Online Appendix

A Proofs

A.1 Derivation of the investor's modified problem

We start by providing a derivation of the investor's modified problem.

Proof. First, we adopt a change of variables and write the investor's problem as follows

$$V_{i,t} = \max_{\{\tilde{C}_{i,t}, h_{i,t}, B_{i,t}, S_{i,t}\}} (1-\beta)U\left(\tilde{C}_{i,t}\right) + \beta U\left(\Psi^{-1}\left(\mathbb{E}_{i,t}\left[\Psi\left(U^{-1}(V_{i,t+1})\right)\right]\right)\right), \tag{A.1}$$

subject to

$$\tilde{C}_{i,t} + Q_t S_{i,t} + B_{i,t} = R_{e,t} Q_{t-1} S_{i,t-1} + R_{b,t} B_{i,t-1} + W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu}, \tag{A.2}$$

and the natural borrowing

$$(Q_t + \pi_t) S_{i,t-1} + R_{b,t} B_{i,t-1} + W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu} \geq -\mathcal{H}_{i,t} \tag{A.3}$$

It is immediate that the optimal value of $h_{i,t}$ satisfies

$$W_t = \xi_t h_{i,t}^{\nu}. \tag{A.4}$$

We show next that, given $h_{i,t}$ satisfying (A.4), if the sequence $(\tilde{C}_{i,t}, B_{i,t}, S_{i,t})$ satisfies (A.2) and (A.3), then there exists $(N_{i,t}, \omega_{i,t})$ such that $(\tilde{C}_{i,t}, N_{i,t}, \omega_{i,t})$ satisfies (10) and $N_{i,t} \geq 0$. Conversely, if $(\tilde{C}_{i,t}, N_{i,t}, \omega_{i,t})$ satisfies (10) and $N_{i,t} \geq 0$, there exists $(B_{i,t}, S_{i,t})$ such that $(\tilde{C}_{i,t}, B_{i,t}, S_{i,t})$ satisfies (A.2) and (A.3).

From the definition of the return on human wealth, we have that $W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu} = R_{h,t-1}\mathcal{H}_{i,t-1} - \mathcal{H}_{i,t}$, which allow us to write (A.2) and (A.3) as follows:

$$\tilde{C}_{i,t} + Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t} = N_{i,t}, \qquad N_{i,t} \geq 0. \tag{A.5}$$

We consider next the law of motion of total wealth:

$$N_{i,t+1} = \left[R_{e,t+1}\frac{Q_t S_{i,t}}{Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t}} + R_{b,t+1}\frac{B_{i,t}}{Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t}} + R_{h_i,t+1}\frac{\mathcal{H}_{i,t}}{Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t}}\right]\left(N_{i,t} - \tilde{C}_{i,t}\right). \tag{A.6}$$

As markets are dynamically complete, there exists replicating portfolios $(\omega_{h_i,t}, \omega_{e,t})$ such that

$$R_{k,t+1} = \omega_{k,t} R_{r,t+1} + (1 - \omega_{k,t}) R_{b,t+1}, \tag{A.7}$$

for $k \in \{h_i, e\}$.

Combining the previous two conditions, we obtain

$$N_{i,t+1} = \left[R_{r,t+1} \frac{\omega_{e,t} Q_t S_{i,t} + \omega_{h_i,t} \mathcal{H}_{i,t}}{N_{i,t} - \tilde{C}_{i,t}} + R_{b,t+1} \frac{B_{i,t} + (1 - \omega_{e,t}) Q_t S_{i,t} + (1 - \omega_{h_i,t}) \mathcal{H}_{i,t}}{N_{i,t} - \tilde{C}_{i,t}} \right] \left(N_{i,t} - \tilde{C}_{i,t} \right). \tag{A.8}$$

Using the first condition in (A.5) to solve for $B_{i,t}$, we obtain

$$N_{i,t+1} = \left[\left(R_{r,t+1} - R_{b,t+1} \right) \omega_{i,t} + R_{b,t+1} \right] \left(N_{i,t} - \tilde{C}_{i,t} \right), \tag{A.9}$$

where $\omega_{i,t} \equiv \frac{\omega_{e,t} Q_t S_{i,t} + \omega_{h_i,t} \mathcal{H}_{i,t}}{N_{i,t} - \tilde{C}_{i,t}}$.

□

A.2 Proof of Lemma 1

The next lemma characterizes the value function, the consumption function, and the Euler equations of each investor.

Lemma 1 (Consumption and Euler equations)**.** *The household's value function takes the form: $V_i(N, X, s) = U\left(v_i(X, s) N\right)$, where $v_i(X, s)$ denotes the wealth multiplier. The consumption-wealth ratio $c_i(X, s) = \frac{\tilde{C}_i(N, X, s)}{N}$ and Euler equations for investor $i \in \mathcal{I}$ are given by*

(i) Consumption-wealth ratio:

$$c_i(X, s) = \frac{(\beta^{-1} - 1)^{\psi} \mathcal{R}_i(X, s)^{1-\psi}}{1 + (\beta^{-1} - 1)^{\psi} \mathcal{R}_i(X, s)^{1-\psi}}, \tag{A.10}$$

where $\mathcal{R}_i(X, s) \equiv \Psi^{-1} \left(\mathbb{E}_i \left[\Psi \left(v(X', s') R_{i,n}(X, s, s') \right) | X \right] \right)$.

(ii) Euler equation for an asset $j \in \{r, b\}$:

$$1 = \mathbb{E}_i \left[\Lambda_i(X, s, s') R_j(X, s, s') \right], \tag{A.11}$$

where, for $\theta \equiv \frac{1-\gamma}{1-\psi^{-1}}$, the investor's SDF is given by

$$\Lambda_i(X, s, s') = \beta^{\theta} \left(\frac{c_i(\chi(X, s, s'), s') N'}{c_i(X, s) N} \right)^{-\frac{\theta}{\psi}} R_{i,n}(X, s, s')^{-(1-\theta)}. \tag{A.12}$$

(iii) The wealth multipliers satisfy:

$$v_i(X,s) = U^{-1}\left[U\left(c_i(X,s)\right) + \beta U\left(\mathcal{R}_i(X,s)\left(1 - c_i(X,s)\right)\right)\right]. \tag{A.13}$$

Proof. First, we verify that the value function takes the form (**??**). Given the conjecture about the value function, the Bellman equation for investor i can be written as

$$\frac{(v_i(X,s)N)^{1-\psi^{-1}} - 1}{1-\psi^{-1}} = \max_{\tilde{c}_i,\omega_i}(1-\beta)\frac{(\tilde{c}_i N)^{1-\psi^{-1}} - 1}{1-\psi^{-1}} + \beta\frac{\mathbb{E}_i\left[(v_i(X',s')N')^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}} - 1}{1-\psi^{-1}}, \tag{A.14}$$

subject to $N' = R_{i,n}(X,s,s')(1-\tilde{c}_i)N$ and $N' \geq 0$.

The first-order conditions for the consumption-wealth ratio and the portfolio share are given by

$$(1-\beta)\tilde{c}_i^{-\psi^{-1}} = \beta\mathcal{R}_i(X,s)^{1-\psi^{-1}}(1-\tilde{c}_i)^{-\psi^{-1}} \tag{A.15}$$

$$0 = \mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s'))^{-\gamma}v_i(X')(R_r(X,s,s') - R_b(X,s))\right] \tag{A.16}$$

where $\mathcal{R}_i(X,s) = \mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s'))^{1-\gamma}|X,s\right]^{\frac{1}{1-\gamma}}$.

Given $\mathcal{R}_i(X,s)$, we can solve for the consumption-wealth ratio:

$$\tilde{c}_i(X,s) = \frac{(\beta^{-1}-1)^{\psi}\mathcal{R}_i(X,s)^{1-\psi}}{1+(\beta^{-1}-1)^{\psi}\mathcal{R}_i(X,s)^{1-\psi}}. \tag{A.17}$$

The envelope condition with respect to N is given by

$$v_i(X)^{1-1/\psi} = \beta\mathcal{R}_i(X)^{1-1/\psi}(1-\tilde{c}_i(X))^{-1/\psi} \Rightarrow \tilde{c}_i(X) = (1-\beta)^{\psi}v_i(X)^{1-\psi}. \tag{A.18}$$

From the optimality condition for the risky asset, we obtain

$$\mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s'))^{1-\gamma}\right] = \mathbb{E}_i\left[v_i(X',s')^{1-\gamma}R_{i,n}(X,s,s')^{-\gamma}R_j(X,s,s')\right], \tag{A.19}$$

for $j \in \{r,b\}$.

Raising the envelope condition (A.18) to the power $\theta \equiv \frac{1-\gamma}{1-\psi^{-1}}$, using the definition of $\mathcal{R}_i(X)$ and condition (A.19), we obtain

$$1 = \mathbb{E}_i\left[\beta^{\theta}\left(\frac{v_i(X',s')}{v_i(X,s')}\right)^{1-\gamma}R_{i,n}(X,s,s')^{-\gamma}R_j(X,s,s')(1-\tilde{c}_i(X,s))^{-\theta/\psi}\right]. \tag{A.20}$$

Using the condition $v_i(X) = (1-\beta)^{\frac{1}{1-\psi^{-1}}} \tilde{c}_i(X)^{-\frac{\psi^{-1}}{1-\psi^{-1}}}$, we obtain the Euler equations

$$1 = \mathbb{E}_i \left[\beta^\theta \left(\frac{\tilde{c}_i(X',s')N'}{\tilde{c}_i(X,s)N} \right)^{-\frac{\theta}{\psi}} R_{i,n}(X,s,s')^{-(1-\theta)} R_j(X,s,s') \right]. \tag{A.21}$$

This concludes the derivation of the consumption-wealth ratio and the Euler equations for the two assets. It remains to check that the value function takes the form (**??**), which amounts to show that $v_i(X)$ indeed does not depend on N. Notice that $\tilde{c}_i(X,s)$ and $\omega_i(X,s)$ do not depend on N. We can then write the Bellman equation as follows:

$$v_i(X,s)^{1-\psi^{-1}} = (1-\beta)\tilde{c}_i(X,s)^{1-\psi^{-1}} + \beta \mathbb{E}_i \left[(v_i(X',s')R_{i,n}(X,s,s')(1-\tilde{c}_i(X)))^{1-\gamma} \right]^{\frac{1-\psi^{-1}}{1-\gamma}}, \tag{A.22}$$

for $\psi \neq 1$ and

$$\log v_i(X,s) = (1-\beta)\log \tilde{c}_i(X,s) + \beta \log \mathbb{E}_i \left[(v_i(X',s')R_{i,n}(X,s,s')(1-\tilde{c}_i(X,s)))^{1-\gamma} \right]^{\frac{1}{1-\gamma}}. \tag{A.23}$$

which is independent of N, which confirms our conjecture for the value function (**??**).

□

A.3 Proof of Lemma 2

The following lemma characterizes households' portfolio weight in the surplus claim in terms of the economy-wide SDF, the market prices, and their beliefs.

Lemma 2 (Portfolio share)**.** *The shares of total wealth invested in the risky asset are*

$$\omega_i(X,s) = \frac{1}{\Delta R_r(X,s)} \left[\frac{\tilde{p}_i(X,s,H)}{p_{s,H}\Lambda(X,s,H)} - \frac{\tilde{p}_i(X,s,L)}{p_{s,L}\Lambda(X,s,L)} \right], \tag{A.24}$$

where $\tilde{p}_i(X,s,s')$ *is*

$$\tilde{p}_i(X,s,s') = \frac{(p^i_{ss'})^{\frac{1}{\gamma}} \left[v_i(\chi(X,s,s'),s') | R^e_r(X,s,s') | \right]^{\frac{1}{\gamma}-1}}{\sum_{\tilde{s}' \in \{L,H\}} (p^i_{s\tilde{s}'})^{\frac{1}{\gamma}} \left[v_i(\chi(X,s,\tilde{s}'),\tilde{s}') | R^e_r(X,s,\tilde{s}') | \right]^{\frac{1}{\gamma}-1}}.$$

Lemma 2, describes how portfolio shares depend on distorted probabilities, $\tilde{p}_i(X,s,s')$, and $p_{ss'} \times \Lambda(X,s,s')$. The portfolio share $\omega_i(X,s)$ in (A.24) is increasing in p^i_{sH}. This means that relatively optimistic investors hold more of the risky surplus claim.

Proof. The optimal portfolio share satisfies the condition

$$\frac{p^i_{sL}}{p^i_{sH}}\frac{v_i(\chi(X,s,L),L)^{1-\gamma}}{v_i(\chi(X,s,H),H)^{1-\gamma}}\left(\frac{\omega_i(X,s)R^e_r(X,s,L)+1}{\omega_i(X,s)R^e_r(X,s,H)+1}\right)^{-\gamma}\frac{|R^e_r(X,s,L)|}{R^e_r(X,s,H)}=1 \tag{A.25}$$

Raising both sides to $-\frac{1}{\gamma}$, we obtain

$$\left(\frac{p^i_{sL}}{p^i_{sH}}\right)^{-\frac{1}{\gamma}}\frac{v_i(\chi(X,s,L),L)^{1-\frac{1}{\gamma}}}{v_i(\chi(X,s,H),H)^{1-\frac{1}{\gamma}}}\frac{\omega_i(X,s)R^e_r(X,s,L)+1}{\omega_i(X,s)R^e_r(X,s,H)+1}\frac{|R^e_r(X,s,L)|^{-\frac{1}{\gamma}}}{R^e_r(X,s,H)^{-\frac{1}{\gamma}}}=1 \tag{A.26}$$

Rearranging the expression above, we obtain

$$\omega_i(X,s)=\frac{\tilde{p}_i(X,s,H)}{|R^e_r(X,s,L)|}-\frac{\tilde{p}_i(X,s,L)}{R^e_r(X,s,H)}, \tag{A.27}$$

where

$$\tilde{p}_i(X,s,s')=\frac{(p^i_{ss'})^{\frac{1}{\gamma}}\left[v_i(\chi(X,s,s'),s')|R^e_r(X,s,s')|\right]^{\frac{1}{\gamma}-1}}{\sum_{s'\in\{L,H\}}(p^i_{ss'})^{\frac{1}{\gamma}}\left[v_i(\chi(X,s,s'),s')|R^e_r(X,s,s')|\right]^{\frac{1}{\gamma}-1}}. \tag{A.28}$$

The SDF in this economy is given by

$$\Lambda(X,s,s')=\frac{1}{p_{ss'}}\frac{1}{R_b(X,s)}\frac{|R_r(X,s,-s')-R_b(X,s)|}{\Delta R_r(X,s)}, \tag{A.29}$$

where $\Delta R_r(X,s)=R_r(X,s,H)-R_r(X,s,L)$.

We can then write $\omega_i(X,s)$ as follows

$$\omega_i(X,s)=\frac{1}{\Delta R_r(X,s)}\left[\frac{\tilde{p}_i(X,s,H)}{p_{s,H}\Lambda(X,s,H)}-\frac{\tilde{p}_i(X,s,L)}{p_{s,L}\Lambda(X,s,L)}\right]. \tag{A.30}$$

Diffusion-like approximation. To better interpret the expression for the portfolio share, it is useful to consider an approximation analogous to the continuous-time limit for diffusion processes. Given $R_r(X,s,s')$, probabilities $p^i_{ss'}$ for household i, and a small parameter $\epsilon>0$, we can find $\mu_{i,r}(X,s)$ and $\sigma_{i,r}(X,s)$ that satisfies the conditions

$$R^e_r(X,s,H)=\mu_{i,r}(X,s)\epsilon+\sqrt{\frac{p_{sL}}{p_{sH}}}\sigma_{i,r}(X,s)\sqrt{\epsilon},\qquad R^e_r(X,s,L)=\mu_{i,r}(X,s)\epsilon-\sqrt{\frac{p_{sH}}{p_{sL}}}\sigma_{i,r}(X,s)\sqrt{\epsilon}, \tag{A.31}$$

which gives us the expected value and variance for household i:

$$\mathbb{E}_i[R^e_r(X,s,s')|X,s]=\mu_{i,r}(X,s)\epsilon,\qquad Var_i[R^e_r(X,s,s')|X,s]=\sigma^2_{i,r}(X,s)\epsilon. \tag{A.32}$$

Similarly, we can write $R_b(X,s) = 1 + r_b(X,s)\epsilon$.

From Equation (B.6), and assuming $\gamma = 1$, we obtain

$$\omega_i(X,s) = R_b(X,s)\frac{p^i_{s,H}R^e_r(X,s,H) + p^i_{s,L}R^e_r(X,s,L)}{|R^e_r(X,s,L)|R^e_r(X,s,H)}$$
$$= (1 + r_b(X,s)\epsilon)\frac{\mu_{i,r}(X,s)\epsilon}{\left(\sqrt{\frac{p_{sH}}{p_{sL}}}\sigma_{i,r}(X,s)\sqrt{\epsilon} - \mu_{i,r}(X,s)\epsilon\right)\left(\mu_{i,r}(X,s)\epsilon + \sqrt{\frac{p_{sL}}{p_{sH}}}\sigma_{i,r}(X,s)\sqrt{\epsilon}\right)}, \tag{A.33}$$

where we used the fact that $R^e_r(X,s,L) < 0$ by no-arbitrage.

In general, $(\mu_{i,r}(X,s), \sigma_{i,r}(X,s))$ and $p^i_{ss'}$ are functions of ϵ. Assuming that $\mu_{i,r}(X,s) = \mathcal{O}(1)$, $\sigma_{i,r}(X,s)) = \mathcal{O}(1)$, and $p^i_{ss'} = \mathcal{O}(1)$, we can write the expression $\omega_i(X,s)$ as follows:[26]

$$\omega_i(X,s) = \frac{\mu_{i,r}(X,s)}{\sigma^2_{i,r}(X,s)} + \mathcal{O}(\epsilon). \tag{A.34}$$

□

A.4 Proof of Proposition 1

Proof. First, we compute the Sharpe ratio on the risky asset. We will compute expectations using the objective measure, but a similar calculation gives the Sharpe ratio using the investors' subjective beliefs. The expected excess return is given by

$$\mathbb{E}\left[R^e_r(X,s,s')\right] = p_{sL}R^e_r(X,s,L) + p_{sH}R^e_r(X,s,H). \tag{A.35}$$

The variance of excess returns is given by

$$Var[R^e_r(X,s,s')] = p_{sL}p_{sH}\Delta R^e_r(X,s)^2. \tag{A.36}$$

The Sharpe ratio in the risky asset is then given by

$$\frac{\mathbb{E}[R^e_r(X,s,s')]}{\sqrt{Var[R^e_r(X,s,s')]}} = \sqrt{\frac{p_{sL}}{p_{sH}}}\frac{R^e_r(X,s,L)}{\Delta R^e_r(X,s)} + \sqrt{\frac{p_{sH}}{p_{sL}}}\frac{R^e_r(X,s,H)}{\Delta R^e_r(X,s)}. \tag{A.37}$$

We can write the expression above in terms of the economy's SDF. The SDF under the

[26]These assumptions are analogous to the ones used by e.g. Merton (1992) to derive the continuous-time limit with diffusion processes. Allowing for rare events, $p^i_{ss'} = \mathcal{O}(\epsilon)$ for some s', would lead to a jump-diffusion process.

objective measure can be written as

$$\Lambda(X,s,L) = \frac{\mathbb{E}[\Lambda(X,s,s')]}{p_{sL}} \frac{R_r^e(X,s,H)}{\Delta R_r^e(X,s)}, \qquad \Lambda(X,s,H) = -\frac{\mathbb{E}[\Lambda(X,s,s')]}{p_{sH}} \frac{R_r^e(X,s,L)}{\Delta R_r^e(X,s)}. \tag{A.38}$$

Combining the expressions above, we obtain

$$\frac{\mathbb{E}[R_r^e(X,s,s')]}{\sqrt{Var[R_r^e(X,s,s')]}} = \sqrt{p_{sL}p_{sH}} \frac{\Lambda(X,s,L) - \Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]}. \tag{A.39}$$

We consider next how the Sharpe ratio affects the risk-neutral expectation of future productivity growth. The risk-neutral expectation of productivity is given by

$$\mathbb{E}^Q\left[x_{t+1}\right] = p_{sL}\frac{\Lambda(X,s,L)}{\mathbb{E}[\Lambda(X,s,s')]}x_L + p_{sH}\frac{\Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]}x_H. \tag{A.40}$$

The difference between the expected value of productivity under the physical measure and the risk-neutral measure is given by

$$\mathbb{E}[x_{t+1}] - \mathbb{E}^Q[x_{t+1}] = p_{sL}\frac{\mathbb{E}[\Lambda(X,s,s')] - \Lambda(X,s,L)}{\mathbb{E}[\Lambda(X,s,s')]}x_L + p_{sH}\frac{\mathbb{E}[\Lambda(X,s,s')] - \Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]}x_H. \tag{A.41}$$

Rearranging the expression above, we obtain

$$\mathbb{E}[x_{t+1}] - \mathbb{E}^Q[x_{t+1}] = p_{sL}p_{sH}\frac{\Lambda(X,s,L) - \Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]}\Delta x, \tag{A.42}$$

where $\Delta x = x_H - x_L$.

Using the expression for the Sharpe ratio, we obtain

$$\mathbb{E}^Q[x_{t+1}] = \mathbb{E}[x_{t+1}] - \sqrt{p_{sL}p_{sH}}\frac{\mathbb{E}[R_r^e(X,s,s')]}{\sqrt{Var[R_r^e(X,s,s')]}}\Delta x. \tag{A.43}$$

□

A.5 Proof of Proposition 2

Proof. We start by deriving the process for returns. From the market clearing condition for goods, we obtain

$$\frac{x_s h(\mathcal{L})^\alpha - \xi\frac{h(\mathcal{L})^{1+\nu}}{1+\nu}}{P(X,s)} = 1 - \beta \tag{A.44}$$

The return on the surplus claim is given by

$$R_p(X,s,s') = \frac{x_s P(\chi(X,s,s'),s')}{P(X,s) - \left(x_s h(\mathcal{L})^\alpha - \xi \frac{h(\mathcal{L})^{1+\nu}}{1+\nu}\right)} = \frac{x_s}{\beta} \frac{x_{s'} h(\mathcal{L}'(X,s))^\alpha - \xi \frac{h(\mathcal{L}'(X,s))^{1+\nu}}{1+\nu}}{x_s h(\mathcal{L})^\alpha - \xi \frac{h(\mathcal{L})^{1+\nu}}{1+\nu}}. \tag{A.45}$$

Using the conditions in (14), we can rewrite the expression as follows

$$R_p(X,s,s') = \frac{x_s}{\beta} \frac{x_{s'} \mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} \mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} \mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{A.46}$$

Note that the denominator in the expression above is positive if and only if $\mathcal{L} < \frac{1+\nu}{\alpha} x_s$. A sufficient condition is given by $\alpha x_H < x_L$, as shown below

$$\mathcal{L} \leq x_H < \frac{x_L}{\alpha} < \frac{1+\nu}{\alpha} x_s, \tag{A.47}$$

and, similarly, this condition guarantees that the numerator is also positive.

Interest rate. The interest rate satisfies the condition $R_b(X,s) = \mathbb{E}\left[\frac{\Lambda(X,s,s')}{\mathbb{E}[\Lambda(X,s,s')]} R_p(X,s,s')\right]$, so $R_b(X,s)$ is given by

$$R_b(X,s) = \left(1 - \frac{\alpha}{1+\nu}\right) \frac{x_s}{\beta} \frac{\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} \mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{A.48}$$

using the fact that $\mathbb{E}\left[\frac{\Lambda(X,s,s')}{\mathbb{E}[\Lambda(X,s,s')]} x_{s'}\right] = \mathcal{L}'(X,s)$.

The expression above is increasing in $\mathcal{L}'(X,s)$, decreasing in x_s, and it is increasing in $\mathcal{L}$ for $s = L$.

Risk premium. The risk asset's excess return is given by

$$\frac{R_p(X,s,s')}{R_b(X,s)} = \frac{1}{1 - \frac{\alpha}{1+\nu}} \frac{x_{s'} - \frac{\alpha}{1+\nu} \mathcal{L}'(X,s)}{\mathcal{L}'(X,s)}. \tag{A.49}$$

The conditional risk premium is then given by

$$\mathbb{E}_s[R_p^e(X,s,s')] = \frac{1}{1 - \frac{\alpha}{1+\nu}} \frac{\mathbb{E}_s[x_{s'}] - \mathcal{L}'(X,s)}{\mathcal{L}'(X,s)}, \tag{A.50}$$

given the definition $R_p^e(X,s,s') \equiv \frac{R_p(X,s,s') - R_b(X,s)}{R_b(X,s)}$.

□

A.6 Proof of Proposition 3 and Corollary 1

Proof. We start by deriving the expression for the SDF. Note that we can express the SDF in terms of $R_e(X,s,s')$ and $R_b(X,s)$ instead of $R_r(X,s,s')$ and $R_b(X,s)$, as we can always obtain the SDF in terms of any two (linearly independent) assets. The SDF is then given by

$$\Lambda(X,s,s') = \frac{1}{p_{ss'}}\frac{1}{R_b(X,s)}\frac{|R_p^e(X,s,-s')|}{\Delta R_p^e(X,s)}. \tag{A.51}$$

The excess return on the surplus claim is given by

$$R_p^e(X,s,s') = \frac{1}{1-\frac{\alpha}{1+\nu}}\frac{x_{s'} - \mathcal{L}'(X,s)}{\mathcal{L}'(X,s)}. \tag{A.52}$$

Combining the previous two expressions, we obtain

$$\Lambda(X,s,s') = \frac{1}{p_{ss'}}\frac{1}{R_b(X,s)}\frac{|x_{-s'} - \mathcal{L}'(X,s)|}{\Delta x} \tag{A.53}$$

using the fact that $\frac{R_p^e(X,s,s')}{\Delta R_p^e(X,s)} = \frac{1}{R_b(X,s)}\frac{x_{s'}-\mathcal{L}'(X,s)}{\Delta x}$.

Demand for risk. The demand for risk in this economy is given by

$$\sum_{i=1}^{I}\eta_i\sigma_s[R_{i,n}(X,s,s')] = \sqrt{p_{sL}p_{sH}}\left[\frac{p_{sH}(X,s)}{p_{sH}\Lambda(X,s,H)} - \frac{p_{sL}(X,s)}{p_{sL}\Lambda(X,s,L)}\right], \tag{A.54}$$

where $p_{ss'}(X,s) = \sum_{i=1}^{I}\eta_{i,t}p_{ss'}^i$, using the fact that $\sigma_s[R_r(X,s,s')] = \sqrt{p_{sH}p_{sL}}\Delta R_r(X,s)$ and the results in Lemma 2.

Using the expression for the SDF, the demand for risk can be written as

$$\sum_{i=1}^{I}\eta_i\sigma_s[R_{i,n}(X,s,s')] = \sigma_s[x_{s'}]\frac{\frac{1+\nu-\alpha}{1+\nu}\frac{x_s}{\beta}}{x_s\mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}\left[\frac{p_{sH}(X,s)\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{\mathcal{L}'(X,s)-x_L} - \frac{p_{sL}(X,s)\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_H-\mathcal{L}'(X,s)}\right], \tag{A.55}$$

given $\sigma_s[x_{s'}] = \sqrt{p_{sL}p_{sH}}\Delta x$.

The first term inside brackets in the expression above is decreasing in $\mathcal{L}'(X,s)$ if and only if the following condition holds

$$\frac{1+\nu}{1+\nu-\alpha}\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}-1}(\mathcal{L}'(X,s)-x_L) - \mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}} < 0 \iff \mathcal{L}'(X,s) < \frac{x_L}{\alpha}, \tag{A.56}$$

which holds, given that $\mathcal{L}'(X,s) \leq x_H < \frac{x_L}{\alpha}$.

Therefore, the demand for risk is decreasing in $\mathcal{L}'(X,s)$. As $\mathcal{L}'(X,s)$ is decreasing in the Sharpe ratio of the risky asset, then the demand for risk is increasing in the Sharpe ratio.

Supply of risk. The volatility of returns is given by

$$\sigma_s[R_p(X,s,s')] = \frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{A.57}$$

which is increasing in $\mathcal{L}'(X,s)$ and decreasing in x_s.

Equilibrium. Combining supply and demand for risk, we obtain

$$\frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}} = \frac{1+\nu-\alpha}{1+\nu}\frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}} \left[\frac{p_{sH}(X,s)}{\mathcal{L}'(X,s)-x_L} - \frac{p_{sL}(X,s)}{x_H - \mathcal{L}'(X,s)}\right]. \tag{A.58}$$

The left-hand side is strictly increasing in $\mathcal{L}'(X,s)$, while the right-hand side is strictly decreasing in $\mathcal{L}'(X,s)$ in the interval $x_L < \mathcal{L}'(X,s) < x_H$. The right-hand side converges to $+\infty$ as $\mathcal{L}'(X',s)$ approaches x_L from above, and it converges to $-\infty$ as $\mathcal{L}'(X,s)$ approaches x_H from below. Therefore, there exists a unique value of $\mathcal{L}'(X,s)$ solving the equation above in this interval. Note that the two curves intersect again for $\mathcal{L}'(X,s) > x_H$, which can be seen by noticing that the right-hand is decreasing in $\mathcal{L}'(X,s)$ for $\mathcal{L}'(X,s) > x_H$ and converges to $+\infty$ as $\mathcal{L}'(X,s)$ approaches x_H from above. Therefore, the economically relevant solution corresponds to the smallest of the two points of intersection.

Rearranging the expression above, we obtain

$$1 = \frac{1+\nu-\alpha}{1+\nu}\mathcal{L}'(X,s)\frac{p_{sH}(X,s)(x_H - \mathcal{L}'(X,s)) - p_{sL}(X,s)(\mathcal{L}'(X,s) - x_L)}{(\mathcal{L}'(X,s) - x_L)(x_H - \mathcal{L}'(X,s))}. \tag{A.59}$$

We then obtain a quadratic equation for $\mathcal{L}'(X,s)$:

$$\frac{\alpha}{1+\nu}\mathcal{L}'(X,s)^2 - \left[\left(1 - \frac{1+\nu-\alpha}{1+v}p_{sH}(X,s)\right)x_H + \left(1 - \frac{1+\nu-\alpha}{1+v}p_{sL}(X,s)\right)x_L\right]\mathcal{L}'(X,s) + x_L x_H = 0 \tag{A.60}$$

The equilibrium value is given by the smallest root of the equation above.

□

A.7 Proof of Proposition 4

Proof. We start by deriving the return on the investor's portfolio. Given that markets are complete, there exists a replicating portfolio $\omega^r(X,s)$ such that

$$R_r(X,s,s') = \omega^r(X,s)R_p(X,s,s') + (1-\omega^r(X,s))R_b(X,s) \Rightarrow R_r^e(X,s,s') = \omega^r(X,s)R_p^e(X,s,s'), \tag{A.61}$$

where $\omega^r(X,s) = \frac{\sigma_s[R_r(X,s,s')]}{\sigma_s[R_p(X,s,s')]} = \frac{\Delta R_r(X,s)}{\Delta R_p(X,s)}$ We can then write the return on the portfolio of investor i as follows:

$$R_{i,n}(X,s,s') = \omega_i(X,s)R_{r,t}^e(X,s,s') + R_b(X,s) = \omega_i(X,s)\frac{\Delta R_r(X,s)}{\Delta R_p(X,s)}R_{p,t}^e(X,s,s') + R_b(X,s). \tag{A.62}$$

Using condition (A.24) and the expression for the SDF, we obtain

$$R_{i,n}^e(X,s,s') = R_b(X,s)\left[\frac{p_{sH}^i}{\mathcal{L}'(X,s)-x_L} - \frac{p_{sL}^i}{x_H-\mathcal{L}'(X,s)}\right]\frac{\Delta x R_p^e(X,s,s')}{\Delta R_p(X,s)} \tag{A.63}$$

$$= R_b(X,s)\left[\frac{p_{sH}^i}{\mathcal{L}'(X,s)-x_L} - \frac{p_{sL}^i}{x_H-\mathcal{L}'(X,s)}\right](x_{s'}-\mathcal{L}'(X,s)). \tag{A.64}$$

The return on the portfolio is then given by

$$R_{i,n}(X,s,L) = \frac{\Delta x R_b(X,s)}{x_H-\mathcal{L}'(X,s)}p_{sL}^i, \qquad R_{i,n}(X,s,H) = \frac{\Delta x R_b(X,s)}{\mathcal{L}'(X,s)-x_L}p_{sH}^i. \tag{A.65}$$

Wealth share dynamics. The share of wealth of investor i is given by

$$\eta_i'(X,s,s') = \frac{\eta_i R_{i,n}(X,s,s')}{\sum_{j=1}^I \eta_j R_{j,n}(X,s,s')} = \frac{\eta_i p_{ss'}^i}{\sum_{j=1}^I \eta_j p_{ss'}^j} = \eta_i\frac{p_{ss'}^i}{p_{ss'}(X)}. \tag{A.66}$$

Long-run wealth dynamics. Note that the wealth share is a (bounded) martingale under market beliefs:

$$p_{sH}(X)\eta_i'(X,s,H) + p_{sL}(X)\eta_i'(X,s,L) = \eta_i(p_{sH}^i + p_{sL}^i) = \eta_i. \tag{A.67}$$

Therefore, from the martingale convergence theorem, the wealth share of investor i

converges. This implies that, for every ϵ, there exits T such that

$$|\eta_{i,T+1} - \eta_{i,T}| < \epsilon \iff \eta_i \left| \frac{p^i_{ss'} - p_{ss'}(X)}{p_{ss'}(X)} \right| < \epsilon, \tag{A.68}$$

almost surely, where the economy is at the state (X, s) at period T.

This implies that either η_i converges to zero or $p^i_{ss'}$ converges to $p_{ss}(X)$. If $p^i_{ss'} \neq p^j_{ss'}$ for any $i, j \in \mathcal{I}$, $i \neq j$, then the wealth share of a single investor converges to one.

By definition, $p^i_{sH} > p_{sH}(X)$ for an optimistic investor in state (X, s), then the wealth share of optimists increase in the good state and decline in the bad state. This implies that market beliefs evolve according to

$$p_{s'H}(X') = \sum_{i=1}^{I} \eta'_i p^i_{s'H} = \sum_{i=1}^{I} \eta_i \frac{p^i_{ss'}}{p_{ss'}(X)} p^i_{s'H}, \tag{A.69}$$

where

$$p_{HH}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{sH}}{p_{sH}(X)} p^i_{HH} \geq p_{sH}(X), \qquad p_{LH}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{sL}}{p_{sL}(X)} p^i_{LH} \leq p_{sH}(X). \tag{A.70}$$

This implies that the relative wealth share for investors i and j is given by

$$\frac{\eta'_i(X, s, s')}{\eta'_j(X, s, s')} = \frac{\eta_i}{\eta_j} \frac{p^i_{ss'}}{p^j_{ss'}}. \tag{A.71}$$

Suppose investor j beliefs coincides with the objective measure. Then, the ratio above is a martingale:

$$\mathbb{E}_s \left[\frac{\eta'_i}{\eta'_j} \right] = p_{sL} \frac{\eta_i}{\eta_j} \frac{p^i_{sL}}{p_{sL}} + p_{sH} \frac{\eta_i}{\eta_j} \frac{p^i_{sH}}{p_{sH}} = \frac{\eta_i}{\eta_j}. \tag{A.72}$$

If the wealth of investor j is bounded away from zero, then the above martingale is bounded and, from the martingale convergence theorem, it converges almost surely. □

A.8 Proof of Corollary 2

Proof. Consider an economy that starts at $s = H$ with wealth distribution $\{\eta_i\}_{i=1}^I$ which switches to the low state after either one period (early transition) or two periods (late transition). Market beliefs on the low state in the case of an early transition are given by

$$p_{LH}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{HL}}{p_{HL}(X)} p^i_{LH}, \tag{A.73}$$

and market beliefs on the low state in the case of a late transition are given by

$$p_{LH}(X'') = \sum_{i=1}^{I} \eta'_i \frac{p^i_{HL}}{p_{HL}(X')} p^i_{LH}, \tag{A.74}$$

where $\eta'_i = \eta_i \frac{p^i_{HH}}{p_{HH}(X)}$.

Note that if investor i is optimistic, $p^i_{HH} > p_{HH}(X)$, then $\eta'_i > \eta_i$ and $p^{-i}_{HL}(X') \leq p^{-i}_{HL}(X)$, where $p^{-i}_{HL}(X) \equiv \frac{1}{1-\eta_i} \sum_{j \neq i} \eta_j p^j_{HL}$. This implies that the following inequality holds:

$$\eta'_i \frac{p^i_{HL}}{p_{HL}(X')} = \frac{\eta'_i p^i_{HL}}{\eta'_i p^i_{HL} + (1-\eta'_i) p^{-i}_{HL}(X')} > \frac{\eta_i p^i_{HL}}{\eta_i p^i_{HL} + (1-\eta_i) p^{-i}_{HL}(X)} = \eta_i \frac{p^i_{HL}}{p_{HL}(X)}. \tag{A.75}$$

Therefore, there is more weight on the beliefs of investors who were optimistic in the original state in the case of a late transition. In the case of rank-preserving beliefs, these agents are also optimistic in the low state, so the market is more optimistic under a late transition:

$$p_{LH}(X'') > p_{LH}(X'). \tag{A.76}$$

Alternatively, the market is now more pessimistic after a late transition in the case of rank-alternating beliefs:

$$p_{LH}(X'') < p_{LH}(X'). \tag{A.77}$$

A similar argument shows that, under rank-preserving beliefs, the market is more pessimistic after a late transition when the economy starts at state $s = L$:

$$p_{HH}(X'') = \sum_{i=1}^{I} \eta'_i \frac{p^i_{LH}}{p_{LH}(X')} p^i_{HH} < \sum_{i=1}^{I} \eta_i \frac{p^i_{LH}}{p_{LH}(X)} p^i_{HH} = p_{HH}(X), \tag{A.78}$$

where $\eta'_i = \eta_i \frac{p^i_{LL}}{p_{LL}(X)}$. Alternatively, the market is more optimistic under a late transition in the case of rank-alternating beliefs.

□

B Trading volume

B.1 Belief Taxanomy and Trading Volume

We consider next the implications of heterogeneous beliefs regarding stock turnover, a measure of trading volume. To compute the stock turnover, we first map the portfolio holdings of the surplus claim, ω_i, into the effective number of shares on firm equity in the primitive economy. This mapping is straightforward in the case of linear labor disutility, $\nu = 0$ because human wealth is zero in this case.[27] To simplify the exhibition, we adopt this assumption for the rest of the section. The traded volume in this case is:

$$\tau_t = \frac{1}{2}\sum_{i=1}^{I} |\omega_{i,t}\eta_{i,t} - \omega_{i,t-1}\eta_{i,t-1}|.$$

This formula shows that the volume traded depends on the level of disagreement.

We consider a small deviation from homogeneous beliefs to study the effect of belief dispersion on volume. We express investor i's beliefs as follows

$$p^i_{ss'} = p^*_{ss'} + \delta^i_{ss'}\epsilon,$$

where $\delta^i_{sH} + \delta^i_{sL} = 0$ and ϵ captures belief heterogeneity as discussed earlier. Also, for parsimony, set $p^*_{ss'} = \frac{1}{2}$, such that, in the absence of heterogeneity, beliefs are iid and symmetric—the proofs hold for general common belief case.

The portfolio share of investor i is:

$$\omega_i(X, s; \epsilon) = 1 + \kappa_\omega \left[p^i_{sH} - \overline{p}^m_{sH}(X)\right] + \mathcal{O}(\epsilon^2),$$

where κ_ω is a positive constant. This expression showcases how optimistic investors, for whom $p^i_{sH} > \overline{p}^m_{sH}(X)$, are levered up in stocks.

The following lemma characterizes the trading behavior of a given investor.

Lemma 3. *Consider current and future states s and s'. The effect of a perturbation in ϵ on the trades of investor i is:*

$$\Delta S_i(X, s, s'; \epsilon) = \underbrace{\Delta\eta_i(X, s, s')}_{\textit{rebalancing effect}} + \underbrace{\Delta\omega_i(X, s, s')\eta_i}_{\textit{change-in-beliefs effect}} + \mathcal{O}(\epsilon^2), \tag{B.1}$$

[27] The share of wealth invested in stocks is ω_i, given that human wealth is equal to zero, $\mathcal{H}_i = 0$, and $R_e(X, s, s') = R_r(X, s, s')$ under this assumption.

as the economy switches from state (X, s) *to* (X', s')*, where*

$$\Delta\eta_i(X,s,s') \equiv \eta_i \frac{p^i_{ss'} - \overline{p}^m_{ss'}(X)}{p^*_{ss'}}, \qquad \Delta\omega_i(X,s,s') \equiv \kappa_\omega \left[p^i_{s'H} - \overline{p}^m_{s'H}(X) - (p^i_{sH} - \overline{p}^m_{sH}(X)) \right].$$

Expression (B.1) reveals two effects. The *rebalancing effect* captures the extent to which investors trade after a change in the state to keep portfolio shares constant: investors who put more likelihood on the realized state relative to the market belief, increased (decreased) their wealth share. Thus, they must buy (sell) the risky asset when that state is realized, to keep the portfolio share constant. Of course, as the economy evolves from s to s', portfolio shares themselves change as beliefs are modified. The *change-in-beliefs effect* captures the trade that follows the change in portfolio shares as the state changes. The change-in-beliefs effect equals zero if $s = s'$, as individual beliefs are constant.

In tandem, the effects of rebalancing and change-in-beliefs determine equilibrium turnover.

Proposition 5 (Turnover)**.** *The economy's turnover, as it switches from state* (X, s) *to state* (X', s')*, is given by*

$$\tau(X,s,s';\epsilon) = \frac{1}{2}\sum_{i=1}^{I} \eta_i \left| \frac{p^i_{ss'} - \overline{p}^m_{ss'}(X)}{p^*_{ss'}} + \kappa_\omega \left[p^i_{s'H} - \overline{p}^m_{s'H}(X) - (p^i_{sH} - \overline{p}^m_{sH}(X)) \right] \right| + \mathcal{O}(\epsilon^2). \quad \text{(B.2)}$$

Proposition 5 provides a characterization of turnover. When $s = s'$, the change-in-beliefs effect vanishes; turnover is driven solely by the rebalancing effect:

$$\tau(X,s,s';\epsilon) = \frac{1}{2}\sum_{i=1}^{I} \eta_i \frac{|p^i_{ss'} - \overline{p}^m_{ss'}(X)|}{p^*_{ss'}} + \mathcal{O}(\epsilon^2).$$

Thus, when there is no change in the state of the economy, turnover is proportional to the average absolute deviation of beliefs. The formula is consistent with the evidence in Section C.4, which shows that dispersion on subjective beliefs about cash flows is correlated with stock market turnover.

The change-in-beliefs effect emerges when the economy switches states, that is, when $s \neq s'$. This effect may either amplify or dampen the rebalancing effect, depending on the type of belief and the direction of change in the economy. For instance, suppose that $s = H$ and $s' = L$ and beliefs are rank-alternating. Optimistic investors lose wealth as the economy switches to a bad state. The rebalancing effect implies that they must sell some risky assets to maintain their portfolio shares once stocks lose value. These investors also become pessimists in downturns, leading them to sell even more stocks. Thus, the two effects go in the same direction, amplifying the impact on the turnover when the economy switches from high to low states. The two effects are opposite when $s = L$ and $s' = H$. Pessimists become optimistic as the economy switches to the high state, which induces them to increase their stock portfolio share. At the same time, the rebalancing effect dictates that they sell stocks once stocks appreciate to keep the portfolio balanced.

Connecting with the Turnover Evidence. It is convenient to express heterogeneity in beliefs, p^i_{ss}, in terms of heterogeneity in the perceived *persistence* of fundamentals. Assuming investors agree on the unconditional mean of x_t, $\overline{x}$, we can write $\mathbb{E}_{i,t}[x_{t+1}] - \overline{x} = \theta_i(x_t - \overline{x})$, where θ_i is a function of $p^i_{ss'}$. The following corollary shows heterogeneous beliefs lead to larger turnover rates as the economy switches from booms to recessions.

Corollary 3. *Suppose investors agree on the unconditional mean of x_t, i.e. $p^i_{LH}/p^i_{HL} = \overline{p}_H/\overline{p}_L$ and that the following condition is satisfied: $p^*_{ss'} = \overline{p}_H = \frac{1}{2}$. Turnover as the economy switches from s to s' is given by*

$$\tau(X, H, L; \epsilon) = \frac{\zeta(s, s')}{2} \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| + \mathcal{O}(\epsilon^2), \tag{B.3}$$

where

$$\zeta(s, s') \equiv \begin{cases} \kappa_\omega + 1, & \text{if } s = H \text{ and } s' = L \\ |\kappa_\omega - 1|, & \text{if } s = L \text{ and } s' = H \\ 1, & \text{if } s = s' \end{cases}$$

The key message from Corollary 3 is that turnover increases in belief dispersion and, furthermore, that the effect is more pronounced during busts. Both predictions are in line with the evidence discussed in Section C.4. The assumption of rank-alternating beliefs is important to obtain this asymmetric effect. If investors have rank-preserving beliefs, where they are equally optimistic or pessimistic in both states, so $\tilde{\delta}^i_{s'H} = \tilde{\delta}^i_{sH}$ even for $s' \neq s$, then the change-in-beliefs effect will be equal to zero and we would not obtain a stronger response of turnover to disagreement during bad times. Therefore, rank-alternating beliefs are key to capturing the dynamics of stock market turnover.

B.2 Proof of Lemma 3

Proof. The portfolio share of stocks for a type-i investor is defined as $\omega_{i,t} \equiv \frac{Q_t S_{i,t}}{N_{i,t}(1-c_{i,t})}$, so $S_{i,t} = \frac{\omega_{i,t} N_{i,t}(1-c_{i,t})}{Q_t}$. Given that $1 - c_{i,t} = \beta$ and $Q_t = \beta P_t$, we obtain $\mu_{i,t} S_{i,t} = \frac{\omega_{i,t}\mu_i N_{i,t}}{P_t} = \omega_{i,t}\eta_{i,t}$. Shares traded by type-$i$ investors are given by $\mu_{i,t}|S_{i,t} - S_{i,t-1}| =$

$|\omega_{i,t}\eta_{i,t} - \omega_{i,t-1}\eta_{i,t-1}|$. Trading volume is then given by

$$\tau_t = \frac{1}{2}\sum_{i=1}^{I} |\omega_{i,t}\eta_{i,t} - \omega_{i,t-1}\eta_{i,t-1}|. \tag{B.4}$$

In recursive notation, we can write

$$\tau(X,s,s') = \frac{1}{2}\sum_{i=1}^{I} |\omega_i(X',s')\eta_i'(X,s,s') - \omega_i(X,s)\eta_i|, \tag{B.5}$$

where $X' = \chi(X,s,s')$ and $\eta_i'(X,s,s') = \eta_i \frac{p^i_{ss'}}{p_{ss'}(X)}$.

Solving for the portfolio share. Using the expression for the economy-wide SDF and Equation (A.24), we can write the portfolio share as follows

$$\omega_i(X,s) = p^i_{sH}\frac{R_b(X,s)}{|R^e_r(X,s,L)|} - p^i_{sL}\frac{R_b(X,s)}{R^e_r(X,s,H)}. \tag{B.6}$$

The return on the risky and riskless assets can be written as follows:

$$R^e_r(X,s,s') = \frac{x_s}{\beta}\frac{(x_{s'} - \mathcal{L}'(X,s))\mathcal{L}'(X,s)^{\frac{\alpha}{1-\alpha}}}{x_s\mathcal{L}^{\frac{\alpha}{1-\alpha}} - \alpha\mathcal{L}^{\frac{1}{1-\alpha}}}, \qquad R_b(X,s) = (1-\alpha)\frac{x_s}{\beta}\frac{\mathcal{L}'(X,s)^{\frac{1}{1-\alpha}}}{x_s\mathcal{L}^{\frac{\alpha}{1-\alpha}} - \alpha\mathcal{L}^{\frac{1}{1-\alpha}}}, \tag{B.7}$$

Combining the previous expressions, we obtain

$$\omega_i(X,s) = (1-\alpha)\left[p^i_{sH}\frac{\mathcal{L}'(X,s)}{\mathcal{L}'(X,s) - x_L} - p^i_{sL}\frac{\mathcal{L}'(X,s)}{x_H - \mathcal{L}'(X,s)}\right], \tag{B.8}$$

which is strictly decreasing in $\mathcal{L}'(X,s)$ and $\omega_i(X,s) > 1$ if and only if $p^i_{sH} > p_{sH}(X)$.

Turnover is then given by

$$\tau(X,s,s') = (1-\alpha)\sum_{i=1}^{I}\eta_i\left|\left(\frac{p^i_{s'H}\mathcal{L}'(X',s')}{\mathcal{L}'(X',s') - x_L} - \frac{p^i_{s'L}\mathcal{L}'(X',s')}{x_H - \mathcal{L}'(X',s')}\right)\frac{p^i_{ss'}}{p_{ss'}(X)} - \left(\frac{p^i_{sH}\mathcal{L}'(X,s)}{\mathcal{L}'(X,s) - x_L} - \frac{p^i_{sL}\mathcal{L}'(X,s)}{x_H - \mathcal{L}'(X,s)}\right)\right| \tag{B.9}$$

Perturbation. It is useful to parameterize the dispersion in beliefs as follows:

$$p^i_{ss'} = p^*_{ss'} + \epsilon\delta^i_{ss'}, \tag{B.10}$$

where $\delta^i_{sH} + \delta^i_{sL} = 0$. If $\epsilon = 0$, then there is no belief heterogeneity and $\tau(X,s,s') = 0$. We consider next how turnover depends on belief heterogeneity for small deviations of this benchmark, that is, for ϵ close to zero.

Notice that all equilibrium variables now depend on ϵ. For instance, the average probability of the high state can be written as

$$p_{sH}(X;\epsilon) = p^*_{sH} + \delta_{sH}(X)\epsilon + \mathcal{O}(\epsilon^2), \tag{B.11}$$

where $\delta_{sH}(X) \equiv \sum_{i=1} \eta_i \delta^i_{sH}$. Risk-neutral expectation of productivity growth is a function of $\mathcal{L}'(X,s;\epsilon) = f_s(p_{sH}(X))$, where $f_s(p)$ satisfies the condition

$$1 = (1-\alpha)\left[p\frac{f_s(p)}{f_s(p)-x_L} - (1-p)\frac{f_s(p)}{x_H - f_s(p)}\right] \Rightarrow f'_s(p) = \frac{\frac{f_s(p)}{f_s(p)-x_L} + \frac{f_s(p)}{x_H - f_s(p)}}{p\frac{x_L}{(f(p)-x_L)^2} + (1-p)\frac{x_H}{(x_H - f_s(p))^2}}. \tag{B.12}$$

Let $\mathcal{L}^*(X,s) \equiv \mathcal{L}'(X,s;0)$ denote the value of $\mathcal{L}'(X,s)$ when $\epsilon = 0$. In this case, we can drop the dependence on X and simply write $\mathcal{L}^*(s)$, as $\mathcal{L}'(X,s)$ would only depend on the state s. We can then expand $\mathcal{L}'(X,s;\epsilon)$ in ϵ to obtain:

$$\mathcal{L}'(X,s;\epsilon) = \mathcal{L}^*(s) + \tilde{\mathcal{L}}(X,s)\epsilon + \mathcal{O}(\epsilon^2), \tag{B.13}$$

where $\tilde{\mathcal{L}}(X,s) = f'(p^*_{sH}) \sum_{i=1}^{I} \eta_i \delta^i_{sH}$, where $f'(\cdot) > 0$.

We can then write the portfolio share of investor i as follows

$$\omega_i(X,s;\epsilon) = 1 + \left[\theta_{\omega,1}(s)\delta^i_{sH} - \theta_{\omega,2}(s)\delta_{sH}(X)\right]\epsilon + \mathcal{O}(\epsilon^2), \tag{B.14}$$

where $\theta_{\omega,1}(s) > 0$ and $\theta_{\omega,2}(s) > 0$

$$\theta_{\omega,1}(s) \equiv (1-\alpha)\left(\frac{\mathcal{L}^*(s)}{\mathcal{L}^*(s) - x_L} + \frac{\mathcal{L}^*(s)}{x_H - \mathcal{L}^*(s)}\right) \tag{B.15}$$

$$\theta_{\omega,2}(s) \equiv (1-\alpha)\left[\frac{p^*_{sH}x_L}{(\mathcal{L}^*(s) - x_L)^2} + \frac{p^*_{sL}x_H}{(x_H - \mathcal{L}^*(s))^2}\right] f'(p^*_{sH}). \tag{B.16}$$

Using the expression for $f'(\cdot)$, we obtain that $\theta_{\omega,1} = \theta_{\omega,2}$. We can then write $\omega_i(X,s;\epsilon)$ as follows:

$$\omega_i(X,s;\epsilon) = 1 + \theta_{\omega,1}(s)\left[\delta^i_{sH} - \delta_{sH}(X)\right]\epsilon + \mathcal{O}(\epsilon^2), \tag{B.17}$$

The evolution of wealth is given by

$$\eta'_i(X,s,s';\epsilon) = \eta_i + \eta_i \frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}}\epsilon + \mathcal{O}(\epsilon^2) \tag{B.18}$$

Let $p_H(X,s,s';\epsilon) = \sum_{i=1}^{I} \eta'_i(x,s,s';\epsilon) p^i_{s'H}$ denote the market-implied probability of the high state after a transition to state s', then

$$p_H(X, s, s'; \epsilon) = p^*_{s'H} + \delta_{s'H}(X)\epsilon + \mathcal{O}(\epsilon^2), \tag{B.19}$$

where $\delta_{s'H}(X) \equiv \sum_{i=1}^{I} \eta_i \frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}} p^*_{s'H} + \sum_{i=1}^{I} \eta_i \delta^i_{s'H} = \sum_{i=1}^{I} \eta_i \delta^i_{s'H}$.

The portfolio share next period is given by

$$\omega'_i(X, s, s'; \epsilon) = 1 + \theta_{\omega,1}(s') \left[\delta^i_{s'H} - \delta_{s'H}(X)\right] \epsilon + \mathcal{O}(\epsilon^2). \tag{B.20}$$

Investor i's net purchases of shares is given by

$$\Delta S_i(X, s, s'; \epsilon) = \eta_i \left[\frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}} + \theta_{\omega,1}(s') \left(\delta^i_{s'H} - \delta_{s'H}(X)\right)\right] \epsilon \\ - \theta_{\omega,1}(s)\eta_i \left[\delta^i_{sH} - \delta_{sH}(X)\right] \epsilon + \mathcal{O}(\epsilon^2) \tag{B.21}$$

For simplicity, suppose that investors believe productivity growth to be iid in the reference economy, that is, $p^*_{Ls'} = p^*_{Hs'}$. We can then write the

$$\Delta S_i(X, s, s'; \epsilon) = \left[\underbrace{\Delta\tilde{\omega}_i(X, s, s')\eta_i}_{\text{change-in-beliefs effect}} + \underbrace{\Delta\tilde{\eta}_i(X, s, s')}_{\text{rebalancing effect}}\right] \epsilon + \mathcal{O}(\epsilon^2), \tag{B.22}$$

where

$$\Delta\tilde{\omega}_i(X, s, s') \equiv \theta_{\omega,1} \left[\left(\delta^i_{s'H} - \delta_{s'H}(X, s)\right) - \left(\delta^i_{sH} - \delta_{sH}(X)\right)\right] \tag{B.23}$$

$$\Delta\tilde{\eta}_i(X, s, s') \equiv \eta_i \frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}}. \tag{B.24}$$

□

B.3 Proof of Proposition 5 and Corollary 3

Proof. Turnover is given by

$$\tau(X, s, s'; \epsilon) = \frac{1}{2} \sum_{i=1}^{I} \eta_i \left| \frac{\tilde{\delta}^i_{ss'}}{p^*_{ss'}} + \kappa_\omega \left(\tilde{\delta}^i_{s'H} - \tilde{\delta}^i_{sH}\right) \right| \epsilon + \mathcal{O}(\epsilon^2), \tag{B.25}$$

where $\tilde{\delta}^i_{ss'} = \delta^i_{ss'} - \delta_{ss'}(X)$.

Suppose $s = s'$, then

$$\tau(X, s, s'; \epsilon) = \frac{1}{2} \sum_{i=1}^{I} \eta_i \frac{\left|\tilde{\delta}^i_{ss'}(X)\right|}{p^*_{ss'}} \epsilon + \mathcal{O}(\epsilon^2) \tag{B.26}$$

$$= \frac{1}{2} \left[\sum_{i=1}^{I} \eta_i \frac{\tilde{\delta}^i_{ss'}(X)}{p^*_{ss'}} \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) \geq 0} - \sum_{i=1}^{I} \eta_i \frac{\tilde{\delta}^i_{ss'}(X)}{p^*_{ss'}} \mathbf{1}_{\tilde{\delta}^i_{ss'} < 0} \right] \epsilon + \mathcal{O}(\epsilon^2) \tag{B.27}$$

$$= \frac{1}{2} \left[\eta_B \frac{\tilde{\delta}^B_{ss'}(X)}{p^*_{ss'}} + \eta_S \frac{|\tilde{\delta}^S_{ss'}(X)|}{p^*_{ss'}} \right] \epsilon + \mathcal{O}(\epsilon^2). \tag{B.28}$$

where

$$\eta_B \equiv \sum_{i=1}^{I} \eta_i \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) \geq 0}, \qquad \tilde{\delta}^B_{ss'}(X) \equiv \frac{1}{\eta_B} \sum_{i=1}^{I} \eta_i \tilde{\delta}^i_{ss'}(X) \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) \geq 0}, \tag{B.29}$$

$$\eta_S \equiv \sum_{i=1}^{I} \eta_i \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) < 0}, \qquad \tilde{\delta}^S_{ss'}(X) \equiv \frac{1}{\eta_S} \sum_{i=1}^{I} \eta_i \tilde{\delta}^i_{ss'}(X) \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) < 0}. \tag{B.30}$$

We can write turnover in this case as follows

$$\tau(X, s, s'; \epsilon) = \eta_B \eta_S \frac{\delta^B_{ss'}(X) + |\delta^S_{ss'}(X)|}{p^*_{ss'}} \epsilon + \mathcal{O}(\epsilon^2), \tag{B.31}$$

using the fact that $\delta_{ss'}(X) = \eta_B \delta^B_{ss'}(X) + \eta_S \delta^S_{ss'}(X)$.

Heterogeneous persistence. We consider next the special case where investors agree about the unconditional mean of x, but they disagree about the persistence of the aggregate productivity growth.

The stationary distribution of beliefs for investor i is given by

$$p^i_L = \frac{p^i_{HL}}{p^i_{LH} + p^i_{HL}}. \tag{B.32}$$

We assume that p^i_L is equalized across investors, so all investors agree about the unconditional mean of x_t. Note this implies that the likelihood ratio p^i_{LH}/p^i_{HL} is equalized across investors. The unconditional mean is given by

$$\overline{x} = \frac{p^i_{HL}}{p^i_{LH} + p^i_{HL}} x_L + \frac{p^i_{LH}}{p^i_{LH} + p^i_{HL}} x_H. \tag{B.33}$$

The expected value of x_{t+1} relative to the mean $\overline{x}$ conditional on $x_t = x_L$ is given by

$$\mathbb{E}_i[x_{t+1} - \overline{x} | x_t = x_L] = p^i_{LL}(x_L - \overline{x}) + p^i_{LH}(x_H - \overline{x}) \tag{B.34}$$

$$= \left[1 + p^i_{LH} \frac{x_H - x_L}{x_L - \overline{x}}\right](x_L - \overline{x}) \tag{B.35}$$

$$= \left[1 - (p^i_{LH} + p^i_{HL})\right](x_L - \overline{x}), \tag{B.36}$$

using the fact that $\overline{x} - x_L = \frac{p^i_{LH}}{p^i_{LH} + p^i_{HL}}(x_H - x_L)$

We obtain a similar expression conditioning on $x_t = x_H$ instead:

$$\mathbb{E}_i[x_{t+1} - \overline{x} | x_t = x_H] = p^i_{HL}(x_L - \overline{x}) + p^i_{HH}(x_H - \overline{x}) \tag{B.37}$$

$$= \left[1 - p^i_{HL} \frac{x_H - x_L}{x_H - \overline{x}}\right](x_H - \overline{x}) \tag{B.38}$$

$$= \left[1 - (p^i_{LH} + p^i_{HL})\right](x_H - \overline{x}), \tag{B.39}$$

using the fact that $x_H - \overline{x} = \frac{p^i_{HL}}{p^i_{LH} + p^i_{HL}}(x_H - x_L)$.

Let $\hat{x}_t = x_t - \overline{x}$, we can then write

$$\mathbb{E}_i[\hat{x}_{t+1} | \hat{x}_t] = \theta_i \hat{x}_t, \tag{B.40}$$

where $\theta_i \equiv 1 - (p^i_{LH} + p^i_{HL}) = p^i_{HH} - p^i_{LH}$.

Given that investors agree about the unconditional mean of x, we are able to pin down beliefs as a function of θ_i:

$$p^i_{LH} = \overline{p}_H(1 - \theta_i), \qquad p^i_{HH} = \overline{p}_H + \overline{p}_L \theta_i. \tag{B.41}$$

Corollary. Under the assumption investors agree about the unconditional mean of x_t, we have that

$$p^i_{LH} - p_{LH}(X) = -\overline{p}_H(\theta_i - \theta(X)), \qquad p^i_{HH} - p_{HH}(X) = \overline{p}_L(\theta_i - \theta(X)), \tag{B.42}$$

where $\overline{\theta}(X) \equiv \sum_{i=1}^{I} \eta_i \theta_i$.

Notice that we have that $\tilde{\delta}^i_{ss'}(X)\epsilon = p^i_{ss'} - p_{ss'}(X)$, which gives us

$$\tilde{\delta}^i_{LH}(X)\epsilon = -\overline{p}_H(\theta_i - \theta(X)), \qquad \tilde{\delta}^i_{HH}(X)\epsilon = (1 - \overline{p}_H)(\theta_i - \theta(X)). \tag{B.43}$$

We can then write turnover in the case $s = L$ and $s' = H$ as follows:

$$\tau(X, L, H; \epsilon) = \frac{1}{2}\left|\kappa_\omega - \frac{\overline{p}_H}{p^*_H}\right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| + \mathcal{O}(\epsilon^2). \tag{B.44}$$

Consider now the case $s = H$ and $s' = L$:

$$\tau(X, H, L; \epsilon) = \frac{1}{2}\left|\kappa_\omega + \frac{\overline{p}_L}{p^*_L}\right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| + \mathcal{O}(\epsilon^2), \tag{B.45}$$

Suppose now that $s = s' = L$, then

$$\tau(X, H, H; \epsilon) = \frac{1}{2}\left|\frac{\overline{p}_L}{p^*_H}\right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)|\epsilon + \mathcal{O}(\epsilon^2) \tag{B.46}$$

$$\tau(X, L, L; \epsilon) = \frac{1}{2}\left|\frac{\overline{p}_H}{p^*_L}\right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)|\epsilon + \mathcal{O}(\epsilon^2). \tag{B.47}$$

□

C Estimating the Heterogeneity in Beliefs

C.1 The process for realized and expected earnings

Let $i \in \mathcal{I}$ denote a firm-analyst pair. We index both firm-level outcomes and the expectations of the analyst covering this firm by i. We denote (realized) earnings for firm i at period t by $e_{i,t}$ and the first-difference of realized earnings by $\Delta e_{i,t} = e_{i,t} - e_{i,t-1}$.[28] We denote aggregate earnings by e_t and the first-difference of aggregate earnings by Δe_t. Realized earnings follows the process:

$$\Delta e_{i,t} = \beta_i \Delta e_t + u_{i,t}, \tag{C.1}$$

where $u_{i,t} = \rho_i u_{i,t-1} + \epsilon_{i,t}$ and $\epsilon_{i,t} \sim \mathcal{N}(0, \sigma^2_\epsilon)$. The error term $\epsilon_{i,t}$ is assumed to be i.i.d. and independent of Δe_t. We assume that $\Delta e_{i,t}$ and Δe_t have already been de-meaned, so we can omit the intercept. We also assume that $\Delta e_{i,t}$ and Δe_t have been normalized to have unit variance.

[28] As $e_{i,t}$ can potentially be negative, we work with first differences instead of proportional differences, $\frac{\Delta e_{i,t}}{e_{i,t}}$, or log-differences, $\Delta \log(e_{i,t})$. By focusing on first differences, we do not have to drop firms which experience negative earnings, which is a significant fraction of our sample.

Given the formulation above, individual earnings depend on aggregate shocks, i.e. shocks that affect aggregate earnings, as well as idiosyncratic shocks, as captured by $u_{i,t}$. The parameters ρ_i controls the persistence of idiosyncratic shocks. Hence, firms are allowed to be heterogeneous on their exposure to the aggregate shock as well as the persistence of idiosyncratic shocks.

We assume that analysts understand that individual earnings follows the process (C.1), but they potentially disagree on the process followed by aggregated earnings. In particular, we assume that analyst i believe (in a dogmatic fashion) that Δe_t follows the following process:

$$\Delta e_t = \theta_i \Delta e_{t-1} + \nu_{i,t}, \tag{C.2}$$

where $\nu_{i,t}$ is an i.i.d. process given by $\nu_{i,t} \sim \mathcal{N}(0, \sigma_\nu^2)$. We assume that analysts agree on the unconditional mean for Δe_t, which we normalize to zero. This allow us to focus only on disagreement about the persistence of shocks to aggregate earnings.

The expected change in aggregate earnings using the subjective beliefs of analyst i is given by

$$\mathbb{E}_{i,t}[\Delta e_{t+1}] = \theta_i \Delta e_t, \tag{C.3}$$

where $\mathbb{E}_{i,t}[\cdot]$ denote the conditional expectation at t according to the subjective beliefs of analyst i.

We assume that Δe_t is perfectly observed by investors at time t, so differences in beliefs are controlled by θ_i. A relatively high value for θ_i implies that analyst i is more optimistic about aggregate earnings after a positive shock and more pessimistic after a negative shock, capturing a form of belief extrapolation.

Notice that expectations of changes in *individual* earnings depend on the degree of persistence of shocks to *aggregate* earnings θ_i:

$$\mathbb{E}_{i,t}[\Delta e_{i,t+1}] = \beta_i \theta_i \Delta e_t + \rho_i u_{i,t}. \tag{C.4}$$

Equation (C.4) shows that we can infer properties of the process for subjective beliefs on *aggregate* earnings using information on subjective beliefs about *individual* earnings. This is important as beliefs on aggregate earnings are not directly available.

C.2 Estimation procedure

We show next how to estimate $(\beta_i, \rho_i, \theta_i)$ in two stages. First, we estimate the parameters in Equation (C.1). In a second stage, we obtain the distribution of θ_i, using Equation (C.4) and the parameters estimated in the first stage.

First stage. Consider first Equation (C.1). We can rewrite the process for $\Delta e_{i,t}$ as follows:

$$\Delta e_{i,t} = \beta_i \Delta e_t + \rho_i \left(\Delta e_{i,t-1} - \beta_i \Delta e_{t-1} \right) + \epsilon_{i,t}, \tag{C.5}$$

where we used the fact that $u_{i,t} = \Delta e_{i,t} - \beta_i \Delta e_t$.

To ensure that $-1 < \rho_i < 1$, we consider the following change of variables. Assume that ρ_i is given by the a non-linear transformation of the parameter $\tilde{\rho}_i \in \mathbb{R}$: $\rho_i = -1 + 2\frac{\exp(\tilde{\rho}_i)}{1+\exp(\tilde{\rho}_i)} \in (-1, 1)$. The parameters $(\beta_i, \tilde{\rho}_i)$ can in principle be estimated using, for instance, non-linear least squares for each company i. We proceed instead by estimating the parameters simultaneously for all i using Bayesian methods. The Bayesian approach is useful as it allow us to regularize the individual estimates and avoid overfitting, which can be a concern in settings where the length of the time series is not particularly long.[29]

Formally, we consider the following multi-level priors:

$$\beta_i \sim \mathcal{N}(\overline{\beta}, \sigma_\beta^2), \qquad \tilde{\rho}_i \sim \mathcal{N}(\overline{\rho}, \sigma_\rho^2), \tag{C.6}$$

The coefficients $(\overline{\beta}, \overline{\rho})$ and $(\sigma_\beta, \sigma_\rho)$ are referred to as *hyperparameters* and they have their own priors, which are given by

$$\overline{\beta} \sim \mathcal{N}(0, 1.50^2), \qquad \overline{\rho} \sim \mathcal{N}(0, 0.50^2), \tag{C.7}$$

and the standard-deviation for each parameter is assumed to follow a Half Student-t distribution with 3 degrees of freedom, a standard value for this class of models. These priors are set to be wide enough to capture the range of plausible values for the parameters.

The multi-level structure allow us to obtain a form of adaptive regularization. If (say) σ_β is very large, then the prior on β_i is not very informative, and this would be analogous to estimate β_i independently for each i. If $\sigma_\beta \approx 0$, then we have effectively a pooling estimator, where β_i will be the same for all i. For intermediate values of σ_β, the parameters are allowed to vary across units, but they are partially shrunk towards the population mean. The shrinkage of the parameters limits the effect of noise or measurement error, as the model is essentially skeptical of extreme values. Because σ_β is also an estimated parameter, the extent to which estimates are regularized is directly informed by the data.[30]

[29] This procedure is analogous to a ridge regression, where the estimates are regularized using a L2 penalty (see e.g. Hastie, Tibshirani, Friedman and Friedman, 2009). For a discussion of how regularized regressions can be reinterpreted as a Bayesian procedure, see e.g. Nagel (2021).

[30] For more details on how multi-level models provide a form of adaptive regularization, see e.g. the discussion in McElreath (2020).

Second stage. Consider next Equation (C.4), which relates subjective beliefs about individual earnings to realized aggregate and individual earnings. To capture the fact that (subjective) expectations are potentially measured with error, we assume that only a noisy version of the analyst's expectation is observed, which is given by $\hat{\mathbb{E}}_{i,t}[\Delta e_{i,t+1}] = \mathbb{E}_{i,t}[\Delta e_{i,t+1}] + \tilde{w}_{i,t}$. The measurement error $\tilde{w}_{i,t}$ is assumed to be a mean-zero normally distributed i.i.d. process with variance given by σ_w^2. Combining this measurement equation with Equation (C.4) and isolating the terms estimated in the first stage, we obtain the following estimating equation:

$$z_{i,t} = \alpha_i + \theta_i x_{i,t} + w_{i,t}, \tag{C.8}$$

where $z_{i,t} \equiv \hat{\mathbb{E}}_{i,t}[\Delta e_{i,t+1}] - \rho_i u_{i,t}$ and $x_{i,t} \equiv \beta_i \Delta e_t$. Notice that $z_{i,t}$ and $x_{i,t}$ are known at this stage, so it only remains to estimate $\theta_{i,t}$.

As before, we use a Bayesian multi-level model to adaptively regularize our estimates. We also consider the transformation $\theta_i = -1 + 2\frac{\exp(\tilde{\theta}_i)}{1+\exp(\tilde{\theta}_i)}$, where $\tilde{\theta}_i \in \mathbb{R}$, such that we can ensure that $\theta_i \in (-1, 1)$. We assume the following prior for $\tilde{\theta}_{i,t}$:

$$\theta_{i,t} \sim \mathcal{N}(\overline{\theta}, \sigma_\theta^2), \tag{C.9}$$

where $\overline{\theta} \sim \mathcal{N}(0, 0.5^2)$ and σ_θ follows a half Student-t distribution with 3 degrees of freedom.

C.3 Data and estimation results

Data. We use data from I/B/E/S on analysts expectations about firms' future earnings. For firms with coverage of more than one analyst, we use the consensus expectation for that firm. We drop firms with missing values for realized or expected earnings in more than 20% of the sample. We ended up with 579 firms covering the time period from March 1977 until December 2020, with a total of 44,267 company-quarter pairs.

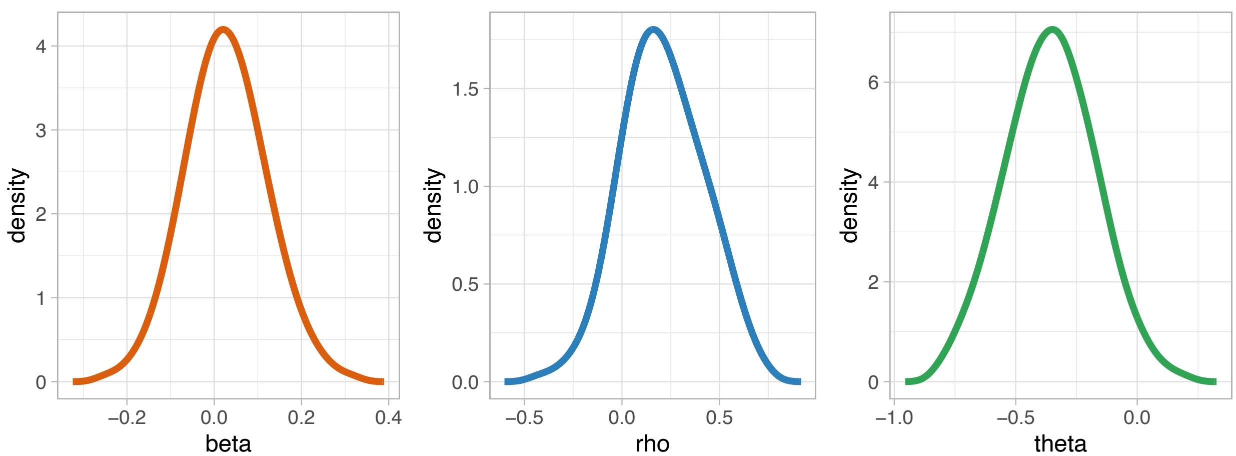
Model fitting and results. We sample the model using an extension of Hamiltonian Monte Carlo, the no-U-turn sampler (NUTS) by Hoffman, Gelman et al. (2014), as implemented in R Stan. Table 5 reports the posterior mean and 95% credible intervals for the cross-sectional mean and dispersion of parameters $(\beta_i, \rho_i, \theta_i)$. Because we have standardized all the variables, the parameter β_i captures the correlation between individual and aggregate earnings. The correlation is close to zero reflecting the fact that typically most of the variation in a company's earnings reflect idiosyncratic shocks. However, there

Table 5: Cross-sectional mean and dispersion of parameters

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat
$\overline{\mathbb{E}}[\beta_i]$	0.03	0.01	0.01	0.04	1.00
$\overline{\mathbb{E}}[\rho_i]$	0.45	0.02	0.41	0.50	1.00
$\overline{\mathbb{E}}[\theta_i]$	-0.48	0.12	-0.72	-0.24	1.00
$\overline{\sigma}[\beta_i]$	0.09	0.01	0.08	0.10	1.00
$\overline{\sigma}[\rho_i]$	0.47	0.02	0.43	0.51	1.00
$\overline{\sigma}[\theta_i]$	0.19	0.13	0.01	0.49	1.00

Note: Posterior mean and credible intervals (CI) for the cross-sectional mean, $\overline{\mathbb{E}}[x_i]$, and cross-sectional standard-deviation, $\overline{\sigma}[x_i]$, for parameters $x \in \{\beta, \rho, \theta\}$. Rhat is an indicator of the convergence of the chains during sampling. Rhat = 1 indicates convergence.

Figure 8: Kernel estimate of cross-sectional distribution of the different parameters



Note: Posterior mean of the kernel density for the cross-section of θ_i (left panel), ρ_i (middle panel), and θ_i (right panel).

is substantial heterogeneity in this parameter, with the cross-sectional dispersion being three times the average β_i. This can be seen in the left panel of Figure 8, which shows the posterior mean of the kernel density for β_i, where β_i ranges from -0.3 to 0.4. The average autocorrelation coefficient ρ_i is positive, but it is also very dispersed across firms, as shown in the middle panel of Figure 8. Finally, we have that θ_i is on average negative, which is consistent with the fact that Δe_t has a negative autocorrelation. However, the average subjective coefficient of autocorrelation is more negative than its objective counterpart, as $\mathbb{E}[\theta_i] = -0.48$ and we obtain a coefficient of autocorrelation of -0.28 for Δe_t using aggregate data. As before, we observe substantial heterogeneity in θ_i, as shown in the right panel of Figure 8.

C.4 Belief disagreement

We consider next a measure of belief disagreement. Notice that the expectation of analyst of aggregate earnings growth is given by $\mathbb{E}_i[\Delta e_{t+1}] = \theta_i \Delta e_t$. This motivates our definition of a *disagreement index* DI_t, which corresponds to the cross-sectional dispersion in beliefs about aggregate earnings growth:

$$DI_t = \underbrace{\overline{\sigma}[\theta_i] \times |\Delta e_t|}_{\overline{\sigma}[\mathbb{E}_i[\Delta e_{t+1}]]}. \tag{C.10}$$

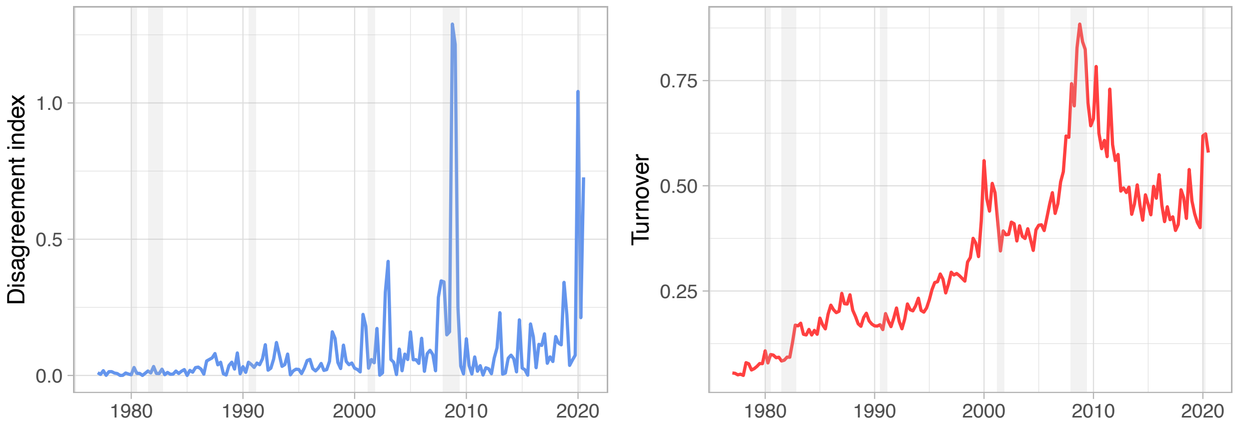
The disagreement index has two components. First, the cross-sectional dispersion in the parameter θ_i. If all analysts agree on the persistence of aggregate earnings growth, such that $\overline{\sigma}[\theta_i] = 0$, then the disagreement index would be equal to zero. Second, the absolute value of aggregate earnings growth, $|\Delta e_t|$. Given that Δe_t has been already demeaned, $|\Delta e_t|$ captures the distance of aggregate earnings growth to its mean. If aggregate earnings growth is already at its average value, $|\Delta e_t| = 0$, then disagreement on how Δe_t reverts to its plays no role in determining expectations. Therefore, the level of disagreement in the economy depends on the interaction between dispersion in beliefs and deviations of aggregate earnings growth from its mean.

The left panel of Figure 9 shows the time series of the disagreement index. The disagreement index is typically low during normal times, and it significantly spikes in periods of crises, where aggregate earnings growth deviates substantially from its average value.

Turnover. One important implication of theories with heterogeneous beliefs is that the level of disagreement is related to the amount of trading in the economy. To test this implication, we consider next a measure of trading activity, the (value-weighted) stock market turnover.[31] We measure the stock turnover—shares traded divided by shares outstanding—for individual securities on the New York and American Stock Exchanges from January 1977 to December 2021. We measure turnover at the quarterly frequency and compute an aggregate turnover measure using a value-weighted average (similar results are obtained by using an equal-weight measure). The right panel of Figure 9 shows the time series of turnover. We can observe that the turnover level changed significantly over time and that turnover has an important cyclical component.

[31] For a discussion of turnover as a measure of trading volume and its connection with standard portfolio theory, see Lo and Wang (2010).

Figure 9: Time series of the disagreement index and stock market turnover



Note: Left panel shows the time series of the disagreement index and the right panel shows the time-series of stock market turnover. The smooth line in the right panel is the HP-filter trend of turnover. The vertical bars represent NBER recessions.

Belief disagreement and turnover. We consider next the relationship between belief disagreement and turnover. Table 6 shows the result of a time-series regression of turnover on the disagreement index. As shown in Figure 9, the disagreement index series has a few outliers, in particular, during crisis periods. To ensure that the relationship between turnover and disagreement is not driven only by these extreme periods, we consider a sample where we exclude observations where the disagreement index is below the 2.5% percentile or above the 97.5% percentile. Column (1) shows that there is a strong statistically significant association between DI and turnover, where we compute Newey-West standard-errors with four lags. If the disagreement index goes from its 25% percentile to its 75% percentile, turnover increases by 8.0 percentage points, an increase of almost 30%. Column (2) tests whether this relationship is nonlinear by introducing a quadratic term, again in the example where we exclude outliers. We find that the quadratic term is not significant, consistent with a linear relationship. This can be verified visually in Figure 10, which shows the scatterplot of turnover and the disagreement index for this sample. Column (3) shows the regression of turnover on DI and DI^2 for the full sample. We find that the quadratic term is now statistically significant, indicating the necessity of considering a nonlinear relationship to capture the effect of the extreme crisis-level disagreement. The magnitude of the marginal effect of changing DI is similar to the linear case for large of values for the disagreement index. Therefore, we conclude that belief disagreement is strongly associated with stock market turnover.

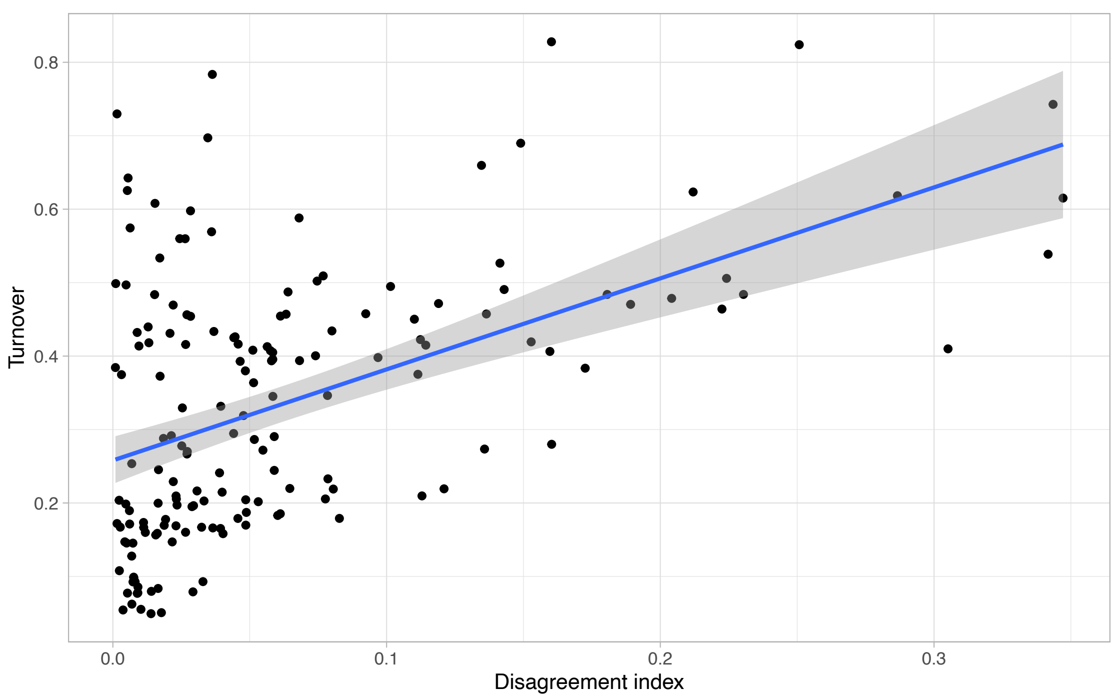
Table 6: Regression of turnover on disagreement index

Dependent Variable:		*turnover*	
Model:	(1)	(2)	(3)
Variables			
(Intercept)	0.2580***	0.2420***	0.2549***
	(0.03373)	(0.04375)	(0.0369)
DI	1.239***	1.798**	1.260***
	(0.2277)	(0.6277)	(0.2898)
DI^2		-2.068	-0.6879**
		(1.6920)	(0.2094)
Fit statistics			
Observations	165	165	175
R^2	0.24084	0.24786	0.30386
Adjusted R^2	0.23618	0.23857	0.29576

Newey-West standard-errors in parentheses (4 lags)
*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

Note: Columns (1) and (2) .

Figure 10: Scatterplot of the disagreement index and stock market turnover



Note: Scatterplot of disagreement index and turnover for a sample without outliers.