Delta t \downarrow 0} j_{\Delta t}(z; w), \quad \theta(z; w) = \lim_{\Delta t \downarrow 0} \theta_{\Delta t}(z; w). \end{aligned}$$

Below, we use the fact that for any function $u_{\Delta t}(z)$ the following two properties hold:

$$\lim_{\Delta t \downarrow 0} \frac{\mathbb{E}_{z'} [u_{\Delta t}(z') | z] - u(z)}{\Delta t} = \mathcal{A}^z u,$$

where \mathcal{A}^z is the characteristic operator of $dz_t = \gamma dt + \sigma d\mathcal{W}_t^z$, and

$$\lim_{\Delta t \downarrow 0} \mathbb{E}_{z'} [u_{\Delta t}(z') | z] = u(z).$$

Similar properties apply to $h_{\Delta t}(z; w)$ and $j_{\Delta t}(z; w)$. For details regarding the convergence of the limit when $\Delta t \downarrow 0$, see Chapters 9 and 10 of [Kushner and Dupuis \(2001\)](#).

Unemployed worker's HJB equation. Using the fact that $e^{-\rho\Delta t} = 1 - \rho\Delta t + o(\Delta t^2)$ and $e^{-f(\theta(z; w))\Delta t} = 1 - f(\theta(z; w))\Delta t + o(\Delta t^2)$, from (I.15) we have that $0 = \tilde{B}e^z + \frac{\mathbb{E}_{z'} [u_{\Delta t}(z') | z] - u_{\Delta t}(z)}{\Delta t} - \rho\mathbb{E}_{z'} [u_{\Delta t}(z') | z] + \max_w f(\theta_{\Delta t}(z; w))\mathbb{E}_{z'} [(h_{\Delta t}(z'; w) - u_{\Delta t}(z')) | z] + o(\Delta t)$. Using the fact that $\lim_{\Delta t \downarrow 0} \frac{\mathbb{E}_{z'} [u_{\Delta t}(z') | z] - u_{\Delta t}(z)}{\Delta t} =$

$\gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 u(z)}{\partial z^2}$, $\lim_{\Delta t \downarrow 0} \mathbb{E}_{z'} [u_{\Delta t}(z') | z] = u(z)$, and $\lim_{\Delta t \downarrow 0} \mathbb{E}_{z'} [h_{\Delta t}(z'; w) | z] = h(z; w)$, we have that $\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 u(z)}{\partial z^2} + \max_w f(\theta(z; w)) (h(z; w) - u(z))$.

Free entry condition. For free entry in (I.16), notice that $[1 - e^{-q(\theta_{\Delta t}(z; w))\Delta t}] = q(\theta_{\Delta t}(z; w))\Delta t + o(\Delta t^2)$. Thus, taking the limit, we obtain $\min \{ \tilde{K}e^z - q(\theta(z; w))j(z; w), \theta(z; w) \} = 0$.

Nontrivial Nash Equilibrium. First, assume that $\mathbb{I}_{\Delta t}^{j*}(z; w) = 0$. Then, $h_{\Delta t}^{\mathbb{I}_{\Delta t}^{j*}}(z; w) = h_{\Delta t}(z; w) = u_{\Delta t}(z)$. Taking the limit, $\mathbb{I}^{j*}(z; w) = 0$, then $h(z; w) = u(z)$.

If $\mathbb{I}_{\Delta t}^{j*}(z; w) = 1$ and $\mathbb{I}_{\Delta t}^{h*}(z; w) = 1$, then

$$h_{\Delta t}(z; w) = e^{-\delta\Delta t} \left[e^{w\Delta t} + e^{-\rho\Delta t} \mathbb{E}_{z'} [h_{\Delta t}(z'; w) | z] \right] + (1 - e^{-\delta\Delta t})u_{\Delta t}(z)$$

and $h_{\Delta t}(z; w) \geq u_{\Delta t}(z)$. Or equivalently,

$$0 = e^w + \frac{\mathbb{E}_{z'} [h_{\Delta t}(z'; w) | z] - h_{\Delta t}(z; w)}{\Delta t} - \rho \mathbb{E}_{z'} [h_{\Delta t}(z'; w) | z] + \delta [(u_{\Delta t}(z) - h_{\Delta t}(z; w))] + o(\Delta t)$$

and $h_{\Delta t}(z; w) \geq u_{\Delta t}(z)$. Taking the limit, if $\mathbb{I}^{j*}(z; w) = 1$ and $\mathbb{I}^{h*}(z; w) = 1$, then

$$\begin{aligned} \rho h(z; w) &= e^w + \gamma \frac{\partial h(z; w)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 h(z; w)}{\partial z^2} + \delta (u(z) - h(z; w)), \\ h(z; w) &\geq u(z). \end{aligned}$$

If $\mathbb{I}_{\Delta t}^{j*}(z; w) = 1$ and $\mathbb{I}_{\Delta t}^{h*}(z; w) = 0$, then

$$h_{\Delta t}(z; w) \geq e^{-\delta\Delta t} \left[e^{w\Delta t} + e^{-\rho\Delta t} \mathbb{E}_{z'} [h_{\Delta t}(z'; w) | z] \right] + (1 - e^{-\delta\Delta t})u_{\Delta t}(z)$$

and

$$h_{\Delta t}(z; w) = u_{\Delta t}(z).$$

In the limit, $\mathbb{I}^{j*}(z; w) = 1$ and $\mathbb{I}^{h*}(z; w) = 0$,

$$\begin{aligned} \rho h(z; w) &\geq e^w + \gamma \frac{\partial h(z; w)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 h(z; w)}{\partial z^2} + \delta (u(z) - h(z; w)), \\ h(z; w) &= u(z). \end{aligned}$$

Therefore, we can summarize the worker's optimality condition as

$$\rho h(z; w) = \max \left\{ e^w + \gamma \frac{\partial h(z; w)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 h(z; w)}{\partial z^2} + \delta (u(z) - h(z; w)), \rho u(z) \right\} \text{ if } \mathbb{I}^{j*}(z; w) = 1.$$

Applying the same argument to the firm's problem, we have that

$$\rho j(z; w) = \max \left\{ e^z - e^w + \gamma \frac{\partial j(z; w)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 j(z; w)}{\partial z^2} + \delta (-j(z; w)), 0 \right\} \text{ if } \mathbb{I}^{h^*}(z; w) = 1$$

Finally, we characterize agents' continuation sets. We show that the worker's continuation set is

$$\mathcal{Z}_{\Delta t}^h(w) = \{z : h_{\Delta t}(z; w) > u_{\Delta t}(z) \text{ or } e^w \Delta t + e^{-\rho \Delta t} \mathbb{E}_{z'} [u_{\Delta t}(z') - u_{\Delta t}(z) | z] > 0\}.$$

Clearly, the worker will continue in the match if $h_{\Delta t}(z; w) > u_{\Delta t}(z)$. We now derive the equilibrium condition for continuation to be a weakly dominating strategy at $(z; w)$. Let us start from the definition of a weakly dominating strategy: Continuing in the match weakly dominates separating when the state is $(z; w)$ if, for all firm's policies $\mathbb{I}_{\Delta t}^j$, we have that

$$e^{-\delta \Delta t} \left[e^w \Delta t + e^{-\rho \Delta t} \mathbb{E}_{z'} \left[h_{\Delta t}^{\mathbb{I}^j}(z'; w) | z \right] \right] + (1 - e^{-\delta \Delta t}) u_{\Delta t}(z) \geq u_{\Delta t}(z),$$

with strict inequality for at least one policy $\mathbb{I}_{\Delta t}^j$. Operating

$$e^{-\delta \Delta t} \left[e^w \Delta t + e^{-\rho \Delta t} \mathbb{E}_{z'} \left[h_{\Delta t}^{\mathbb{I}^j}(z'; w) - u(z) | z \right] \right] \geq 0.$$

Since this holds for all $\mathbb{I}_{\Delta t}^j(z; w)$ at $(z; w)$, it also holds for the infimum of the firm's policy function.

Thus,

$$e^{-\delta \Delta t} \left[e^w \Delta t + e^{-\rho \Delta t} \mathbb{E}_{z'} \left[\inf_{\mathbb{I}^j} h_{\Delta t}^{\mathbb{I}^j}(z'; w) - u_{\Delta t}(z) | z \right] \right] \geq 0$$

Since worker's optimality imposes that $h_{\Delta t}^{\mathbb{I}^j}(z; w) \geq u_{\Delta t}(z)$, with equality when $\mathbb{I}^j(z; w) = 0$, we have that $\inf_{\mathbb{I}^j} h_{\Delta t}^{\mathbb{I}^j}(z; w) = u_{\Delta t}(z)$ and

$$e^w \Delta t + e^{-\rho \Delta t} \mathbb{E}_{z'} [u_{\Delta t}(z') - u_{\Delta t}(z) | z] \geq 0.$$

Define the productivity set

$$\text{WD}_{\Delta t}^h(w) = \left\{ z : e^w \Delta t + e^{-\rho \Delta t} \mathbb{E}_{z'} [u_{\Delta t}(z') - u_{\Delta t}(z) | z] \geq 0 \right\}.$$

Observe that if $\mathbb{I}_{\Delta t}^j(z; w) = 1$ for a given w and all z , it is easy to check that $h_{\Delta t}^{\mathbb{I}^j=1}(z; w) > u_{\Delta t}(z) \forall z \in \text{WD}_{\Delta t}^h(w)$. Thus, the set $\text{WD}_{\Delta t}^h(w)$ characterizes the productivity levels for which continuation is

a weakly dominating strategy for the worker. Taking the limit, we have that

$$\mathcal{Z}^h(w) = \{z : h(z; w) > u(z) \text{ or } e^w - \rho u(z) + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 u(z)}{\partial z^2} \geq 0\}.$$

Applying the same argument to the firm's problem, we have that

$$\mathcal{Z}^h(w) = \{z : j(z; w) > 0 \text{ or } e^z - e^w \geq 0\}.$$

II Additional Results for Section 3: Aggregate Shocks in Labor Markets with Inefficient Turnover

II.1 A Monetary Economy with Exogenous Money Supply

We modify four aspects of the baseline model. First, we introduce preferences over real money holdings:

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} \left(C_{it} + \mu \log \left(\frac{\hat{M}_{it}}{P_t} \right) \right) dt \right], \quad (\text{II.1})$$

where \hat{M}_{it} denotes the money holdings of worker i , P_t is the relative price of the consumption good in terms of money, and μ is a preference weight on real money holdings.

Second, workers face a budget constraint that reflects ownership of firms and access to complete financial markets. Given a history of labor market decisions regarding job search, job acceptance, and job dissolution, $lm_i^t := \{lm_{it'}\}_{t'=0}^t$, a worker's private income is $Y_t(lm_i^t)$, which equals the nominal value of the wage while employed and the nominal value of home production while unemployed. In addition, each worker receives transfers of T_{it} from the government and fully diversified claims on firms' profits. On the spending side, a worker pays for consumption expenditures $P_t C_{it}$ and the opportunity cost of holding money $i_t \hat{M}_{it}$ at a given interest rate $i_t \geq 0$. Letting Q_t denote the time-0 Arrow-Debreu price under complete markets, the worker's budget constraint is

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} Q_t (P_t C_{it} + i_t \hat{M}_{it} - Y_t(lm_i^t) - T_{it}) dt \right] \leq M_{i0}. \quad (\text{II.2})$$

The worker's problem is to choose a consumption stream $\{C_{it}\}_{t=0}^{\infty}$, labor market decisions $\{lm_{it}\}_{t=0}^{\infty}$, and money holdings $\{\hat{M}_{it}\}_{t=0}^{\infty}$ to maximize utility (II.1) subject to the budget constraint (II.2) at time 0.

Third, the economy is subject to shocks to the aggregate money supply M_t . We assume that the

log of the aggregate money supply m_t follows a Brownian motion with drift π and volatility ζ :

$$dm_t = \pi dt + \zeta d\mathcal{W}_t^m,$$

where \mathcal{W}_t^m is a Wiener process. Fourth and finally, we assume that the vacancy posting cost $\tilde{K}Z_t$ and the value of home production $\tilde{B}Z_t$ are both denominated in real terms.

Given these modifications, the market-clearing conditions for goods and money, respectively, are

$$\int_0^1 (C_{it} + \theta_{it} \mathbb{1}[E_{it} = u] \tilde{K}Z_{it}) di = \int_0^1 (Z_{it} \mathbb{1}[E_{it} = h] + \tilde{B}Z_{it} \mathbb{1}[E_{it} = u]) di, \quad (\text{II.3})$$

$$\int_0^1 \hat{M}_{it} di = M_t, \quad (\text{II.4})$$

where $\mathbb{1}[\cdot]$ is an indicator function that takes a logical expression as its argument. Equation (II.3) states that the sum of real consumption and recruiting expenses must equal the total market and home production of the good. Equation (II.4) states that the total demand of nominal money holdings across workers equals the aggregate money supply.

The following proposition characterizes the worker's problem in this monetary economy.

Proposition II.1. *Let $Q_0 = 1$ be the numéraire and assume $\mu = \rho + \pi - \zeta^2/2$. Then, $P_t = M_t$ and the value of a worker at time 0 is*

$$V_0 = \max_{\{lm_{it}\}_{i=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} \frac{Y(lm_i^t)}{P_t} dt \right] + k,$$

where k is a constant independent of the worker's choices, capturing the present discounted value of financial wealth.

Proposition II.1 shows that the price level equals the aggregate money supply and that maximizing (II.1) subject to (II.2) is equivalent to maximizing expected discounted real income. The result relies on three assumptions: (i) complete markets, (ii) worker preferences that are quasi-linear in consumption, and (iii) the log of aggregate money supply following a random walk with drift. The first two assumptions imply a constant marginal value of nominal wealth, which, combined with the last assumption, leads to a constant real interest rate and a one-for-one pass-through of money shocks to inflation.

Proof. Let V_0 be the present discounted value of the optimal plan. The worker's value is given by

$$V_0 = \max_{\{C_{it}, \hat{M}_{it}, lm_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} \left(C_{it} + \mu \log \left(\frac{\hat{M}_{it}}{P_t} \right) \right) dt \right],$$

subject to

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} Q_t (P_t C_{it} + i_t \hat{M}_{it} - Y(lm_{it}^t) - T_{it}) dt \right] \leq M_{i0}. \quad (\text{II.5})$$

The first-order conditions for consumption and money holdings, combined with the definition of the nominal interest rate, are given by

$$e^{-\rho t} = \Lambda_i Q_t P_t, \quad (\text{II.6})$$

$$\mu \frac{e^{-\rho t}}{\hat{M}_{it}} = \Lambda_i Q_t i_t, \quad (\text{II.7})$$

$$\mathbb{E}[dQ_t] = -i_t Q_t dt. \quad (\text{II.8})$$

Here, Λ_i is the Lagrange multiplier of (II.5) for each worker. Equation (II.6) shows that $\Lambda_i = \Lambda$ for all i . Taking integrals over (II.7), we can replace $\hat{M}_{it} = M_t$. With these results, we guess and verify the following equilibrium outcomes

$$P_t = A^p M_t, \quad (\text{II.9})$$

$$i_t = A^i,$$

$$Q_t = \frac{A^Q e^{-\rho t}}{M_t}.$$

given a set of constants A^p , A^i , and A^Q . Using the guess in (II.6) and (II.7)

$$1 = \Lambda A^Q A^p, \quad (\text{II.10})$$

$$\mu = \Lambda A^Q A^i. \quad (\text{II.11})$$

Equations (II.10) and (II.11) provide the equilibrium values for A^Q and A^p given A^i . Applying Ito's lemma and using the guess over (II.8)

$$\begin{aligned} dQ_t &= A^Q d \left(\frac{e^{-\rho t}}{e^{\log(M_t)}} \right), \\ &= -\rho A^Q \left(\frac{e^{-\rho t}}{e^{\log(M_t)}} \right) dt - A^Q \frac{e^{-\rho t}}{e^{\log(M_t)}} d\log(M_t) + A^Q \frac{e^{-\rho t}}{2e^{\log(M_t)}} (d\log(M_t))^2, \end{aligned}$$

$$= -\rho Q_t dt - \pi Q_t dt - \zeta Q_t d\mathcal{W}_t^m + \frac{\zeta^2}{2} Q_t dt.$$

Thus, using the guess (II.9) and $\mathbb{E}[d\mathcal{W}_t^m] = 0$

$$\mathbb{E}[dQ_t] = - \underbrace{\left(\rho + \pi - \frac{\zeta^2}{2}\right)}_{=A^i} Q_t dt.$$

If we take as numéraire $Q_0 = 1$, then we verify the guess with $\mu = \rho + \pi - \frac{\zeta^2}{2}$:

$$\begin{aligned} A^Q &= M_0, \\ A^i &= \rho + \pi - \frac{\zeta^2}{2} = \mu, \\ \Lambda &= \frac{\mu}{M_0(\rho + \pi - \zeta^2/2)} = \frac{1}{M_0}, \\ A^p &= \frac{\rho + \pi - \zeta^2/2}{\mu} = 1. \end{aligned}$$

Using the budget constraint (II.5)

$$\begin{aligned} \mathbb{E}_0 \left[\int_0^\infty Q_t (P_t C_{it} + i_t \hat{M}_{it} - Y(lm_i^t) - T_{it}) dt \right] &= M_{i0} \iff \\ \mathbb{E}_0 \left[\int_0^\infty \frac{M_0 e^{-\rho t}}{M_t} (M_t C_{it} + \mu M_t - Y(lm_i^t) - T_{it}) dt \right] &= M_{i0} \iff \\ M_0 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} C_{it} dt \right] &= M_{i0} + M_0 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{Y(lm_i^t)}{M_t} dt \right] + M_0 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{T_{it}}{M_t} dt \right] - \frac{M_0}{\rho} \mu \iff \\ \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} C_{it} dt \right] &= \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{Y(lm_i^t)}{M_t} dt \right] + k_i, \end{aligned}$$

where k_i is a constant independent of the worker's policies. Thus,

$$\begin{aligned} V_0 &= \max_{\{C_{it}, \hat{M}_{it}, lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(C_{it} + \mu \log \left(\frac{\hat{M}_{it}}{P_t} \right) \right) dt \right], \\ &= \max_{\{C_{it}, lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(C_{it} + \mu \log \left(\frac{\mu}{\rho + \pi - \zeta^2/2} \right) \right) dt \right], \\ &= \max_{\{C_{it}, lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} C_{it} dt \right], \\ &= \max_{\{lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{Y(lm_i^t)}{M_t} dt \right] + k_i. \end{aligned}$$

□

II.2 A Monetary Economy with a Taylor Rule and Interest Rate Shocks

We now show that our previous environment is isomorphic to an economy in which the monetary authority sets the interest rate by following a Taylor rule. As in Galí (2015), we study a cashless economy in discrete time where $t \in \mathbb{T} = \{0, \Delta, 2\Delta, 3\Delta, \dots\}$.

Preferences are the same as in Section 2:

$$\mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} e^{-\rho t} C_{it} \Delta \right], \quad (\text{II.12})$$

where $e^{-\rho t}$ denotes the discount factor. Workers face a budget constraint that reflects ownership of firms and access to complete financial markets. Given a history of labor market decisions regarding job search, job acceptance, and job dissolution, $lm_i^t := \{lm_{it'}\}_{t'=0}^t$, a worker's private income is $Y_t(lm_i^t)$, which equals the nominal value of the wage while employed and the nominal value of home production while unemployed. In addition, each worker receives transfers of T_{it} from the government and fully diversified claims on firms' profits. On the spending side, a worker pays for consumption expenditures $P_t C_{it}$. Letting Q_t denote the time-0 Arrow-Debreu price under complete markets, the worker's budget constraint is

$$\mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} Q_t (P_t C_{it} - Y_t(lm_i^t) - T_{it}) \Delta \right] \leq 0. \quad (\text{II.13})$$

The worker's problem is to choose a consumption stream $\{C_{it}\}_{t=0}^{\infty}$ and labor market decisions $\{lm_{it}\}_{t=0}^{\infty}$ to maximize utility (II.12) subject to the budget constraint (II.13) at time 0.

In this microfoundation, the central bank sets the nominal interest rate following a Taylor rule given by

$$i_t = \rho + \bar{\pi} + \phi_{\pi}(\pi_t - \bar{\pi}) + \iota_t$$

Here, i_t is the nominal interest rate, $\bar{\pi}$ is the inflation target, and ι_t is a compound Poisson process such that with probability $e^{-\lambda \Delta}$ it is equal to zero and with probability $1 - e^{-\lambda \Delta}$ it is equal to $\epsilon_t \sigma_i / \Delta$, where ϵ_t is an i.i.d. random variable with mean zero and standard deviation of 1.

Finally, we assume that the vacancy posting cost $\tilde{K}Z_t$ and the value of home production $\tilde{B}Z_t$ are both denominated in real terms. The market-clearing condition for the goods market is still given by (II.3).

The following proposition characterizes the worker's problem in this monetary economy.

Proposition II.2. *Take the limit $\Delta \downarrow 0$. Then,*

$$d\log(P_t) = \bar{\pi} dt - \frac{\sigma_i}{\phi_\pi} \epsilon_t d\mathcal{N}_t,$$

where \mathcal{N}_t is a Poisson process with intensity λ . The value of a worker at time 0 is

$$V_0 = \max_{\{lm_{it}\}_{i=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{Y(lm_i^t)}{P_t} dt \right] + k_i,$$

where k_i is a constant independent of the worker's choices, which captures the present discounted value of financial wealth.

Proof. Let V_0 be the present discounted value of the optimal plan. The worker's value is given by

$$V_0 = \max_{\{C_{it}, lm_{it}\}_{i=0}^\infty} \mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} e^{-\rho t} C_{it} \Delta \right],$$

subject to

$$\mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} Q_t (P_t C_{it} - Y_t(lm_i^t) - T_{it}) \Delta \right] \leq 0. \quad (\text{II.14})$$

The first-order condition for consumption is given by

$$e^{-\rho t} = \Lambda_i Q_t P_t, \quad (\text{II.15})$$

Here, Λ_i is the Lagrange multiplier of (II.14) for worker i . Equation (II.15) shows that $\Lambda_i = \Lambda$ for all i . Evaluating (II.15) at periods t and $t + \Delta$ and taking their ratio, we have

$$e^{-\rho \Delta} = \frac{Q_{t+\Delta} P_{t+\Delta}}{Q_t P_t} \iff \mathbb{E}_t \left[\frac{P_t}{P_{t+\Delta}} e^{-\rho \Delta} \right] = \mathbb{E}_t \left[\frac{Q_{t+\Delta}}{Q_t} \right]$$

By definition of the interest rate, the lack of arbitrage opportunities with the nominal bond offered by the monetary authority $\mathbb{E}_t \left[\frac{Q_{t+\Delta}}{Q_t} \right] = e^{-i_t \Delta}$, and the Taylor rule, we have the following system of equations:

$$1 = e^{i_t \Delta} e^{-\rho \Delta} \mathbb{E}_t \left[\frac{P_t}{P_{t+\Delta}} \right] \quad \text{and} \quad i_t = \rho + \bar{\pi} + \phi_\pi (\pi_t - \bar{\pi}) + \iota_t.$$

Since $P_{t+\Delta} = P_t e^{\pi_{t+\Delta} \Delta}$, from the first equation we have that $e^{\rho \Delta} = e^{i_t \Delta} \mathbb{E}_t \left[e^{-\pi_{t+\Delta} \Delta} \right]$. Making a first

order Taylor approximation when $\Delta \downarrow 0$, we obtain $\rho = i_t - \mathbb{E}_t[\pi_{t+\Delta}]$. Replacing in the expression for i_t from the Taylor rule,

$$\pi_t = \bar{\pi} - \frac{\iota_t}{\phi_\pi} + \frac{1}{\phi_\pi} \mathbb{E}_t[\pi_{t+\Delta} - \bar{\pi}].$$

Iterating this equation forward, inflation can be expressed as a function of the current and future shocks:

$$\pi_t - \bar{\pi} = -\mathbb{E}_t \left[\sum_{j=0}^{\infty} \left(\frac{1}{\phi_\pi} \right)^{j\Delta} \frac{\iota_{t+j}}{\phi_\pi} \right].$$

Since $\mathbb{E}_t[\iota_{t+j}] = 0$, we have that $\pi_t - \bar{\pi} = -\frac{\iota_t}{\phi_\pi}$, and therefore

$$\log(P_{t+\Delta}) = \log(P_t) + \pi_{t+\Delta}\Delta = \log(P_t) + \bar{\pi}\Delta - \frac{\iota_t\Delta}{\phi_\pi} = \log(P_t) + \bar{\pi}\Delta - \mathbb{B}_t \frac{\sigma_i}{\phi_\pi} \epsilon_t,$$

where \mathbb{B}_t is a random variable equal to one with probability $1 - e^{-\lambda\Delta}$ and zero otherwise. Taking the limit $\Delta \downarrow 0$, we have a continuous-time compound Poisson process for the aggregate consumer price index

$$d\log(P_t) = \bar{\pi} dt - \frac{\sigma_i}{\phi_\pi} \epsilon_t dN_t,$$

where N_t is a Poisson process with intensity λ . Combining the fact that $e^{-\rho t} = \Lambda Q_t P_t$ with the worker's budget constraint:

$$\begin{aligned} \mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} e^{-\rho t} C_{it} \Delta \right] &= \Lambda \mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} Q_t P_t C_{it} \Delta \right] \\ &= \Lambda \mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} Q_t (Y_t(lm_i^t) + T_{it}) \Delta \right] \\ &= \Lambda \mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} Q_t Y_t(lm_i^t) \Delta \right] + \underbrace{\Lambda \mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} Q_t T_{it} \Delta \right]}_{=k_i} \\ &= \mathbb{E}_0 \left[\sum_{t \in \mathbb{T}} e^{-\rho t} \frac{Y_t(lm_i^t)}{P_t} \Delta \right] + k_i. \end{aligned}$$

Taking the limit when $\Delta \downarrow 0$, we have the desired result. □

II.3 Model Extension: Staggered Wage Renegotiations

In this section, we generalize our model to the case of staggered wage renegotiations, which we assume follow a Nash bargaining protocol with worker weight α and to occur at rate $\delta^r \geq 0$ à la Calvo (1983). The generalized model nests as a special case the economy with fully rigid wages ($\delta^r \rightarrow 0$) presented in the main text and also the polar opposite case with fully flexible wages ($\delta^r \rightarrow \infty$). By convexifying between these two cases, the generalized model allows for arbitrary frequencies of wage changes in employment that can be matched to the data. The generalized model with staggered wage renegotiations yields several results but our main conclusion is that all of our key insights extend to an environment with $0 < \delta^r < \infty$ subject to minor modifications.

In the baseline model, wages are completely rigid within a worker-firm match. We have chosen to present and analyze this simplified model to enhance clarity and provide a first theoretical foundation for analyzing several core aspects of the model. Nevertheless, this assumption is unrealistic and affects our results. To address this limitation, we follow the approach of Gertler and Trigari (2009) and extend the baseline model by incorporating staggered wage renegotiations à la Calvo (1983) in a way consistent with the Hosios (1990) condition. The resulting insight is that all our key findings can be extended to this more general environment. Importantly, almost all the proofs can be extended to the model involving wage adjustments within a match with only minor modifications. Next, we present a summary of how wage renegotiations within a match affect our results.

Environment.

Here, we modify the baseline setup exclusively to permit wage renegotiations within a match. The preferences, technology, and search frictions remain the same as those outlined in the main text. Wage bargaining within a match is modeled by a Poisson process with a rate denoted by $\delta^r \geq 0$. We assume that these renegotiations entail setting the wage within a worker-firm match according to Nash bargaining, with the worker's weight over the prevailing surplus at the time of renegotiation given by α (i.e., equal to the elasticity of the matching function to satisfy Hosios condition). As $\delta^r \rightarrow 0$, we recover the baseline model with fully rigid wages within a match. As $\delta^r \rightarrow \infty$, the model transitions to the opposite extreme where wages are flexible and continuously reset.

Recursive Formulation.

Value Functions. The Hamilton-Jacobi-Bellman (HJB) equation of an unemployed worker is

still given by

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 u(z)}{\partial z^2} + \max_w f(\theta(z; w)) [h(z; w) - u(z)].$$

The HJB equation of a worker employed at log wage w with log productivity $z \in \mathcal{Z}^j(w)$, for which the firm prefers to continue, is now

$$\rho h(z; w) = \max \left\{ e^w + \gamma \frac{\partial h(z; w)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 h(z; w)}{\partial z^2} + \delta^r [h(z; w^*(z)) - h(z; w)] - \delta [h(z; w) - u(z)], \rho u(z) \right\},$$

and for productivities $z \notin \mathcal{Z}^j(w)$ (i.e., at productivity for which the firm prefers to dissolve the match), the HJB is given by

$$h(z; w) = u(z) \quad \forall z \in (\mathcal{Z}^{j^*}(w))^c,$$

with $h(\cdot; w) \in \mathbf{C}^1(\mathcal{Z}^{j^*}(w)) \cap \mathbf{C}(\mathbb{R})$. The only difference relative to the baseline model is the term $\delta^r [h(z; w^*(z)) - h(z; w)]$, which captures the expected capital gain from wage renegotiation.

Similarly, the HJB equation of a firm employing a worker at log wage w with log productivity $z \in \mathcal{Z}^h(w)$, for which the worker prefers to continue the match, is now given by

$$\rho j(z; w) = \max \left\{ e^z - e^w + \gamma \frac{\partial j(z; w)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 j(z; w)}{\partial z^2} + \delta^r [j(z; w^*(z)) - j(z; w)] - \delta j(z; w), 0 \right\}.$$

and the HJB equation evaluated at log productivity $z \notin \mathcal{Z}^h(w)$, when the worker prefers to dissolve the match, is given by

$$j(z; w) = 0 \quad \forall z \in (\mathcal{Z}^{h^*}(w))^c,$$

with $j(\cdot; w) \in \mathbf{C}^1(\mathcal{Z}^{h^*}(w)) \cap \mathbf{C}(\mathbb{R})$. Again, the only difference is the term $\delta^r [j(z; w^*(z)) - j(z; w)]$, which captures the expected capital gain that a firm experiences when renegotiating the wage with the worker.

Continuation Sets. The firm's and worker's optimal continuation sets are

$$\begin{aligned} \mathcal{Z}^{j^*}(w) &= \text{int} \{z \in \mathbb{R} : j(z; w) > 0 \text{ or } e^z - e^w > 0\}, \\ \mathcal{Z}^{h^*}(w) &= \text{int} \left\{ z \in \mathbb{R} : h(z; w) > u(z) \text{ or } 0 < e^w - \rho u(z) + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 u(z)}{\partial z^2} \right\}. \end{aligned}$$

Observe the following property: Allowing for wage renegotiation within a match does not

affect the equilibrium conditions characterizing the continuation sets. We first explain the intuition behind this result and then we formalize it.

To understand the intuition, we follow the same steps as in Online Appendix I.3. We do so by focusing on the continuation set of the worker—a similar logic applies to the continuation set of the firm. Take a discrete time approximation Δt of our model. Let $\mathbb{I}_{\Delta t}^j(z; w)$ denote the firm's continuation policy, which is equal to 1 if the firm continues the match and zero if the firm lays the worker off. Continuing in the match dominates separating from it whenever

$$h_{\Delta t}(z; w) = e^w \Delta t + e^{-\rho \Delta t} \mathbb{E}_{(z; w)} [h_{\Delta t}(z'; w') \mathbb{I}^j(z'; w') + u_{\Delta t}(z')(1 - \mathbb{I}^j(z'; w'))] > u_{\Delta t}(z), \quad (\text{II.16})$$

for the firm's stopping policy $\mathbb{I}^j(z'; w')$. Here, $\mathbb{E}_{(z; w)}[\cdot]$ denotes the conditional expectation given wage renegotiations, the law of motion of productivity, and exogenous separations. Since the worker is optimally choosing to stay or quit, we have $h^\Delta(z; w) \geq u^\Delta(z)$ for all z . Since condition (II.16) holds for any policy $\mathbb{I}^j(z'; w')$, it must also hold for $\mathbb{I}^j(z'; w') = 0$ for all $(z'; w')$. Therefore,

$$e^w + \frac{e^{-\rho \Delta t} \mathbb{E}_{(z; w)} [u_{\Delta t}(z; w)(z'; w')] - u(z)}{\Delta t} > 0.$$

Taking the limit as $\Delta t \downarrow 0$, we have

$$e^w - \rho u(z) + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 u(z)}{\partial z^2} > 0,$$

which is the same condition derived from our baseline model.

The reason why wage renegotiations do not affect the condition for continuation to be a weakly dominating strategy is that they do not directly affect the worker's or the firm's value conditional on a separation (i.e., $u(z)$ and 0). Formally, define the law motion of the worker's state variables as

$$\begin{aligned} dw_t &= (\hat{w}^*(z) - \hat{w}_{t-}) dN_t, \\ dz_t &= \gamma dt + \sigma d\mathcal{W}_t^z, \end{aligned}$$

where \mathcal{N}_t is a Poisson counter with arrival rate δ^r and $\hat{w}^*(z)$ is the bargained wage (see below). Given the law of motion of the state and a stopping time τ^m , the value functions of the worker and

the firm are

$$h(w, z) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} e^{w_t} dt + e^{-\rho \tau^m} u(z_{\tau^m}) \right],$$

$$j(w, z) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} [e^{z_t} - e^{w_t}] + e^{-\rho \tau^m} \times 0 dt \right].$$

Let \mathcal{A} be the characteristic operator of (w_t, z_t) adjusted by discounting—i.e., $\mathcal{A}(\phi) = -\rho\phi(w, z) + \delta^r(\phi(w^*(z), z) - \phi(w, z)) + \gamma \frac{\partial \phi(\cdot)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 \phi(\cdot)}{\partial z^2}$. Then, the pairs (w, z) for which the worker and the firm prefer to continue for every stopping time are given by

$$e^w + \mathcal{A}(u) > 0,$$

$$1 - e^w + \mathcal{A}(0) > 0.$$

Wage Renegotiations. Let $w^*(z)$ be the solution to a Nash bargaining problem with worker's bargaining weight given by α , which satisfies the [Hosios \(1990\)](#) condition:

$$w^*(z) = \arg \max_w \left\{ (h(z; w) - u(z))^\alpha j(z; w)^{1-\alpha} \right\}.$$

We conclude this section with a discussion on the wage-renegotiation protocol. Our aim is to expand our model by incorporating the on-the-job bargaining framework as presented in [Gertler and Trigari \(2009\)](#), while keeping the economic environment unchanged. However, it is important to make some comments. First, the opportunity cost for each agent during bargaining is the corresponding value of separation ($u(z)$ for the worker and 0 for the firm), not the corresponding value at the current wage. This implies that agents commit to separate from the match in the off-equilibrium event that bargaining fails—the conventional assumption adopted in the literature (e.g., [Shimer, 2005](#); [Gertler and Trigari, 2009](#)). For a deviation from this assumption, refer to [Blanco and Drenik \(2022\)](#). Second, we adhere to the [Hosios \(1990\)](#) condition. The consequence of this assumption is that the entry wage coincides with the bargained wage. While a deviation from the [Hosios \(1990\)](#) condition would break this equality, the economic mechanisms affecting the entry and bargained wages remain the same.

Equilibrium Characterization.

Using the change of notation adopted in Lemma 1— $\hat{w} := w - z$, $\hat{\rho} := \rho - \gamma - \sigma^2/2$ and

$\hat{\gamma} := \gamma + \sigma^2$ —we define

$$(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{\theta}(w-z)) = \left(\frac{u(z)}{e^z}, \frac{j(z;w)}{e^z}, \frac{h(z;w) - u(z)}{e^z}, \theta(z;w) \right).$$

Rewriting the HJB equations using this change of notation, we get

$$\begin{aligned} \hat{\rho}\hat{U} &= \bar{B} + \max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w}) \\ (\hat{\rho} + \delta)\hat{W}(\hat{w}) &= \max \left\{ e^{\hat{w}} - \hat{\rho}\hat{U} + \delta^r (\hat{W}(\hat{w}^*) - \hat{W}(\hat{w})) - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}), 0 \right\} \\ (\hat{\rho} + \delta)\hat{J}(\hat{w}) &= \max \left\{ 1 - e^{\hat{w}} + \delta^r (\hat{J}(\hat{w}^*) - \hat{J}(\hat{w})) - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}), 0 \right\}. \end{aligned}$$

Here, the terms $\delta^r\hat{W}(\hat{w}^*)$ and $\delta^r\hat{J}(\hat{w}^*)$ are constant, since the reset wage \hat{w}^* does not depend on the current value of \hat{w} . Therefore, the problem is identical to that in the baseline model with completely rigid wages (i.e., with $\delta^r = 0$), with the exception of three aspects. First, the effective discount rate for both workers and firms becomes $\hat{\rho} + \delta + \delta^r$ instead of the previous expression $\hat{\rho} + \delta$. Second, the worker's flow value is now given by $e^{\hat{w}} + \delta^r\hat{W}(\hat{w}^*) - \hat{\rho}\hat{U}$ instead of the previous expression $e^{\hat{w}} - \hat{\rho}\hat{U}$. Third, the firm's flow value is now given by $1 - e^{\hat{w}} + \delta^r\hat{J}(\hat{w}^*)$ instead of the previous expression $1 - e^{\hat{w}}$. It is important to note that these expressions simplify to those from the baseline model as $\delta^r \rightarrow 0$.

We now extend our key results to the case of on-the-job wage renegotiations.

Equilibrium Policies. We now analyze how on-the-job wage renegotiation affects equilibrium policies associated with job creation and job destruction. In particular, we extend Proposition 2-Parts 1 to 3, which focus on job creation. We skip Part 4 of that proposition since its extension to a setting with wage renegotiation is trivial.

Proposition II.3. *With wage renegotiations à la Calvo (1983), the BRE has the following properties:*

1. *The joint match surplus satisfies*

$$\hat{S}(\hat{w}) = \frac{1 - \hat{\rho}\hat{U}}{1 - \delta^r\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)}\mathcal{T}(\hat{w}, \hat{\rho} + \delta^r),$$

where

$$\mathcal{T}(\hat{w}, \hat{\rho} + \delta^r) := \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-(\hat{\rho} + \delta^r)t} dt \right]$$

is the expected discounted duration of current wages and $1 > \hat{\rho}\hat{U} > \tilde{B}$. The following properties hold:

$$\lim_{\delta^r \downarrow 0} \hat{S}(\hat{w}) = (1 - \hat{\rho}\hat{U})\mathcal{T}(\hat{w}, \hat{\rho}) \text{ and } \lim_{\delta^r \rightarrow \infty} \hat{S}(\hat{w}) = \frac{(1 - \hat{\rho}\hat{U})}{\hat{\rho} + \delta} \quad (\text{II.17})$$

i.e., endogenous separations do not affect the surplus (and thus, the entry wage) when $\delta^r \rightarrow \infty$.

2. The competitive entry wage—i.e., $\hat{w}^* = \arg \max_{\hat{w}} f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w})$ —exists and is unique. Moreover, it solves:

$$\hat{w}^* = \arg \max_{\hat{w}} \left\{ \hat{W}(\hat{w})^\alpha \hat{J}(\hat{w})^{1-\alpha} \right\} = \arg \max_{\hat{w}} \left\{ \eta(\hat{w})^\alpha (1 - \eta(\hat{w}))^{1-\alpha} \mathcal{T}(\hat{w}, \hat{\rho} + \delta^r) \right\},$$

with optimality condition

$$\underbrace{\eta'(\hat{w}^*) \left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)} \right)}_{\text{Share channel}} = - \underbrace{\frac{\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho} + \delta^r)}{\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)}}_{\text{Surplus channel}}.$$

with $\eta(\hat{w}^*) = \alpha$ as $\delta^r \rightarrow \infty$.

3. Given $\eta(\hat{w}^*)$ and $\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)$, the equilibrium job finding rate $f(\hat{\theta}(\hat{w}^*))$ and the flow opportunity cost of employment $\hat{\rho}\hat{U}$ are given by

$$f(\hat{\theta}(\hat{w}^*)) = \left[(1 - \eta(\hat{w}^*)) \frac{1 - \hat{\rho}\hat{U}}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)} \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r) / \tilde{K} \right]^{\frac{1-\alpha}{\alpha}},$$

$$\hat{\rho}\hat{U} = \tilde{B} + \left(\tilde{K}^{\alpha-1} (1 - \eta(\hat{w}^*))^{1-\alpha} \eta(\hat{w}^*)^\alpha \frac{1 - \hat{\rho}\hat{U}}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)} \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r) \right)^{\frac{1}{\alpha}}.$$

Proof. Now, we prove each equilibrium property.

1. The fact that $\hat{\rho}U \geq \tilde{B}$ follows from the same argument as before. Combining the sequence and recursive formulations of the value functions, we have

$$\hat{W}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-(\hat{\rho} + \delta^r)t} (e^{\hat{w}t} + \delta^r \hat{W}(\hat{w}^*) - \hat{\rho}\hat{U}) dt \right]$$

$$\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-(\hat{\rho} + \delta^r)t} (1 - e^{\hat{w}t} + \delta^r \hat{J}(\hat{w}^*)) dt \right]$$

where τ^{m*} is the optimal stopping time that determines match duration. Summing up, we

have

$$\hat{S}(\hat{w}) = \hat{W}(\hat{w}) + \hat{J}(\hat{w}) = (1 + \delta^r \hat{S}(\hat{w}^*) - \hat{\rho} \hat{U}) \mathcal{T}(\hat{w}, \hat{\rho} + \delta^r).$$

Evaluating the expression for match surplus $\hat{S}(\hat{w})$ at \hat{w}^* , we get

$$\hat{S}(\hat{w}^*) = (1 + \delta^r \hat{S}(\hat{w}^*) - \hat{\rho} \hat{U}) \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)$$

and thus

$$\hat{S}(\hat{w}^*) = \frac{1 - \hat{\rho} \hat{U}}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)} \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r).$$

Plugging this back into the above expression, we obtain

$$\hat{S}(\hat{w}) = \frac{1 - \hat{\rho} \hat{U}}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)} \mathcal{T}(\hat{w}, \hat{\rho} + \delta^r),$$

which is an expression for $\hat{S}(\hat{w})$ that depends only on \hat{U} , $\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)$ and $\mathcal{T}(\hat{w}, \hat{\rho} + \delta^r)$, but not on $\hat{S}(\hat{w}^*)$. Since $\hat{W}(\hat{w}), \hat{J}(\hat{w}) \geq 0$, then $\hat{S}(\hat{w}) \geq 0$ and thus

$$0 \leq \hat{S}(\hat{w}^*) = (1 - \hat{\rho} \hat{U}) \underbrace{\frac{\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)}}_{>0} \iff 0 \leq 1 - \hat{\rho} \hat{U} \iff 1 \geq \hat{\rho} \hat{U}.$$

Therefore, $1 \geq \hat{\rho} \hat{U} \geq \tilde{B}$. To go from weak to strict inequalities, we follow the same steps as in the baseline model.

Observe that, if $\hat{\tau}^{m*}$ denotes only the stopping times arising from endogenous separations, then

$$\mathcal{T}(\hat{w}, \hat{\rho} + \delta^r) = \frac{1}{\hat{\rho} + \delta + \delta^r} \mathbb{E}_{\hat{w}} \left[1 - e^{-(\hat{\rho} + \delta + \delta^r) \hat{\tau}^{m*}} \right]$$

Using these results and the algebra of limits, we obtain

$$\lim_{\delta^r \rightarrow \infty} \hat{S}(\hat{w}) = \frac{1 - \hat{\rho} \hat{U}}{\hat{\rho} + \delta}.$$

2. The proof is analogous. For log-concavity of the value functions, the Ricatti equation continues

to hold following the redefinition of variables, but now

$$F(t) \equiv \frac{2}{\sigma^2} \left[(\hat{\rho} + \delta + \delta^r) - \frac{(e^{t+\hat{w}} - \hat{\rho}\hat{U}) + \delta^r \hat{W}(\hat{w}^*)}{\hat{W}(t + \hat{w})} \right].$$

3. The same equilibrium conditions apply. □

Discussion of the Effect of Wage Renegotiations on Job Creation and the Entry Wage. Before discussing Proposition II.3, it is important to highlight a distinction between the baseline model and the model with on-the-job bargaining. In the baseline model, $\mathcal{T}(\hat{w}, \hat{\rho})$ represented the expected discounted duration of a match, but now, $\mathcal{T}(\hat{w}, \hat{\rho} + \delta^r)$ denotes the expected discounted duration of the current wage. While these objects are trivially identical when $\delta^r = 0$, as shown in equation (II.17), they differ when $\delta^r > 0$.

Proposition II.3 formalizes a simple intuition: As the frequency of bargaining increases, the economic mechanisms influencing job creation resemble those in existing models of directed search. To illustrate this, Figure III shows the value functions of the worker and the firm, the surplus of the match, and the objective function $\hat{J}(\hat{w})^{1-\alpha} \hat{W}(\hat{w})^\alpha$ that the bargained wage \hat{w}^* maximizes. As we can see in Panel A, when $\delta^r = 0$, the surplus function exhibits curvature at $\hat{w} = \hat{w}^*$ since the entry wage affects the likelihood of future separations. Instead, when $\delta^r = 0.2$, the surplus function becomes independent of the wage for \hat{w} close to \hat{w}^* since the probability of an inefficient separation is small (i.e., it is quite likely that the wage will be renegotiated before the match gets endogenously dissolved). In the limit, as $\delta^r \rightarrow \infty$, the surplus function becomes a constant independent of \hat{w} .

Figure II2 plots the worker's share evaluated at the entry wage and the flow opportunity cost as a function of the renegotiation rate δ^r . As proposition II.3 shows, when $\delta^r \rightarrow \infty$, the worker's share of surplus $\eta(\hat{w}^*) \rightarrow \alpha$ and the flow opportunity cost $\hat{\rho}\hat{U}^*$ converges to the solution of the following equation

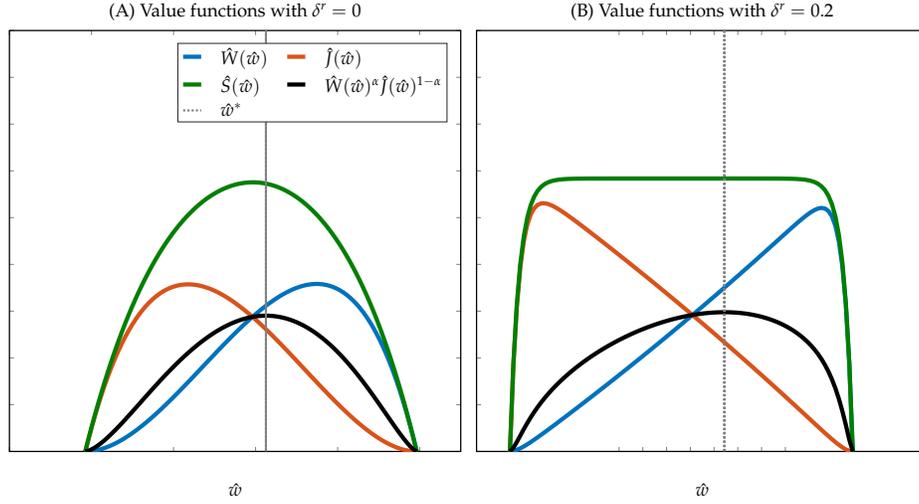
$$\hat{\rho}\hat{U}^{fb} = \tilde{B} + \left(\tilde{K}^{\alpha-1} (1 - \alpha)^{1-\alpha} \alpha^\alpha \frac{1 - \hat{\rho}\hat{U}^{fb}}{\hat{\rho} + \delta} \right)^{\frac{1}{\alpha}},$$

i.e., the flow opportunity cost when there are no inefficient separations.

Static and Dynamic Considerations Behind Equilibrium Policies. We now extend Propositions 3, 4, and 5 within the model with wage renegotiations.

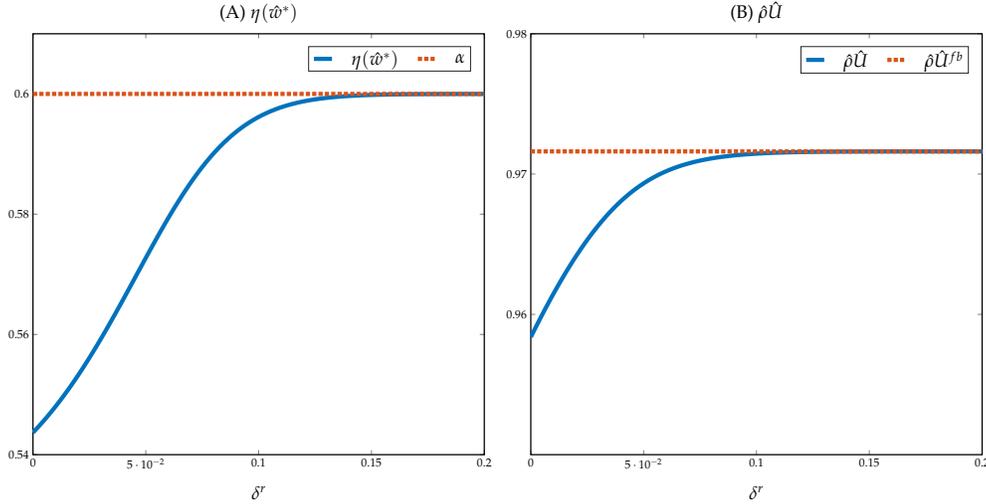
Proposition II.4. *The following properties hold:*

FIGURE II1. EQUILIBRIUM VALUES AND CONTINUATION SETS IN \hat{w} -SPACE



Notes: The figure plots the equilibrium value functions of the firm $\hat{J}(\hat{w})$ (red line) and the employed worker $\hat{W}(\hat{w})$ (blue line), the surplus of the match $\hat{S}(\hat{w})$ (green line), and the “Nash bargaining” objective function $\hat{J}(\hat{w})^{1-\alpha} \hat{W}(\hat{w})^\alpha$ (black line) as a function of $\hat{w} = w - z$ for $\delta^r = 0$ and $\delta^r = 0.2$. The vertical lines mark the boundaries of the firms’ continuation set $(-\infty, \hat{w}^+)$ and the worker’s continuation set (\hat{w}^-, ∞) .

FIGURE II2. EQUILIBRIUM POLICIES FOR DIFFERENT VALUES OF δ^r



Notes: Panels A and B plot the worker’s share evaluated at the entry wage $\eta(\hat{w}^*)$ and the flow opportunity cost $\hat{\rho}\hat{U}$ as a function of the renegotiation rate δ^r , respectively. The solid blue lines show the equilibrium values of $\eta(\hat{w}^*)$ and $\hat{\rho}\hat{U}$, while the dotted red lines show the corresponding values when $\delta^r \rightarrow \infty$.

1. If $\gamma = \sigma = 0$, then the optimal policies are given by

$$(\hat{w}^-, \hat{w}^*, \hat{w}^+) = \log \left(\hat{\rho}\hat{U} - \frac{\delta^r}{\hat{\rho} + \delta} \alpha (1 - \hat{\rho}\hat{U}), \alpha + (1 - \alpha) \hat{\rho}\hat{U}, 1 + \frac{\delta^r}{\hat{\rho} + \delta} (1 - \alpha) (1 - \hat{\rho}\hat{U}). \right)$$

with $\eta(\hat{w}^*) = \alpha$ and $\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r) = 1/(\hat{\rho} + \delta)$.

2. Assume $\hat{\gamma} = 0$ and $\alpha = 1/2$, and a first-order approximation of the flow payoffs around \hat{w}^* . Then

$$\hat{w}^\pm = \hat{w}^* \pm h(\varphi, \Phi, \frac{\delta^r}{\delta + \hat{\rho}}) \text{ with } \varphi = \sqrt{2 \frac{\hat{\rho} + \delta + \delta^r}{\sigma^2}} \text{ and } \Phi = \frac{1 - \hat{\rho}\hat{U}}{1 + \hat{\rho}\hat{U}}. \text{ If } \delta^r \rightarrow \infty, \text{ then } h\left(\sqrt{2 \frac{\hat{\rho} + \delta + \delta^r}{\sigma^2}}, \Phi, \frac{\delta^r}{\delta + \hat{\rho}}\right) \rightarrow$$

∞ . Fix φ such that $\varphi h\left(\varphi, \Phi, \frac{\delta^r}{\delta + \hat{\rho}}\right) < 1.606107734475270$, then for a given φ , $h(\varphi, \Phi, \cdot)$ is decreasing in $\frac{\delta^r}{\delta + \hat{\rho}}$. Furthermore, $\eta(\hat{w}^*) = \alpha$ and

$$\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r) = \frac{1 - \operatorname{sech}\left(\varphi h\left(\varphi, \Phi, \frac{\delta^r}{\delta + \hat{\rho}}\right)\right)}{\delta + \hat{\rho} + \delta^r}.$$

3. Assume $\sigma^2 = 0$ and $\hat{\gamma} \geq 0$. Then,

$$\hat{w}^- = \log(\hat{\rho}\hat{U} - \delta^r\hat{W}(\hat{w}^*)).$$

There exists a $\bar{\delta}^r$ satisfying $\frac{\hat{\rho}\hat{U}}{e^{\hat{w}^* - \hat{\rho}\hat{U}}}(\hat{\rho} + \delta) < \bar{\delta}^r < \frac{\hat{\rho}\hat{U}e^{(\hat{\rho} + \delta)T(\hat{w}^*)}}{\int_0^{T(\hat{w}^*)}(e^{\hat{w}^* - \hat{\gamma}s - \hat{\rho}\hat{U}}) ds}$ such that if $\delta^r \uparrow \bar{\delta}^r$, then $\hat{w}^- \rightarrow -\infty$.

Proof. We depart from the equilibrium conditions:

$$\begin{aligned} (\hat{\rho} + \delta + \delta^r)\hat{W}(\hat{w}) &= e^{\hat{w}} - \hat{\rho}\hat{U} + \delta^r\hat{W}(\hat{w}^*) - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}), \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+) \\ (\hat{\rho} + \delta + \delta^r)\hat{J}(\hat{w}) &= 1 - e^{\hat{w}} + \delta^r\hat{J}(\hat{w}^*) - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}), \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+) \\ (1 - \alpha)\frac{d\log \hat{J}(\hat{w}^*)}{d\hat{w}} &= -\alpha\frac{d\log \hat{W}(\hat{w}^*)}{d\hat{w}}, \\ \hat{W}(\hat{w}^-) &= \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0 \\ \hat{W}'(\hat{w}^-) &= \hat{J}'(\hat{w}^+) = 0 \end{aligned}$$

whenever $\hat{\gamma} \neq 0$ or $\sigma > 0$. When $\hat{\gamma} = \sigma = 0$, we have the variation inequality holding without a smooth pasting condition. We now show the properties of equilibrium policies.

Case $\gamma = \sigma = 0$: When $\gamma = \sigma = 0$, we have that

$$(\hat{\rho} + \delta + \delta^r)\hat{W}(\hat{w}) = \max\{e^{\hat{w}} - \hat{\rho}\hat{U} + \delta^r\hat{W}(\hat{w}^*), 0\} \tag{II.18}$$

$$(\hat{\rho} + \delta + \delta^r)\hat{J}(\hat{w}) = \max\{1 - e^{\hat{w}} + \delta^r\hat{J}(\hat{w}^*), 0\} \tag{II.19}$$

$$(1 - \alpha)\frac{d\log \hat{J}(\hat{w}^*)}{d\hat{w}} = -\alpha\frac{d\log \hat{W}(\hat{w}^*)}{d\hat{w}}. \tag{II.20}$$

Evaluating the equilibrium conditions (II.18) and (II.19) at \hat{w}^*

$$\hat{W}(\hat{w}^*) = \frac{e^{\hat{w}^*} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta}, \quad \hat{J}(\hat{w}^*) = \frac{1 - e^{\hat{w}^*}}{\hat{\rho} + \delta}.$$

and using the equilibrium conditions again, we obtain

$$\hat{W}(\hat{w}) = \frac{e^{\hat{w}} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta + \delta^r} + \frac{\delta^r}{\hat{\rho} + \delta + \delta^r} \frac{e^{\hat{w}^*} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta}, \quad \hat{J}(\hat{w}) = \frac{1 - e^{\hat{w}}}{\hat{\rho} + \delta + \delta^r} + \frac{\delta^r}{\hat{\rho} + \delta + \delta^r} \frac{1 - e^{\hat{w}^*}}{\hat{\rho} + \delta}.$$

Next, we compute the reset wage. Given value functions, the equilibrium condition (II.20) yields

$$\begin{aligned} (1 - \alpha) \frac{\frac{-e^{\hat{w}^*}}{\hat{\rho} + \delta + \delta^r}}{\frac{1 - e^{\hat{w}^*}}{\hat{\rho} + \delta + \delta^r} + \frac{\delta^r}{\hat{\rho} + \delta + \delta^r} \frac{1 - e^{\hat{w}^*}}{\hat{\rho} + \delta}} &= -\alpha \frac{\frac{e^{\hat{w}^*}}{\hat{\rho} + \delta + \delta^r}}{\frac{e^{\hat{w}^*} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta + \delta^r} + \frac{\delta^r}{\hat{\rho} + \delta + \delta^r} \frac{e^{\hat{w}^*} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta}} \iff \\ \left(1 + \frac{\delta^r}{\hat{\rho} + \delta}\right) (\alpha + (1 - \alpha)\hat{\rho}\hat{U}) &= e^{\hat{w}^*} (\alpha + (1 - \alpha)) \left(1 + \frac{\delta^r}{\hat{\rho} + \delta}\right) \\ e^{\hat{w}^*} &= \alpha + (1 - \alpha)\hat{\rho}\hat{U}. \end{aligned}$$

The boundaries of the continuation region \hat{w}^- and \hat{w}^+ are given by:

$$\begin{aligned} \hat{W}(\hat{w}^-) &= 0 \\ \frac{e^{\hat{w}^-} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta + \delta^r} + \frac{\delta^r}{\hat{\rho} + \delta + \delta^r} \frac{e^{\hat{w}^*} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta} &= 0 \\ \frac{e^{\hat{w}^-} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta + \delta^r} + \frac{\delta^r}{\hat{\rho} + \delta + \delta^r} \frac{\alpha + (1 - \alpha)\hat{\rho}\hat{U} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta} &= 0 \\ e^{\hat{w}^-} - \hat{\rho}\hat{U} + \frac{\delta^r}{\hat{\rho} + \delta} \alpha (1 - \hat{\rho}\hat{U}) &= 0 \\ e^{\hat{w}^-} &= \hat{\rho}\hat{U} - \frac{\delta^r}{\hat{\rho} + \delta} \alpha (1 - \hat{\rho}\hat{U}). \end{aligned}$$

Similarly, \hat{w}^+ is given by

$$e^{\hat{w}^+} = 1 + \frac{\delta^r}{\hat{\rho} + \delta} (1 - \alpha) (1 - \hat{\rho}\hat{U}).$$

The rest of the proof is similar to the proof in the baseline model without renegotiation.

Case $\hat{\gamma} = 0$ and $\alpha = 1/2$: We follow the same strategy as in Proposition 4. Let us guess and verify that $e^{\hat{w}^*} = \frac{1 + \hat{\rho}\hat{U}}{2}$, $\hat{w}^- = \hat{w}^* - h$ and $\hat{w}^+ = \hat{w}^* + h$ for a given h . Using a Taylor approximation of the flow profits around \hat{w}^* , we have that

$$(\hat{\rho} + \delta + \delta^r) \hat{W}(\hat{w}) = \frac{1 - \hat{\rho}\hat{U}}{2} + e^{\hat{w}^*} (\hat{w} - \hat{w}^*) + \delta^r \hat{W}(\hat{w}^*) + \frac{\sigma^2}{2} \hat{W}''(\hat{w}), \quad \forall \hat{w} \in (\hat{w}^* - h, \hat{w}^* + h)$$

$$(\hat{\rho} + \delta + \delta^r)\hat{f}(\hat{w}) = \frac{1 - \hat{\rho}\hat{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \delta^r\hat{f}(\hat{w}^*) + \frac{\sigma^2}{2}\hat{f}''(\hat{w}), \quad \forall \hat{w} \in (\hat{w}^* - h, \hat{w}^* + h),$$

with the border conditions given by the value matching and smooth pasting conditions. It is easy to check that when $\delta^r \rightarrow \infty$, $\hat{W}(\hat{w}) \rightarrow \hat{W}(\hat{w}^*)$, so h converges to ∞ . Define $J(x) = \frac{\hat{f}(x + \hat{w}^*) - \frac{1 - \hat{\rho}\hat{U}}{2(\hat{\rho} + \delta + \delta^r)}}{e^{\hat{w}^*}}$ and $W(x) = \frac{\hat{W}(x + \hat{w}^*) - \frac{1 - \hat{\rho}\hat{U}}{2(\hat{\rho} + \delta + \delta^r)}}{e^{\hat{w}^*}}$. Following the same steps as in Proposition 4, we have that $J(x) = W(-x)$ with

$$\begin{aligned} (\hat{\rho} + \delta + \delta^r)W(x) &= x + \delta^r W(0) + \frac{\sigma^2}{2}W''(x), \quad \forall x \in (-h, h) \\ W(-h) &= W(h) = -\frac{\Phi}{\hat{\rho} + \delta + \delta^r}; \quad W'(-h) = 0. \end{aligned}$$

Notice that an increase in the renegotiation arrival rate δ^r , increases the effective discount factor and, at the same time, it increases the worker's flow value. It is easy to show that the solution of the previous differential equation is given by

$$W(x) = Ae^{\varphi x} + Be^{-\varphi x} + \frac{x}{\hat{\rho} + \delta + \delta^r} + \frac{\delta^r}{\delta + \hat{\rho}}(A + B),$$

where $\varphi = \sqrt{2\frac{\hat{\rho} + \delta + \delta^r}{\sigma^2}}$. Writing the value matching conditions and operating, we obtain

$$\begin{aligned} A \left(e^{\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) + B \left(e^{-\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) &= \frac{-\Phi - h}{\hat{\rho} + \delta + \delta^r} \\ A \left(e^{-\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) + B \left(e^{\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) &= \frac{-\Phi + h}{\hat{\rho} + \delta + \delta^r}. \end{aligned}$$

Solving for A and B,

$$\begin{aligned} A &= -\frac{1}{\hat{\rho} + \delta + \delta^r} \frac{\left(e^{\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) (\Phi + h) + \left(e^{-\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) (h - \Phi)}{\left(e^{\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right)^2 - \left(e^{-\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right)^2} \\ B &= \frac{1}{\hat{\rho} + \delta + \delta^r} \frac{\left(e^{\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) (h - \Phi) + \left(e^{-\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) (h + \Phi)}{\left(e^{\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right)^2 - \left(e^{-\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right)^2}. \end{aligned}$$

Thus,

$$-W(x) = \frac{\frac{\left(e^{\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) (\Phi + h) + \left(e^{-\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right) (h - \Phi)}{\left(e^{\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right)^2 - \left(e^{-\varphi h} + \frac{\delta^r}{\delta + \hat{\rho}} \right)^2} \left(e^{\varphi x} + \frac{\delta^r}{\delta + \hat{\rho}} \right)}{\hat{\rho} + \delta + \delta^r}$$

$$+ \frac{\frac{\left(e^{\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)(h - \Phi) + \left(e^{-\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)(h + \Phi)}{\left(e^{\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)^2 - \left(e^{-\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)^2} \left(e^{-\varphi x + \frac{\delta^r}{\delta + \hat{\rho}}}\right) + x}{\hat{\rho} + \delta + \delta^r}.$$

Evaluating the smooth pasting condition, we obtain

$$\begin{aligned} & - \frac{\left(e^{\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)(\Phi + h) + \left(e^{-\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)(h - \Phi)}{\left(e^{\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)^2 - \left(e^{-\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)^2} \varphi e^{-\varphi h} \\ & - \frac{\left(e^{\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)(h - \Phi) + \left(e^{-\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)(h + \Phi)}{\left(e^{\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)^2 - \left(e^{-\varphi h + \frac{\delta^r}{\delta + \hat{\rho}}}\right)^2} \varphi e^{\varphi h} + 1 = 0. \end{aligned}$$

Operating on this expression and defining $q = \varphi h$, we get

$$\Phi \varphi 2 + \frac{4\delta^r}{\delta + \hat{\rho}} \frac{\sinh(q) - q \cosh(q)}{\cosh(2q) - 1} = - \frac{2 \sinh(2q) - 2q(\cosh(2q) + 1)}{\cosh(2q) - 1}.$$

Thus,

$$\Phi \varphi 2 = - \frac{4\delta^r}{\delta + \hat{\rho}} \frac{\sinh(q) - q \cosh(q)}{\cosh(2q) - 1} - \frac{2 \sinh(2q) - 2q(\cosh(2q) + 1)}{\cosh(2q) - 1}$$

The following properties hold: $-\frac{\sinh(q) - q \cosh(q)}{\cosh(2q) - 1}$ converges to 0 when $q \downarrow 0$, it increases until $q \approx 1.606$ and then decreases to 0 for $q > 1.606$. Since $-\frac{\sinh(q) - q \cosh(q)}{\cosh(2q) - 1}$ is increasing in q if the solution q is lower than 1.606, we have that $q(\Phi \varphi 2, \frac{4\delta^r}{\delta + \hat{\rho}})$ is decreasing in the second argument.

As in the baseline model, due to symmetry $\hat{w}^\pm = \hat{w}^* \pm h(\varphi, \Phi, \frac{\delta^r}{\delta + \hat{\rho}})$, we have that $T'_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$ and $\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r) = \frac{1 - \text{sech}(\varphi h(\varphi, \Phi, \frac{\delta^r}{\delta + \hat{\rho}}))}{\delta + \hat{\rho} + \delta^r}$.

Case $\hat{\gamma} > 0$ and $\sigma = 0$: In this case, the stopping time is a deterministic function; hence, it is easier to work with the sequential formulation:

$$\begin{aligned} \hat{W}(\hat{w}) &= \max_T \int_0^T e^{-(\hat{\rho} + \delta + \delta^r)s} (e^{\hat{w} - \hat{\gamma}s} + \delta^r \hat{W}(\hat{w}^*) - \hat{\rho} \hat{U}) ds, \\ \hat{J}(\hat{w}) &= \int_0^{T(\hat{w})} e^{-(\hat{\rho} + \delta + \delta^r)s} (1 + \delta^r \hat{J}(\hat{w}^*) - e^{\hat{w} - \hat{\gamma}s}) ds. \end{aligned}$$

In equation (A.28), $T(\hat{w})$ is the worker's optimal policy. Taking the the FOC with respect to $T(\hat{w})$:

$$e^{\hat{w} - \hat{\gamma}T(\hat{w})} = \hat{\rho} \hat{U} - \delta^r \hat{W}(\hat{w}^*).$$

Solving the previous equation,

$$T(\hat{w}) = \frac{\hat{w} - \log(\hat{\rho}\hat{U} - \delta^r W(\hat{w}^*))}{\hat{\gamma}}.$$

Thus, if $\hat{w} = \hat{w}^*$, we have that $\hat{w}^- = \hat{w}^* - \hat{\gamma}T(\hat{w}^*)$ satisfies

$$\hat{w}^- = \log(\hat{\rho}\hat{U} - \delta^r \hat{W}(\hat{w}^*)).$$

Following similar steps as in the baseline model, it is easy to show how $\hat{W}(\hat{w}^*)$ depends on $\hat{\gamma}$ and $\hat{\rho} + \delta$. Now, we prove the last property—i.e., there exists a $\bar{\delta}^r < \infty$ such that $\lim_{\delta^r \rightarrow \bar{\delta}^r} \hat{w}^- = -\infty$. That is, if the bargaining probability is high enough but finite, then the continuation region becomes unbounded. Observe that if $\delta^r \uparrow \bar{\delta}^r$, where $\bar{\delta}^r := \frac{\hat{\rho}\hat{U}}{\hat{W}(\hat{w}^*)}$, then $\hat{w}^- \rightarrow -\infty$. Since $\hat{W}(\hat{w}^*) < \frac{e^{\hat{w}^*} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta}$, we have that

$$0 = \hat{\rho}\hat{U} - \bar{\delta}^r \hat{W}(\hat{w}^*) > \hat{\rho}\hat{U} - \bar{\delta}^r \frac{e^{\hat{w}^*} - \hat{\rho}\hat{U}}{\hat{\rho} + \delta}, \iff \bar{\delta}^r > \frac{\hat{\rho}\hat{U}}{e^{\hat{w}^*} - \hat{\rho}\hat{U}}(\hat{\rho} + \delta).$$

Thus, we have a lower bound for $\bar{\delta}^r$. To find an upper bound, we compute the value function without on-the-job bargaining. In this case, we have that

$$\hat{W}(\hat{w}^*) > \int_0^{T(\hat{w}^*)} e^{-(\hat{\rho} + \delta)s} (e^{\hat{w} - \hat{\gamma}s} - \hat{\rho}\hat{U}) ds > e^{-(\hat{\rho} + \delta)T(\hat{w}^*)} \int_0^{T(\hat{w}^*)} (e^{\hat{w} - \hat{\gamma}s} - \hat{\rho}\hat{U}) ds$$

Thus,

$$\bar{\delta}^r < \frac{\hat{\rho}\hat{U}e^{(\hat{\rho} + \delta)T(\hat{w}^*)}}{\int_0^{T(\hat{w}^*)} (e^{\hat{w} - \hat{\gamma}s} - \hat{\rho}\hat{U}) ds}.$$

□

Discussion of Static and Dynamic Consideration for Equilibrium Policies. Intuition suggests that an increase in the frequency of bargaining will lead to a change in the quit and layoff triggers, ultimately resulting in an extended match duration. Proposition II.4-Part 1 demonstrates that this intuition holds true whenever there is no drift or shocks in idiosyncratic productivity. Furthermore, it illustrates how the continuation set of the match changes as a function of the primitives $(\delta^r, \hat{\rho} + \delta, \alpha, 1 - \hat{\rho}\hat{U})$. Importantly, the size of the surplus does not affect the separation thresholds when $\delta^r = 0$, but also affects the marginal effect of the frequency of bargaining since both terms appear multiplicatively. Also, observe that when $\delta^r \rightarrow \infty$, then $\hat{w}^- \rightarrow -\infty$ and $\hat{w}^+ \rightarrow \infty$. This property

also holds for $\hat{\gamma} > 0$ and $\sigma > 0$.

Proposition II.4-Part 2 characterizes the interaction between the option value effect and the frequency of on-the-job bargaining. The width of the continuation region depends on $\frac{\sigma^2}{\hat{\rho} + \delta + \delta^r}$, the surplus, and the frequency of on-the-job bargaining. The first result, where we fix the value of σ^2 , shows that if the frequency of bargaining increases, then the width of the continuation set converges to infinity.

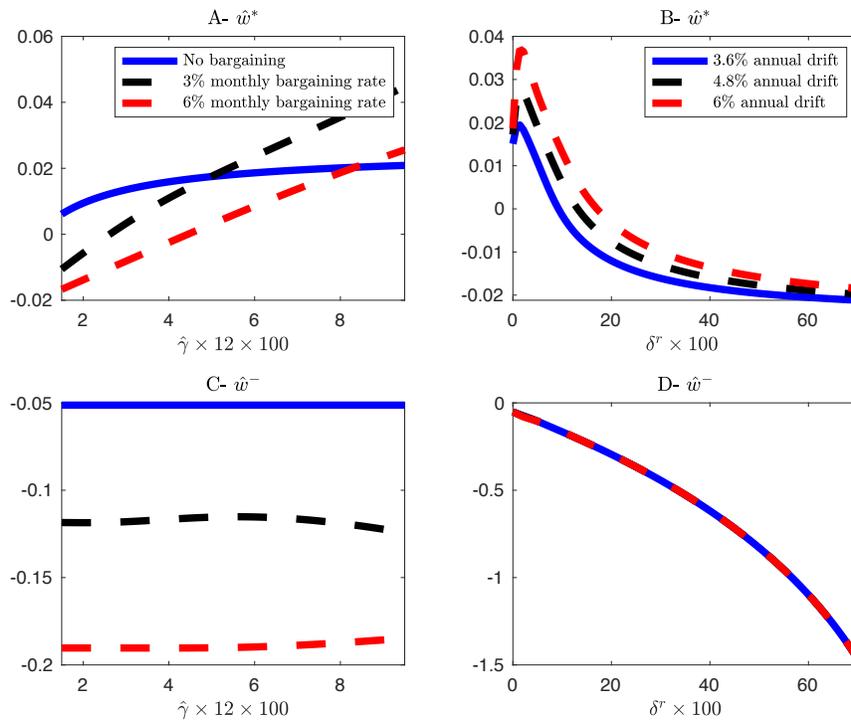
In the second result, we construct a specific case that highlights a counter-intuitive finding: If there is an increase in δ^r and σ^2 while keeping $\varphi = \sqrt{2 \frac{\hat{\rho} + \delta + \delta^r}{\sigma^2}}$ fixed, then the width of the continuation region decreases. The intuition behind this result is not straightforward. To understand it, first recall that in the baseline model, the width of the inaction region does not grow unboundedly with the volatility of shocks due to lack of commitment. A firm paying a high \hat{w} is not willing to wait before dissolving the match when the volatility of productivity shocks is high because of the associated high probability of a sufficiently large and positive shock that would make the worker quit. In this version of the model, this willingness to wait before dissolving the match decreases when the frequency of bargaining increases. This is because a higher bargaining rate is associated with a \hat{w} that tends to fluctuate well within the boundaries of the continuation region of the match. Therefore, for the same increase in the volatility of productivity shocks, the probability of experiencing a shock positive enough that makes the worker quit is much higher; thus, the firm decides to dissolve the match sooner.

Proposition II.4-Part 3 characterizes the anticipatory effect of the drift on the quit threshold. When $\delta^r = 0$, we recover the result presented for the baseline model: The quit threshold is fully static; i.e., the worker quits when $\hat{w} < \log(\hat{\rho}\hat{U})$. When $\delta^r > 0$, the quit threshold is dynamic and depends on the value of renegotiating the wage. A novel result arises: If the renegotiation frequency is large enough, but not necessarily infinitely large, then the worker will never quit her job. The intuition is that if the incentives to wait offered by wage renegotiation are significantly larger than the opportunity cost, then the worker will never find it optimal to quit.

We finish this discussion with a joint analysis of equilibrium policies when $\sigma = 0$ and the worker's opportunity cost is kept constant. Figure II3-Panels A and C show the effect of the drift for different values of the monthly frequency of wage renegotiation. As we can see, the entry wage is increasing in the drift—as in the main text. In addition, the quit threshold is almost independent of the value of the drift—as in the main text—and it depends mainly on the probability of resetting the wage within the match. This is the result of two opposing forces almost balancing

each other perfectly: i) for a fixed bargained wage, a larger drift and the associated higher quit probability reduce the value of the worker, and ii) a larger drift results in a higher bargained wage and, thus, a higher value when evaluated at that wage. Figure II3-Panels B and D show the effect of the frequency of bargaining for different values of the annual drift. As we can see, for a frequency large enough, the reset wage decreases—due a weaker anticipatory effect—and it converges to the static Nash bargaining solution. Once the entry wage converges, the elasticity of the quit threshold with respect to the renegotiation rate will become unboundedly large (recall that $\hat{w}^- = \log(\hat{\rho}\hat{U} - \delta^r \hat{W}(\hat{w}^*))$).

FIGURE II3. COMPARATIVE STATICS WITH RESPECT TO THE DRIFT



Notes: Panels A and C plot the entry wage and the quit threshold as a function of the drift for three values of the monthly frequency of wage renegotiation, respectively. Panels B and D plot the entry wage and the quit threshold as a function of the frequency of bargaining for three values of the annual drift, respectively.

The CIR of Employment with Flexible and Sticky Entry Wages.

Before presenting the new results, we define new notation. Let τ^m denote the duration of the current wage spell. Let τ^r denote the time elapsed until the arrival of an opportunity to renegotiate the wage. Observe that, if $\tau^m < \tau^r$, then the match finishes in a separation. Otherwise, if $\tau^m = \tau^r$, then the current wage is renegotiated. Furthermore, as in the baseline model, let $g^h(\Delta z)$ and $g^u(\Delta z)$ be the distributions of Δz across employed and unemployed workers, respectively. Observe that

Δz now represents the cumulative shocks to revenue productivity $z + p$ that the match experienced since either its inception or its last wage renegotiation. The support of $g^h(\Delta z)$ is given by $[-\Delta^-, \Delta^+]$, where $\Delta^- := \hat{w}^* - \hat{w}^-$ and $\Delta^+ := \hat{w}^+ - \hat{w}^*$. We denote by $\mathbb{E}_h[\cdot]$ and $\mathbb{E}_u[\cdot]$ the expectation operators under the distributions $g^h(\Delta z)$ and $g^u(\Delta z)$, respectively.

Below, we describe the KFEs characterizing $g^h(\Delta z)$ and $g^u(\Delta z)$:

$$(\delta + \delta^r) g^h(\Delta z) = (\gamma + \chi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \text{for all } \Delta z \in (-\Delta^-, \Delta^+) / \{0\}, \quad (\text{II.21})$$

$$f(\hat{\theta}(\hat{w}^*))g^u(\Delta z) = (\gamma + \chi)(g^u)'(\Delta z) + \frac{\sigma^2}{2}(g^u)''(\Delta z) \quad \text{for all } \Delta z \in (-\infty, \infty) / \{0\}. \quad (\text{II.22})$$

$$g^h(\Delta z) = 0, \quad \text{for all } \Delta z \notin (-\Delta^-, \Delta^+) \quad (\text{II.23})$$

$$\lim_{\Delta z \rightarrow -\infty} g^u(\Delta z) = \lim_{\Delta z \rightarrow \infty} g^u(\Delta z) = 0. \quad (\text{II.24})$$

$$1 = \int_{-\infty}^{\infty} g^u(\Delta z) d\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z) d\Delta z, \quad (\text{II.25})$$

$$f(\hat{\theta}(\hat{w}^*))(1 - \mathcal{E}) = \delta \mathcal{E} + \frac{\sigma^2}{2} \left[\lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+} (g^h)'(\Delta z) \right], \quad (\text{II.26})$$

$$g^u(\Delta z) \in \mathbf{C}, \mathbf{C}^2((-\infty, \infty) / \{0\}), \quad g^h(\Delta z) \in \mathbf{C}, \mathbf{C}^2((-\Delta^-, \Delta^+) / \{0\})$$

The main difference with our baseline analysis is the additional δ^r term in the KFE for $g^h(\Delta z)$. For this to hold, the renegotiated wage must be the same as the entry wage from unemployment, which results from the assumption that the worker's bargaining weight equals the elasticity of the matching function.

We divide the proof of the extension of Proposition 6 to the case of wage renegotiations into three propositions. Proposition II.5 relates the CIR to a perturbation of two Bellman equations describing future employment fluctuations for initially employed and unemployed workers. This proposition covers both the case with flexible and sticky entry wages. Proposition II.6 relates steady-state moments of the perturbed Bellman equations to steady-state moments of the distribution of Δz . Finally, Proposition II.7 related the steady-state moments of Δz to observable moments in the steady-state.

Taken together, Propositions II.6 and II.7 extend Proposition 6 for the case with wage changes within a job. Finally, Proposition II.8 extends Lemma 2 for the case with wage changes within a job. Whenever the steps of the proof are the same as those in the baseline model, we omit them.

Proposition II.5. *Given steady-state policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ and distributions $(g^h(\Delta z), g^u(\Delta z))$, the CIR*

is

$$CIR_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta) d\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, \zeta) g^u(\Delta z + \zeta) d\Delta z,$$

where the value functions $m_{\mathcal{E},h}(\Delta z)$ and $m_{\mathcal{E},u}(\Delta z, \zeta)$ are characterized by:

$$0 = 1 - \mathcal{E}_{ss} - (\gamma + \chi) \frac{dm_{\mathcal{E},h}(\Delta z)}{d\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},h}(\Delta z)}{d\Delta z^2} + \delta(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\Delta z)) + \delta^r(m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\Delta z)), \quad (\text{II.27})$$

$$0 = -\mathcal{E}_{ss} - (\gamma + \chi) \frac{dm_{\mathcal{E},u}(\Delta z, \zeta)}{d\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},u}(\Delta z, \zeta)}{d\Delta z^2} + f(\hat{\theta}(\hat{w}^* - \zeta))(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(\Delta z, \zeta)) \quad (\text{II.28})$$

$$m_{\mathcal{E},u}(0,0) = m_{\mathcal{E},h}(\Delta z), \quad \forall \Delta z \notin (-\Delta^-, \Delta^+)$$

$$0 = \lim_{\Delta z \rightarrow -\infty} \frac{dm_{\mathcal{E},u}(\Delta z, \zeta)}{d\Delta z} = \lim_{\Delta z \rightarrow \infty} \frac{dm_{\mathcal{E},u}(\Delta z, \zeta)}{d\Delta z} \quad (\text{II.29})$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) d\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, 0) g^u(\Delta z) d\Delta z. \quad (\text{II.30})$$

Proof. We define the CIR of aggregate employment to an aggregate TFPR shock as

$$CIR_{\mathcal{E}}(\zeta) = \int_0^{\infty} \int_{-\infty}^{\infty} (g^h(\Delta z, \zeta, t) - g^h(\Delta z)) d\Delta z dt.$$

Here, $\mathcal{E}_t = \int_{-\infty}^{\infty} g^h(\Delta z, \zeta, t) d\Delta z$ is a function of ζ since aggregate shocks affect net flows into employment. As in the main proof of Proposition 6, starting from the definition of the CIR, we can derive

$$CIR_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} \lim_{\mathcal{T} \rightarrow \infty} m_{\mathcal{E},h}(\Delta z, \mathcal{T}) g^h(\Delta z + \zeta) d\Delta z + \int_{-\infty}^{\infty} \lim_{\mathcal{T} \rightarrow \infty} m_{\mathcal{E},u}(\Delta z, \zeta, \mathcal{T}) g^u(\Delta z + \zeta) d\Delta z,$$

where we define

$$m_{\mathcal{E},h}(\Delta z_0, \mathcal{T}) \equiv \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss}) g^h(\Delta z, t | \Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t | \Delta z_0, h) \right] d\Delta z dt \right]$$

$$m_{\mathcal{E},u}(\Delta z_0, \zeta, \mathcal{T}) \equiv \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss}) g^h(\Delta z, \zeta, t | \Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t | \Delta z_0, u) \right] d\Delta z dt \right].$$

Taking the limit as $\mathcal{T} \rightarrow \infty$, we have that the value functions $m_{\mathcal{E},h}(\Delta z_0)$ and $m_{\mathcal{E},u}(\Delta z_0, \zeta)$ satisfy the condition in equation (II.27) to (II.30). \square

Notice that the main difference between equations (II.27)–(II.30) and equations (B.4)–(B.8) is

the extra term in the HJB characterizing $m_{\mathcal{E},h}(\Delta z_0)$, which takes into account staggered wage renegotiations.

The CIR of Employment with Flexible Entry Wages and Wage Renegotiations.

Proposition II.6. *Assume flexible entry wages. Up to first order, the CIR of employment is given by:*

$$\begin{aligned} \frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = & -(1 - \mathcal{E}_{ss}) \frac{[(\gamma + \chi)\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2} \\ & - \frac{\mathcal{E}_{ss}}{\sigma^2 f(\hat{\theta}(\hat{w}^*))} \left((\gamma + \chi)\mathcal{E}_{ss} \left(1 - \frac{\mathbb{E}_{\mathcal{D}}[\tau^m \mathbb{1}\{\tau^m < \tau^{\delta^r}\}]}{\mathbb{E}_{\mathcal{D}}[\tau^m]} \right) + \mathbb{E}_h[\Delta z] \mathcal{F}^{\Delta w} \right) + o(\zeta), \end{aligned}$$

where $\mathcal{F}^{\Delta w}$ denotes the observed frequency of wage renegotiation within a match in the data.

Proof. The proof proceeds in three steps. Step 1 computes the value function for an unemployed worker $m_{\mathcal{E},u}(\Delta z)$ (when entry wages are flexible, the job-finding rate and this value function are independent of the shock ζ , so we omit this argument). Step 2 computes the value for the employed worker at $\Delta z = 0$ —i.e., $m_{\mathcal{E},h}(0)$. Step 3 characterizes the CIR as a function of steady-state aggregate variables and moments.

Step 1. The CIR is given by

$$CIR_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta) d\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) \right) (1 - \mathcal{E}_{ss}),$$

with

$$\begin{aligned} 0 = & 1 - \mathcal{E}_{ss} - (\gamma + \chi) \frac{dm_{\mathcal{E},h}(\Delta z)}{d\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},h}(\Delta z)}{d\Delta z^2} + \delta \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\Delta z) \right) \\ & + \delta^r (m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\Delta z)), \\ m_{\mathcal{E},h}(\Delta z) = & -\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0), \quad \forall \Delta z \notin (-\Delta^-, \Delta^+) \\ 0 = & \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) d\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) \right) (1 - \mathcal{E}_{ss}). \end{aligned} \tag{II.31}$$

Proof of Step 1. To show this result, observe that the solution to (II.28) and (II.29) is

$$m_{\mathcal{E},u}(\Delta z) = m_{\mathcal{E},u}(0), \quad \forall \Delta z.$$

Thus,

$$0 = -\mathcal{E}_{ss} + f(\hat{\theta}(\hat{w}^*)) (m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0)) \iff m_{\mathcal{E},u}(0) = -\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0). \quad (\text{II.32})$$

Replacing (II.32) into the CIR, we have the result.

Step 2. We show that $m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} \left((1 - \mathcal{E}_{ss}) + \mathcal{E}_{ss} \frac{\mathbb{E}_{\mathcal{D}}[\tau^m \mathbb{1}\{\tau^m < \tau^{\delta^r}\}]}{\mathbb{E}_{\mathcal{D}}[\tau^m]} \right) - (1 - \mathcal{E}_{ss}) \mathbb{E}_h[a]$, where $\mathbb{E}_h[a]$ is the cross-sectional expected age of the match or the worker's tenure at the current match.

Proof of Step 2. Observe that $m_{\mathcal{E},h}(\Delta z)$ satisfies the following recursive representation

$$m_{\mathcal{E},h}(\Delta z) = \mathbb{E} \left[\int_0^{\tau^m} (1 - \mathcal{E}_{ss}) dt + \left(-\mathbb{1}\{\tau^m < \tau^{\delta^r}\} \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) \right) \middle| \Delta z_0 = \Delta z \right]. \quad (\text{II.33})$$

Define the following auxiliary function

$$\Psi(\Delta z|\varphi) = \mathbb{E} \left[\int_0^{\tau^m} e^{\varphi t} (1 - \mathcal{E}_{ss}) dt + e^{\varphi \tau^m} \left(-\mathbb{1}\{\tau^m < \tau^{\delta^r}\} \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) \right) \middle| \Delta z_0 = \Delta z \right]. \quad (\text{II.34})$$

and note that $\Psi(\Delta z|0) = m_{\mathcal{E},h}(\Delta z)$. Then $\Psi(\Delta z|\varphi)$ satisfies the following HJB and border conditions:

$$\begin{aligned} & -\varphi \Psi(\Delta z|\varphi) + (\delta + \delta^r) (\Psi(\Delta z|\varphi) - m_{\mathcal{E},h}(0)) + \delta \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} \\ & = (1 - \mathcal{E}_{ss}) - (\gamma + \chi) \frac{\partial \Psi(\Delta z|\varphi)}{\partial \Delta z} + \frac{\sigma^2}{2} \frac{\partial^2 \Psi(\Delta z|\varphi)}{\partial \Delta z^2}, \quad (\text{II.35}) \\ & \Psi(\Delta z, \varphi) = \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) \right) \forall \Delta z \notin (-\Delta^-, \Delta^+). \end{aligned}$$

Taking the derivative with respect to φ in (II.35), we have that

$$\begin{aligned} (\delta + \delta^r - \varphi) \frac{\partial \Psi(\Delta z|\varphi)}{\partial \varphi} - \Psi(\Delta z|\varphi) &= -(\gamma + \chi) \frac{\partial^2 \Psi(\Delta z, \varphi)}{\partial \Delta z \partial \varphi} + \frac{\sigma^2}{2} \frac{\partial^3 \Psi(\Delta z|\varphi)}{\partial \Delta z^2 \partial \varphi}, \\ \frac{\partial \Psi(\Delta z|\varphi)}{\partial \varphi} &= 0 \forall \Delta z \notin (-\Delta^-, \Delta^+). \end{aligned}$$

Using Schwarz's Theorem to exchange partial derivatives, evaluating at $\varphi = 0$, and using $\Psi(\Delta z|0) = m_{\mathcal{E},h}(\Delta z)$, we obtain

$$(\delta + \delta^r) \frac{\partial \Psi(\Delta z|0)}{\partial \varphi} - m_{\mathcal{E},h}(\Delta z) = -(\gamma + \chi) \frac{\partial}{\partial \Delta z} \left(\frac{\partial \Psi(\Delta z|0)}{\partial \varphi} \right) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial \Delta z^2} \left(\frac{\partial \Psi(\Delta z|0)}{\partial \varphi} \right). \quad (\text{II.36})$$

$$\frac{\partial \Psi(-\Delta^-|0)}{\partial \varphi} = \frac{\partial \Psi(\Delta^+|0)}{\partial \varphi} = 0. \quad (\text{II.37})$$

Equations (II.36) and (II.37) correspond to the HJB and border conditions of the function $\frac{\partial \Psi(\Delta z|0)}{\partial \varphi} = \mathbb{E} \left[\int_0^{\tau^m} m_{\mathcal{E},h}(\Delta z_t) dt \mid \Delta z_0 = \Delta z \right]$. Evaluating $\frac{\partial \Psi(\Delta z|0)}{\partial \varphi}$ at $\Delta z = 0$, using the occupancy measure and result (II.31), we write the previous equation as:

$$\begin{aligned} \frac{\partial \Psi(0|0)}{\partial \varphi} &= \mathbb{E} \left[\int_0^{\tau^m} m_{\mathcal{E},h}(\Delta z_t) dt \mid \Delta z_0 = 0 \right] \\ &= \mathbb{E}_{\mathcal{D}}[\tau^m] \frac{\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) d\Delta z}{\mathcal{E}_{ss}} \\ &= \mathbb{E}_{\mathcal{D}}[\tau^m] \left(\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} - m_{\mathcal{E},h}(0) \right) \frac{(1 - \mathcal{E}_{ss})}{\mathcal{E}_{ss}}, \end{aligned} \quad (\text{II.38})$$

where $\mathbb{E}_{\mathcal{D}}[\tau^m]$ is the mean duration of completed *wage* spells (the subscript highlights that the moment can be computed from the data).²¹ From (II.34), we also have that

$$\frac{\partial \Psi(0|0)}{\partial \varphi} = \mathbb{E}_{\mathcal{D}}[\tau^m] \left[(1 - \mathcal{E}_{ss}) \frac{\mathbb{E}_h[a]}{\mathcal{E}_{ss}} + m_{\mathcal{E},h}(0) \right] - \mathbb{E} \left[\tau^m \mathbb{1}\{\tau^m < \tau^{\delta^r}\} \mid \Delta z_0 = 0 \right] \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} \quad (\text{II.39})$$

Combining (II.38) and (II.39), and solving for $m_{\mathcal{E},h}(0)$ we obtain:

$$m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} \left((1 - \mathcal{E}_{ss}) + \mathcal{E}_{ss} \frac{\mathbb{E} \left[\tau^m \mathbb{1}\{\tau^m < \tau^{\delta^r}\} \mid \Delta z_0 = 0 \right]}{\mathbb{E}_{\mathcal{D}}[\tau^m]} \right) - (1 - \mathcal{E}_{ss}) \mathbb{E}_h[a].$$

Observe that $\mathbb{E} \left[\tau^m \mathbb{1}\{\tau^m < \tau^{\delta^r}\} \mid \Delta z_0 = 0 \right]$ is equal to $\mathbb{E}_{\mathcal{D}} \left[\tau^m \mathbb{1}\{\tau^m < \tau^{\delta^r}\} \right]$ —i.e., the average duration of wage spells that ended in a job separation.

Step 3. Up to a first-order approximation, the CIR is given by:

$$\begin{aligned} \text{CIR}_{\mathcal{E}}(\zeta) &= -(1 - \mathcal{E}_{ss}) \frac{[(\gamma + \chi) \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2} \zeta \\ &\quad - \frac{\mathcal{E}_{ss}}{\sigma^2 f(\hat{\theta}(\hat{w}^*))} \left((\gamma + \chi) \mathcal{E}_{ss} \left(1 - \frac{\mathbb{E}_{\mathcal{D}} \left[\tau^m \mathbb{1}\{\tau^m < \tau^{\delta^r}\} \right]}{\mathbb{E}_{\mathcal{D}}[\tau^m]} \right) + \mathbb{E}_h[\Delta z] \mathcal{F}^{\Delta w} \right) \zeta + O(\zeta^2). \end{aligned}$$

Proof of Step 3. To help the reader, we summarize below the conditions used in this step of the proof.

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta) d\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) \right) (1 - \mathcal{E}_{ss}) \quad (\text{II.40})$$

²¹A completed wage spell starts when the worker earns a new wage (i.e., finding a new job or renegotiating the wage with the current employer) and ends either when the match dissolves or the wage is renegotiated.

with

$$(\delta + \delta^r) m_{\mathcal{E},h}(\Delta z) = 1 - \mathcal{E}_{ss} - (\gamma + \chi) \frac{dm_{\mathcal{E},h}(\Delta z)}{d\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},h}(\Delta z)}{d\Delta z^2} + \delta m_{\mathcal{E},u}(0) + \delta^r m_{\mathcal{E},h}(0) \quad (\text{II.41})$$

$$m_{\mathcal{E},u}(0) = m_{\mathcal{E},h}(\Delta z) \quad \forall \Delta z \notin (-\Delta^-, \Delta^+)$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) d\Delta z + m_{\mathcal{E},u}(0)(1 - \mathcal{E}_{ss}). \quad (\text{II.42})$$

1. **Zeroth Order:** If $\zeta = 0$, condition (II.42) implies

$$\text{CIR}_{\mathcal{E}}(0) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) d\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + m_{\mathcal{E},h}(0) \right) (1 - \mathcal{E}_{ss}) = 0.$$

2. **First Order:** Taking the derivative of (II.40) we obtain

$$\text{CIR}'_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z + \zeta) d\Delta z,$$

which evaluated at $\zeta = 0$ becomes

$$\text{CIR}'_{\mathcal{E}}(0) = \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) d\Delta z.$$

Using condition (II.21) to replace $(\delta + \delta^r) = \frac{(\gamma + \chi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)}$ into equation (II.41), we obtain

$$\begin{aligned} & \frac{(\gamma + \chi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)} m_{\mathcal{E},h}(\Delta z) \\ = & 1 - \mathcal{E}_{ss} - (\gamma + \chi) m'_{\mathcal{E},h}(\Delta z) + \frac{\sigma^2}{2} m''_{\mathcal{E},h}(\Delta z) + \left(\frac{(\gamma + \chi)g'(\Delta z) + \frac{\sigma^2}{2}g''(\Delta z)}{g(\Delta z)} - \delta^r \right) m_{\mathcal{E},u}(0) \\ & + \delta^r m_{\mathcal{E},h}(0). \end{aligned}$$

Multiplying both sides by $g^h(\Delta z)\Delta z$ and integrating between $-\Delta^-$ and Δ^+ ,

$$0 = (1 - \mathcal{E}_{ss}) \mathbb{E}_h[\Delta z] - (\gamma + \chi) T_1 + \frac{\sigma^2}{2} T_2 + m_{\mathcal{E},u}(0) T_3 + \delta^r (m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0)) \mathbb{E}_h[\Delta z] \quad (\text{II.43})$$

$$T_1 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) \right] d\Delta z$$

$$T_2 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z) (g^h)''(\Delta z) \right] d\Delta z$$

$$T_3 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left((\gamma + \chi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \right) d\Delta z.$$

Operating on the terms T_1 , T_2 , and T_3 , we get

$$T_1 = m_{\mathcal{E},u}(0)(1 - \mathcal{E}_{ss}), \quad (\text{II.44})$$

$$T_2 = -m_{\mathcal{E},u}(0) \Delta z (g^h)'(\Delta z) \Big|_{-\Delta^-}^{\Delta^+} + 2 \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) g'(\Delta z) d\Delta z, \quad (\text{II.45})$$

$$T_3 = -(\gamma + \chi)\mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z) \Big|_{-\Delta^-}^{\Delta^+} \right]. \quad (\text{II.46})$$

Combining (II.43), (II.44), (II.45), (II.46), and the results from Step 2, we obtain

$$\begin{aligned} 0 &= (1 - \mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma + \chi)T_1 + \frac{\sigma^2}{2}T_2 + m_{\mathcal{E},u}(0)T_3 + \delta^r (m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0)) \mathbb{E}_h[\Delta z] \\ &= (1 - \mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma + \chi)m_{\mathcal{E},u}(0)(1 - \mathcal{E}_{ss}) \\ &\quad + \frac{\sigma^2}{2} \left[-m_{\mathcal{E},u}(0) \Delta z (g^h)'(\Delta z) \Big|_{-\Delta^-}^{\Delta^+} + 2 \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) d\Delta z \right] + \dots \\ &\quad + m_{\mathcal{E},u}(0) \left[-(\gamma + \chi)\mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z) \Big|_{-\Delta^-}^{\Delta^+} \right] \right] + \delta^r \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} \mathbb{E}_h[\Delta z] \\ &= (1 - \mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma + \chi)m_{\mathcal{E},u}(0) + \sigma^2 \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) d\Delta z + \delta^r \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} \mathbb{E}_h[\Delta z], \end{aligned}$$

which implies

$$\begin{aligned} &\sigma^2 \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) d\Delta z \\ &= (\gamma + \chi) \left(-\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} + \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} \left((1 - \mathcal{E}_{ss}) + \mathcal{E}_{ss} \frac{\mathbb{E}_{\mathcal{D}}[\tau^m \mathbb{1}\{\tau^m < \tau^{\delta^r}\}]}{\mathbb{E}_{\mathcal{D}}[\tau^m]} \right) - (1 - \mathcal{E}_{ss})\mathbb{E}_h[a] \right) \\ &\quad - (1 - \mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - \delta^r \frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^*))} \mathbb{E}_h[\Delta z], \end{aligned}$$

and

$$\begin{aligned} &\int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) d\Delta z \\ &= - (1 - \mathcal{E}_{ss}) \frac{[(\gamma + \chi)\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2} \end{aligned}$$

$$-\frac{\mathcal{E}_{ss}}{\sigma^2 f(\hat{\theta}(\hat{w}^*))} \left((\gamma + \chi) \mathcal{E}_{ss} \left(1 - \frac{\mathbb{E}_{\mathcal{D}} [\tau^m \mathbb{1}\{\tau^m < \tau^{\delta^r}\}]}{\mathbb{E}_{\mathcal{D}} [\tau^m]} \right) + \mathbb{E}_h[\Delta z] \delta^r \right).$$

Finally, since the probability of wage renegotiation is independent of the state of the match, we have

$$\mathcal{F}^{\Delta w} dt = \int_{-\Delta^-}^{\Delta^+} Pr(\text{bargaining in } [t, t + dt] | \Delta z) \frac{g(\Delta z)}{\mathcal{E}_{ss}} d\Delta z = \delta^r dt$$

and, therefore,

$$\mathcal{F}^{\Delta w} = \delta^r.$$

□

Next, we write the CIR as a function of observable moments. Let Δw_B denote the log wage change following a wage renegotiation and let $l^B(\Delta w)$ denote its distribution. In addition, let $l^{EUE}(\Delta w)$ be the distribution of wage changes following a separation (i.e., wage changes between two consecutive jobs). Our objective is to recover $\mathbb{E}_h[a]$ and $\mathbb{E}_h[\Delta z]$ from observable micro-data on wage changes. To simplify the discussion, from now on we focus on the case with $\gamma + \chi = 0$. Under this parametric restriction, we only need to recover one moment: $\mathbb{E}_h[\Delta z]$. The CIR of employment is given by:

$$\begin{aligned} \frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} &= -(1 - \mathcal{E}_{ss}) \frac{[\mathbb{E}_h[\Delta z]]}{\sigma^2} - \frac{\mathcal{E}_{ss}}{\sigma^2 f(\hat{\theta}(\hat{w}^*))} \left(\mathbb{E}_h[\Delta z] \mathcal{F}^{\Delta w} \right) + o(\zeta), \\ &= -\frac{\mathbb{E}_h[\Delta z]}{\sigma^2} \left(1 - \mathcal{E}_{ss} + \mathcal{E}_{ss} \frac{\mathcal{F}^{\Delta w}}{f(\hat{\theta}(\hat{w}^*))} \right). \end{aligned}$$

Two moments are informative of $\mathbb{E}_h[\Delta z]$. The first one is obtained from the distribution of wage changes within a match $\Delta \hat{w}_B$. Under our assumptions that the bargaining process satisfies the [Hosios \(1990\)](#) condition and the renegotiation hazard is independent of the idiosyncratic state, we have that

$$\frac{\mathbb{E}_h[\Delta z]}{\mathcal{E}_{ss}} = -\mathbb{E}_{\mathcal{D}}[\Delta w_B].$$

In addition, the next proposition provides an alternative expression that remains valid when these assumptions do not hold. Importantly, it is easy to show that the following proposition still holds when the hazard rate for renegotiation is state-dependent.

Proposition II.7. *Assume that $\gamma + \chi = 0$. Up to first order, the $CIR_{\mathcal{E}}(\zeta)$ can be expressed in terms of data*

moments as follows:

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = \frac{1}{3f(\hat{\theta}(\hat{w}^*))} \frac{\left(1 - \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}}\right) \mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^3] + \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}} \mathbb{E}_{\mathcal{D}}[\Delta w_B^3]}{\left[\left(1 - \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}}\right) \mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^2] + \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}} \mathbb{E}_{\mathcal{D}}[\Delta w_B^2]\right]^2} + o(\zeta).$$

Proof. The goal is to express the sufficient statistics of the CIR, $\mathbb{E}_h[\Delta z]$, in terms of moments of the distribution of Δw^{EUE} , Δw^B , and (τ^u, τ^m) when $(\gamma + \chi) = 0$. Let $\tilde{x} \equiv x / \mathbb{E}_{\mathcal{D}}[x]$ denote random variable x relative to its mean in the data.

Our starting point is the KFE for the distribution of employed workers:

$$\left(\delta + \mathcal{F}^{\Delta w}\right) g_h(\Delta z) = \frac{\sigma^2}{2} g_h''(\Delta z),$$

where we used the result that $\mathcal{F}^{\Delta w} = \delta^r$. Since the arrival rate of bargaining opportunities is independent of the state of the match, we have that $\mathcal{F}^{\Delta w} g_h(\Delta z) = \mathcal{F}^{\Delta w} \mathcal{E}_{ss} l^B(-\Delta z)$. Thus,

$$\delta g_h(\Delta z) + \mathcal{F}^{\Delta w} \mathcal{E}_{ss} l^B(-\Delta z) = \frac{\sigma^2}{2} g_h''(\Delta z).$$

Multiplying both sides of the equation by Δz^2 and integrating, we

$$\delta \int_{-\Delta^-}^{\Delta^+} \Delta z^2 g_h(\Delta z) d\Delta z + \mathcal{F}^{\Delta w} \mathcal{E}_{ss} \int_{-\Delta^-}^{\Delta^+} \Delta z^2 l^B(-\Delta z) d\Delta z = \frac{\sigma^2}{2} \int_{-\Delta^-}^{\Delta^+} \Delta z^2 g_h''(\Delta z) d\Delta z.$$

Notice that $\mathcal{E}_{ss} \int_{-\Delta^-}^{\Delta^+} \Delta z^2 l^B(-\Delta z) d\Delta z = \mathcal{E}_{ss} \mathbb{E}_{\mathcal{D}}[\Delta w_B^2]$. Integrating $\int_{-\Delta^-}^{\Delta^+} \Delta z^2 g_h''(\Delta z) d\Delta z$ by parts, we have

$$\begin{aligned} \int_{-\Delta^-}^{\Delta^+} \Delta z^2 g_h''(\Delta z) d\Delta z &= \Delta z^2 g_h'(\Delta z) \Big|_{-\Delta^-}^{\Delta^+} - 2 \int_{-\Delta^-}^{\Delta^+} \Delta z g_h'(\Delta z) d\Delta z \\ &= \Delta z^2 g_h'(\Delta z) \Big|_{-\Delta^-}^{\Delta^+} - 2 \underbrace{\Delta z g_h(\Delta z) \Big|_{-\Delta^-}^{\Delta^+}}_{=0} + 2 \underbrace{\int_{-\Delta^-}^{\Delta^+} g_h(\Delta z) d\Delta z}_{= \mathcal{E}_{ss}} \end{aligned}$$

Operating, we have that

$$\begin{aligned} \delta \int_{-\Delta^-}^{\Delta^+} \Delta z^2 g_h(\Delta z) d\Delta z - \frac{\sigma^2}{2} \Delta z^2 g_h'(\Delta z) \Big|_{-\Delta^-}^{\Delta^+} + \mathcal{E}_{ss} \mathcal{F}^{\Delta w} \mathbb{E}_{\mathcal{D}}[\Delta w_B^2] &= \sigma^2 \mathcal{E}_{ss} \iff \\ s \mathcal{E}_{ss} \bar{\mathbb{E}}_h[\Delta z^2] + \mathcal{E}_{ss} \mathcal{F}^{\Delta w} \mathbb{E}_{\mathcal{D}}[\Delta w_B^2] &= \sigma^2 \mathcal{E}_{ss}, \iff \end{aligned}$$

$$\bar{\mathbb{E}}_h[\Delta z^2] + \frac{\mathcal{F}^{\Delta w}}{s} \mathbb{E}_{\mathcal{D}}[\Delta w_B^2] = \frac{\sigma^2}{s}$$

Following Proposition B.3 when $\gamma + \chi = 0$, $\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^2] = [\bar{\mathbb{E}}_u[\Delta z^2] + 2\bar{\mathbb{E}}_h[\Delta z] \bar{\mathbb{E}}_u[\Delta z] + \bar{\mathbb{E}}_h[\Delta z^2]] = \bar{\mathbb{E}}_u[\Delta z^2] + \bar{\mathbb{E}}_h[\Delta z^2]$ with $\bar{\mathbb{E}}_u[\Delta z^2] = \frac{\sigma^2}{f(\hat{\theta}(\hat{w}^*))}$. Thus,

$$\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^2] = \bar{\mathbb{E}}_h[\Delta z^2] + \frac{\sigma^2}{f(\hat{\theta}(\hat{w}^*))}.$$

Combining the previous two results, we have that

$$\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^2] + \frac{\mathcal{F}^{\Delta w}}{s} \mathbb{E}_{\mathcal{D}}[\Delta w_B^2] = \frac{\sigma^2}{s} + \frac{\sigma^2}{f(\hat{\theta}(\hat{w}^*))},$$

or equivalently

$$\sigma^2 = \frac{s\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^2] + \mathcal{F}^{\Delta w} \mathbb{E}_{\mathcal{D}}[\Delta w_B^2]}{1 + \frac{s}{f(\hat{\theta}(\hat{w}^*))}}.$$

With this expression, we are ready to compute $\mathbb{E}_h[\Delta z]$. Repeating the same steps as before but with Δz^3 , we have that

$$s\mathcal{E}_{ss}\bar{\mathbb{E}}_h[\Delta z^3] - \mathcal{F}^{\Delta w}\mathcal{E}_{ss}\mathbb{E}_{\mathcal{D}}[\Delta w_B^3] = \frac{\sigma^2}{2}6\mathbb{E}_h[\Delta z]$$

Following similar steps as in Proposition B.3, we have that

$$\bar{\mathbb{E}}_h[\Delta z^3] = -\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^3]$$

Thus,

$$\mathbb{E}_h[\Delta z] = -\mathcal{E}_{ss} \frac{s\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^3] + \mathcal{F}^{\Delta w}\mathbb{E}_{\mathcal{D}}[\Delta w_B^3]}{3\sigma^2}.$$

Combining this expression with the one for σ^2 , we have that

$$\begin{aligned} \frac{\mathbb{E}_h[\Delta z]}{\sigma^2} &= -\mathcal{E}_{ss} \frac{s\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^3] + \mathcal{F}^{\Delta w}\mathbb{E}_{\mathcal{D}}[\Delta w_B^3]}{3(\sigma^2)^2} \\ &= -\frac{\mathcal{E}_{ss}}{3} \left(1 + \frac{s}{f(\hat{\theta}(\hat{w}^*))}\right)^2 \frac{s\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^3] + \mathcal{F}^{\Delta w}\mathbb{E}_{\mathcal{D}}[\Delta w_B^3]}{[s\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^2] + \mathcal{F}^{\Delta w}\mathbb{E}_{\mathcal{D}}[\Delta w_B^2]]^2}. \end{aligned}$$

Therefore, the CIR for employment is given by

$$\frac{\text{CIR}_{\mathcal{E}}(\zeta)}{\zeta} = -\frac{\mathbb{E}_h[\Delta z]}{\sigma^2} \left(1 - \mathcal{E}_{ss} + \mathcal{E}_{ss} \frac{\mathcal{F}^{\Delta w}}{f(\hat{\theta}(\hat{w}^*))}\right) + o(\zeta)$$

$$= \frac{1}{3f(\hat{\theta}(\hat{w}^*))} \frac{\left(1 - \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}}\right) \mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^3] + \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}} \mathbb{E}_{\mathcal{D}}[\Delta w_B^3]}{\left[\left(1 - \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}}\right) \mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^2] + \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}} \mathbb{E}_{\mathcal{D}}[\Delta w_B^2]\right]^2} + o(\zeta).$$

□

Discussion of Employment Dynamics With Flexible Entry Wages. How do infrequent wage adjustments within the match affect the business cycle dynamics of employment when entry wages are flexible? The CIR of employment provides valuable theoretical insight to answer this question. Intuitively, with all other parameters in the model held constant, a higher frequency of wage renegotiation decreases the CIR of employment: Some workers will be able to reset their wages before transitioning into unemployment. The theory presented shows how the CIR of employment is affected by on-the-job renegotiation, which is now given by

$$\frac{\text{CIR}_{\mathcal{E}}(\zeta)}{\zeta} = \frac{1}{3f(\hat{\theta}(\hat{w}^*))} \frac{\left(1 - \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}}\right) \mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^3] + \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}} \bar{\mathbb{E}}[\Delta w_B^3]}{\left[\left(1 - \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}}\right) \mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^2] + \frac{\mathcal{F}^{\Delta w}}{s + \mathcal{F}^{\Delta w}} \mathbb{E}[\Delta w_B^2]\right]^2} + o(\zeta).$$

Notice that, without wage bargaining (i.e., $\delta^r = \mathcal{F}^{\Delta w} = 0$), we recover the expression in Proposition 6 for the baseline case:

$$\frac{\text{CIR}_{\mathcal{E}}(\zeta)}{\zeta} = \frac{1}{3f(\hat{\theta}(\hat{w}^*))} \frac{\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^3]}{\mathbb{E}_{\mathcal{D}}[\Delta w_{EUE}^2]^2} + o(\zeta).$$

A key result from this analysis is that the relevant micro-moments in the labor market are the second and third moments of the distributions of wage changes within and across jobs, alongside the probabilities of wage renegotiation and job-finding. The intuition for the relevance of the job-finding rate is the same as in the baseline model. The reason the weighted sum of moments of wage changes within and across jobs appear in the CIR is the simple application of Bayes' law:

$$\frac{\mathbb{E}[\Delta w^3]}{\mathbb{E}[\Delta w^2]^2} = \frac{\mathbb{E}[\Delta w^3|EUE]Pr(EUE) + \mathbb{E}[\Delta w^3|bargaining]Pr(bargaining)}{(\mathbb{E}[\Delta w^2|EUE]Pr(EUE) + \mathbb{E}[\Delta w^2|bargaining]Pr(bargaining))^2},$$

where $Pr(EUE) + Pr(bargaining) = 1$. Thus, the CIR of employment following a TFPR shock is fully captured by the moments of the distribution of Δz , which can be recovered with micro-data on wage changes workers experience within and across jobs.

The CIR of Employment with Sticky Entry Wages and Wage Renegotiations.

As in the previous subsection, we focus on the case with no drift; i.e., $\gamma + \chi = 0$. Furthermore,

for pedagogical purposes, we consider the case with symmetric separation thresholds $\Delta^- = \Delta^+$.

Proposition II.8. *Assume sticky entry wages. Up to first order, the CIR of employment is given by*

$$\frac{\text{CIR}_\varepsilon(\zeta)}{\zeta} = \frac{1}{f(\hat{\theta}(\hat{w}^*)) + s} \frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + o(\zeta), \quad (\text{II.47})$$

where

1. If $\Delta^+ \rightarrow \infty$, then

$$\left. \frac{d \log(\eta(\hat{w}))}{d \hat{w}} \right|_{\hat{w}=\hat{w}^*} = \frac{\hat{\rho} + \delta}{\hat{\rho} + \delta + \mathcal{N}^{\Delta w}} \frac{[\alpha + (1 - \alpha)\hat{\rho}\hat{U}]}{\alpha(1 - \hat{\rho}\hat{U})}.$$

2. If $\delta^r \rightarrow \infty$, then

$$\left. \frac{d \log(\eta(\hat{w}))}{d \hat{w}} \right|_{\hat{w}=\hat{w}^*} = 0.$$

3. If Δ^+ small enough, then

$$\left. \frac{d \log(\eta(\hat{w}))}{d \hat{w}} \right|_{\hat{w}=\hat{w}^*} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}.$$

Proof. We divide the proof into three steps. Step 1 characterizes $m_{\varepsilon,\mu}(\Delta z, \zeta)$. Step 2 uses the equilibrium conditions to show (II.47). Step 3 extends proposition B.5 for the case with on-the-job bargaining.

The starting point is the CIR for employment, which is given by

$$\text{CIR}_\varepsilon(\zeta) = \int_{-\infty}^{\infty} m_{\varepsilon,h}(\Delta z) g^h(\Delta z + \zeta) d\Delta z + \int_{-\infty}^{\infty} m_{\varepsilon,\mu}(\Delta z, \zeta) g^u(\Delta z + \zeta) d\Delta z, \quad (\text{II.48})$$

with

$$\begin{aligned} 0 &= 1 - \mathcal{E}_{ss} + \frac{\sigma^2}{2} \frac{d^2 m_{\varepsilon,h}(\Delta z)}{d\Delta z^2} + \delta(m_{\varepsilon,\mu}(0,0) - m_{\varepsilon,h}(\Delta z)) \\ &\quad + \delta^r(m_{\varepsilon,h}(0) - m_{\varepsilon,h}(\Delta z)), \quad \forall \Delta z \in (-\Delta^+, \Delta^+) \\ 0 &= -\mathcal{E}_{ss} + \frac{\sigma^2}{2} \frac{d^2 m_{\varepsilon,\mu}(\Delta z, \zeta)}{d\Delta z^2} + f(\hat{\theta}(\hat{w}^* - \zeta))(m_{\varepsilon,h}(-\zeta) - m_{\varepsilon,\mu}(\Delta z, \zeta)) \end{aligned} \quad (\text{II.49})$$

$$m_{\varepsilon,\mu}(0,0) = m_{\varepsilon,h}(\Delta z), \quad \forall \Delta z \notin (-\Delta^+, \Delta^+) \quad (\text{II.50})$$

$$0 = \lim_{\Delta z \rightarrow -\infty} \frac{d m_{\varepsilon,\mu}(\Delta z, \zeta)}{d \Delta z} = \lim_{\Delta z \rightarrow \infty} \frac{d m_{\varepsilon,\mu}(\Delta z, \zeta)}{d \Delta z} \quad (\text{II.51})$$

$$0 = \int_{-\infty}^{\infty} m_{\varepsilon,h}(\Delta z) g^h(\Delta z) d\Delta z + \int_{-\infty}^{\infty} m_{\varepsilon,\mu}(\Delta z) g^u(\Delta z) d\Delta z \quad (\text{II.52})$$

Step 1. The value function $m_{\mathcal{E},u}(\Delta z, \zeta)$ is independent of Δz and satisfies

$$m_{\mathcal{E},u}(\Delta z, \zeta) = -\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^* - \zeta))} + m_{\mathcal{E},h}(-\zeta).$$

Proof of Step 1. We guess and verify that $m_{\mathcal{E},u}(\Delta z, \zeta) = m_{\mathcal{E},u}(0, \zeta)$ for all Δz . From the equilibrium conditions (II.49) and (II.51),

$$0 = -\mathcal{E}_{ss} + f(\hat{\theta}(\hat{w}^* - \zeta))(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(0, \zeta)).$$

Thus,

$$m_{\mathcal{E},u}(0, \zeta) = m_{\mathcal{E},u}(\Delta z, \zeta) = -\frac{\mathcal{E}_{ss}}{f(\hat{\theta}(\hat{w}^* - \zeta))} + m_{\mathcal{E},h}(-\zeta).$$

Step 2. Up to a first-order approximation, the CIR is given by:

$$\text{CIR}_{\mathcal{E}}(\zeta) = \frac{1}{f(\hat{\theta}(\hat{w}^*)) + s} \frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} \zeta + O(\zeta^2).$$

Proof of Step 2. From Step 1, we have that

$$\text{CIR}'_{\mathcal{E}}(0) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) d\Delta z + \left(-\frac{\mathcal{E}_{ss} f_{\hat{w}}(\hat{\theta}(\hat{w}^*))}{f(\hat{\theta}(\hat{w}^*))^2} - m'_{\mathcal{E},h}(0) \right) (1 - \mathcal{E}_{ss}).$$

Since $\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) d\Delta z$ satisfies the same system of functional equations as the CIR of employment with flexible entry wages and wage renegotiations characterized in Online Appendix II.3,

$$\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) d\Delta z = -\frac{\mathbb{E}_h[\Delta z]}{\sigma^2} \left(1 - \mathcal{E}_{ss} + \mathcal{E}_{ss} \frac{\mathcal{F}^{\Delta w}}{f(\hat{\theta}(\hat{w}^*))} \right).$$

By the symmetry of separation thresholds, we have that

$$\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) (g^h)'(\Delta z) d\Delta z = 0.$$

and $m'_{\mathcal{E},h}(0) = 0$. Thus,

$$\text{CIR}'_{\mathcal{E}}(0) = -\frac{\mathcal{E}_{ss} f_{\hat{w}}(\hat{\theta}(\hat{w}^*))}{f(\hat{\theta}(\hat{w}^*))^2} (1 - \mathcal{E}_{ss})$$

Since under symmetry $\eta(\hat{w}^*) = \alpha$ and $\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho} + \delta^r) = 0$, from free entry and optimality of \hat{w}^* ,

$$\frac{f_{\hat{w}}(\hat{\theta}(\hat{w}^*))}{f(\hat{\theta}(\hat{w}^*))} = -\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)}.$$

Thus,

$$\text{CIR}'_{\mathcal{E}}(0) = \frac{1}{f(\hat{\theta}(\hat{w}^*)) + s} \frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)}.$$

Step 3. Define

$$\tau^{end} = \inf\{t \geq 0 : \Gamma_t \notin (\hat{w}^-, \hat{w}^+)\}$$

where (\hat{w}^-, \hat{w}^+) is a Nash equilibrium. Then, the worker's share $\eta(\hat{w})$ satisfies the Bellman equation

$$\eta(\hat{w}) = \mathbb{E} \left[\int_0^{\tau^{end}} e^{-(\hat{\rho} + \delta + \delta^r)t} (\hat{\rho} + \delta + \delta^r) \left(\frac{e^{\Gamma_t} - 1}{1 - \hat{\rho}\hat{U}} (1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)) + 1 \right) dt \right. \\ \left. + e^{-(\hat{\rho} + \delta + \delta^r)\tau^{end}} \mathbb{1}[\Delta z_{\tau^{end}} = \Delta^+] | \Gamma_0 = \hat{w} \right]$$

with

$$d\Gamma_t = (\hat{\rho} + \delta + \delta^r) \sigma^2 \mathcal{T}'_{\hat{w}}(\Gamma_t, \hat{\rho} + \delta^r) dt + \sigma \sqrt{\mathcal{T}(\Gamma_t, \hat{\rho} + \delta^r) (\hat{\rho} + \delta + \delta^r)} d\mathcal{W}_t^z.$$

Proof of step 3. The HJB equations for the worker's value and the surplus of the match are

$$(\hat{\rho} + \delta + \delta^r) \hat{W}(\hat{w}) = e^{\hat{w}} + \delta^r \hat{W}(\hat{w}^*) - \hat{\rho}\hat{U} + \frac{\sigma^2}{2} \hat{W}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+) \\ (\hat{\rho} + \delta + \delta^r) \hat{S}(\hat{w}) = 1 + \delta^r \hat{S}(\hat{w}^*) - \hat{\rho}\hat{U} + \frac{\sigma^2}{2} \hat{S}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+),$$

respectively. Replacing $\eta(\hat{w}) = \hat{W}(\hat{w})/\hat{S}(\hat{w})$ in the worker's value function, we have $\forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$:

$$(\hat{\rho} + \delta + \delta^r) (\eta(\hat{w}) \hat{S}(\hat{w})) = e^{\hat{w}} + \delta^r \eta(\hat{w}^*) \hat{S}(\hat{w}^*) - \hat{\rho}\hat{U} + \frac{\sigma^2}{2} (\eta(\hat{w}) \hat{S}''(\hat{w}) + 2\eta'(\hat{w}) \hat{S}'(\hat{w}) + \eta''(\hat{w}) \hat{S}(\hat{w})).$$

Using the HJB equation of the surplus to replace $(\hat{\rho} + \delta) \hat{S}(\hat{w})$ on the left hand side,

$$(1 - \hat{\rho}\hat{U} + \delta^r \hat{S}(\hat{w}^*)) \eta(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} + \delta^r \hat{S}(\hat{w}^*) + \eta'(\hat{w}) \sigma^2 \hat{S}'(\hat{w}) + \eta''(\hat{w}) \frac{\sigma^2}{2} \hat{S}(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Since $\hat{S}(\hat{w}) = \frac{1-\hat{\rho}\hat{U}}{1-\delta^r\mathcal{T}(\hat{w}^*,\hat{\rho}+\delta^r)}\mathcal{T}(\hat{w},\hat{\rho}+\delta^r)$, operating from the left hand side $\forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$, we have

$$\begin{aligned} & (1 - \hat{\rho}\hat{U} + \delta^r\hat{S}(\hat{w}^*))\eta(\hat{w}) \\ &= e^{\hat{w}} - 1 + \frac{1 - \hat{\rho}\hat{U}}{1 - \delta^r\mathcal{T}(\hat{w}^*,\hat{\rho} + \delta^r)} + \eta'(\hat{w})\sigma^2 \frac{(1 - \hat{\rho}\hat{U})\mathcal{T}'(\hat{w},\hat{\rho} + \delta^r)}{1 - \delta^r\mathcal{T}(\hat{w}^*,\hat{\rho} + \delta^r)} + \eta''(\hat{w})\frac{\sigma^2}{2} \frac{(1 - \hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho} + \delta^r)}{1 - \delta^r\mathcal{T}(\hat{w}^*,\hat{\rho} + \delta^r)} \end{aligned}$$

In conclusion, we arrive at

$$\eta(\hat{w}) = (1 - \delta^r\mathcal{T}(\hat{w}^*,\hat{\rho} + \delta^r)) \frac{e^{\hat{w}} - 1}{1 - \hat{\rho}\hat{U}} + 1 + \eta'(\hat{w})\sigma^2\mathcal{T}'(\hat{w},\hat{\rho} + \delta^r) + \eta''(\hat{w})\frac{\sigma^2}{2}\mathcal{T}(\hat{w},\hat{\rho} + \delta^r).$$

Multiplying by $(\hat{\rho} + \delta + \delta^r)$, we have, $\forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$:

$$\begin{aligned} (\hat{\rho} + \delta + \delta^r)\eta(\hat{w}) &= (\hat{\rho} + \delta + \delta^r) \left((1 - \delta^r\mathcal{T}(\hat{w}^*,\hat{\rho} + \delta^r)) \frac{e^{\hat{w}} - 1}{1 - \hat{\rho}\hat{U}} + 1 \right) \\ &\quad + \eta'(\hat{w})\sigma^2(\hat{\rho} + \delta + \delta^r)\mathcal{T}'(\hat{w},\hat{\rho} + \delta^r) + \eta''(\hat{w})\frac{\sigma^2}{2}(\hat{\rho} + \delta + \delta^r)\mathcal{T}(\hat{w},\hat{\rho} + \delta^r). \end{aligned}$$

Finally, recall the value-matching and smooth-pasting conditions

$$\hat{W}(\hat{w}^-) = \hat{f}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{f}(\hat{w}^+) = 0, \quad \hat{W}'(-\Delta^-) = \hat{f}'(\Delta^+) = 0.$$

By L'Hôpital's rule,

$$\begin{aligned} \lim_{\hat{w} \downarrow \hat{w}^-} \eta(\hat{w}) &= \lim_{\hat{w} \downarrow \hat{w}^-} \frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w} \downarrow \hat{w}^-} \frac{\hat{W}'(\hat{w})}{\hat{f}'(\hat{w})} = 0 \\ \lim_{\hat{w} \uparrow \hat{w}^+} \eta(\hat{w}) &= \lim_{\hat{w} \uparrow \hat{w}^+} \frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w} \uparrow \hat{w}^+} \frac{\hat{W}'(\hat{w})}{\hat{W}'(\hat{w})} = 1, \end{aligned}$$

which are the boundary values for the worker's share at the separation triggers.

Finally, the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping from the corresponding HJB equations and boundary conditions of $\eta(\hat{w})$ to the sequential formulation) gives us the following Bellman equation

$$\begin{aligned} \eta(\hat{w}) &= \mathbb{E} \left[\int_0^{\tau^{end}} e^{-(\hat{\rho}+\delta+\delta^r)t} (\hat{\rho} + \delta + \delta^r) \left(\frac{e^{\Gamma t} - 1}{1 - \hat{\rho}\hat{U}} (1 - \delta^r\mathcal{T}(\hat{w}^*,\hat{\rho} + \delta^r)) + 1 \right) dt \right. \\ &\quad \left. + e^{-(\hat{\rho}+\delta+\delta^r)\tau^{end}} \mathbb{1}[\Delta z_{\tau^{end}} = \Delta^+] | \Gamma_0 = \hat{w} \right] \end{aligned}$$

where

$$\tau^{end} = \inf\{t \geq 0 : \Gamma_t \notin (\hat{w}^-, \hat{w}^+)\}$$

and

$$d\Gamma_t = (\hat{\rho} + \delta + \delta^r)\sigma^2 \mathcal{T}'_{\hat{w}}(\Gamma_t, \hat{\rho} + \delta^r) dt + \sigma \sqrt{\mathcal{T}(\Gamma_t, \hat{\rho} + \delta^r)(\hat{\rho} + \delta + \delta^r)} d\mathcal{W}_t^z.$$

Step 4. The following results hold:

1. If $\Delta^+ \rightarrow \infty$, then

$$\left. \frac{d\log(\eta(\hat{w}))}{d\hat{w}} \right|_{\hat{w}=\hat{w}^*} = \frac{\hat{\rho} + \delta}{\hat{\rho} + \delta + \mathcal{N}^{\Delta w}} \frac{[\alpha + (1 - \alpha)\hat{\rho}\hat{U}]}{\alpha(1 - \hat{\rho}\hat{U})}.$$

2. If $\delta^r \rightarrow \infty$, then

$$\left. \frac{d\log(\eta(\hat{w}))}{d\hat{w}} \right|_{\hat{w}=\hat{w}^*} = 0.$$

3. If Δ^+ small enough, then

$$\left. \frac{d\log(\eta(\hat{w}))}{d\hat{w}} \right|_{\hat{w}=\hat{w}^*} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}.$$

Proof of step 4.

Next, we prove the first two results. If $\Delta^+ \rightarrow \infty$, then $\mathcal{T}(\hat{w}, \hat{\rho}) = \int_0^\infty e^{-(\hat{\rho} + \delta + \delta^r)t} dt = \frac{1}{\hat{\rho} + \delta + \delta^r}$ and $\mathcal{T}'_{\hat{w}^*}(\hat{w}^*, \hat{\rho} + \delta^r) = 0$. Similarly, if $\delta^r \rightarrow \infty$, then $\lim_{\delta^r \rightarrow \infty} \frac{\mathcal{T}(\hat{w}, \hat{\rho} + \delta^r)}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)} = \frac{1}{\hat{\rho} + \delta}$; thus, $\mathcal{T}'_{\hat{w}^*}(\hat{w}^*, \hat{\rho} + \delta^r) = 0$. Therefore, by the definition of $\eta(\hat{w})$,

$$\alpha = \eta(\hat{w}^*) = \frac{\hat{W}(\hat{w}^*)}{S(\hat{w}^*)} = \frac{\mathbb{E} \left[\int_0^{\tau^m} e^{-(\hat{\rho} + \delta^r)t + w_t} | \hat{w}_0 = \hat{w}^* \right] + (\delta^r \overbrace{\hat{W}(\hat{w}^*)}^{\alpha \hat{S}(\hat{w}^*)} - \hat{\rho}\hat{U}) \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)}{\frac{1 - \hat{\rho}\hat{U}}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)} \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)}$$

which gives, for both limits $\Delta^+ \rightarrow \infty$ and $\delta^r \rightarrow \infty$:

$$\mathbb{E} \left[\int_0^{\tau^m} e^{-(\hat{\rho} + \delta^r)t + w_t} | \hat{w}_0 = \hat{w}^* \right] = (\alpha + (1 - \alpha)\hat{\rho}\hat{U}) \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)$$

Take the limit $\Delta^+ \rightarrow \infty$ and following similar steps as in the proof in the baseline model, we obtain

$$\eta'(\hat{w}^*) = (\hat{\rho} + \delta + \delta^r)(1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)) \frac{\mathbb{E} \left[\int_0^{\tau^m} e^{-(\hat{\rho} + \delta^r)t + \hat{w}_t} dt | \hat{w}_0 = \hat{w}^* \right]}{1 - \hat{\rho}\hat{U}}.$$

Combining all these results, we finally obtain

$$\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{\hat{\rho} + \delta}{\hat{\rho} + \delta + \delta^r} \frac{(\alpha + (1 - \alpha)\hat{\rho}\hat{U})}{\alpha(1 - \hat{\rho}\hat{U})}.$$

Regarding the second limit $\delta^r \rightarrow \infty$, it is easy to show that $\eta'(\hat{w}^*) = 0$.

Following similar steps as in the proof for the baseline model, we have that if Δ^+ is small, then

$$\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{1}{2\alpha\Delta^+}.$$

Since $\frac{1}{s + \mathcal{F}\Delta w} = \mathcal{T}(0, \delta^r) \approx \frac{1}{\delta + \delta^r + (\sigma/\Delta^+)^2}$ and letting $s^{end} = \frac{\sigma}{\Delta^+}$ denote the rate of endogenous separations,

$$\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}.$$

□

Discussion of Employment Dynamics With Sticky Entry Wages. Several insights emerge from this analysis. From the first part of Proposition II.8, we learn that the presence of on-the-job wage renegotiations affects the response of the job-finding rate to the TFPR shock. The intuition is that the possibility of wage renegotiations reduces the effect of the TFPR shock on job creation: Shocks stop affecting the real normalized wage following the first wage renegotiation. In the extreme case, when the frequency of bargaining tends to ∞ , job creation does not respond to the shock since wages become renegotiated and reflect the occurrence of the shock immediately after the match is created. The last part of Proposition II.8 shows that the possibility of wage renegotiations does not affect the shape of the sufficient statistic for the elasticity of the worker's share to the entry wage, which continues to be determined by the frequency of endogenous separations. However, wage renegotiations does affect the value of the elasticity of the worker's share to the entry wage because a higher frequency of wage renegotiations reduces the frequency of endogenous separations.

III Additional Results for Section 4: Mapping the Model to Labor Market Microdata

III.1 Characterizing $g^h(\Delta z)$ and $g^u(\Delta z)$

Proposition III.1. Assume $\delta > 0$. Then, $g^h(\Delta z)$ and $g^u(\Delta z)$ are given by

$$g^h(\Delta z) = \mathcal{E} \mathcal{G}_h \begin{cases} \frac{e^{\beta_1(\delta)(\Delta z + \Delta^-)} - e^{\beta_2(\delta)(\Delta z + \Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} & \text{if } \Delta z \in (-\Delta^-, 0] \\ \frac{e^{\beta_1(\delta)(\Delta z - \Delta^+)} - e^{\beta_2(\delta)(\Delta z - \Delta^+)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} & \text{if } \Delta z \in [0, \Delta^+) \end{cases}$$

$$g^u(\Delta z) = (1 - \mathcal{E}) \mathcal{G}_u \begin{cases} e^{\beta_2(f(\hat{\theta}(\hat{w}^*))\Delta z)} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{\theta}(\hat{w}^*))\Delta z)} & \text{if } \Delta z \in [0, \infty) \end{cases}$$

where

$$\beta_1(x) = \frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}, \beta_2(x) = \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2},$$

$$\mathcal{E} = \frac{f(\hat{\theta}(\hat{w}^*))}{f(\hat{\theta}(\hat{w}^*)) + \delta + \frac{\sigma^2}{2} \mathcal{G}_h \left[\frac{\beta_1(\delta) - \beta_2(\delta)}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} - \frac{\beta_1(\delta) - \beta_2(\delta)}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} \right]},$$

$$\mathcal{G}_h = \left[\frac{e^{\beta_1(\delta)\Delta^-} - 1}{\beta_1(\delta)} - \frac{e^{\beta_2(\delta)\Delta^-} - 1}{\beta_2(\delta)} + \frac{1 - e^{-\beta_1\Delta^+}}{\beta_1(\delta)} - \frac{1 - e^{-\beta_2\Delta^+}}{\beta_2(\delta)} \right]^{-1},$$

$$\mathcal{G}_u = \left[-\beta_1(f(\hat{\theta}(\hat{w}^*)))^{-1} + \beta_2(f(\hat{\theta}(\hat{w}^*)))^{-1} \right]^{-1}.$$

Proof. Let us write the KFE and border conditions:

$$\delta g^h(\Delta z) = \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2} (g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+) / \{0\} \quad (\text{III.1})$$

$$g^h(-\Delta^-) = g^h(\Delta^+) = 0, \quad (\text{III.2})$$

$$f(\hat{\theta}(\hat{w}^*)) g^u(\Delta z) = \gamma (g^u)'(\Delta z) + \frac{\sigma^2}{2} (g^u)''(\Delta z) \quad \forall \Delta z \in (-\infty, \infty) / \{0\}, \quad (\text{III.3})$$

$$\lim_{\Delta z \rightarrow -\infty} g^u(\Delta z) = \lim_{\Delta z \rightarrow \infty} g^u(\Delta z) = 0, \quad (\text{III.4})$$

$$1 = \int_{-\infty}^{\infty} g^u(\Delta z) d\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z) d\Delta z, \quad (\text{III.5})$$

$$f(\hat{\theta}(\hat{w}^*))(1 - \mathcal{E}) = \delta \mathcal{E} + \frac{\sigma^2}{2} \left[\lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+} (g^h)'(\Delta z) \right], \quad (\text{III.6})$$

$$g^h(\Delta z), g^u(\Delta z) \in \mathbb{C}.$$

We guess and verify the proposed solution. Substituting the guess for $g^h(\Delta z)$ in (III.1) for $\Delta z < 0$, we have

$$0 = -\delta \mathcal{E} \mathcal{G}_h \frac{e^{\beta_1(\delta)(\Delta z + \Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \gamma \beta_1(\delta) \mathcal{E} \mathcal{G}_h \frac{e^{\beta_1(\delta)(\Delta z + \Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \mathcal{E} \mathcal{G}_h \frac{\sigma^2}{2} \beta_1(\delta)^2 \frac{e^{\beta_1(\delta)(\Delta z + \Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} \iff$$

$$0 = -\delta + \gamma \beta_1(\delta) + \frac{\sigma^2}{2} \beta_1(\delta)^2,$$

mutatis mutandis, for the terms that include $\beta_2(\delta)$. Given the definition of $\beta_1(\delta)$, the guess satisfies (III.1). A similar argument applies when (III.1) is evaluated at $\Delta z > 0$. It is easy to verify that the boundary conditions (III.2) are satisfied and that $g^h(\Delta z)$ is continuous at $\Delta z = 0$. Following the same steps for $g^u(\Delta z)$, we verify conditions (III.3) and (III.4). Next, we verify condition (III.5):

$$\begin{aligned} & \int_{-\infty}^{\infty} g^u(\Delta z) d\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z) d\Delta z \\ &= (1 - \mathcal{E}) \mathcal{G}_u \left[\int_{-\infty}^0 e^{\beta_2(f(\hat{\theta}(\hat{w}^*)))\Delta z} d\Delta z + \int_0^{\infty} e^{\beta_1(f(\hat{\theta}(\hat{w}^*)))\Delta z} d\Delta z \right] + \dots \\ & \quad + \mathcal{E} \mathcal{G}_h \left[\int_{-\Delta^-}^0 \frac{e^{\beta_1(\delta)(\Delta z + \Delta^-)} - e^{\beta_2(\delta)(\Delta z + \Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} d\Delta z + \int_0^{\Delta^+} \frac{e^{\beta_1(\delta)(\Delta z - \Delta^+)} - e^{\beta_2(\delta)(\Delta z - \Delta^+)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} d\Delta z \right] \\ &= (1 - \mathcal{E}) \mathcal{G}_u \left[\frac{1 - \lim_{\Delta z \rightarrow -\infty} e^{\beta_2(f(\hat{\theta}(\hat{w}^*)))\Delta z}}{\beta_2(f(\hat{\theta}(\hat{w}^*)))} + \frac{\lim_{\Delta z \rightarrow \infty} e^{\beta_1(f(\hat{\theta}(\hat{w}^*)))\Delta z} - 1}{\beta_1(f(\hat{\theta}(\hat{w}^*)))} \right] + \dots \\ & \quad + \mathcal{E} \mathcal{G}_h \left[\frac{e^{\beta_1(\delta)\Delta^-} - 1}{\beta_1(\delta)} - \frac{e^{\beta_2(\delta)\Delta^-} - 1}{\beta_2(\delta)} + \frac{1 - e^{-\beta_1\Delta^+}}{\beta_1(\delta)} - \frac{1 - e^{-\beta_2\Delta^+}}{\beta_2(\delta)} \right] = (1 - \mathcal{E}) + \mathcal{E} = 1. \end{aligned}$$

Finally, combining condition (III.6) with the definition of $g^h(\Delta z)$, the employment rate is

$$\mathcal{E} = \frac{f(\hat{\theta}(\hat{w}^*))}{f(\hat{\theta}(\hat{w}^*)) + \delta + \frac{\sigma^2}{2} \mathcal{G}_h \left[\frac{\beta_1(\delta) - \beta_2(\delta)}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} - \frac{\beta_1(\delta) - \beta_2(\delta)}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} \right]}.$$

□

III.2 Characterizing $l^w(\Delta w)$

Proposition III.2. *The distribution of log nominal wage changes satisfies*

$$l^w(\Delta w) = \mathcal{G}_u \left[\beta_2(f(\hat{\theta}(\hat{w}^*))) e^{-\beta_2(f(\hat{\theta}(\hat{w}^*)))\Delta w} \Gamma_2(\Delta w) + \beta_1(f(\hat{\theta}(\hat{w}^*))) e^{-\beta_1(f(\hat{\theta}(\hat{w}^*)))\Delta w} \Gamma_1(\Delta w) \right]$$

with

$$(\Gamma_1(c), \Gamma_2(c)) = \left(\int_{-\infty}^{-c} e^{-\beta_1(f(\hat{w}^*))x} \bar{G}^h(x) dx, \int_{-c}^{\infty} e^{-\beta_2(f(\hat{w}^*))x} \bar{G}^h(x) dx \right).$$

Proof. Fix a date t_0 and focus on a newly hired worker. Then, the distribution of wage changes between two new jobs is given by

$$\begin{aligned} Pr(\Delta w \leq c) &= Pr(w_{t_0+\tau^m+\tau^u} - w_{t_0} \leq c) \\ &=^{(1)} Pr(w_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m+\tau^u} - (w_{t_0} - z_{t_0}) + (z_{t_0+\tau^m+\tau^u} - z_{t_0}) \leq c) \\ &=^{(2)} Pr(\hat{w}^* - \hat{w}^* + (z_{t_0+\tau^m+\tau^u} - z_{t_0}) \leq c) \\ &=^{(3)} Pr(-(\Delta z^h + \Delta z^u) \leq c), \end{aligned}$$

where Δz^h and Δz^u denote cumulative productivity shocks during completed employment and unemployment spells, respectively. Here, step (1) adds and subtracts productivity at the beginning of both job spells. In step (2), we use the result that \hat{w}^* is constant across jobs. Step 3 uses the facts that τ^u and the Brownian motion increments are independent of the filtration \mathcal{F}_{τ_u} . Therefore, the distributions of cumulative productivity shocks for completed employment and unemployment spells are given by

$$\bar{G}^h(\Delta z) = \begin{cases} 1 & \text{if } \Delta z \in [\Delta^+, \infty) \\ \frac{1}{s\bar{\mathcal{E}}} \left[\frac{\sigma^2}{2} \lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z) + \delta \int_{-\Delta^-}^{\Delta z} g^h(x) dx \right] & \text{if } \Delta z \in [-\Delta^-, \Delta^+) \\ 0 & \text{if } \Delta z \in (-\infty, -\Delta^-) \end{cases}$$

$$\bar{g}^u(\Delta z) = \mathcal{G}_u \begin{cases} e^{\beta_2(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}$$

Thus,

$$\begin{aligned} Pr(\Delta w \leq c) &= Pr(-(\Delta z^u + \Delta z^h) \leq c) \\ &= 1 - Pr(\Delta z^u + \Delta z^h \leq -c) \\ &=^{(1)} 1 - \int_{-\infty}^{\infty} \bar{G}^h(-(c + \Delta z)) \bar{g}^u(\Delta z) d\Delta z \\ &=^{(2)} 1 - \mathcal{G}_u \left[\int_{-\infty}^0 e^{\beta_2(f(\hat{w}^*))\Delta z} \bar{G}^h(-(c + \Delta z)) d\Delta z + \int_0^{\infty} e^{\beta_1(f(\hat{w}^*))\Delta z} \bar{G}^h(-(c + \Delta z)) d\Delta z \right] \end{aligned}$$

$$\begin{aligned}
&=^{(3)} 1 + \mathcal{G}_u \left[\int_{\infty}^{-c} e^{-\beta_2(f(\hat{\theta}(\hat{w}^*))(c+x))} \bar{G}^h(x) \, dx + \int_{-c}^{-\infty} e^{-\beta_1(f(\hat{\theta}(\hat{w}^*))(c+x))} \bar{G}^h(x) \, dx \right] \\
&=^{(4)} 1 - \mathcal{G}_u \left[e^{-\beta_2(f(\hat{\theta}(\hat{w}^*)))c} \int_{-c}^{\infty} e^{-\beta_2(f(\hat{\theta}(\hat{w}^*)))x} \bar{G}^h(x) \, dx + e^{-\beta_1(f(\hat{\theta}(\hat{w}^*)))c} \int_{-\infty}^{-c} e^{-\beta_1(f(\hat{\theta}(\hat{w}^*)))x} \bar{G}^h(x) \, dx \right].
\end{aligned}$$

In step (1), we use the independence of Δz^u and Δz^h . In step (2), we use the definition of $\bar{g}^u(\Delta z)$. In step (3), we integrate by substituting $x = -c - \Delta z$, and in step (4), we use the properties of an integral. The last step involves defining

$$(\Gamma_1(c), \Gamma_2(c)) = \left(\int_{-\infty}^{-c} e^{-\beta_1(f(\hat{\theta}(\hat{w}^*)))x} \bar{G}^h(x) \, dx, \int_{-c}^{\infty} e^{-\beta_2(f(\hat{\theta}(\hat{w}^*)))x} \bar{G}^h(x) \, dx \right).$$

□

III.3 Characterizing $\mathbb{E}_h[\Delta z^n]$

Let $\bar{\mathbb{E}}_h[\cdot]$ and $\bar{\mathbb{E}}_u[\cdot]$ be the expectation operators under the distributions $\bar{g}^h(\Delta z)$ and $\bar{g}^u(\Delta z)$, respectively.

Proposition III.3. Define the weights $\omega^{hm}(\Delta z) = \frac{\Delta z^m}{\bar{\mathbb{E}}_h[\Delta z^m]}$ with the property that

$$\bar{\mathbb{E}}_h \left[\omega^{hm}(\Delta z) \right] = 1.$$

If $\gamma + \chi = 0$, then $\mathbb{E}_h[(\Delta z)^n]$ can be recovered from

$$\mathbb{E}_h[(\Delta z)^n] = \frac{2\mathcal{E}}{(n+1)(n+2)} \bar{\mathbb{E}}_h \left[(\Delta z)^n \omega^{h2}(\Delta z) \right]. \quad (\text{III.7})$$

If $\gamma + \chi \neq 0$, then $\mathbb{E}_h[(\Delta z)^n]$ can be recovered recursively from

$$\mathbb{E}_h[(\Delta z)^n] = \frac{\mathcal{E}}{n+1} \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)] + \frac{\sigma^2 n}{2\gamma} \mathbb{E}_h[(\Delta z)^{n-1}].$$

The moments $\bar{\mathbb{E}}_h[(\Delta z)^n \omega^{hk}(\Delta z)] = \frac{\bar{\mathbb{E}}_h[(\Delta z)^{n+k}]}{\bar{\mathbb{E}}_h[(\Delta z)^k]}$ can be recovered from the following linear system of equations:

$$\mathbb{E}_{\mathcal{D}}[\Delta w^n] = (-1)^n \sum_{i=0}^n \binom{n}{i} \bar{\mathbb{E}}_h[\Delta z^i] \bar{\mathbb{E}}_u[\Delta z^{n-i}],$$

$$\bar{\mathbb{E}}_u[(\Delta z)^{n-i}] = \frac{(n-i)!}{\mathcal{L}_1^{n-i}(\mathcal{L}_2 + \mathcal{L}_2^{-1})} \left(\mathcal{L}_2^{-(n-i+1)} - (-\mathcal{L}_2)^{(n-i+1)} \right),$$

where

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{\theta}(\hat{w}^*))}{\sigma^2}} \text{ and } \mathcal{L}_2 = \sqrt{\frac{(\gamma + \chi) + \sqrt{(\gamma + \chi)^2 + 2\sigma^2 f(\hat{\theta}(\hat{w}^*))}}{- (\gamma + \chi) + \sqrt{(\gamma + \chi)^2 + 2\sigma^2 f(\hat{\theta}(\hat{w}^*))}}.$$

Proof. We divide the proof into 3 steps.

Step 1. We first show that

$$\mathbb{E}_h[(\Delta z)^n] = \frac{\mathcal{E}}{n+1} \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)] - \frac{\sigma^2 n}{2(\gamma + \chi)} \mathbb{E}_h[(\Delta z)^{n-1}].$$

when $(\gamma + \chi) \neq 0$. For the case with $(\gamma + \chi) = 0$, see [Baley and Blanco \(2021\)](#).

Let us define $Y_t = (\Delta z_t)^n$. The law of motion for Δz_t is given by $d\Delta z_t = -(\gamma + \chi) dt + \sigma d\mathcal{W}_t^z$.

Applying Itô's Lemma, we obtain

$$\begin{aligned} dY_t &= n(\Delta z_t)^{n-1} d\Delta z_t + \frac{1}{2}n(n-1)(\Delta z_t)^{n-2}(d\Delta z_t)^2 \\ &= \left[-(\gamma + \chi)n(\Delta z_t)^{n-1} + \frac{\sigma^2}{2}n(n-1)(\Delta z_t)^{n-2} \right] dt + n\sigma(\Delta z_t)^{n-1} d\mathcal{W}_t^z \end{aligned}$$

Thus,

$$(\Delta z_{\tau^m})^n = -(\gamma + \chi)n \int_0^{\tau^m} (\Delta z_t)^{n-1} dt + \frac{\sigma^2}{2}n(n-1) \int_0^{\tau^m} (\Delta z_t)^{n-2} dt + n \int_0^{\tau^m} (\Delta z_t)^{n-1} \sigma d\mathcal{W}_t^z.$$

Following the same arguments as in the proof of [Proposition C.1](#) and using the Renewal Principle to have $\mathbb{E}_{\mathcal{D}}[\tau^m] = 1/s$, we obtain

$$\bar{\mathbb{E}}_h[(\Delta z)^n] = -(\gamma + \chi)n \mathbb{E}_{\mathcal{D}}[\tau^m] \frac{\bar{\mathbb{E}}_h[(\Delta z)^{n-1}]}{\mathcal{E}} + \frac{\sigma^2 n(n-1)}{2s} \frac{\bar{\mathbb{E}}_h[(\Delta z)^{n-2}]}{\mathcal{E}}$$

or equivalently

$$\bar{\mathbb{E}}_h[(\Delta z)^n] = -\frac{\mathcal{E}}{(\gamma + \chi)\mathbb{E}_{\mathcal{D}}[\tau^m]} \frac{\bar{\mathbb{E}}_h[(\Delta z)^{n+1}]}{n+1} + \frac{\sigma^2 n}{2(\gamma + \chi)} \bar{\mathbb{E}}_h[(\Delta z)^{n-1}].$$

From [Proposition C.1](#), we have $(\gamma + \chi)\mathbb{E}_{\mathcal{D}}[\tau^m] = -\bar{\mathbb{E}}_h[(\Delta z)]$ and $\frac{\bar{\mathbb{E}}_h[(\Delta z)^{n+1}]}{\bar{\mathbb{E}}_h[(\Delta z)]} = \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)]$.

Thus,

$$\mathbb{E}_h[(\Delta z)^n] = \frac{\mathcal{E}}{n+1} \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)] + \frac{\sigma^2 n}{2(\gamma + \chi)} \mathbb{E}_h[(\Delta z)^{n-1}].$$

Step 2. Here we show that

$$\mathbb{E}_{\mathcal{D}}[\Delta w^n] = (-1)^n \sum_{i=0}^n \binom{n}{i} \bar{\mathbb{E}}_h[\Delta z^i] \bar{\mathbb{E}}_u[\Delta z^{n-i}].$$

Using the independence of cumulative productivity shocks during employment and unemployment,

$$\begin{aligned} \mathbb{E}_{\mathcal{D}}[\Delta w^n] &= \bar{\mathbb{E}}[(-\Delta z^h - \Delta z^u)^n], \\ &= \sum_{i=0}^n \binom{n}{i} \bar{\mathbb{E}}[(-\Delta z^h)^i (-\Delta z^u)^{n-i}], \\ &= \sum_{i=0}^n \binom{n}{i} \bar{\mathbb{E}}_h[(-\Delta z)^i] \bar{\mathbb{E}}_u[(-\Delta z)^{n-i}], \\ &= (-1)^n \sum_{i=0}^n \binom{n}{i} \bar{\mathbb{E}}_h[\Delta z^i] \bar{\mathbb{E}}_u[\Delta z^{n-i}], \end{aligned}$$

Step 3. Here we show that

$$\bar{\mathbb{E}}_u[(\Delta z)^{n-i}] = \frac{(n-i)!}{\mathcal{L}_1^{n-i} (\mathcal{L}_2 + \mathcal{L}_2^{-1})} \left(\mathcal{L}_2^{-(n-i+1)} - (-\mathcal{L}_2)^{(n-i+1)} \right).$$

Let us depart from the definition of $\bar{g}^u(\Delta z)$, which is given by

$$\bar{g}^u(\Delta z) = \left[-\beta_1(f(\hat{\theta}(\hat{w}^*)))^{-1} + \beta_2(f(\hat{\theta}(\hat{w}^*)))^{-1} \right]^{-1} \begin{cases} e^{\beta_2(f(\hat{\theta}(\hat{w}^*)))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{\theta}(\hat{w}^*)))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}$$

where $\beta_1(x) = \frac{-(\gamma+\chi) - \sqrt{(\gamma+\chi)^2 + 2\sigma^2 x}}{\sigma^2}$ and $\beta_2(x) = \frac{-(\gamma+\chi) + \sqrt{(\gamma+\chi)^2 + 2\sigma^2 x}}{\sigma^2}$. This step consist of showing that $\bar{g}^u(\Delta z)$ is an asymmetric Laplace distribution with parameters

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{\theta}(\hat{w}^*))}{\sigma^2}} \quad \text{and} \quad \mathcal{L}_2 = \sqrt{\frac{(\gamma + \chi) + \sqrt{(\gamma + \chi)^2 + 2\sigma^2 f(\hat{\theta}(\hat{w}^*))}}{-\gamma - \chi + \sqrt{(\gamma + \chi)^2 + 2\sigma^2 f(\hat{\theta}(\hat{w}^*))}}$$

The ratio between \mathcal{L}_1 and \mathcal{L}_2 is

$$\frac{\mathcal{L}_1}{\mathcal{L}_2} = \beta_2(f(\hat{\theta}(\hat{w}^*))).$$

The negative of the product between \mathcal{L}_1 and \mathcal{L}_2 is

$$-\mathcal{L}_1\mathcal{L}_2 = \beta_1(f(\hat{\theta}(\hat{w}^*))).$$

Therefore, we can write $\bar{g}^u(\Delta z)$

$$\bar{g}^u(\Delta z) = \frac{\mathcal{L}_1}{\mathcal{L}_2 + \mathcal{L}_2^{-1}} \begin{cases} e^{\frac{\mathcal{L}_1}{\mathcal{L}_2}\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{-\mathcal{L}_1\mathcal{L}_2\Delta z} & \text{if } \Delta z \in [0, \infty), \end{cases}$$

which is the probability distribution function of an asymmetric Laplace distribution. It is a standard result that the n -th moment for an asymmetric Laplace distribution is given by

$$\bar{\mathbb{E}}_u[(\Delta z)^n] = \frac{n!}{\mathcal{L}_1^n (\mathcal{L}_2 + \mathcal{L}_2^{-1})} \left(\mathcal{L}_2^{-(n+1)} - (-\mathcal{L}_2)^{(n+1)} \right).$$

□

References for the Online Appendix

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