A Additional Proofs

A.1 Proof of Lemma 1

Beginning with the asset price equation:

$$R_{t+1} = \frac{D_{t+1} + V_{t+1}}{V_t}$$

Divide both sides by $N_{t+1} = (1+n)N_t$ to obtain per-capita quantities and rearranging:

$$v_t = \frac{1+n}{R_{t+1}}(d_{t+1} + v_{t+1})$$

Now impose the steady state $v_t = v_{t+1}$ and simplify:

$$v = \frac{1+n}{r-n}((1-\tau)\pi - (1-\psi\tau)(n+\delta)k)$$

where we used the definition of per-capita dividends and the steady state investment rate $i = (n+\delta)k$. Noting that $\pi = \left(1 - \frac{1-\alpha}{\mu}\right)k^\alpha$, we can simplify as follows:

$$v = \frac{1+n}{r-n}((1-\tau)\left[1 - \frac{1-\alpha}{\mu}\right]k^\alpha - (1-\psi\tau)(n+\delta)k$$

$$v = (1+n)(1-\tau)\underbrace{\frac{\left[1 - \frac{1}{\mu}\right]k^\alpha}{r-n}}_{\equiv \xi} + \frac{1+n}{r-n}\left((1-\tau)\frac{\alpha}{\mu}k^\alpha - (1-\psi\tau)(n+\delta)k\right)$$

$$v = (1+n)(1-\tau)\xi + \frac{1+n}{r-n}(1-\psi\tau)k\underbrace{\left(\frac{1-\tau}{1-\psi\tau}\frac{\alpha}{\mu}k^{\alpha-1} - \delta - n\right)}_{\equiv r-n}$$

$$v = (1+n)(1-\tau)\xi + (1+n)(1-\psi\tau)k$$

Rearranging the last expression, we obtain equation (10).

A.2 Proof of Lemma 3

Begin by noting that the effective rate of return is:

$$r - n = \left(\frac{1-\tau}{1-\psi\tau}\right)\left(\frac{\alpha}{\mu}\right)k^{\alpha-1} - \delta - n$$

The proof follows from the observation that when $k = k_{GR}$, the return on equity is less than the population growth rate $r(k_{GR}) < n$. However, a fundamental steady state

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requires $r(k_{low}^*) > n$, which implies that $k_{low}^* \neq k_{GR}$. Since $r'(k) < 0$, it follows that $k_{low}^* < k_{GR}$.

A.3   Proof of Proposition 3

The first remark (i) follows directly from Lemma 3, which states that k_{low}^* is always below the Golden Rule k_{GR}. Since $C(k)$ is maximized at k_{GR}, any corporate tax rate that maximizes k_{low}^* will also maximize $C(k)$ in a constrained sense.

The second result (ii) is more involved. We need to show that $k'(\tau^*) = 0$ for $\tau^* \in (0,1)$ and $k'(\tau) > 0$ for $\tau < \tau^*$, $k'(\tau) < 0$ for $\tau > \tau^*$. Begin with the market clearing condition in the equity market:

$$s(k, \boldsymbol{z}) - v(k, \boldsymbol{z}) = 0$$

where $\boldsymbol{z} = (\tau, \bar{\psi})$ and $s(\cdot) = \theta(W(k) + \tau b(k, \bar{\psi}))$. Implicitly differentiating in the neighborhood of the fundamental steady state, we get:

$$k'(\tau)|_{r>n} = -\frac{s_\tau - v_\tau}{s_k - v_k} \tag{20}$$

Since the denominator is strictly negative, $k'(\tau)$ will be positive if $s_\tau - v_\tau > 0$ and $k'(\tau^*) = 0$ when $s_\tau = v_\tau$. Let $\tilde{r} = r - n$. Using (10) and (11), we can express the numerator as:

$$s_\tau - v_\tau = \theta b + (1 + n)\left[\bar{\psi}k + \xi - (1 - \tau)\xi_\tau\right] \tag{21}$$

Define the elasticity of ξ with respect to the corporate tax rate τ as:

$$\varepsilon = \xi_\tau \frac{\tau}{\xi} > 0$$

Using this definition in (21), the optimal tax τ^* for which $s_\tau = v_\tau$ is defined implicitly by

$$\frac{1 - \tau^*}{\tau^*} = \frac{\theta b + (1 + n)(\bar{\psi}k + \xi)}{\varepsilon(1 + n)\xi} > 0 \tag{22}$$

Note that the tax base $b > 0$ and the elasticity $\varepsilon 0$ are strictly positive. Moreover, in a fundamental steady state market power wealth is also strictly positive $\xi > 0$. It follows that the right side of equation (22) is strictly positive. Since $(1 - \tau^*)/\tau^* > 0$ for $\tau \in (0,1)$, it follows that equation (22) is only satisfied when τ^* is positive.

The last step is to show that $k'(\tau) > 0$ when $\tau < \tau^*$ and vice-versa. From (20) we can see that $k'(\tau) > 0$ if $s_\tau - v_\tau > 0$. Since $s_\tau > 0 \ \forall \tau \in [0,1]$, a sufficient condition is $v_\tau < 0$ for $0 < \tau < \tau^*$. Note that

$$\xi_\tau = -\frac{\xi}{r-n}r_\tau = \xi\frac{1-\psi}{(1-\psi\tau)^2}\left(\frac{\alpha}{\mu}\right)k^{\alpha-1}\frac{1}{r-n} = \xi\left(\frac{1-\psi}{1-\psi\tau}\right)\left(\frac{r+\delta}{r-n}\right)\frac{1}{1-\tau}$$

It follows that v_τ can be expressed as:

$$v_\tau = -(1+n)\left[\psi k + \xi\left(1 - \left(\frac{1-\psi}{1-\psi\tau}\right)\frac{r+\delta}{r-n}\right)\right]$$

Thus, if $\tau = 0$ then $v_\tau < 0$ if

$$\xi\left[\left(\frac{1-\psi}{1-\psi\tau}\right)\frac{r+\delta}{r-n} - 1\right] < \psi$$

which is satisfied for a sufficiently high expensing rate $\psi > \left(\frac{n+\delta}{r+\delta}\right)$. This is the condition described in the proposition. It is easy to see that $k'(\tau) < 0$ when $1 > \tau > \tau^*$ since $\lim_{\tau\to 1}\xi = \infty$, which implies $\lim_{\tau\to 1}v_\tau = \infty$.

B   Bubbles, Indeterminacy, and Dynamics

This appendix provides additional results related to the existence of non-fundamental equilibria in our framework. In particular, we show that the combination of market power and asset pricing can lead to exotic macroeconomic properties in our OLG setting. For instance, the existence of MPW leads to two distinct steady states. The first is the unique *fundamental* steady state studied in the main text, in which the stock market price equals the discounted value of future dividends and the capital-labor ratio is well below the Golden Rule due to crowding out effects from MPW. The second steady state features a high capital-labor ratio sustained by an asset price bubble. In addition, the model features equilibrium indeterminacy along rational expectations transition paths (i.e. sunspot equilibria).

We start by describing the steady state valuation of the firm when bubbles are possible.

Lemma 4 (Bubbly firm valuation) *Let $\tilde{k}$ denote the k such that $r(\tilde{k}) = n$. In addition to the fundamental region described above, the steady state equity price $v(k, \boldsymbol{z})$ has a "bubbly" region in $k > \tilde{k}$. In the bubbly region, the NPV of market power wealth is negative $\xi(k, \boldsymbol{z}) < 0$. The equity price $v(k, \boldsymbol{z})$ is upward sloping, concave, and satisfies $\lim_{k\to\tilde{k}^+} v = \infty^-$.*

Lemma 4 states that when bubbles are possible, the equity supply schedule has two distinct regions depending on whether $k^* < \tilde{k}$ or $k^* > \tilde{k}$. The following result establishes that this property gives rise to multiple steady states.

Proposition 4 (Steady State Multiplicity) *Suppose $\tau \in [0, 1)$. There exists two steady state equilibria: k^*_{low} and k^*_{high}.*
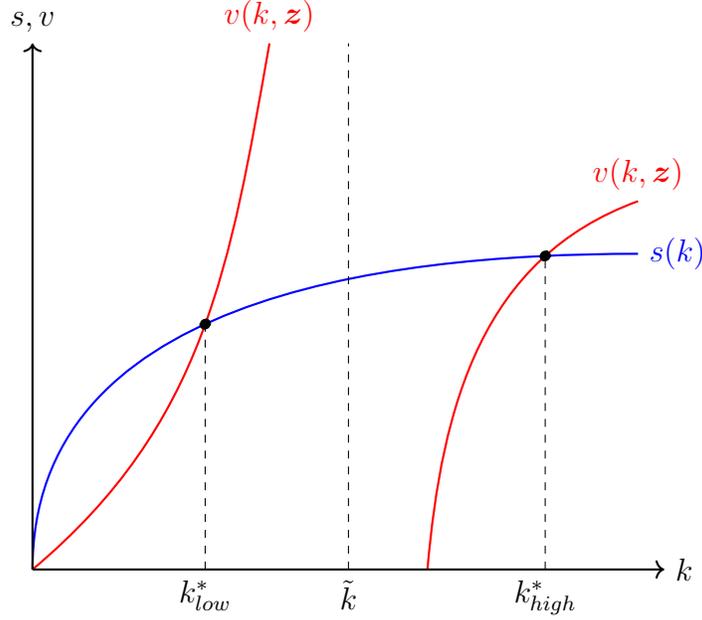
Figure 5: Equity market clearing in capital-wealth space

Proof. We need to establish that two steady state capital-labor ratios $k^* = \{k^*_{low}, k^*_{high}\}$ exist and satisfy $k^*_{low} < \tilde{k} < k^*_{high}$. The existence of k^*_{low} follows directly from Lemma 1. Since $v(k)$ monotonically increases in k, is convex, and $v(k) \to \infty$ as $k \to \tilde{k}$, it will intersect the saving schedule once in the region $k \in (0, \tilde{k})$.

The existence of k^*_{high} requires that $v(k)$ grow faster than $s(k)$ in the region $k \in (\tilde{k}, \infty^+)$. This implies that $v(k)/s(k) > 1$ for $k > k^*_{high}$. Let $A > 0$ and $B > 0$ denote positive constants. Note that as $k \to \infty$, we have

$$\frac{v(k)}{s(k)} = Ak^{1-\alpha} + \frac{B}{r(k) - n}$$

$$\lim_{k \to \infty} \frac{v(k)}{s(k)} = \infty - \frac{B}{\delta + n}$$

$$\lim_{k \to \infty} \frac{v(k)}{s(k)} = \infty$$

where the second step used $\lim_{k \to \infty} r(k) = -\delta$. ■

The intuition for Proposition 4 is depicted in Figure 5, which shows the market clearing condition for the equity market in capital-wealth space. The savings schedule is drawn in blue and is upward sloping and concave. From Lemma 4, the equity value schedule $v(k, \boldsymbol{z})$ has two distinct regions depending on whether $r(k) > n$ or $r(k) < n$. As can be seen, $v(k, \boldsymbol{z})$ intersects the savings schedule twice, implying that two steady state equilibria exist $k^* = \{k^*_{low}, k^*_{high}\}$. Since MPW is positive $\xi > 0$ when $r(k) > n$, the steady state k^*_{low} will feature significant crowding out effects and a long-run capital-

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labor ratio below the perfect competition benchmark. In contrast, since $\xi < 0$ when $r(k) < n$, the second steady state k_{high}^* will feature *crowding in effects* and an elevated capital-labor ratio.

Multiple steady states are possible because of the finite life-time of agents in an OLG and the institutional structure of firms. Since households do not directly own the capital stock and instead save by purchasing equity shares in firms, capital can never be "unbolted" and converted into consumption. Instead, firms make investment decisions in order to maximize the returns to their current shareholders, which consist of dividend payments and the resale price of the firms' equity. Thus, firms inherit the finite planning horizon of their shareholders and do not satisfy the typical transversality condition that would rule out a path resulting in $r(k) < n$ in the long-run.

Considering that multiplicity arises due to MPW, it is not surprising that the steady state can be unique in special cases with zero MPW. Indeed, the steady state will be unique, for example, when the corporate tax rate is set at 100% and MPW is thus completely "taxed away." Similarly, the steady state is unique if the economy features perfect competition (i.e. $\mu = 1$). These observations are summarized below.

Corollary 1 (Uniqueness Conditions) *The economy features a unique steady state if either of the following conditions are met:*

1. *The corporate tax rate is 100% ($\tau = 1$) and there is partial expensing ($\psi \in [0, 1)$);*

2. *Firms are perfectly competitive ($\mu = 1$).*

Proof. Informally, both observations follow directly from noting that the asset price schedule $v(k)$ becomes linear in k when either (i) $\tau = 1$ or (ii) $\mu = 1$. For instance, when $\mu = 1$, we have:
$$v(k) = (1+n)(1-\psi\tau)k$$
which clearly intersects the savings schedule $s(k)$ only once. ∎

Bubble Characterization An implication of the existence of two long-run steady states satisfying either $r(k_{low}^*) > n$ or $r(k_{high}^*) < n$ is that the stock price V_t may or may not feature a bubble depending on which steady state the economy converges to.[16] To see this, iterate the asset pricing equation forward T periods and take the limit $T \to \infty$ to obtain:
$$V_t = \sum_{h=1}^{\infty} \frac{D_{t+h}}{\prod_{m=1}^{h}(1+r_{t+m})} + \lim_{T \to \infty} \frac{V_{t+T}}{\prod_{m=1}^{T}(1+r_{t+m})}$$

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[16]See Hirano and Toda (2024) for a recent survey on the definition and existence of bubbles.

Dividing through by N_t and using $N_{t+T} = (1+n)^T N_t$, we can express the stock price in per capita terms as follows:

$$v_t = \underbrace{\sum_{h=1}^{\infty} \prod_{m=1}^{h} \left(\frac{1+n}{R_{t+m}} \right) d_{t+h}}_{\equiv f_t} + \underbrace{\lim_{T \to \infty} \prod_{m=1}^{T} \left(\frac{1+n}{R_{t+m}} \right) v_{t+T}}_{\equiv b_t} \tag{23}$$

Here, the term f_t refers to the stock price's fundamental value and b_t denotes the bubble component. Notice that if the transversality condition $\lim_{T \to \infty} \prod_{m=1}^{T} \left(\frac{1+n}{R_{t+m}} \right) v_{t+T} = 0$ is satisfied, then $b_t = 0$ and $v_t = f_t$, in which case we say that the stock price reflects the fundamental value. Alternatively, if the transversality condition is violated and $b_t > 0$, then $v_t > f_t$ and the stock price is said to contain a bubble component.

Proposition 5 (Existence of Bubbles in the Long Run) *Suppose the economy is in a long-run steady-state $k^* \in \{k_{low}^*, k_{high}^*\}$. The following results are true:*

1. *The stock price equals its fundamental value $v = f$ when $k^* = k_{low}^*$;*

2. *The stock price contains a bubble when $k^* = k_{high}^*$.*

Proof. In a steady-state we have $d_t = d$ and $v_t = v$. The bubble term can be expressed as:

$$b = \lim_{T \to \infty} \left(\frac{1+n}{R} \right)^T v$$

Beginning with case 1, notice that $r(k_{low}^*) > n$ implies $(1+n)/R < 1$. Therefore, the bubble term is clearly equal to zero $b = 0$ in this case.

Now let's deal with case 2. From the expression above, we can see that $b > 0$ if $(1+n)/R > 1$, which is the case for $r(k_{high}^*) < n$. ∎

A peculiar property of the bubbly steady state is that dividend payments to share-holders are *negative* in the long-run. This property follows directly by noting that the steady state equity price can be alternatively expressed as

$$v = \left(\frac{1+n}{r(k_{high}^*) - n} \right) d$$

Since the stock price must be non-negative and the bubbly steady state satisfies $r(k_{high}^*) < n$, it follows that $d < 0$. Intuitively, in the bubbly steady state the firm's gross profits are entirely exhausted by the amount of investment needed to sustain the large capital stock. Thus, the demand for the firm's stock is driven by its growing price, despite paying negative dividends.

Transitional Dynamics   We now characterize the transitional dynamics of this economy. For notational convenience, the following exposition abstracts from the role of taxes and sets $\tau = 0$ and $\psi = 0$. Let $i_t = I_t/N_t$ denote per-capita investment. The model's transitional dynamics can be represented by a first-order two dimensional system in (i_t, k_t). The law of motion for per-capita investment can be obtained by solving for i_{t+1} from the asset pricing equation:

$$i_{t+1} = \left(1 - \frac{1-\alpha}{\mu}\right) k_{t+1}^\alpha + v_{t+1} - \frac{R_{t+1}}{1+n} v_t \tag{24}$$

The capital-labor ratio must satisfy $(1+n)k_{t+1} = (1-\delta)k_t + i_t$. Using this fact with equation (24) and the stock market clearing condition $v_t = s(k_t)$, we can define our dynamic system as:

$$i_{t+1} = F(i_t, k_t) \tag{25a}$$

$$k_{t+1} = \frac{(1-\delta)k_t + i_t}{1+n} \tag{25b}$$

Lemma 5 *The Jacobian matrix associated with system (25) is given by:*

$$J = \begin{bmatrix} F_i & F_k \\ \frac{1}{1+n} & \frac{1-\delta}{1+n} \end{bmatrix} \tag{26}$$

where F_i and F_k denote the partial derivatives of $F(i_t, k_t)$ with respect to i_t and k_t, respectively. In addition, F_i and F_k satisfy:

$$F_i = \left(\frac{1}{1+n}\right) \left[\alpha(n+\delta) + \frac{\alpha + r + (1-\alpha)\delta}{1+n}\right] \frac{v}{k} \tag{27}$$

$$F_k = (1-\delta)F_i - \alpha \left(\frac{1+r}{1+n}\right) \frac{v}{k} \tag{28}$$

Conjecture 1 (Transition Paths) *Let λ_i^F and λ_i^B denote the Eigenvalues of the dynamic system (25) evaluated at the fundamental and bubbly steady states, respectively.*

- *The Eigenvalues for the fundamental steady state satisfy $|\lambda_1^F| < 1$ and $|\lambda_2^F| > 1$. It follows that k_{low}^* is a saddle point.*

- *Both Eigenvalues for the bubbly steady state lie within the unit circle $|\lambda_1^B| < 1$ and $|\lambda_2^B| < 1$, indicating that k_{high}^* is a stable sink.*

The conjecture states that the fundamental steady state k_{low}^* is a saddle path while the bubbly steady state k_{high}^* is a stable sink. The economy therefore features equilibrium indeterminacy. This can be seen in Figure 6, which depicts the phase diagram for the dynamic system (25). Consider first the fundamental steady state k_{low}^*. Since

Figure 6: Phase diagram in (k_t, i_t)-space
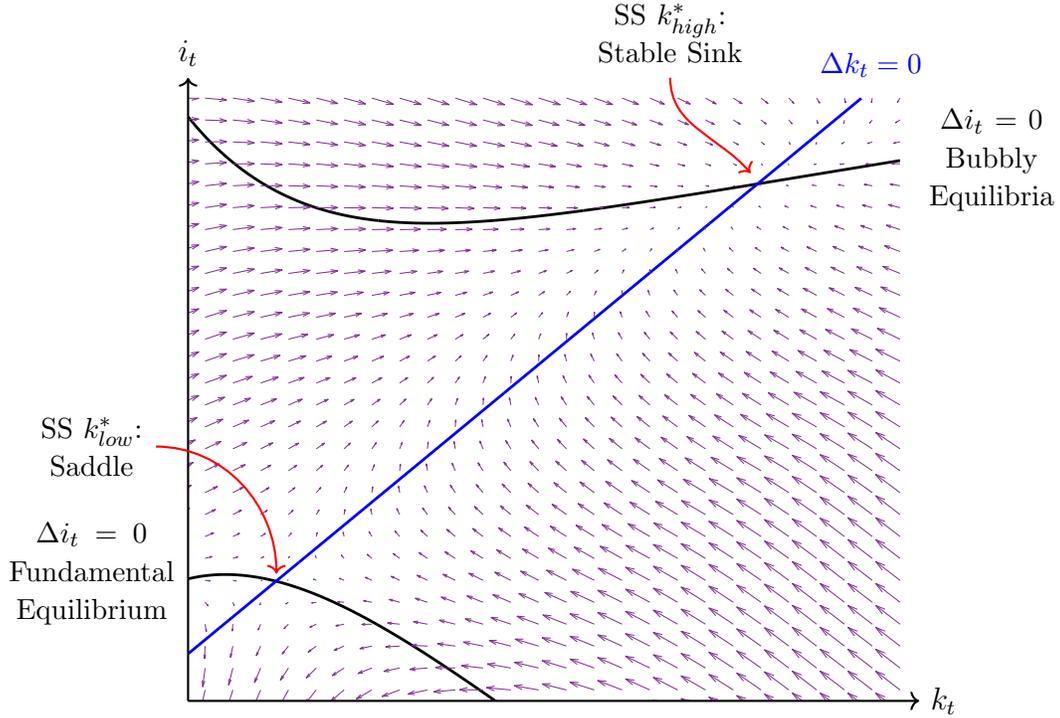
k_t is predetermined while i_t is a jump variable, we can see from the phase arrows that there is a unique transition path that ensures convergence to k_{low}^* given an initial k_0.

Matters are qualitatively different for transition paths to the bubbly steady state. Since k_{high}^* is a stable sink, for a given initial capital-labor ratio k_0, *there exists a continuum of rational expectation equilibrium paths* that converge to the bubbly steady state. The dynamic properties of this model are therefore compatible with sunspots and an independent role for self-fulfilling expectations to coordinate investment behavior and select a transition path.