Wage Adjustment in Efficient Long-Term Employment Relationships: Appendices

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A. Proofs of main results

Proof of Proposition 1. (i) See online appendix.

(ii) Expected remaining tenure at productivity x satisfies

$$\left(\delta + s\lambda \mathbf{1}_{\{x<x_0\}}\right)\bar{\tau}(x) = 1 + \mu x \bar{\tau}'(x) + \frac{1}{2}\sigma^2 x^2 \bar{\tau}''(x). \tag{33}$$

The boundary conditions are $\bar{\tau}(x_l) = 0$, $\bar{\tau}(x_0^-) = \bar{\tau}(x_0^+)$, $\bar{\tau}'(x_0^-) = \bar{\tau}'(x_0^+)$, and $\lim_{x\to\infty} \bar{\tau}(x) = 1/\delta$. The marginal response of the total surplus to rU satisfies

$$\left(r + \delta + \beta s\lambda \mathbf{1}_{\{x<x_0\}}\right) S_{rU}(x) = -1 + \mu x S_{rUx} + \frac{1}{2}\sigma^2 x^2 S_{rUxx}, \tag{34}$$

with boundary conditions $S_{rU}(x_l) = 0$, $S_{rU}(x_0^-) = S_{rU}(x_0^+)$, $S'_{rU}(x_0^-) = S'_{rU}(x_0^+)$, and $\lim_{x\to\infty} S_{rU}(x) = 1/(r+\delta)$. Symmetry of (33) and (34), and their boundary conditions yields the result.

Proof of Lemma 1. (i) and (ii) See online appendix.

(iii) It can be verified from (4) that $rU \leq \beta s\lambda S(x_0)$ implies $S(0) \geq 0$, which in turn implies that $x_l \leq 0$, so that matches almost surely never reach x_l. Instead, they separate at exogenous rate δ, and additionally at rate $s\lambda$ whenever $x < x_0$. If $\mu = \sigma^2/2$, then $\ln x_t$ is a driftless Brownian motion started at $\ln x_0$. Defining $\Gamma_+(\tau) \equiv \int_0^\tau \mathbf{1}_{\{\ln x_s > \ln x_0\}} \mathrm{d}s$, Lévy's Arcsine Law for the occupation time of Brownian motion states that

$$\Pr(\Gamma_+(\tau) \leq \theta) = \frac{2}{\pi} \cdot \arcsin\sqrt{\theta/\tau}. \tag{35}$$

See Karatzas and Shreve (1998, page 273). By symmetry, $\Gamma_-(\tau) \equiv \int_0^\tau \mathbf{1}_{\{\ln x_s < \ln x_0\}} \mathrm{d}s$ has the same distribution. It follows that the distribution of completed tenure spells is

$$H(\tau) = 1 - \int_0^\tau \exp(-\delta\tau - s\lambda\theta) \frac{\partial}{\partial\theta}\left(\frac{2}{\pi}\arcsin\sqrt{\theta/\tau}\right) d\theta$$
$$= 1 - \exp(-\delta\tau)\frac{1}{\pi}\int_0^\tau \frac{\exp(-s\lambda\theta)}{\sqrt{\theta(\tau-\theta)}} d\theta = 1 - \exp\left[-\left(\delta + \frac{s\lambda}{2}\right)\tau\right] I_0\left(\frac{s\lambda\tau}{2}\right), \quad (36)$$

as stated. Akahori (1995) generalizes Lévy's Arcsine Law to Brownian motion with drift.

Proof of Proposition 2. The firm surplus satisfies the Bellman equation in (10), where the $S(x)$ is given by Proposition 1. Note from the boundary conditions (12), (13), and (14) that $x \in \mathrm{N}_i(w)$ implies $x \geq x_0$. Thus, for $x \in \mathrm{N}_i(w)$, the general solution to (10) is

$$J(w,x) = \frac{x}{r+\delta-\mu} - \frac{w}{r+\delta} + J_{i1}(w)x^{\gamma_1} + J_{i2}(w)x^{\gamma_2}, \quad (37)$$

where $\gamma_1 < 0$ and $\gamma_2 > 1$ are the roots of $\rho(\gamma) = 0$, and

$$\rho(\gamma) \equiv -\frac{1}{2}\sigma^2\gamma^2 - \left(\mu - \frac{1}{2}\sigma^2\right)\gamma + r + \delta = 0. \quad (38)$$

For $x \in \bar{\mathrm{N}}_i(w)$, the general solution to (10) is

$$J(w,x) = \frac{x}{r+\delta-\mu+s\lambda} - \frac{w}{r+\delta+s\lambda} + s\lambda\mathcal{P}(x) + \mathcal{J}_{i1}(w)x^{\psi_1} + \mathcal{J}_{i2}(w)x^{\psi_2} \quad (39)$$

where $\psi_1 < 0$ and $\psi_2 > 1$ are the roots of $\rho(\psi) + s\lambda = 0$, and $\mathcal{P}(x)$ is a particular solution determined by the method of variation of parameters,

$$\mathcal{P}(x) = -\frac{1-\beta(1-\Delta_e)}{(\sigma^2/2)(\psi_2-\psi_1)}\int_{x_0}^{\max\{x_0,x\}} \left[\left(\frac{x}{\tilde{x}}\right)^{\psi_2} - \left(\frac{x}{\tilde{x}}\right)^{\psi_1}\right] \frac{S(\tilde{x}) - S(x_0)}{\tilde{x}} \mathrm{d}\tilde{x}. \quad (40)$$

Setting $\mu = \sigma^2/2$ implies $\psi_2 = \sqrt{(r+\delta+s\lambda)/\mu} = -\psi_1$, $\gamma_2 = \sqrt{(r+\delta)/\mu} = -\gamma_1$ and thereby the stated solution.

(ii) The expected duration until next wage adjustment satisfies

$$\left(\delta + s\lambda\mathbf{1}_{\{x\in\bar{\mathrm{N}}_i(w)\}}\right)\bar{\tau}^w(w,x) = 1 + \mu x\bar{\tau}_x^w + \frac{1}{2}\sigma^2x^2\bar{\tau}_{xx}^w, \quad (41)$$

subject to $\bar{\tau}^w(w,x_f(w)) = 0 = \bar{\tau}^w(w,x_e(w))$, $\bar{\tau}^w(w,x_n^-(w)) = \bar{\tau}^w(w,x_n^+(w))$, and $\bar{\tau}_x^w(w,x_n^-(w)) = \bar{\tau}_x^w(w,x_n^+(w))$. The marginal value of the wage to the firm satisfies

$$\left(r + \delta + s\lambda\mathbf{1}_{\{x\in\bar{\mathrm{N}}_i(w)\}}\right)J_w(x) = -1 + \mu xJ_{wx} + \frac{1}{2}\sigma^2x^2J_{wxx}, \quad (42)$$

subject to $J_w(w,x_f(w)) = 0 = J_w(w,x_e(w))$, $J_w(w,x_n^-(w)) = J_w(w,x_n^+(w))$, and $J_{wx}(w,x_n^-(w)) = J_{wx}(w,x_n^+(w))$. The result follows from the symmetry of (41) and (42), and their boundary conditions.

Proof of Lemma 2. See online appendix.

Proof of Proposition 3. The general solution to (10), evaluated at $s = 0$, is

$$J(w,x) = \frac{x}{\rho(1)} - \frac{w}{\rho(0)} + J_1(w)x^{\gamma_1} + J_2(w)x^{\gamma_2}, \tag{43}$$

where $\gamma_1 < 0$ and $\gamma_2 > 1$ are the roots of $\rho(\gamma) = 0$ in (38). $J_1(w)$ and $J_2(w)$, and the boundaries $x_e(w)$ and $x_f(w)$, that satisfy the boundary conditions (13) and (14) are inferred from an extension of the method of Abel and Eberly (1996) in the online appendix.

Proof of Proposition 4. We propose and verify a solution characterized by the four regions in Figure 4B, with cutoffs $w_{\rm I} < w_{\rm II} < w_{\rm III} < w_{\rm IV}$. $w_{\rm IV}$ satisfies $x_f(w_{\rm IV}) \equiv x_n(w_{\rm IV}) = x_0$. For $w \geq w_{\rm IV}$, the general solution for the firm's surplus is as in (37). The boundary conditions are $J\big(w, x_f(w)\big) \equiv 0$ and $J_x\big(w, x_f(w)\big) = 0$. Since $\gamma_2 > 1$, it must be that $J_2(w) = 0$. $J_1(w)$, the boundary $x_f(w)$, and the wage cutoff $w_{\rm IV}$ are then

$$J_1(w) = -\frac{1}{\gamma_1}\frac{\left[x_f(w)\right]^{1-\gamma_1}}{\rho(1)},\ x_f(w) = -\frac{\gamma_1}{1-\gamma_1}\frac{\rho(1)}{\rho(0)}w,\ \text{and,}\ \ w_{\rm IV} = -\frac{1-\gamma_1}{\gamma_1}\frac{\rho(0)}{\rho(1)}x_0. \tag{44}$$

$w_{\rm III}$ satisfies $\lim_{w\to w_{\rm III}} x_n(w) = \infty$. In this limit, the prospect of any future wage adjustment becomes negligible, $\lim_{w\to w_{\rm III}} V\big(w, x_n(w)\big) = (w_{\rm III} - rU)/(r+\delta) = S(x_0)$, and

$$w_{\rm III} = rU + (r+\delta)S(x_0) = \frac{\rho(0)}{\rho(1)}x_0\left[1 - \frac{1}{\gamma_1}\left(\frac{x_0}{x_l}\right)^{\gamma_1 - 1}\right] < w_{\rm IV}, \tag{45}$$

where the second equality follows from the solutions for $S(x)$ and x_l in Lemma 1 (i).

$w_{\rm I}$ satisfies $x_e^{-1}(x) = w_{\rm I}$ for all $x \geq x_0$. Under the latter, the worker value satisfies $V(w_{\rm I}, x) = [w_{\rm I} - rU + s\lambda S(x_0)]/(r+\delta) = 0$ for all $x \geq x_0$, and $w_{\rm I} = rU - s\lambda S(x_0) < w_{\rm III}$.

Finally, $w_{\rm II}$ satisfies $x_f(w_{\rm II}) \equiv x_e(w_{\rm II}) = x_l$. To establish that $w_{\rm II} < w_{\rm III}$ note that, since $V(w_{\rm II}, x_l) = V_x(w_{\rm II}, x_l) = 0$, it must be that $V_{xx}(w_{\rm II}, x_l) \geq 0$, and so (11) implies $w_{\rm II} = rU - (1/2)\sigma^2 V_{xx}(w_{\rm II}, x_l) \leq rU < w_{\rm III}$. To establish that $w_{\rm II} > w_{\rm I}$, consider two matches, match A currently at $(w_{\rm II}, x_l)$, and match B currently at $(w_{\rm I}, x_0)$. Note that $V^A \equiv V(w_{\rm II}, x_l) = 0 = V^B \equiv V(w_{\rm I}, x_0)$. Fix a sequence of innovations to x, arrivals of job offers, and job destruction shocks that are the same for both matches, and denote by T the first time one of the following events is realized: (i) x falls below x_l for match A, such that it is endogenously destroyed; (ii) an outside offer arrives; and (iii) the job is exogenously destroyed. Then, one can write $V^B - V^A = \mathbb{E}\big[\int_0^T \mathrm{e}^{-rt}(w_{\rm I} - w_{\rm II})\mathrm{d}t + \mathrm{e}^{-rT}\big(\tilde{V}_T^B - \tilde{V}_T^A\big)\big] = 0$, where, with a slight abuse of notation, $\tilde{V}_T^j$ is the terminal surplus at T of the worker currently employed in match $j \in \{A, B\}$. It must be that $\tilde{V}_T^B > 0 = \tilde{V}_T^A$: If event (i) occurs at T, $\tilde{V}_T^B \geq 0 = \tilde{V}_T^A$ since $x_0 > x_l$, and the sequence of innovations to x is the same for both matches; if event (ii) occurs at T, $\tilde{V}_T^B \geq \tilde{V}_T^A$, with strict inequality if productivity in match A is below at x_0 at T; if event (iii) occurs at T, $\tilde{V}_T^B = 0 = \tilde{V}_T^A$. It follows that $w_{\mathrm{I}} < w_{\mathrm{II}}$.

Given the ordering of the wage cutoffs, the coefficients of the firm surplus and the boundaries are solved for region by region in the online appendix.

Proof of Proposition 5. For $x \in \overline{\mathrm{N}}_i(w;\pi)$, we can rewrite (28) as

$$\begin{aligned}&(r+\delta+s\lambda)J(w,x;\pi)\\&= x - w + s\lambda\mathbf{1}_{\{x\geq x_0\}}[1-\beta(1-\Delta_e)][S(x)-S(x_0)] - \pi w J_w + \mu x J_x + \frac{1}{2}\sigma^2 x^2 J_{xx}.\end{aligned} \tag{46}$$

A Taylor series approximation to the firm surplus is $J(w,x;\pi) = J(w,x;0) + J_\pi(w,x;0)\pi + O(\pi^2)$. Recall that the general solution for $J(w,x;0)$ takes the form in (39). Differentiating the Bellman equation, the marginal value of inflation satisfies

$$(r+\delta+s\lambda)J_\pi(w,x;0) = \left[-wJ_w + \mu x J_{\pi x} + \frac{1}{2}\sigma^2 x^2 J_{\pi xx}\right]_{\pi=0}. \tag{47}$$

The general solution to the latter takes the form

$$J_\pi(w,x;0) = \mathbb{P}_{\overline{\mathrm{N}}}(w,x) + \jmath_{i1}(w)x^{\psi_1} + \jmath_{i2}(w)x^{\psi_2}, \tag{48}$$

for some coefficients $\jmath_{i1}(w)$ and $\jmath_{i2}(w)$, where $\mathbb{P}_{\overline{\mathrm{N}}}(w,x)$ is a particular solution to (47). It follows that the general solution to the Bellman equation takes the form

$$J(w,x;\pi) = J(w,x;0) + \pi\mathbb{P}_{\overline{\mathrm{N}}}(w,x) + \pi\jmath_{i1}(w)x^{\psi_1} + \pi\jmath_{i2}(w)x^{\psi_2} + O(\pi^2). \tag{49}$$

Observing that $J_w(w,x;0)$ in (47) is provided by Proposition 2, the particular solution $\mathbb{P}_{\overline{\mathrm{N}}}(w,x)$ then follows from application of the method of variation of parameters,

$$\mathbb{P}_{\overline{\mathrm{N}}}(w,x) = -\frac{w}{\sigma^2/2}\frac{1}{\psi_2-\psi_1}\int^x \left[\left(\frac{x}{\tilde{x}}\right)^{\psi_2} - \left(\frac{x}{\tilde{x}}\right)^{\psi_1}\right]\frac{J_w(w,\tilde{x};0)}{\tilde{x}}\,\mathrm{d}\tilde{x}. \tag{50}$$

Now consider the case in which $x \in \mathrm{N}_i(w;\pi)$. Then we can rewrite (28) as

$$(r+\delta)J(w,x;\pi) = x - w - \pi w J_w + \mu x J_x + \frac{1}{2}\sigma^2 x^2 J_{xx}. \tag{51}$$

Recall that the general solution for $J(w,x;0)$ takes the form in (37). Applying the same steps as above, the general solution satisfies

$$J(w,x;\pi) = J(w,x;0) + \pi\mathbb{P}_{\mathrm{N}}(w,x) + \pi j_{i1}(w)x^{\gamma_1} + \pi j_{i2}(w)x^{\gamma_2} + O(\pi^2), \tag{52}$$

for some coefficients $j_{i1}(w)$ and $j_{i2}(w)$, where $\mathbb{P}_{\mathrm{N}}(w,x)$ is the particular solution

$$\mathbb{P}_{\mathrm{N}}(w,x) = -\frac{w}{\sigma^2/2}\frac{1}{\gamma_2-\gamma_1}\int^x \left[\left(\frac{x}{\tilde{x}}\right)^{\gamma_2} - \left(\frac{x}{\tilde{x}}\right)^{\gamma_1}\right]\frac{J_w(w,\tilde{x};0)}{\tilde{x}}\,\mathrm{d}\tilde{x}. \tag{53}$$

B. Online appendix: Additional proofs and derivations

Proof of Proposition 1, (i). We first establish that the surplus $S(x)$ is monotonically increasing in productivity x. Fix a separation boundary x_l at which $S(x_l) = 0$, and the surplus in new jobs $S(x_0) = \bar{S} > 0$, and conjecture that the implied $S(x)$ is monotonically increasing in x. Consider two matches with different initial productivities $x' > x$. Fix, for both matches, a given sample path for changes in idiosyncratic productivity, arrivals of outside job offers, and job destruction shocks. Denote by T the first time one of the following events occurs for match x: (i) the job is destroyed endogenously; (ii) an outside offer arrives; and (iii) the job is destroyed exogenously. Further denote by S_T the continuation value thereafter for match x, and S'_T the continuation value for match x'. Since we have fixed the sample path of shocks, and the arrivals of outside job offers, and job destruction shocks, it follows $S'_T \geq S_T$: If event (i) or (ii) is realized at T, $S'_T > S_T$; if event (iii) $S'_T = S_T = 0$. Since the match surplus is based on expectations over sample paths, $S(x') = \mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}(x'_t - rU)\mathrm{d}t + \mathrm{e}^{-rT} S'_T | x'\right] > \mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}(x_t - rU)\mathrm{d}t + \mathrm{e}^{-rT} S_T | x\right] = S(x)$. This confirms the conjecture. Continuity of the coefficients of the differential equation for $S(x)$ in (2) further implies that there exists a unique solution for given $S(x_0)$ and x_l, by the Picard-Lindelöf theorem. Therefore, for any $\bar{S}$ and x_l, there is a unique, monotonically increasing solution for $S(x)$.

The general solution to (2) is then given by

$$S(x) = \begin{cases} \dfrac{x}{r+\delta-\mu+\beta s\lambda} - \dfrac{rU - \beta s\lambda S(x_0)}{r+\delta+\beta s\lambda} + \mathcal{S}_1 x^{\tilde{\gamma}_1} + \mathcal{S}_2 x^{\tilde{\gamma}_2} & \text{if} \quad x < x_0, \\ \dfrac{x}{r+\delta-\mu} - \dfrac{rU}{r+\delta} + S_1 x^{\gamma_1} + S_2 x^{\gamma_2} & \text{if} \quad x \geq x_0, \end{cases} \tag{54}$$

where $\tilde{\gamma}_1 < \gamma_1 < 0$, and $\tilde{\gamma}_2 > \gamma_2 > 1$ are the roots of $\rho(\tilde{\gamma}) + \beta s\lambda = 0$ and $\rho(\gamma) = 0$, where

$$\rho(\gamma) \equiv -\frac{1}{2}\sigma^2\gamma^2 - \left(\mu - \frac{1}{2}\sigma^2\right)\gamma + r + \delta = 0. \tag{55}$$

The boundary conditions are given in (3). Since $\gamma_2 > 1$, the solution will explode as $x \to \infty$ unless $S_2 = 0$. The separation boundary x_l, and the remaining coefficients $\mathcal{S}_1$, $\mathcal{S}_2$, and S_1, can then be recovered from the boundary conditions. Setting $\mu = \sigma^2/2$ implies $\tilde{\gamma} = \pm\sqrt{(r+\delta+\beta s\lambda)/\mu}$ and $\gamma = \pm\sqrt{(r+\delta)/\mu}$, and thereby the stated solution.

Proof of Lemma 1, (i) and (ii). (i) If $\beta s = 0$, the general solution for the surplus is

$$S(x) = \frac{x}{r+\delta-\mu} - \frac{rU}{r+\delta} + S_1 x^{\gamma_1}. \tag{56}$$

Imposing the value-matching and smooth-pasting conditions $S(x_l) \equiv 0$ and $S'(x_l) = 0$,

$$S(x) = \frac{x}{r+\delta-\mu} - \frac{rU}{r+\delta}\left[1 - \frac{(x/x_l)^{\gamma_1}}{1-\gamma_1}\right], \qquad x_l = -\frac{\gamma_1}{1-\gamma_1}\frac{r+\delta-\mu}{r+\delta}rU. \tag{57}$$

Setting $\mu = \sigma^2/2$ implies $\gamma_1 = -\sqrt{(r+\delta)/\mu}$, and thereby the stated solution.

(ii) A standard result on first passage times implies

$$h_0(\tau) = \frac{\ln(x_0/x_l)}{\sigma\tau^{3/2}}\phi\left(\frac{\ln(x_0/x_l) + [\mu - (\sigma^2/2)]\tau}{\sigma\tau^{1/2}}\right). \tag{58}$$

See, for example, Buhai and Teulings (2014). Setting $\mu = \sigma^2/2$ simplifies the solution as stated. Equation (7) follows because exogenous separations are independent.

Proof of Lemma 2. For $\beta \in (0,1]$, and $\Delta_e, \Delta_f \in [0,1)$, the boundary conditions (12), (13), and (14), imply that the boundaries $x_e(w) \to \infty$, $x_f(w) \to \infty$, and $x_n(w) \to \infty$, as $w \to \infty$. Since $\gamma_1 < 0$, it follows from Proposition 1 that $S(x) \to [x/(r+\delta-\mu)] - [rU/(r+\delta)]$ as $x \to \infty$. Furthermore, the definitions of $x_e(w)$ and $x_n(w)$ in (13) and (14) imply that $x_n(w) \to x_e(w)$ as $w \to \infty$. Combining these observations, it follows that

$$J(w,x) \to \frac{x}{r+\delta-\mu} - \frac{w}{r+\delta}, \text{ and, } V(w,x) \to \frac{w-rU}{r+\delta} \tag{59}$$

as $w \to \infty$. Recalling the definitions of the boundaries $x_e(w)$, $x_f(w)$, and $x_n(w)$ in (12), (13), and (14), it follows that these become affine and increasing as $w \to \infty$.

Proof of Proposition 3: Further detail. Since the arguments that follow hold for all levels of the current wage w, we treat w as parametric and, where necessary to avoid clutter, suppress notation for dependence on w.

The coefficients J_1 and J_2, and the boundaries x_e and x_f, that satisfy the boundary conditions (13) and (14) can be inferred from an extension of the solution method of Abel and Eberly (1996) as follows. Define the functions

$$\begin{aligned}\vartheta_1(\mathcal{G},a,b) &\equiv \frac{[1-(r+\delta-\mu)a]\mathcal{G}^{\gamma_2} - [1-(r+\delta-\mu)b]\mathcal{G}}{\mathcal{G}^{\gamma_2} - \mathcal{G}^{\gamma_1}}, \\ \vartheta_2(\mathcal{G},a,b) &\equiv \frac{[1-(r+\delta-\mu)b]\mathcal{G} - [1-(r+\delta-\mu)a]\mathcal{G}^{\gamma_1}}{\mathcal{G}^{\gamma_2} - \mathcal{G}^{\gamma_1}}.\end{aligned} \tag{60}$$

Note that $\vartheta_1(\mathcal{G}^{-1},a,b) = \mathcal{G}^{\gamma_1-1}\vartheta_1(\mathcal{G},b,a)$, and $\vartheta_2(\mathcal{G}^{-1},a,b) = \mathcal{G}^{\gamma_2-1}\vartheta_2(\mathcal{G},b,a)$. Define the minimum surplus shares of the firm $\mathcal{B}_f \equiv (1-\beta)(1-\Delta_f)$, and worker $\mathcal{B}_e \equiv \beta(1-\Delta_e)$. Then the boundary conditions imply the following nonlinear equations in the coefficients,

$$J_1 = -\frac{1}{\gamma_1}\frac{x_f^{1-\gamma_1}}{\rho(1)}\vartheta_1\left(\frac{x_e}{x_f}, \mathcal{B}_f S'(x_f), (1-\mathcal{B}_e)S'(x_e)\right), \text{ and,}$$
$$J_2 = -\frac{1}{\gamma_2}\frac{x_f^{1-\gamma_2}}{\rho(1)}\vartheta_2\left(\frac{x_e}{x_f}, \mathcal{B}_f S'(x_f), (1-\mathcal{B}_e)S'(x_e)\right), \tag{61}$$

and the boundaries

$$\frac{x_f}{\rho(1)}\left[1 - \frac{1}{\gamma_1}\vartheta_1\left(\frac{x_e}{x_f}, \mathcal{B}_f S'(x_f), (1-\mathcal{B}_e)S'(x_e)\right) - \frac{1}{\gamma_2}\vartheta_2\left(\frac{x_e}{x_f}, \mathcal{B}_f S'(x_f), (1-\mathcal{B}_e)S'(x_e)\right)\right] = \frac{w}{\rho(0)} + \mathcal{B}_f S(x_f), \tag{62}$$

and,

$$\frac{x_e}{\rho(1)}\left[1 - \frac{1}{\gamma_1}\vartheta_1\left(\frac{x_f}{x_e}, (1-\mathcal{B}_e)S'(x_e), \mathcal{B}_f S'(x_f)\right) - \frac{1}{\gamma_2}\vartheta_2\left(\frac{x_f}{x_e}, (1-\mathcal{B}_e)S'(x_e), \mathcal{B}_f S'(x_f)\right)\right] = \frac{w}{\rho(0)} + (1-\mathcal{B}_e)S(x_e). \tag{63}$$

Noting that a solution for the total surplus $S(x)$ is provided in Lemma 1, and that the preceding steps hold for any current wage w, completes the solution.

Proof of Proposition 4: Further detail. Since the arguments that follow hold for all levels of the current wage w, we treat w as parametric and, where necessary to avoid clutter, suppress notation for dependence on w.

The total match surplus $S(x)$ is as in Lemma 1, and the firm surplus satisfies the Bellman equation in (10), evaluated at $\beta = 0$. The solution involves repeated use of the quadratic $\rho(\gamma)$ in (38), and its cousin, $\varrho(\psi) \equiv \rho(\psi) + s\lambda$. The latter has roots $\psi_1 < \gamma_1 < 0$ and $\psi_2 > \gamma_2 > 1$. Note that each can be written as $\rho(\gamma) = -(\sigma^2/2)(\gamma-\gamma_1)(\gamma-\gamma_2)$, and $\varrho(\psi) = -(\sigma^2/2)(\psi-\psi_1)(\psi-\psi_2)$.

Region I: $w \in (w_{\mathrm{I}}, w_{\mathrm{II}})$. The general solution for the firm's surplus takes the form

$$J(w,x) = \frac{x}{\varrho(1)} - \frac{w}{\varrho(0)} + s\lambda\mathcal{P}(x) + \mathcal{J}_1(w)x^{\psi_1} + \mathcal{J}_2(w)x^{\psi_2}, \tag{64}$$

where, applying the method of variation of parameters,

$$\mathcal{P}(x) = -\frac{1}{\sigma^2/2}\frac{1}{\psi_2-\psi_1}\int_{x_0}^{\max\{x,x_0\}}\left[\left(\frac{x}{\tilde{x}}\right)^{\psi_2} - \left(\frac{x}{\tilde{x}}\right)^{\psi_1}\right]\frac{S(\tilde{x}) - S(x_0)}{\tilde{x}}\,\mathrm{d}\tilde{x}. \tag{65}$$

The boundary conditions are

$$J\big(w, x_e(w)\big) \equiv S\big(x_e(w)\big), \text{ and, } J_x\big(w, x_e(w)\big) = S'\big(x_e(w)\big). \tag{66}$$

Furthermore, since $\psi_2 > 1$, it must be that all terms in x^{ψ_2} in the general solution cancel. Using the solution for $S(x)$ in Lemma 1, the definitions of the roots, and expanding $\mathcal{P}(x)$, this implies (after some tedious algebra)

$$\mathcal{J}_2 = \frac{s\lambda}{\sigma^2/2}\frac{1}{\psi_2}\frac{1}{\psi_2-\psi_1}\frac{x_0^{1-\psi_2}}{\rho(1)}\left[\frac{1}{\psi_2-1}-\frac{1}{\psi_2-\gamma_1}\left(\frac{x_0}{x_l}\right)^{\gamma_1-1}\right]. \tag{67}$$

Applying the smooth-pasting condition, and observing that, under the proposed solution, $\mathcal{P}(x_e) = \mathcal{P}'(x_e) = 0$, yields the remaining coefficient,

$$\mathcal{J}_1 = -\frac{x_e^{1-\psi_1}}{\psi_1}\left[\frac{1}{\varrho(1)} + \mathcal{J}_2\psi_2 x_e^{\psi_2-1} - S'(x_e)\right]. \tag{68}$$

Further imposing the value-matching condition yields a nonlinear equation in the boundary $x_e(w)$,

$$\frac{x_e}{\varrho(1)}\left(1-\frac{1}{\psi_1}\right) + \mathcal{J}_2 x_e^{\psi_2}\left(1-\frac{\psi_2}{\psi_1}\right) - S(x_e) + \frac{1}{\psi_1}x_e S'(x_e) - \frac{w}{\varrho(0)} = 0. \tag{69}$$

Region II: $w \in (w_{\text{II}}, w_{\text{III}})$. The general solution for the firm's surplus takes the same form as in Region I. The boundary conditions are

$$J\left(w, x_f(w)\right) \equiv 0, \text{ and, } J_x\left(w, x_f(w)\right) = 0. \tag{70}$$

As in Region I, since $\psi_2 > 1$, all terms in x^{ψ_2} in the general solution must cancel, and so the solution for $\mathcal{J}_2$ in (67) again holds. Applying the smooth-pasting condition, and noting that $x_f(w) \le x_0$ for all w in Region II implies $\mathcal{P}(x_f) = \mathcal{P}'(x_f) = 0$,

$$\mathcal{J}_1 = -\frac{x_f^{1-\psi_1}}{\psi_1}\left[\frac{1}{\varrho(1)} + \mathcal{J}_2\psi_2 x_f^{\psi_2-1}\right]. \tag{71}$$

Imposing the value-matching condition,

$$\frac{x_f}{\varrho(1)}\left(1-\frac{1}{\psi_1}\right) + \mathcal{J}_2 x_f^{\psi_2}\left(1-\frac{\psi_2}{\psi_1}\right) - \frac{w}{\varrho(0)} = 0. \tag{72}$$

Region III: $w \in (w_{\text{III}}, w_{\text{IV}})$. We divide Region III into two sub-regions:

Region III(a): $x \in \left(x_f(w), x_n(w)\right)$. The general solution in this case takes the same form as in Regions I and II. The value-matching conditions are

$$J\left(w, x_f(w)\right) \equiv 0, \text{ and, } J\big(w, x_n(w)\big) \equiv S\big(x_n(w)\big) - S(x_0), \tag{73}$$

and the smooth-pasting conditions are

$$J_x\left(w, x_f(w)\right) = 0, \text{ and, } J_x(w, x_n(w)^-) = J_x(w, x_n(w)^+) \equiv \kappa. \tag{74}$$

Mirroring the approach taken in (60), it will be useful to define the functions

$$\begin{aligned}\Theta_1(\mathcal{G}, a, b) &\equiv \frac{[1-(r+\delta+s\lambda-\mu)a]\mathcal{G}^{\psi_2} - [1-(r+\delta+s\lambda-\mu)b]\mathcal{G}}{\mathcal{G}^{\psi_2}-\mathcal{G}^{\psi_1}}, \\ \Theta_2(\mathcal{G}, a, b) &\equiv \frac{[1-(r+\delta+s\lambda-\mu)b]\mathcal{G} - [1-(r+\delta+s\lambda-\mu)a]\mathcal{G}^{\psi_1}}{\mathcal{G}^{\psi_2}-\mathcal{G}^{\psi_1}}.\end{aligned} \tag{75}$$

As before, we have $\Theta_1(\mathcal{G}^{-1}, a, b) = \mathcal{G}^{\psi_1 - 1}\Theta_1(\mathcal{G}, b, a)$, and $\Theta_2(\mathcal{G}^{-1}, a, b) = \mathcal{G}^{\psi_2 - 1}\Theta_2(\mathcal{G}, b, a)$. Observing that $x_f(w) \leq x_0$ for all w in Region III(a) implies $\mathcal{P}\left(x_f(w)\right) = \mathcal{P}'\left(x_f(w)\right) = 0$, yields the coefficients

$$\begin{aligned}\mathcal{J}_1 &= -\frac{1}{\psi_1}\frac{x_f^{1-\psi_1}}{\varrho(1)}\Theta_1\left(\frac{x_n}{x_f}, 0, \kappa - s\lambda\mathcal{P}'(x_n)\right), \text{ and} \\ \mathcal{J}_2 &= -\frac{1}{\psi_2}\frac{x_f^{1-\psi_2}}{\varrho(1)}\Theta_2\left(\frac{x_n}{x_f}, 0, \kappa - s\lambda\mathcal{P}'(x_n)\right).\end{aligned} \tag{76}$$

and the boundaries

$$\frac{x_f}{\varrho(1)}\left[1 - \frac{1}{\psi_1}\Theta_1\left(\frac{x_n}{x_f}, 0, \kappa - s\lambda\mathcal{P}'(x_n)\right) - \frac{1}{\psi_2}\Theta_2\left(\frac{x_n}{x_f}, 0, \kappa - s\lambda\mathcal{P}'(x_n)\right)\right] = \frac{w}{\varrho(0)}, \tag{77}$$

and

$$\begin{aligned}\frac{x_n}{\varrho(1)}\left[1 - \frac{1}{\psi_1}\Theta_1\left(\frac{x_f}{x_n}, \kappa - s\lambda\mathcal{P}'(x_n), 0\right) - \frac{1}{\psi_2}\Theta_2\left(\frac{x_f}{x_n}, \kappa - s\lambda\mathcal{P}'(x_n), 0\right)\right] \\ = \frac{w}{\varrho(0)} + S(x_n) - S(x_0) - s\lambda\mathcal{P}(x_n).\end{aligned} \tag{78}$$

Region III(b): $x > x_n(w)$. The general solution in this case takes the form

$$J(w, x) = \frac{x}{\rho(1)} - \frac{w}{\rho(0)} + J_1(w)x^{\gamma_1} + J_2(w)x^{\gamma_2}. \tag{79}$$

The boundary conditions are

$$J\left(w, x_n(w)\right) \equiv S\left(x_n(w)\right) - S(x_0), \text{ and, } J_x(w, x_n(w)^-) = J_x(w, x_n(w)^+) \equiv \kappa. \tag{80}$$

Since $\gamma_2 > 1$, it must be that $J_2(w) = 0$. Applying the smooth-pasting condition yields the remaining coefficient,

$$J_1 = -\frac{x_n^{1-\gamma_1}}{\gamma_1}\left[\frac{1}{\rho(1)} - \kappa\right]. \tag{81}$$

Applying the value-matching condition yields the boundary,

$$\frac{x_n}{\rho(1)}\left(1 - \frac{1}{\gamma_1}\right) = \frac{w}{\rho(0)} + S(x_n) - S(x_0) - \frac{1}{\gamma_1}\kappa x_n. \tag{82}$$

The latter provides a solution for $\mathcal{J}_1$, $\mathcal{J}_2$, and J_1, and the boundaries x_f and x_n, for a given κ. It remains to pin down $\kappa(w) \equiv J_x\big(w, x_n(w)\big)$. Equating x_n across Region III(a) and III(b) yields the following nonlinear equation x_f, x_n, and κ,

$$\begin{aligned}\frac{x_n}{\varrho(1)}\left[1 - \frac{\varrho(1)}{\rho(1)}\left(1 - \frac{1}{\gamma_1}\right) - \frac{1}{\psi_1}\Theta_1\left(\frac{x_f}{x_n}, \kappa - s\lambda\mathcal{P}'(x_n), 0\right) - \frac{1}{\psi_2}\Theta_2\left(\frac{x_f}{x_n}, \kappa - s\lambda\mathcal{P}'(x_n), 0\right)\right] \\ = \frac{w}{\varrho(0)}\left[1 - \frac{\varrho(0)}{\rho(0)}\right] - s\lambda\mathcal{P}(x_n) + \frac{1}{\gamma_1}\kappa x_n.\end{aligned} \tag{83}$$

Region IV: $w \in (w_{\mathrm{IV}}, \infty)$. The solution is given in (44) in the main appendix.

Derivation of the worker density $\boldsymbol{g(x;\theta)}$. For notational simplicity, we suppress notation for dependence on tightness θ. The Fokker-Planck (Kolmogorov Forward) equation for the steady-state worker density takes the form

$$\begin{aligned}0 = -\mu\frac{\partial}{\partial x}[xg(x)] + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}[x^2 g(x)] - \big(\delta + s\lambda\mathbf{1}_{\{x<x_0\}}\big)g(x) \\ + \left[\lambda\frac{u}{1-u} + s\lambda G(x_0)\right]\mathbf{1}_{\{x=x_0\}}.\end{aligned} \tag{84}$$

The final term captures the inflow of workers due to the creation of new matches. An inflow λu is hired from unemployment into new matches with initial productivity x_0. This inflow is scaled by $1-u$ to conform to the definition of $g(x)$ as the density of *employees* over productivity. A further inflow $s\lambda G(x_0)$ is hired from employment in matches with productivity $x < x_0$.

Noting that, in steady state, $u/(1-u) = \varsigma/\lambda$, and integrating once,

$$\big(\delta + s\lambda\mathbf{1}_{\{x<x_0\}}\big)G(x) + (\mu - \sigma^2)xg(x) = \frac{1}{2}\sigma^2 x^2 g'(x) + \varsigma\mathbf{1}_{\{x\geq x_0\}} + C_1, \tag{85}$$

where C_1 is a constant of integration. The general solution is

$$G(x) = \begin{cases} G_1^- x^{\tilde{\xi}_1} + G_2^- x^{\tilde{\xi}_2} + G_0^- & \text{if} \quad x < x_0, \\ G_1^+ x^{\xi_1} + G_2^+ x^{\xi_2} + G_0^+ & \text{if} \quad x \geq x_0, \end{cases} \tag{86}$$

where $G_0^{-/+}$, $G_1^{-/+}$, and $G_2^{-/+}$ are constants to be determined, and $\tilde{\xi}_1 < \xi_1 < 0$, $\tilde{\xi}_2 > \xi_2 > 0$ are the roots of

$$-\frac{1}{2}\sigma^2\tilde{\xi}^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\tilde{\xi} + \delta + s\lambda = 0, \text{ and, } -\frac{1}{2}\sigma^2\xi^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\xi + \delta = 0. \quad (87)$$

Since $\xi_2 > 0$ (because $\mu = \sigma^2/2$) the solution will explode as $x \to \infty$ unless $G_2^+ = 0$. Furthermore, since x_l is an absorbing barrier, it follows that $g(x_l) = 0$. The remaining boundary conditions are $g(x_0^-) = g(x_0^+)$, $G(x_0^-) = G(x_0^+)$, and $\lim_{x\to\infty} G(x) = 1$. Imposing these, solving for the remaining coefficients, and noting that $\tilde{\xi}_1\tilde{\xi}_2 = -(\delta + s\lambda)/(\sigma^2/2)$ and $\xi_1\xi_2 = -\delta/(\sigma^2/2)$, yields the following solution for the worker distribution

$$G(x) = \begin{cases} g_0\left[\frac{1}{\tilde{\xi}_2}\left(\frac{x}{x_l}\right)^{\tilde{\xi}_2} - \frac{1}{\tilde{\xi}_1}\left(\frac{x}{x_l}\right)^{\tilde{\xi}_1}\right] + G_0^- & \text{if } \quad x \in (x_l, x_0), \\ g_0\left[\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_2} - \left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_1}\right]\frac{1}{\xi_1}\left(\frac{x}{x_0}\right)^{\xi_1} + 1 & \text{if } \quad x \geq x_0, \end{cases} \quad (88)$$

where

$$g_0 = \left[\left(\frac{1}{\tilde{\xi}_2} - \frac{1}{\xi_1}\right)\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_2} - \left(\frac{1}{\tilde{\xi}_1} - \frac{1}{\xi_1}\right)\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_1} - \left(\frac{1}{\tilde{\xi}_2} - \frac{1}{\tilde{\xi}_1}\right)\right]^{-1}, \qquad G_0^- = -g_0\left(\frac{1}{\tilde{\xi}_2} - \frac{1}{\tilde{\xi}_1}\right). \quad (89)$$

Differentiating yields the following solution for the density $g(x)$.

$$g(x) = \begin{cases} g_0 \cdot \frac{1}{x}\left[\left(\frac{x}{x_l}\right)^{\tilde{\xi}_2} - \left(\frac{x}{x_l}\right)^{\tilde{\xi}_1}\right] & \text{if } \quad x \in (x_l, x_0), \\ g_0 \cdot \frac{1}{x}\left[\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_2} - \left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_1}\right]\left(\frac{x}{x_0}\right)^{\xi_1} & \text{if } \quad x \geq x_0, \end{cases} \quad (90)$$

The solution for the separation rate into unemployment is then given by $\varsigma = \delta + (\sigma^2/2)x_l^2 g'(x_l)$—see, for example, Moscarini (2005, section 6.5).

C. Online computational appendix

Baseline model: Solution for wage adjustment boundaries. We implement a solution approach based on Proposition 2 as follows. Recall that we have an analytical solution for the total surplus $S(x)$ from Proposition 1. We then create a grid for the wage w; in practice, we use 120 grid points, with greater density for lower wages, where the boundaries are especially nonlinear. We then implement the following algorithm:

1. For each w on the grid, we guess $x_e(w)$. This approach is informed by two observations. First, since $\Delta_e < 1$, for any given wage w, the worker will initiate a renegotiation if match productivity x is sufficiently high. Second, comparison of the boundary conditions in (13) and (14) implies that the locus $x_e(w)$ is always in the *competitive* region where outside offers are attractive to the employee. Thus, given the guess for $x_e(w)$ and the solution for the total surplus $S(x)$, the boundary conditions in (13) imply solutions for the option value coefficients for the firm's value $J(w,x)$ in (16). This yields a candidate solution $\hat{J}(w,x)$ for the firm's value in the competitive region.

2. Evaluate this candidate solution for lower xs until one of three conditions is met:
   (a) $\hat{J}(w,x)$ traverses the boundary condition for the wage cut boundary (12).
   (b) $\hat{J}(w,x)$ traverses the boundary condition for the wage increase boundary (13).
   (c) $\hat{J}(w,x)$ traverses the boundary condition for the (non)competitive region (14).

3. In case of (a) (respectively (b)), we evaluate the smooth-pasting condition in (12) (respectively (13)). If the latter is satisfied (up to numerical error) we have a solution. If not, we update the initial guess for $x_e(w)$ and repeat.

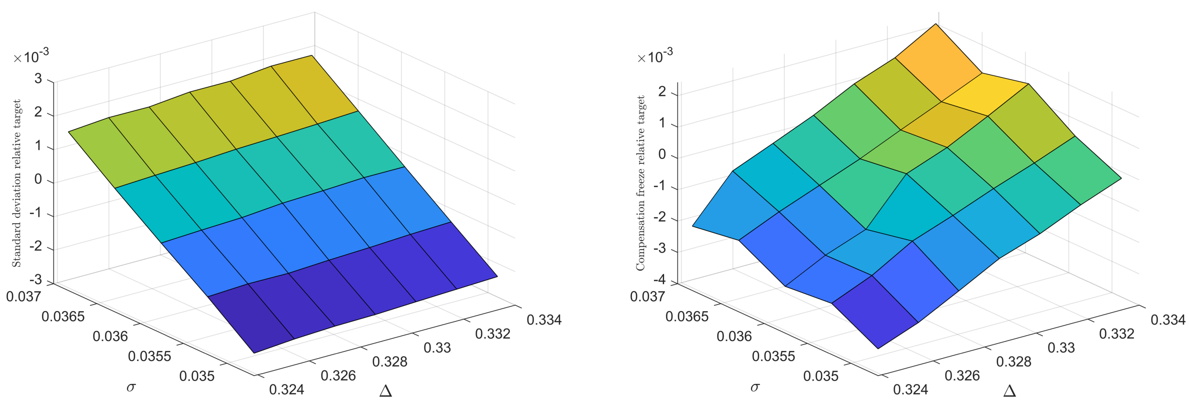
   In case of (c), we apply the value-matching and smooth-pasting conditions in (14), which, in turn, allow us to solve for the option value coefficients for the firm's value in the subsequent (non)competitive region. The algorithm then returns to step 2.[24]

For each w, the latter yields an accurate solution for the firm's value $J(w,x)$. Given our analytical solutions, it is costless to evaluate the latter on a very fine grid for match productivity x for our quantitative results. (Where numerical error might arise is from the relatively sparser grid for w.)

[24] In a final step, the algorithm also checks for the presence of a further inaction interval that lies below the solution identified by steps 1 through 3. This can arise due to the presence of the option value of on-the-job search with offer matching, which can induce nonmonotone bargained wages (e.g., Cahuc et al. 2006).

Figure C. Identification of Δ and σ

A. Std. dev. of log base wage changes B. Incidence of compensation freezes



Baseline model: Identification of Δ and σ. The second stage of our calibration approach uses, respectively, the empirical incidence of compensation freezes and the empirical standard deviation of log base wage changes to pin down, respectively, the breakdown probability parameter Δ and the standard deviation of shocks to match productivity σ.

Of course, both parameters affect both moments. To investigate how well separately identified are Δ and σ, we create a grid for both parameters around their calibrated levels in Table 1 and evaluate the implied incidence of compensation freezes and the implied standard deviation of log base wage changes for each parameter combination. In doing so, we maintain the first stage of our calibration approach that targets the unemployment rate and the E-to-E rate in Panel B of Table 1; this involves a transform of rU to hit these targets for each grid point for σ.[25]

Figure C plots the outcome of this exercise. As described in the main text, the breakdown probability Δ mostly affects the incidence of compensation freezes, whereas σ mostly affects the standard deviation of log base wage changes. As a result, both parameters are separately identified, under the model.

[25] Specifically, suppose we scale σ to $\sigma' = A\sigma$, and transform (x_0/x_l) such that $(x_0'/x_l')^A = (x_0/x_l)$. Then, recalling the solution for the worker distribution in (87) and (88), and recalling that $\mu = \sigma^2/2$ in the calibrated model, note that the shape parameters are transformed to $\tilde{\xi}' = \tilde{\xi}/A$ and $\xi' = \xi/A$. It then follows that the new worker distribution over (x'/x_l') is the same as the original over (x/x_l). Since $x_0 = x_0' = 1$ in the calibrated model, all that is required is to adjust rU such that the rescaling of x_l is satisfied. Finally, observe that the share of workers below x_0 is left unchanged by the transform, and thus so is the implied E-to-E rate. Similarly, one can confirm that E-to-U rate is likewise unchanged under the transform.

Nominal adjustment and inflation: Solution for wage adjustment boundaries. In this case, we use a finite difference scheme (e.g., Achdou et al. 2022) and the penalty method to solve the problem (similar to the algorithm used in Elsby and Gottfries 2022 and Elsby et al. 2024). One difference is that the Bellman equation in (28) is a two-dimensional partial differential equation. As mentioned in the main text, we address the latter using a perturbation approach, whereby the capital gain associated with the decay of the wage J_w is approximated using our solution for the baseline model. As described in the appendix of Elsby et al. (2024), the boundary conditions are then implemented using a penalty method that penalizes deviations above or below the exercise option.

References for online appendices

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