

Appendix for
“Immobility as Memory:
Some New Approaches to Characterizing
Intergenerational Persistence via Markov Chains”

Lawrence E. Blume[†] Neil A. Cholli[‡] Steven N. Durlauf[§]
Aleksandra Lukina[§]

[†]Cornell University and IHS Vienna

[‡]Cornell University

[§]Harris School of Public Policy and Stone Center for Research on Wealth Inequality and Mobility, University of Chicago

Appendix

A Supplemental Tables and Figures

Table A.1: Summary statistics

	Mean	S.D.
<i>Demographics</i>		
Female	0.50	0.50
Year of birth	1960.52	2.26
Non-Hispanic white	0.80	0.40
Non-Hispanic Black	0.14	0.34
Hispanic	0.06	0.24
<i>Parent macroclasses</i>		
1. Not working	0.10	0.30
2. Primary manual (farming)	0.29	0.45
3. Non-farming manual	0.10	0.30
4. Routine nonmanual	0.45	0.50
5. Professional-managerial	0.04	0.19
<i>Child macroclasses</i>		
1. Not working	0.04	0.20
2. Primary manual (farming)	0.41	0.49
3. Non-farming manual	0.13	0.34
4. Routine nonmanual	0.41	0.49
5. Professional-managerial	0.01	0.09
Number of observations	2,847	

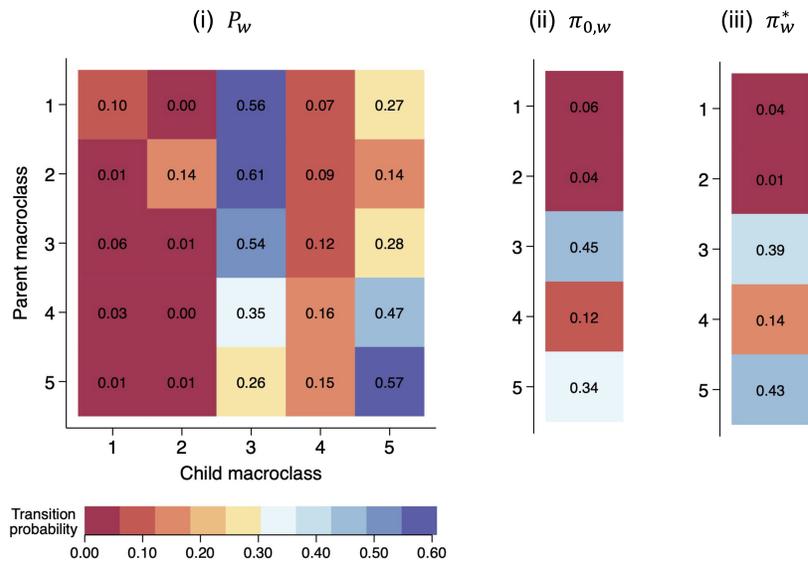
Notes: All statistics apply custom NLSY79 sampling weights. See Table 1 for more details on the Jonsson et al. (2009)-based occupation classification scheme.

Table A.2: $d'(1)$, defined in Equation 3, and second largest eigenvalue modulus $|\lambda_2|$ for macro-, meso- and microclasses

	No. classes	$d'(1)$	$ \lambda_2 $
Macroclasses	5	0.456	0.320
Mesoclasses	10	0.590	0.348
Microclasses	26	0.664	0.360

Figure A.1: Markov chains by racial group: Macroclasses

(a) White



(b) Black

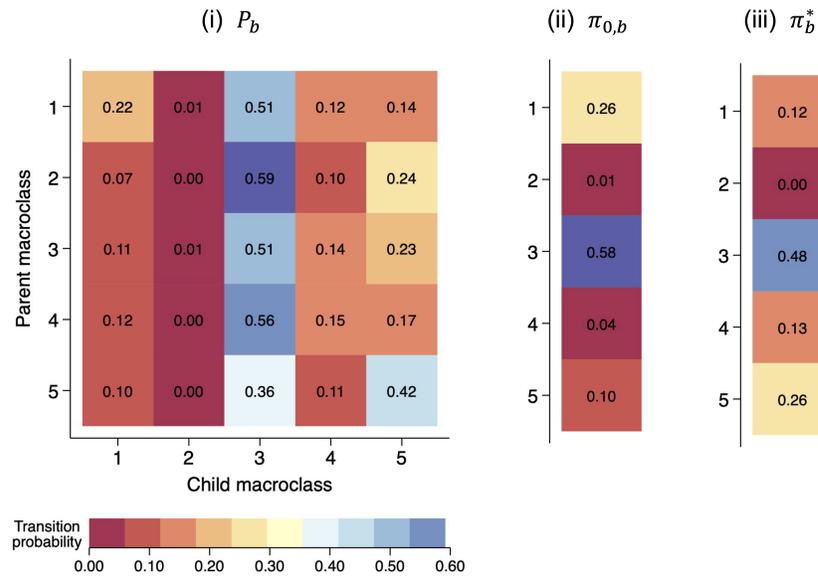
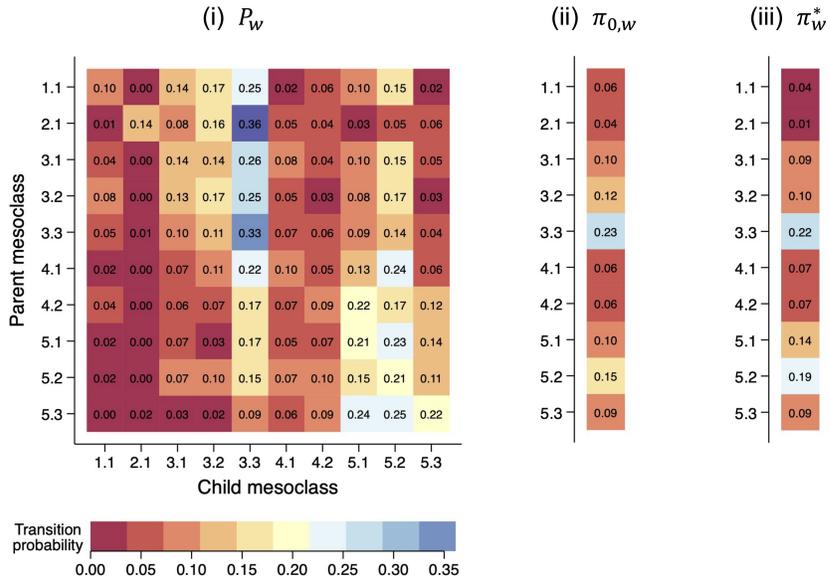


Figure A.2: Markov chains by racial group: Mesoclasses

(a) White



(b) Black

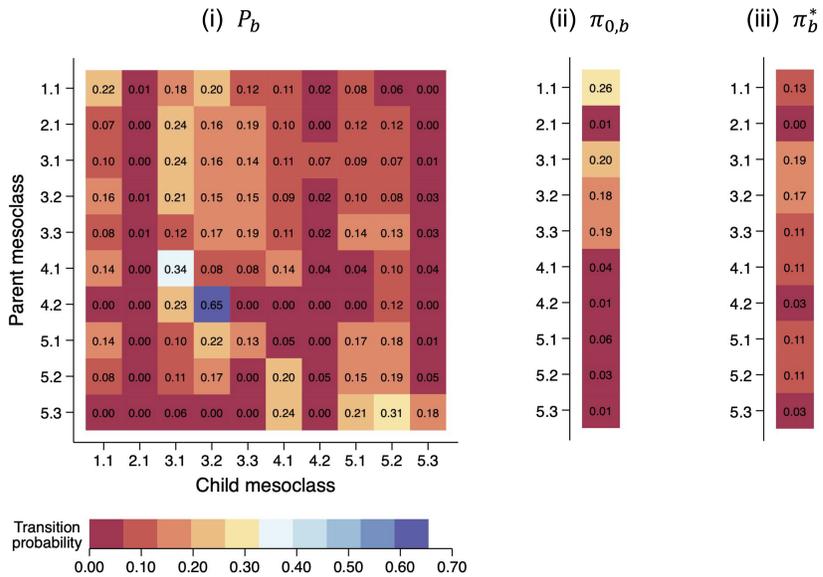
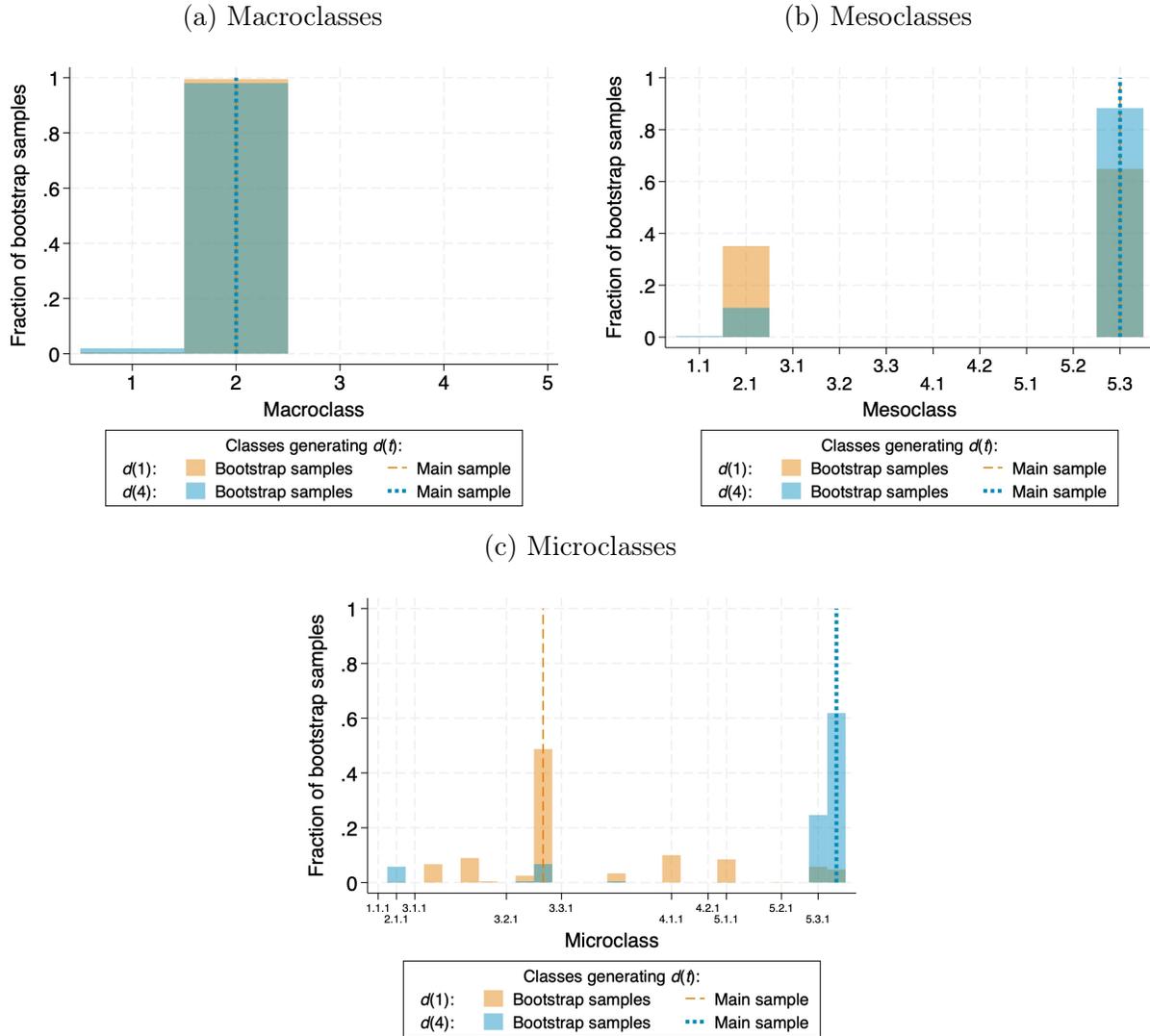
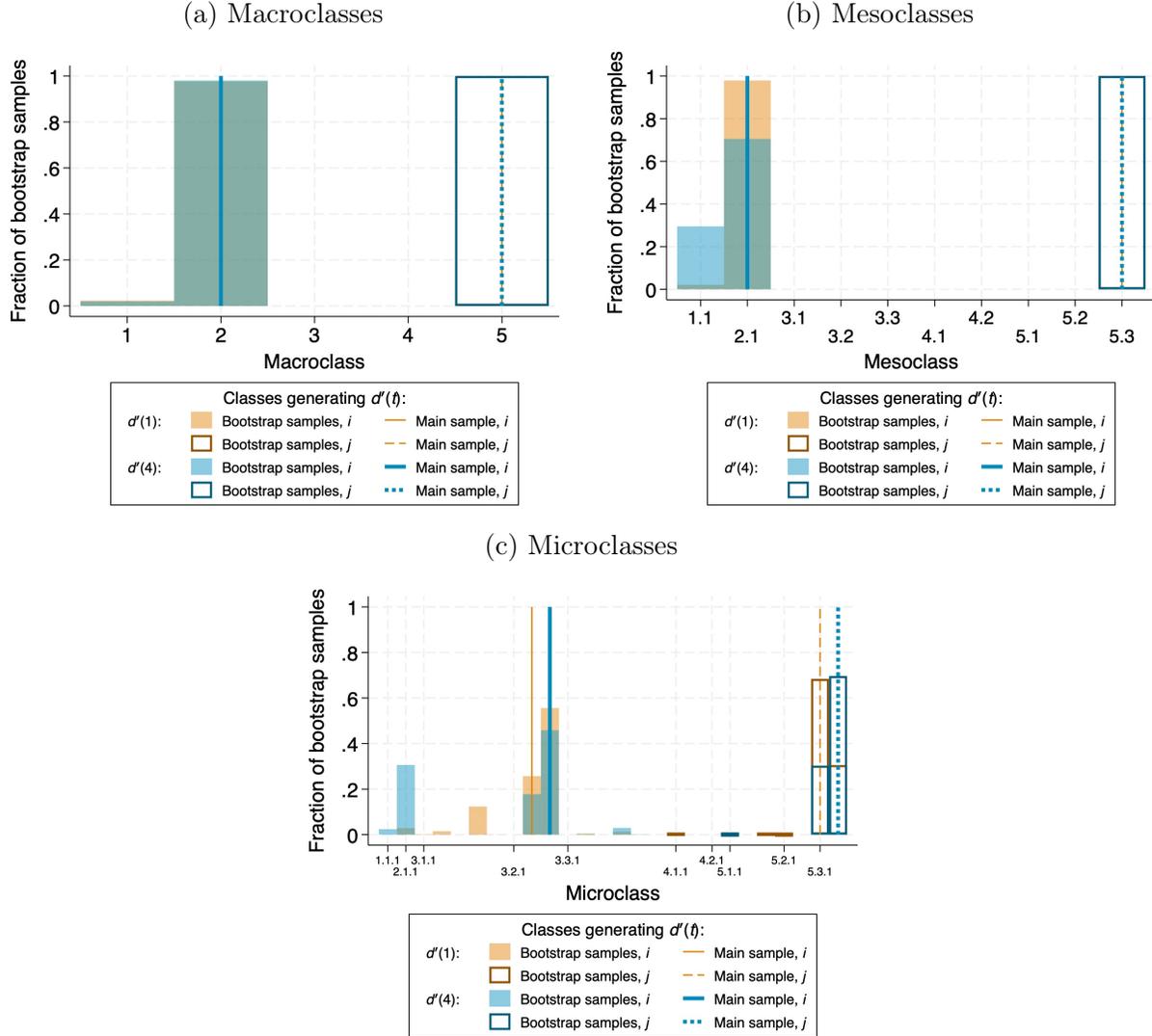


Figure A.3: Histograms of classes generating $d(t)$ for $t = 1, 4$ over bootstrap samples



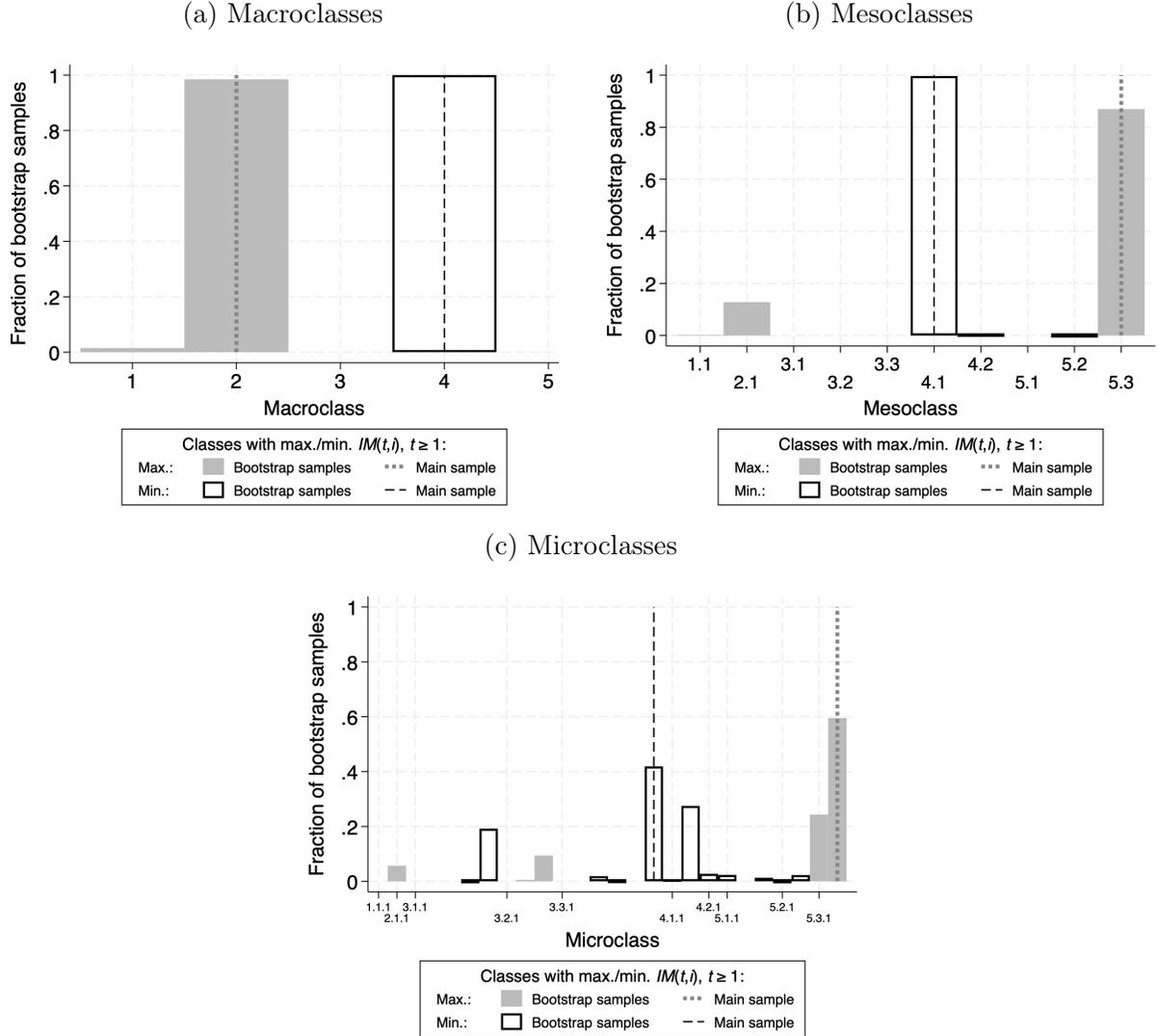
Notes: This figure plots histograms of the occupational classes i^* that generate $d(t)$ (i.e., $i^* = \arg \max_i IM(t, i)$) across 2,000 Bayesian bootstrap replication samples for $t = 1, 4$. The vertical dashed lines indicate the classes that generate $d(t)$ in the main sample.

Figure A.4: Histograms of classes generating $d'(t)$ for $t = 1, 4$ over bootstrap samples



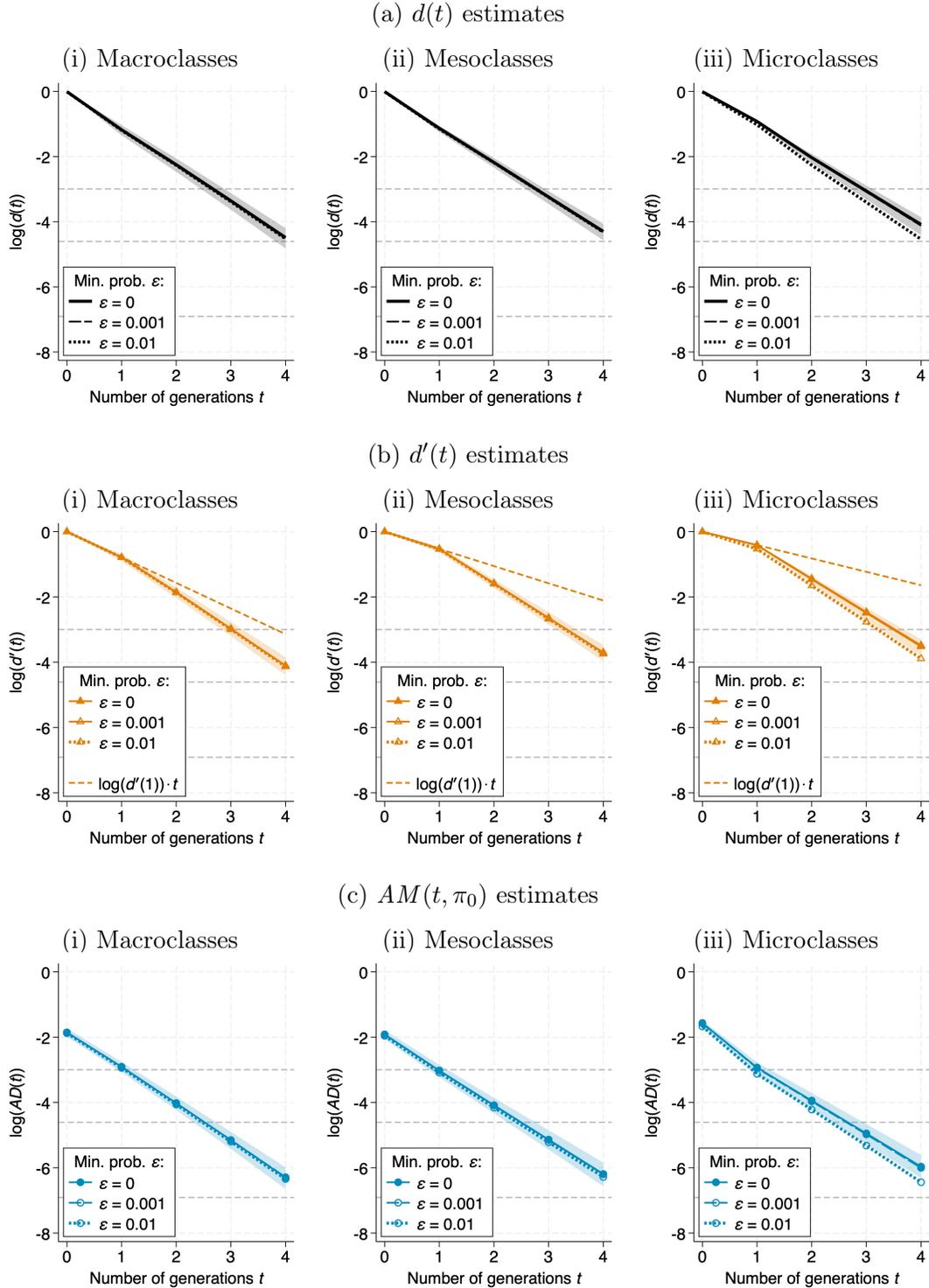
Notes: This figure plots histograms of the occupational classes (i^*, j^*) that generate $d'(t)$ (i.e., $(i^*, j^*) = \arg \max_{(i,j)} d'(t, i, j)$ where $d'(t, i, j) = \| e_i \cdot P^t - e_j \cdot P^t \|$) across 2,000 Bayesian bootstrap replication samples for $t = 1, 4$. The vertical dashed lines indicate the classes that generate $d'(t)$ in the main sample.

Figure A.5: Histograms of classes with maximum and minimum $IM(t, i)$ for $t \geq 1$ over bootstrap samples



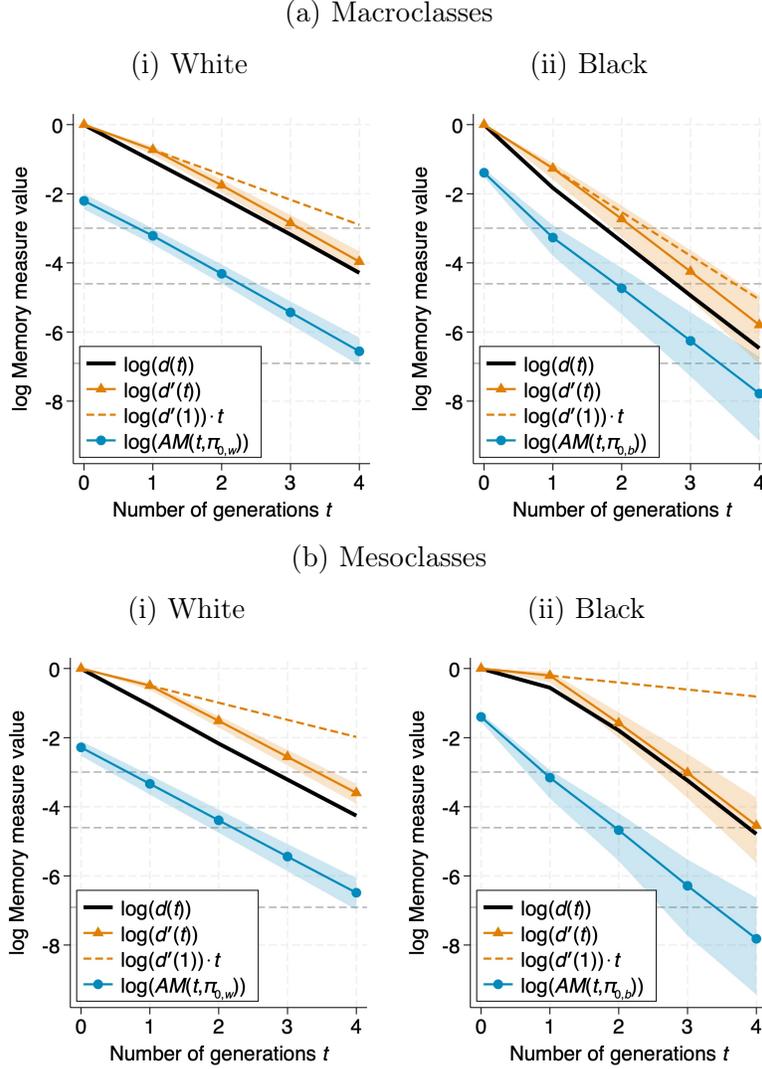
Notes: This figure plots histograms of the occupational classes (i_{max}, i_{min}) that possess the maximum and minimum class-specific memory IM between $1 \leq t \leq 4$ (i.e., $i_{max} = \arg \max_i \sum_{t=1}^8 \log(IM(t, i))$ and $i_{min} = \arg \min_i \sum_{t=1}^8 \log(IM(t, i))$) across 2,000 Bayesian bootstrap replication samples for $t = 1, 4$. The vertical dashed lines indicate the classes that generate the maximum and minimum IM curves in the main sample.

Figure A.6: Robustness of estimates to adjusting small probabilities in transition matrix



Notes: This figure compares memory measure estimates when adjusting the transition matrix P to have minimum probability ϵ . The original memory measure estimates ($\epsilon = 0$) is included as a reference. Shaded regions are 95% BC_a confidence intervals of the original memory measure estimates using 2,000 Bayesian bootstrap replication samples. See the notes of Figure 3 for details on the confidence intervals of Panels (a) and (b).

Figure A.7: Estimates of $d(t)$, $d'(t)$, and $AM(t, \pi_0)$ curves, by racial subgroup



Notes: This figure plots $d(t)$, $d'(t)$, and $AM(t, \pi_{0,r})$ (defined in Equations 1, 2, and 4) in log scale over generation t by racial subgroup $r \in \{w, b\}$. All race-specific memory measures apply their respective race-specific transition matrix P_r . Orange dashed lines $\log(d'(1)) \cdot t$ are also included as reference. Shaded regions are 95% BC_a confidence intervals using 2,000 Bayesian bootstrap replication samples. Confidence intervals are excluded from $\log(d(t))$ for visual clarity but are available upon request. Dashed horizontal lines correspond to $\log(0.05)$, $\log(0.01)$, and $\log(0.001)$.

B Structural and Exchange Mobility in Markov Models

B.1 Additional Formalization

The population interpretation of Markov models suggests a straightforward approach to formalizing the exchange versus structural distinction. The steady state distribution of dynasties over classes is such distribution that, for each class, the inflow of dynasties into this class equals the outflow of dynasties from this class. If this was not true, the cross section distribution would necessarily change. If the population is not in the steady state and structural mobility is present, there must exist at least two classes for which there are imbalances between the inflows and outflows into the classes. One metric for structural mobility may therefore be based on the vector of such imbalances, the magnitude of structural mobility is measured by a norm applied to the vector. Over time the marginal distributions converge to the steady state, so the norm of this vector will converge to 0, meaning that structural mobility disappears.

To be more precise, consider inflows and outflows into the first class observed in period $t + 1$. If in generation t all N dynasties are distributed over n classes as (N_1, N_2, \dots, N_n) , the (expected) inflow of dynasties into the first class may be computed as $N_2 \cdot P_{21} + N_3 \cdot P_{31} + \dots + N_n \cdot P_{n1} = (0, N_2, N_3, \dots, N_n) \cdot P(\cdot, 1)$, where $P(\cdot, 1)$ is the first column of matrix P . The (expected) outflow of dynasties from the first class is $N_1 \cdot (1 - P_{11})$ ($N_1 \cdot P_{11}$ stayed in same first class). The difference between the inflow and the outflow into class 1 is equal to

$$N_1 \cdot (P_{11} - 1) + (0, N_2, N_3, \dots, N_n) \cdot P(\cdot, 1) = -N_1 + (N_1, N_2, \dots, N_n) \cdot P(\cdot, 1).$$

Likewise, the difference between the inflow and the outflow for class i , $1 \leq i \leq n$, is $-N_i + (N_1, N_2, \dots, N_n) \cdot P(\cdot, i)$.

$$\begin{aligned} imb_{vector} &= (-N_1 + (N_1, N_2, \dots, N_n) \cdot P(\cdot, 1), \dots, -N_n + (N_1, N_2, \dots, N_n) \cdot P(\cdot, n)) = \\ &= (N_1, N_2, \dots, N_n) \cdot P - (N_1, N_2, \dots, N_n) = N_{t+1} - N_t, \end{aligned}$$

where N_t is the period- t distribution of the population over the classes. If we move from absolute numbers of dynasties to fractions of the total number of dynasties, we end up with $imb_{vector} = \pi_{t+1} - \pi_t$, where all imbalances are measured in fractions of the population.

One can also evaluate structural mobility at a more disaggregated level by comparing the observed “communication” between any two classes with the steady-state communication between same classes. For each pair of classes (i, j) one can compute the fraction of dynasties which move from i to j in the absence of structural mobility (i.e. in the steady state): $\pi_i^* \cdot P_{ij}$. And then, in presence of structural mobility, we can compare this same fraction computed as $(\pi_t)_i \cdot P_{ij}$ with pure

exchange fraction $\pi_i^* \cdot P_{ij}$. Thus, one can calculate the following imbalance matrix:

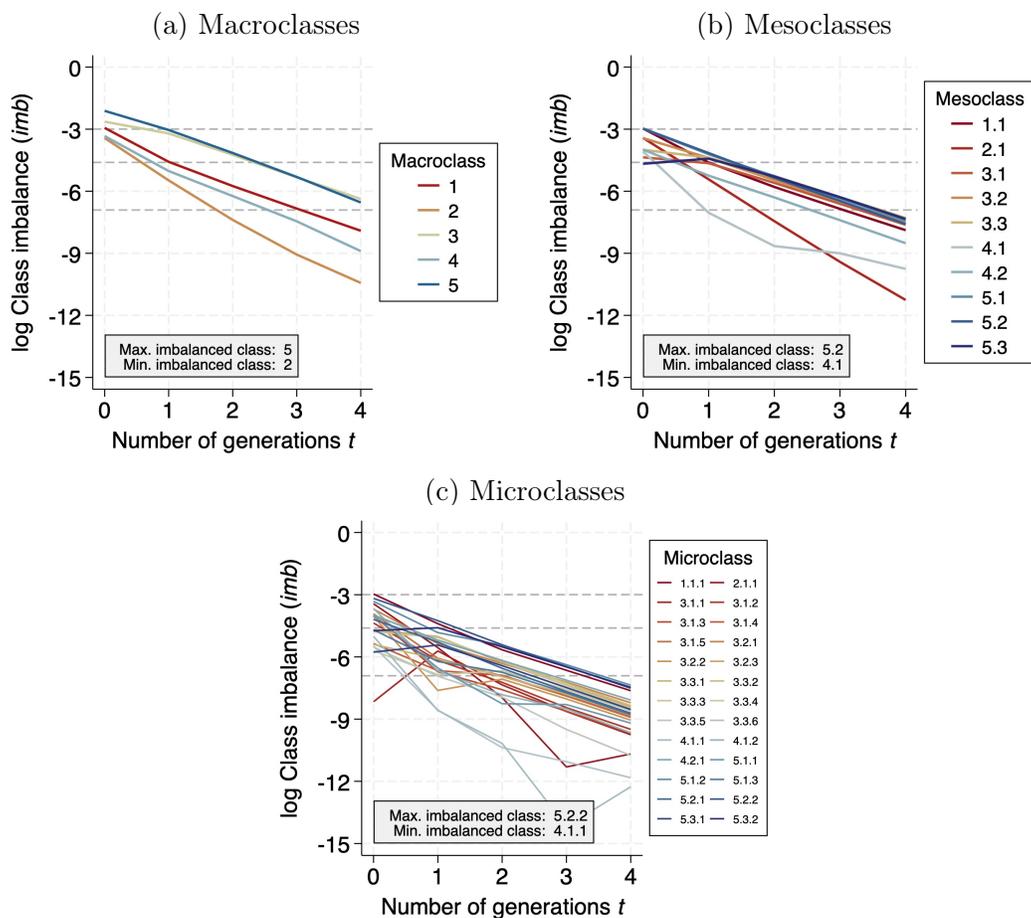
$$imb_{matrix} = \{((\pi_t)_i - \pi_i^*) \cdot P_{ij}\}_{1 \leq i, j \leq n}.$$

This “Imbalance” matrix may be a better object to focus on because it contains the maximal possible information. As a single number metric we may consider its norm.

B.2 Empirical Results

Illustrating Section B and disappearance of structural mobility as time evolves, we report imbalance vectors computed for different classes and different time steps (see Figure B.1) together with imbalance matrices for macro- and mesoclasses (see Tables B.1 and B.2). From Figure B.1, the maximum and the minimum average imbalances over the period from $t = 0$ to $t = 4$ are achieved from the following occupational classes: for macroclasses, professional-managerial (5) and primary manual (farming) occupations (2); for mesoclasses, managers and officials (5.2) and clerical (4.1); for microclasses, other private managers (5.2.2) and postal and mail distribution (4.1.1). By the fourth generation, all such imbalances essentially disappear and we begin observing pure exchange mobility patterns.

Figure B.1: Measuring structural mobility through class-specific imbalances



Notes: This figure plots class-specific imbalances $imb(j)$ (defined in Appendix B) for all classes over generation t in log scale. The gray textbox reports classes j with the maximum and minimum average $imb(j)$ for $t \geq 1$. Dashed horizontal lines correspond to $\log(0.05)$, $\log(0.01)$, and $\log(0.001)$. See Table 1 for definitions of macro- and mesoclass codes.

Table B.1: Imbalance matrices: Macroclasses

(a) $t = 0$

Parent class	Child class				
	1	2	3	4	5
1	0.008	0.000	0.028	0.005	0.011
2	0.001	0.004	0.019	0.003	0.005
3	0.005	0.001	0.038	0.009	0.020
4	-0.001	0.000	-0.013	-0.005	-0.016
5	-0.002	-0.001	-0.032	-0.018	-0.067

(b) $t = 1$

Parent class	Child class				
	1	2	3	4	5
1	0.002	0.000	0.006	0.001	0.002
2	0.000	0.001	0.002	0.000	0.001
3	0.003	0.000	0.022	0.005	0.011
4	-0.000	0.000	-0.002	-0.001	-0.003
5	-0.001	-0.000	-0.013	-0.007	-0.027

(c) $t = 2$

Parent class	Child class				
	1	2	3	4	5
1	0.000	0.000	0.002	0.000	0.001
2	0.000	0.000	0.000	0.000	0.000
3	0.001	0.000	0.008	0.002	0.004
4	-0.000	0.000	-0.001	-0.000	-0.001
5	-0.000	-0.000	-0.004	-0.002	-0.009

(d) $t = 3$

Parent class	Child class				
	1	2	3	4	5
1	0.000	0.000	0.001	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.003	0.001	0.001
4	-0.000	0.000	-0.000	-0.000	-0.000
5	-0.000	-0.000	-0.001	-0.001	-0.003

(e) $t = 4$

Parent class	Child class				
	1	2	3	4	5
1	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.001	0.000	0.000
4	-0.000	0.000	-0.000	-0.000	-0.000
5	-0.000	-0.000	-0.000	-0.000	-0.001

Notes: This table reports imbalance matrices, defined as $\{(\pi_{t,i} - \pi_i^*) \cdot P_{ij}\}_{1 \leq i, j \leq n}$, by generation t . See Table 1 for definitions of class codes.

Table B.2: Imbalance matrices: Mesoclasses

(a) $t = 0$

Parent class	Child class									
	1.1	2.1	3.1	3.2	3.3	4.1	4.2	5.1	5.2	5.3
0	0.008	0.000	0.008	0.009	0.010	0.003	0.002	0.004	0.006	0.001
2.1	0.001	0.004	0.003	0.005	0.011	0.002	0.001	0.001	0.002	0.002
3.1	0.001	0.000	0.002	0.002	0.003	0.001	0.001	0.001	0.002	0.001
3.2	0.003	0.000	0.005	0.005	0.007	0.002	0.001	0.003	0.005	0.001
3.3	0.001	0.000	0.002	0.002	0.006	0.001	0.001	0.002	0.003	0.001
4.1	-0.000	0.000	-0.002	-0.002	-0.004	-0.002	-0.001	-0.002	-0.004	-0.001
4.2	-0.001	0.000	-0.001	-0.002	-0.003	-0.001	-0.002	-0.004	-0.003	-0.002
5.1	-0.002	0.000	-0.004	-0.003	-0.009	-0.003	-0.003	-0.010	-0.012	-0.006
5.2	-0.001	-0.000	-0.004	-0.005	-0.008	-0.004	-0.005	-0.008	-0.011	-0.006
5.3	0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.002	-0.002	-0.002

(b) $t = 1$

Parent class	Child class									
	1.1	2.1	3.1	3.2	3.3	4.1	4.2	5.1	5.2	5.3
1.1	0.002	0.000	0.002	0.002	0.002	0.001	0.000	0.001	0.001	0.000
2.1	0.000	0.001	0.000	0.001	0.001	0.001	0.000	0.000	0.000	0.000
3.1	0.001	0.000	0.002	0.001	0.002	0.001	0.000	0.001	0.001	0.000
3.2	0.001	0.000	0.002	0.002	0.003	0.001	0.000	0.001	0.002	0.000
3.3	0.001	0.000	0.001	0.002	0.004	0.001	0.001	0.001	0.002	0.000
4.1	-0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
4.2	-0.000	0.000	-0.000	-0.000	-0.001	-0.000	-0.000	-0.001	-0.001	-0.001
5.1	-0.001	0.000	-0.001	-0.001	-0.003	-0.001	-0.001	-0.003	-0.003	-0.002
5.2	-0.000	-0.000	-0.001	-0.002	-0.002	-0.001	-0.002	-0.002	-0.003	-0.002
5.3	0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.003	-0.003

(c) $t = 2$

Parent class	Child class									
	1.1	2.1	3.1	3.2	3.3	4.1	4.2	5.1	5.2	5.3
1.1	0.000	0.000	0.000	0.001	0.001	0.000	0.000	0.000	0.000	0.000
2.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.1	0.000	0.000	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
3.2	0.000	0.000	0.001	0.001	0.001	0.000	0.000	0.000	0.001	0.000
3.3	0.000	0.000	0.001	0.001	0.002	0.000	0.000	0.000	0.001	0.000
4.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4.2	-0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
5.1	-0.000	0.000	-0.000	-0.000	-0.001	-0.000	-0.000	-0.001	-0.001	-0.001
5.2	-0.000	-0.000	-0.000	-0.000	-0.001	-0.000	-0.000	-0.001	-0.001	-0.001
5.3	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001

(d) $t = 3$

Parent class	Child class									
	1.1	2.1	3.1	3.2	3.3	4.1	4.2	5.1	5.2	5.3
1.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.3	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000	0.000
4.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4.2	-0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
5.1	-0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
5.2	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
5.3	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

(e) $t = 4$

Parent class	Child class									
	1.1	2.1	3.1	3.2	3.3	4.1	4.2	5.1	5.2	5.3
1.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4.1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4.2	-0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
5.1	-0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
5.2	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
5.3	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

Notes: This table reports imbalance matrices, defined as $\{(\pi_{t,i} - \pi_i^*) \cdot P_{ij}\}_{1 \leq i, j \leq n}$, by generation t . See Table 1 for definitions of class codes.

C Data Appendix

C.1 Constructing the Sample

Table C.1 summarizes the sampling scheme used to construct our main samples. Steps 1 and 2 are commonly taken by the mobility literature that uses the NLSY79 sample in order to observe both parent and child income or occupations during adulthood. Step 3 ensures that at least one parental figure (birth or step-parent) is present in household to ensure consistent definitions of parent and child occupations.

Steps 4 and 5 drop individuals with missing occupation data to retrieve our final occupation samples. More than 5,000 observations are dropped from our occupation sample, namely since parent occupations are observed for only one or two years in the data.

C.2 Constructing Occupation Variables

Parent occupation: We use the qualifying father’s and mother’s occupation reported in 1978 and retrospectively when the child is age 14.

- We first adjust the father and mother’s occupation separately using the available data. We prioritize the 1978 survey year variable over the retrospective age 14 variable. Next, we prioritize the father’s occupation over the mother’s occupation unless the father is not working or the father’s occupation is missing. If neither the father or mother is working, we assign the parent occupation to “not working”; if no occupational data is observed, then it is coded as missing. “Not working” status is elicited from the NLSY family background questionnaire.

Table C.1: Overview of sampling scheme

Step	Description of step	<i>N</i>	% Base sample
0	Base sample	12,686	100.0
1	Drop economically-disadvantaged white and military samples	9,763	77.0
2	Drop observations without Household Interview Versions A or B	8,507	67.1
3	At least one parental figure in household, 1979–1986	8,122	64.0
<i>Income Sample</i>			
4	Drop missing parent income after age restrictions	4,767	37.6
5	Drop missing child income after age restrictions	2,839	22.4
<i>Occupation Sample</i>			
4	Drop missing parent occupation after age restrictions	4,470	35.2
5	Drop missing child occupation after age restrictions	2,752	21.7

- Parent occupations are based on the 1970 Census codes. We map these Census occupation codes to our adaptation of Jonsson et al.’s microclass codes.

Child occupation: We use the NLSY’s constructed variable providing the occupation of the child or spouse/partner’s current or most recent job.

- Occupations are reported since the calendar year *prior* to the survey year.
- The NLSY elicits occupations of the current or *most recent* spouse/partner. To remove the possibility of picking up occupations of previous (divorced or separated) spouses/partners, we only use this data whenever a spouse or partner is cohabiting in the household. Before 1994, occupations were elicited from *only* spouses; after 1994, occupations of partners were also elicited. Therefore, we only can use occupations of the child’s spouse/partner before 1994 if the child is married.
- When the child has a spouse or cohabiting partner, we prioritize the male’s occupation unless the male is not working or the male’s occupation is missing. If neither the male or female is working, we assign the child occupation to “not working”; if no occupational data is observed, then it is coded as missing. “Not working” is defined as having worked 0 weeks in the previous calendar year, either due to unemployment or nonparticipation in the labor force.
- Before 2002, occupations are based on the 1970 Census codes; thereafter, they are based on the 2000 Census codes. We map these Census occupation codes to our adaptation of Jonsson et al.’s microclass codes.
- After assigning one harmonized occupation code for the child’s household per year, we find the modal value across all survey years (subject to all age restrictions described in the main text).

D Technical Appendix

D.1 The Total Variation Norm

The total variation norm is coming from the Markov chains literature, and it has some very clear interpretation. If we have two distributions μ and ν over the same state space S (in our case it is the set of all classes), then the distance between them in the total variation (TV) norm is defined as $\|\mu - \nu\|_{TV} = \max_{A \in \Omega} |\mu(A) - \nu(A)|$, where Ω is the σ -algebra on S . The probabilistic interpretation is consistent with the lottery interpretation. In this case $\|\mu - \nu\|_{TV}$ is the maximum difference between the probabilities assigned to a single event by distributions μ and ν . Namely, we go over all events $A \in \Omega$, i.e. in our framework we should consider events of a single (randomly chosen) dynasty being in all classes and in all possible unions of those classes to find such class or a union of classes that would give the maximum difference between the probabilities of being in this set of classes “under” μ and ν . For example, $\|e_i P^t - e_j P^t\|_{TV}$ is the maximum difference between the probabilities of dynasties with ancestors in classes i and j being in some set of classes in period $t + 1$. Alternatively, $\|\mu - \nu\|_{TV}$ may be computed as follows:

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{1 \leq i \leq n} |\mu_i - \nu_i| = \frac{1}{2} \|\mu - \nu\|_1 = \sum_{i: \mu_i \geq \nu_i} |\mu_i - \nu_i| = \sum_{i: \nu_i \geq \mu_i} |\mu_i - \nu_i|,$$

where $\|\cdot\|_1$ denotes the l_1 -norm. These expressions allow us to interpret the TV norm in marginal population interpretation. First we identify a common part of distributions μ and ν , i.e. distribution θ such that $\theta_i = \min\{\mu_i, \nu_i\}$ and then we sum up fractions of the population in μ or in ν (meaning that those fractions equal each other) which are over this common distribution θ . In other words, $\|\mu - \nu\|_{TV}$ is “overconcentration” of the population in μ or ν relative to the common part of μ and ν . Please see Levin and Peres (2017), pp. 47-48 for details.

D.2 Adjusting Transition Matrices with Small Probabilities

Let P be a $K \times K$ transition matrix where some cells $P_{ij} = 0$. Consider the problem

$$\arg \min_{Q \in \mathcal{P}} \|Q - P\|_{mat} \quad \text{s.t.} \quad Q_{ij} \geq \epsilon > 0; \quad \sum_{j=1}^K Q_{ij} = 1, \quad \forall i = 1, \dots, K, \quad (8)$$

where \mathcal{P} is the set of well-defined transition matrices and the matrix norm operator $\|\cdot\|_{mat}$ is defined as

$$\|M\|_{mat} = \sum_{i=1}^K \sum_{j=1}^K |M_{ij}|. \quad (9)$$

Thus, the optimization problem can be rewritten as

$$\arg \min_{\{Q_{ij} \mid i,j=1,\dots,N\}} \sum_{i=1}^K \sum_{j=1}^K |Q_{ij} - P_{ij}| \quad \text{s.t.} \quad Q_{ij} \geq \epsilon > 0; \quad \sum_{j=1}^K Q_{ij} = 1, \quad \forall i = 1, \dots, K. \quad (10)$$

Since the choice of Q_{ij} values are independent across rows i , this problem is equivalent to the composite of K separate optimization problems corresponding to each row: For each i ,

$$\arg \min_{\{Q_{ij} \mid j=1,\dots,N\}} \sum_{j=1}^K |Q_{ij} - P_{ij}| \quad \text{s.t.} \quad Q_{ij} \geq \epsilon > 0; \quad \sum_{j=1}^K Q_{ij} = 1. \quad (11)$$

There are three possible cases for a given row i :

1. $P_{ij} \geq \epsilon, \forall j = 1, \dots, N$. Then the solution $\{Q_{ij}^* \mid j = 1, \dots, N\}$ of (11) is $Q_{ij}^* = P_{ij}$.
2. Either $P_{ij} < \epsilon$ or $P_{ij} \geq \epsilon + \frac{M_i}{K-z_i}$, where $M_i = \sum_{j=1}^K (\epsilon - P_{ij}) \cdot \mathbf{1}\{P_{ij} < \epsilon\}$ (i.e., the cumulative perturbations required to satisfy only the minimum constraint) and $z_i = \sum_{j=1}^K \mathbf{1}\{P_{ij} < \epsilon\}$ (i.e., the number of elements smaller than ϵ in the i th row of P). Define $\Delta_{ij} = Q_{ij} - P_{ij}$. Note that, for values j such that $P_{ij} < \epsilon$, we must have $\Delta_{ij} \geq \epsilon - P_{ij}$. Without the adding-up constraint, the optimal solution would simply set $Q_{ij} = \epsilon$ whenever $P_{ij} < \epsilon$, which implies $\sum_{j=1}^K \Delta_{ij} = \sum_{j=1}^K |\Delta_{ij}| = \sum_{j=1}^K (\epsilon - P_{ij}) \cdot \mathbf{1}\{P_{ij} < \epsilon\} = M_i$. However, with the adding-up constraint, we must satisfy $\sum_{j=1}^K \Delta_{ij} = 0$, which immediately implies that minimum vector norm value is $\sum_{j=1}^K |\Delta_{ij}| = 2M_i$. Hence, any solution $\{Q_{ij}^* \mid j = 1, \dots, N\}$ that attains this minimum value is an optimal solution of (11). One such solution⁴⁵ is $Q_{ij}^* = \epsilon$ for j s.t. $P_{ij} < \epsilon$ and $Q_{ij}^* = P_{ij} - \frac{M_i}{K-z_i}$ for j s.t. $P_{ij} \geq \epsilon + \frac{M_i}{K-z_i}$. Note that, in the latter case, $Q_{ij}^* \geq \epsilon + \frac{M_i}{K-z_i} - \frac{M_i}{K-z_i} = \epsilon$.
3. $P_{ij} < \epsilon$ and $P_{ik} \in (\epsilon, \epsilon + \frac{M_i}{K-z_i})$ for some $j, k = 1, \dots, K$, where M_i, z_i are defined as before. The solution concept is similar to before, except now the minimum constraint $Q_{ij} \geq \epsilon$ would bind for elements j s.t. $P_{ij} \in (\epsilon, \epsilon + \frac{M_i}{K-z_i})$. One possible solution is the following algorithm (in what follows, suppress the index i to reduce notational burden):

- (a) Set all $q_j^* = \epsilon$ whenever $p_j < \epsilon$. Initialize $k = 0$.
- (b) Define $p_{(j)}$ as the j th order statistic (i.e., $p_{(1)} \leq \dots \leq p_{(K)}$); note that $p_{(j)} > \epsilon$ for $j \geq z$ and $p_{(z+1)} \in (\epsilon, \epsilon + \frac{M_i}{K-z})$. Set $q_j^* = \epsilon$ s.t. $p_j = p_{(z+1)}$ and define $n = |\{j \mid p_j = p_{(z+1)}\}|$. Set k to $k + n$.

45. Note that this proposed solution is *not unique*. Using the L^1 -norm will always leads to a family of possible solutions; the solution Q^* defined above provides one method for preventing large changes occurring for any given element p_{ij} by attempting to undo the perturbations evenly over all remaining non-zero elements of P . Using alternative norms may lead to unique solutions; we employ the L^1 -norm for the sake of consistency.

- (c) Define $\delta_{(z+k)} = n \cdot (p_{(z+k)} - \epsilon)$. Then the total magnitude of perturbations that remain to be distributed across the remaining row elements to satisfy the adding-up constraint is $S_{(z+k)} = M_i - \delta_{(z+k)}$. Check whether $p_{(z+k+1)} \geq \epsilon + \frac{S_{(z+k)}}{K-z-k}$.
- If true, then define $q_j^* = p_j - \frac{S_{(z+k)}}{K-z-k}$ for all remaining entries of the row vector.
 - If false, then proceed to Step (b) except set $M_i = S_{z+k}$.

The solutions recovered from each row can be combined to attain the final solution Q^* .