Online Appendix for

Using Discrepancies to Correct for False Matches in Historical Linked Data

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A Additional tables and figures

(a) False Match Rate (b) Migration Probability (c) Standard Error

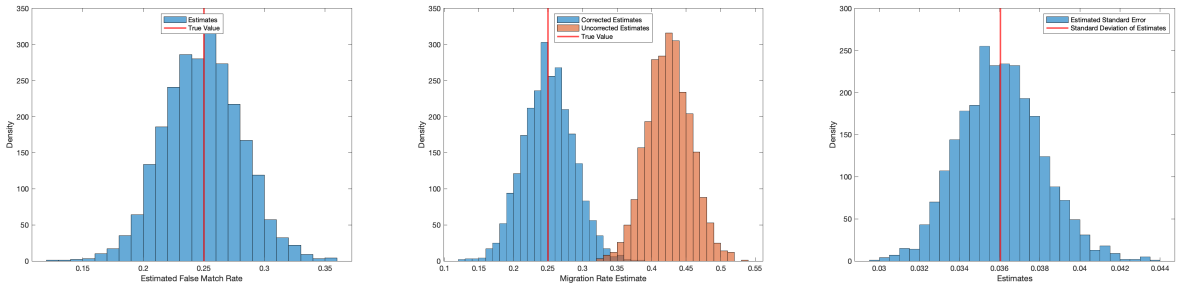

Figure A.1: Results of simulation of the migration model with unconfounded matching

Note: These figures present results of simulating the model described in section 3.1 with unconfounded matching for a false match probability of $\pi = 0.25$. The true migration rate is 0.25 and the probability of a link is 0.25. Panel (a) presents estimates of the false match rate π and the true value. Panel (b) presents the corrected and uncorrected estimates of the migration rate, as well as the true value. Panel (c) presents the estimated standard error of the migration rate, as well as the standard deviation of the estimates as the "true" value.

(a) False Match Rate (b) Migration Probability (c) Standard Error

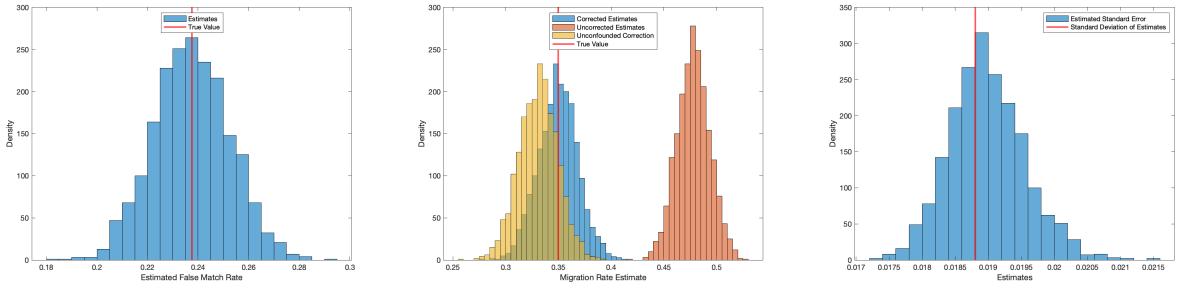

Figure A.2: Results of simulation of the migration model with confounded matching

Note: These figures present results of simulating the model described in section 3.1 with confounded matching. Panel (a) presents estimates of the false match rate π and the true value. Panel (b) presents the corrected and uncorrected estimates of the migration rate, as well as the true value. Panel (c) presents the estimated standard error of the migration rate, as well as the standard deviation of the estimates as the "true" value.

(a) β_0 (b) β_1

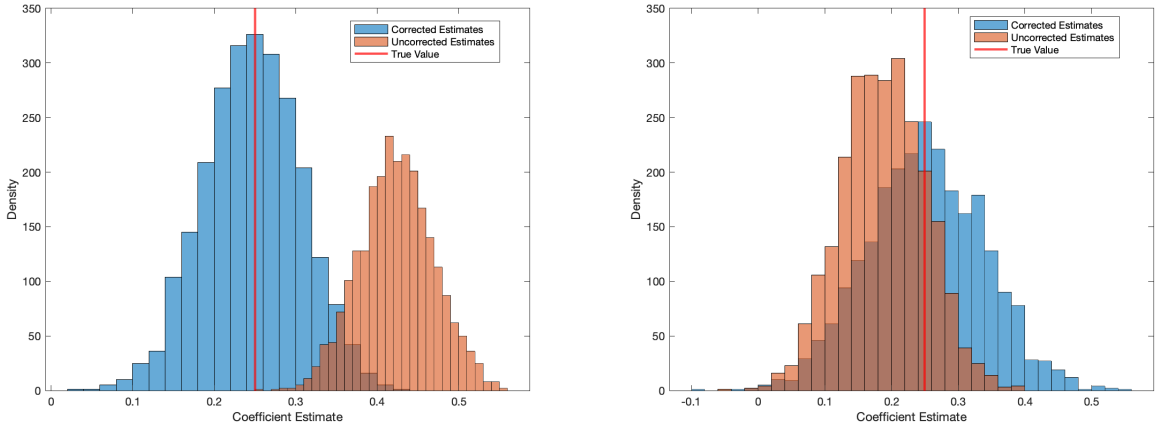

Figure A.3: Results of simulation of the migration regression model with unconfounded matching

Note: These figures present results of simulating the model described in section 3.2 with unconfounded matching.

(a) β_0 (b) β_1

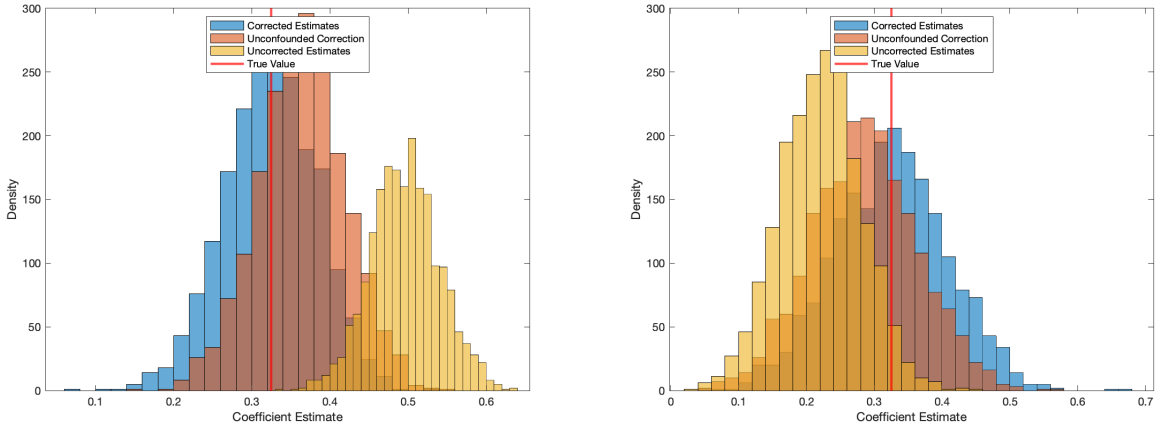

Figure A.4: Results of simulation of the migration regression model with confounded matching

Note: These figures present results of simulating the model described in section 3.2 with confounded matching.

(a) β_0 (b) β_1

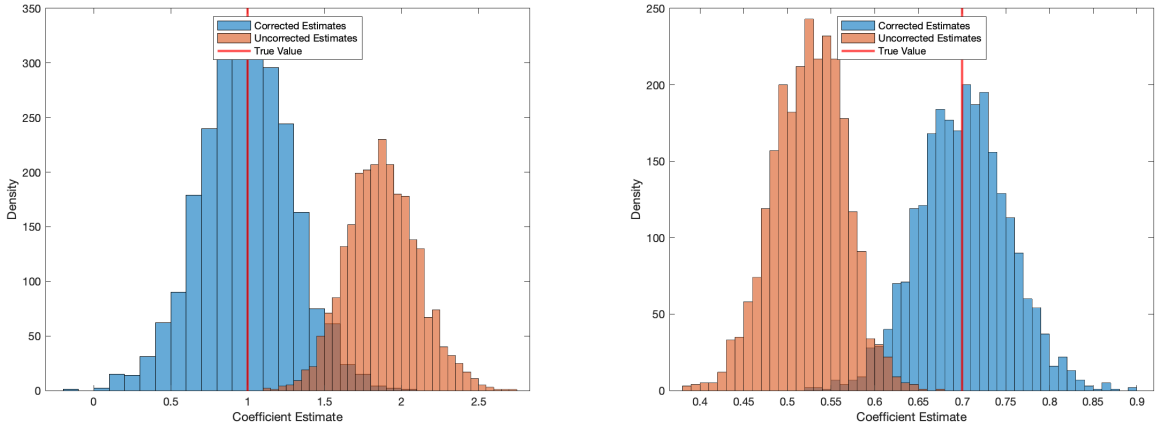

Figure A.5: Results of simulation of the regression model with unconfounded matching

Note: These figures present results of simulating the model described in section 3.3 with unconfounded matching and a false match probability of $\pi = 0.25$.

(a) β_0 (b) β_1 (c) β_2

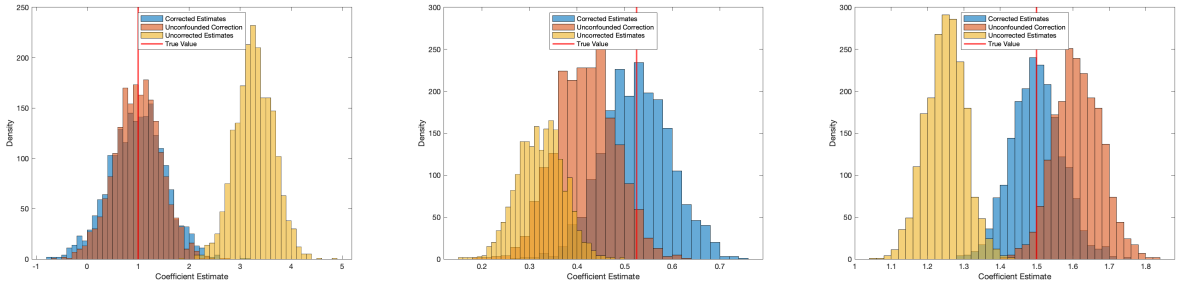

Figure A.6: Results of simulation of the regression model with confounded matching

Note: These figures present results of simulating the model described in section 3.3 with confounded matching.

B Mathematical appendix

B.1 Asymptotic Properties

This appendix section theoretically justifies that (2.3.3) and (2.3.4) consistently estimate the respective components on the right-hand side of (2.3.2). The same argument applies to establishing the consistency of (2.3.10) and (2.3.11) for (2.3.9).

We start by fixing some notations to be used throughout this appendix section. Suppose that there is a set $\mathcal{K}$ of N individuals in common between census $t = 1$ and census $t = 2$. Thus, we have $N = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} R_{ij}$. This leaves the set $\mathcal{K}_1$ of $N_1 - N$ individual appearing only in census $t = 1$ and the set $\mathcal{K}_2$ of $N_2 - N$ individuals appearing only in census $t = 2$. In total, there is a set $\mathcal{K} \bigcup \mathcal{K}_1 \bigcup \mathcal{K}_2$ of $N + (N_1 - N) + (N_2 - N)$ unique individuals in the two census. We write $i = i(k)$ to denote the index of individual $k \in \mathcal{K} \bigcup \mathcal{K}_1$ in census $t = 1$ and write $j = j(k)$ to denote the index of individual $k \in \mathcal{K} \bigcup \mathcal{K}_2$ in census $t = 2$.

Assumption 1. $(R_{i(k)j(\kappa)}, D_{i(k)j(\kappa)}, Y_{i(k)j(\kappa)}, Z_{i(k)j(\kappa)}, W_{i(k)j(\kappa)}) = f(\alpha_k, \alpha_\kappa, \varepsilon_{k\kappa})$ *where the sequences,* $\{\alpha_k\}_{k=1}^{\infty}$ *and* $\{\{\varepsilon_{k\kappa}\}_{k=1}^{\infty}\}_{\kappa=1}^{\infty}$*, of random vectors are i.i.d. and independent from each other.*

This assumption states that R_{ij}, D_{ij}, Y_{ij}, Z_{ij} and W_{ij} for $i = i(k)$ and $j = j(\kappa)$ are generated by individual attributes, α_k and α_κ, possibly with idiosyncratic shocks $\varepsilon_{k\kappa}$ which may reflect data processing error for example. Some coordinates of α_k and $\varepsilon_{k\kappa}$ may be observed by a researcher, but they need not be observable. For example, one can suppose that α_k contains a coordinate storing the true identity α_{0k} of individual k, which generates $R_{i(k)j(\kappa)} = f_1(\alpha_k, \alpha_\kappa, \varepsilon_{k\kappa}) = \mathbb{1}\{\alpha_{0k} = \alpha_{0\kappa}\}$ for all κ, while a researcher does not observe such a latent identity α_{0k}. Under Assumption 1, we have independence across the individuals $\{\alpha_k\}_{k=1}^{\infty}$. However, it does *not* imply the independence across the $N_1 \times N_2$ units in the array $\{(R_{ij}, D_{ij}, Y_{ij}, Z_{ij}, W_{ij}) : 1 \leq i \leq N_1, 1 \leq j \leq N_2\}$, since each individual $k \in \mathcal{K}$ in common between the two censuses show up repeatedly in this array $N_1 + N_2 - 1$ times for example. Hence, we cannot directly apply the standard law of large numbers to a double sum of the form $N_1^{-1} N_2^{-1} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \xi_{ij}$ where $\xi_{ij} = g(R_{ij}, D_{ij}, Y_{ij}, Z_{ij}, W_{ij})$. However, by appropriately decomposing such a double sum into the sums of i.i.d. individuals $k \in \mathcal{K} \bigcup \mathcal{K}_1 \bigcup \mathcal{K}_2$, we can obtain the starndard asymptotic properties. The following lemma uses this idea.

Lemma 1. *Let* $\xi_{ij} = g(R_{ij}, D_{ij}, Y_{ij}, Z_{ij}, W_{ij})$ *such that* $E[|\xi_{i(k)j(\kappa)}|] < \infty$ *for all* k*,* κ *and* $E[\xi_{i(k)j(\kappa)}] = 0$ *for all* k*,* κ *with* $k \neq \kappa$*. If Assumption 1 holds, then* $N_1^{-1} N_2^{-1} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \xi_{ij} \xrightarrow{p} 0$ *as* $N \to \infty$*.*

Proof. Decompose the double sum as

$$\begin{aligned}\frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \xi_{ij} =& \frac{N}{N_1 N_2} \frac{1}{N} \sum_{k \in \mathcal{K}} A_{1k} + \\ & \frac{N(N_2 - 1)}{N_1 N_2} \frac{1}{N} \sum_{k \in \mathcal{K}} A_{2k} + \frac{N(N_2 - 1)}{N_1 N_2} \frac{1}{N} \sum_{k \in \mathcal{K}} A_{3k} + \\ & \frac{1}{N_1} \sum_{k \in \mathcal{K}_1} A_{4k} + \frac{1}{N_1} \sum_{k \in \mathcal{K}_1} A_{5k}, \end{aligned} \tag{B.1}$$

where

$$\begin{aligned}
A_{1k} =& \xi_{i(k)j(k)}, \\
A_{2k} =& E[\xi_{i(k)j(\kappa)}|\alpha_{i(k)}] \qquad (k \neq \kappa), \\
A_{3k} =& \frac{1}{N_2 - 1} \sum_{\substack{\kappa \in \mathcal{K}_2 \\ \kappa \neq k}} (\xi_{i(k)j(\kappa)} - E[\xi_{i(k)j(\kappa)}|\alpha_{i(k)}]), \\
A_{4k} =& E[\xi_{i(k)j(\kappa)}|\alpha_{i(k)}] \qquad (k \neq \kappa), \qquad \text{and} \\
A_{5k} =& \frac{1}{N_2} \sum_{\kappa \in \mathcal{K}_2} (\xi_{i(k)j(\kappa)} - E[\xi_{i(k)j(\kappa)}|\alpha_{i(k)}]).
\end{aligned}$$

By the law of iterated expectations, we have $E[A_{2k}] = E[A_{3k}] = E[A_{4k}] = E[A_{5k}] = 0$. Note also that the assumption of the lemma directly gives $E[|A_{1k}|] = E[|\xi_{i(k)j(k)}|] < \infty$. We also have

$$\begin{aligned}
E[|A_{2k}|] =& E[|E[\xi_{i(k)j(\kappa)}|\alpha_{i(k)}]|] \leq E[|\xi_{i(k)j(\kappa)}|] < \infty \qquad \text{and} \\
E[|A_{3k}|] \leq& E[|\xi_{i(k)j(\kappa)}|] + E[|E[\xi_{i(k)j(\kappa)}|\alpha_{i(k)}]|] \leq 2E[|\xi_{i(k)j(\kappa)}|] < \infty,
\end{aligned}$$

where the second inequality in each line follows from Rao-Blackwell inequality. Similar arguments show $E[|A_{4k}|] < \infty$ and $E[|A_{5k}|] < \infty$. Note that $\{A_{1k}\}_{k \in \mathcal{K}}$, $\{A_{2k}\}_{k \in \mathcal{K}}$, $\{A_{3k}\}_{k \in \mathcal{K}}$, $\{A_{4k}\}_{k \in \mathcal{K}_1}$, $\{A_{5k}\}_{k \in \mathcal{K}_1}$ are i.i.d. Therefore, by Assumption 1, we can apply the weak law of large numbers to each of the five components in (B.1) to conclude $N_1^{-1} N_2^{-1} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} X_{ij} \xrightarrow{p} 0$. □

This lemma is auxiliary for the following theorems.

Theorem 1. *Suppose that Assumption 1 holds, Y has a finite moment, and $Pr(D = 1) > 0$. Then, $\widehat{E}[Y|D = 1] \xrightarrow{p} E[Y|D = 1]$ as $N \to \infty$.*

Proof. Setting $g(R_{ij}, D_{ij}, Y_{ij}, Z_{ij}, W_{ij}) = Y_{ij} D_{ij} - E[YD]$ in Lemma 1 yields

$$\frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} Y_{ij} D_{ij} - E[YD] = o_p(1).$$

Similarly, setting $g(R_{ij}, D_{ij}, Y_{ij}, Z_{ij}, W_{ij}) = D_{ij} - E[D]$ in Lemma 1 yields

$$\frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} D_{ij} - E[D] = o_p(1).$$

Therefore, by the continuous mapping theorem,

$$\widehat{E}[Y|D = 1] = \frac{\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} Y_{ij} D_{ij}}{\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} D_{ij}} \xrightarrow{p} \frac{E[YD]}{E[D]} = E[Y|D = 1]$$

follows as claimed in the theorem. □

Theorem 2. *If Assumption 1 holds and Y has a finite moment, then $\widehat{E}[Y] \xrightarrow{p} E[Y|R = 0]$ as $N \to \infty$.*

Proof. First, observe that the weak law of large numbers yields

$$\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}R_{ij} = \frac{N}{N_1 N_2}\frac{1}{N}\sum_{k\in\mathcal{K}} Y_{i(k)j(k)} = o_p(1)$$

under Assumption 1. Setting $g(R_{ij}, D_{ij}, Y_{ij}, Z_{ij}, W_{ij}) = Y_{ij}(1 - R_{ij}) - E[Y(1 - R)]$ in Lemma 1 yields

$$\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}(1 - R_{ij}) - E[Y(1 - R)] = o_p(1).$$

Similarly, setting $g(R_{ij}, D_{ij}, Y_{ij}, Z_{ij}, W_{ij}) = (1 - R_{ij}) - E[1 - R]$ in Lemma 1 yields

$$\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(1 - R_{ij}) - E[1 - R] = o_p(1).$$

Finally, note that

$$\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(1 - R_{ij}) = \frac{N_1 N_2 - N}{N_1 N_2}$$

Therefore, putting all these pieces together, we have

$$\begin{aligned}
\widehat{E}[Y] =& \frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}R_{ij} + \frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}(1 - R_{ij}) \\
=& \frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}R_{ij} + \frac{\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}(1 - R_{ij})}{\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(1 - R_{ij})} \cdot \frac{N_1 N_2 - N}{N_1 N_2} \\
\xrightarrow{p}& 0 + \frac{E[Y(1 - R)]}{E[1 - R]} \cdot 1 = E[Y|R = 0]
\end{aligned}$$

as claimed in the theorem. □

Theorem 3. *Suppose that Assumption 1 holds and $Pr(D = 1) > 0$. Then, $\widehat{E}[Z|D = 1] \xrightarrow{p} E[Z|D = 1]$ as $N \to \infty$.*

Proof. The theorem can be proved as in Theorem 1, except that Y is replaced by Z. □

Theorem 4. *If Assumption 1 holds, then $\widehat{E}[Z] \xrightarrow{p} E[Z|R = 0]$ as $N \to \infty$.*

Proof. The theorem can be proved as in Theorem 2, except that Y is replaced by Z. □

Theorem 5. *Suppose that Assumption 1 holds, Y has a finite moment, and $Pr(D = 1|W = w) > 0$. Then, $\widehat{E}[Y|D = 1, W = w] \xrightarrow{p} E[Y|D = 1, W = w]$ as $N \to \infty$.*

Proof. Setting $g(R_{ij}, D_{ij}, Y_{ij}, Z_{ij}, W_{ij}) = Y_{ij}D_{ij}\mathbb{1}\{W_{ij} = w\} - E[YD\mathbb{1}\{W = w\}]$ in Lemma 1 yields

$$\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}D_{ij}\mathbb{1}\{W_{ij} = w\} - E[YD\mathbb{1}\{W = w\}] = o_p(1).$$

Similarly, setting $g(R_{ij}, D_{ij}, Y_{ij}, Z_{ij}, W_{ij}) = D_{ij}\mathbb{1}\{W_{ij} = w\} - E[D\mathbb{1}\{W = w\}]$ in Lemma 1 yields

$$\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}\{W_{ij} = w\} - E[D\mathbb{1}\{W = w\}] = o_p(1).$$

Therefore, by the continuous mapping theorem,

$$\begin{aligned}\widehat{E}[Y|D = 1, W = w] &= \frac{\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}D_{ij}\mathbb{1}\{W_{ij} = w\}}{\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}\{W_{ij} = 1\}} \\ &\xrightarrow{p} \frac{E[YD\mathbb{1}\{W = w\}]}{E[D\mathbb{1}\{W = w\}]} = E[Y|D = 1, W = w]\end{aligned}$$

follows as claimed in the theorem. □

Theorem 6. *If Assumption 1 holds and Y has a finite moment, then $\widehat{E}[Y|W = w] \xrightarrow{p} E[Y|R = 0, W = w]$ as $N \to \infty$.*

Proof. Using the same argument as in the proof of Theorem 2 with Y replaced by $Y\mathbb{1}\{W = w\}$, we have

$$\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\mathbb{1}\{W_{ij} = w\} \xrightarrow{p} E[Y\mathbb{1}\{W = w\}|R = 0]$$

as $N \to \infty$. Likewise, using the same argument as in the proof of Theorem 2 with Y replaced by $\mathbb{1}\{W = w\}$, we have

$$\frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}\{W_{ij} = w\} \xrightarrow{p} E[\mathbb{1}\{W = w\}|R = 0]$$

as $N \to \infty$. Therefore, by the continuous mapping theorem,

$$\begin{aligned}\widehat{E}[Y|W = w] &= \frac{\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\mathbb{1}\{W_{ij} = w\}}{\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}\{W_{ij} = w\}} \\ &\xrightarrow{p} \frac{E[Y\mathbb{1}\{W = w\}|R = 0]}{E[\mathbb{1}\{W = w\}|R = 0]} = E[Y|R = 0, W = w]\end{aligned}$$

follows as claimed in the theorem. □

Theorem 7. *Suppose that Assumption 1 holds and $Pr(D = 1|W = w) > 0$. Then, $\widehat{E}[Z|D = 1, W = w] \xrightarrow{p} E[Z|D = 1, W = w]$ as $N \to \infty$.*

Proof. The theorem can be proved as in Theorem 5, except that Y is replaced by Z. □

Theorem 8. *If Assumption 1 holds, then $\widehat{E}[Z|W = w] \xrightarrow{p} E[Z|R = 0, W = w]$ as $N \to \infty$.*

Proof. The theorem can be proved as in Theorem 6, except that Y is replaced by Z. □

B.2 Standard Errors

Assumption 2. *$D_{ij} \geq R_{ij}$ almost surely. $\pi(w) = Pr(R_{ij} = 0|D_{ij} = 1, W_{ij} = w) = Pr(R_{ij} = 0|D_{ij} = 1, W_{ij} = w, Y_{ij'}, Z_{ij'}, W_{ij'}, Y_{i'j}, Z_{i'j}, W_{i'j})$ whenever $i \neq i'$ and $j \neq j'$.*

The first part of this assumption requires a monotone false matching in that, if unit i in census $t = 1$ and unit j in census $t = 2$ are actually an identical individual, then they are matched in data.

Theorem 9. *Let $\xi_{ij} = g(D_{ij}, Y_{ij}, Z_{ij}, W_{ij})$ such that $E[\xi_{ij}] = 0$ and $E[\|\xi_{ij}\|^2] < \infty$. If Assumptions 1 and 2 hold, then*

$$V\left(\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}\right) = E\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}\xi'_{ij'}\right] + E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\xi_{ij}\xi'_{i'j}\right] + E\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}\xi'_{ij}\right]$$
$$+ E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}D_{ij'}(1-\pi(W_{ij'}))\xi'_{i'j'}\right] + E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}D_{i'j}(1-\pi(W_{i'j}))\xi'_{i'j'}\right].$$

Proof. First, we decompose the variance as follows.

$$\begin{aligned}
V\left(\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}\right) =& E\left[\sum_{i=1}^{N_1}\sum_{i'=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'=1}^{N_2}\xi_{ij}\xi'_{i'j'}\right] \\
=& E\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}\xi'_{ij'}\right] + E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\xi_{ij}\xi'_{i'j}\right] \\
&+ E\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}\xi'_{ij}\right] + E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}\xi'_{i'j'}\right].
\end{aligned}$$

The last term on the right-hand size can be rewritten in turn as

$$\begin{aligned}
& E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}\xi'_{i'j'}\right] \\
=& E\left[\sum_{k\in\mathcal{K}}\sum_{i'\neq i(k)}\sum_{j\neq j(k)}\xi_{i(k)j}\xi'_{i'j(k)}\right] + E\left[\sum_{k\in\mathcal{K}}\sum_{j'\neq j(k)}\sum_{i\neq i(k)}\xi_{ij(k)}\xi'_{i(k)j'}\right] \\
=& E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}R_{ij'}\xi'_{i'j'}\right] + E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}R_{i'j}\xi'_{i'j'}\right] \\
=& E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}D_{ij'}(1-\pi(W_{ij'}))\xi'_{i'j'}\right] + E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}D_{i'j}(1-\pi(W_{i'j}))\xi'_{i'j'}\right],
\end{aligned}$$

where the first equality is due to the independence in Assumption 1, the third equality is due to Assumption 2. □

This proposition motivates the following variance estimator for the double sum $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}$.

$$\widehat{V}\left(\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}\right) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}\xi'_{ij'} + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\xi_{ij}\xi'_{i'j} + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}\xi'_{ij}$$
$$+ \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}D_{ij'}(1-\widehat{\pi}(W_{ij'}))\xi'_{i'j'} + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}D_{i'j}(1-\widehat{\pi}(W_{i'j}))\xi'_{i'j'}. \tag{B.1}$$

It is worthy to remark that, if the true match R_{ij} were fully observable, then the data are essentially dyadic so that the existing Central Limit Theorem for dyadic data can be applied to entail the variance of the form

$$V\left(\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}\right) = E\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}\xi'_{ij'}\right] + E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\xi_{ij}\xi'_{i'j}\right] + E\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\xi_{ij}\xi'_{ij}\right]$$
$$+ E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}R_{ij'}\xi'_{i'j'}\right] + E\left[\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\xi_{ij}R_{i'j}\xi'_{i'j'}\right].$$

Since R_{ij} is not observable in our framework, however, this variance formula is not estimable. Proposition 9 shows how to estimate this variance by replacing R_{ij} with $D_{ij}(1-\pi(W_{ij}))$ in the last two terms.

B.2.1 Standard Errors for (2.3.5) Under Conditions (2.3.1) and (2.3.8)

Based on (B.1), the variance of $\widehat{E}_{\text{debiased}}[Y|R=1]$ in (2.3.5) can be estimated by

$$\widehat{V}_{\widehat{E}_{\text{debiased}}[Y|R=1]} = (\widehat{\Delta}^0_D, \widehat{\Delta}^0_Y, \widehat{\Delta}^0_Z, \widehat{\Delta}^0_{DY}, \widehat{\Delta}^0_{DZ})'\widehat{V}(\widehat{\Delta}^0_D, \widehat{\Delta}^0_Y, \widehat{\Delta}^0_Z, \widehat{\Delta}^0_{DY}, \widehat{\Delta}^0_{DZ}),$$

where

$$\begin{aligned}
\widehat{\Delta}_D^0 =& \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2} \\
& - \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)^2\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2}, \\
\widehat{\Delta}_Y^0 =& \frac{-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}}{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}}, \\
\widehat{\Delta}_Z^0 =& \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}}{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}} \\
& + \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2}, \\
& - \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2} \\
\widehat{\Delta}_{DY}^0 =& \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}}{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}}, \\
\widehat{\Delta}_{DZ}^0 =& \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2} \\
& - \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2} \\
& - \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}}{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}},
\end{aligned}$$

and $\widehat{V}$ is a 5×5 matrix defined by

$$\begin{aligned}
\widehat{V} =& \frac{1}{N_1^2N_2^2}\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi_{ij}\Xi_{ij'}' + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\Xi_{ij}\Xi_{i'j}' + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Xi_{ij}\Xi_{ij}'\right. \\
& \left. + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi_{ij}D_{ij'}(1-\widehat{\pi})\Xi_{i'j'}' + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi_{ij}D_{i'j}(1-\widehat{\pi})\Xi_{i'j'}'\right]
\end{aligned}$$

with the short-hand notation

$$\Xi_{i'j'} = \begin{pmatrix} D_{i'j'} - \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} D_{ij} \\ Y_{i'j'} - \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} Y_{ij} \\ Z_{i'j'} - \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} Z_{ij} \\ D_{i'j'} Y_{i'j'} - \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} D_{ij} Y_{ij} \\ D_{i'j'} Z_{i'j'} - \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} D_{ij} Z_{ij} \end{pmatrix}$$

for each (i', j').

B.2.2 Standard Errors for (2.3.16) Under Conditions (2.3.8) and (2.3.14)

Let $\widehat{e}_{ij} = Y_{ij} - X'_{ij}\widehat{\beta}_{\text{debiased}}$ denote the OLS residual, where $\widehat{\beta}_{\text{debiased}}$ is defined in (2.3.16). The k-th coordinate of X_{ij} will be denoted by X_{ijk}. In other words, $X_{ij} = (X_{ij1}, \ldots, X_{ijK})'$ where K denotes the number of coordinates. Let $\widehat{\Omega}$ denote the $K \times K$ matrix whose (k, ℓ)-th entry is defined by

$$\widehat{\Omega}_{k\ell} = (\widehat{\Delta}^0_{D,k}, \widehat{\Delta}^0_Y, \widehat{\Delta}^0_{Z,k}, \widehat{\Delta}^0_{DY}, \widehat{\Delta}^0_{DZ,k})' \widehat{V}_{k\ell} (\widehat{\Delta}^0_{D,\ell}, \widehat{\Delta}^0_Y, \widehat{\Delta}^0_{Z,\ell}, \widehat{\Delta}^0_{DY}, \widehat{\Delta}^0_{DZ,\ell}),$$

where

$$\widehat{\Delta}^0_{D,k} = \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)^2\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2},$$

$$\widehat{\Delta}^0_{Y} = \frac{-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}}{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}},$$

$$\widehat{\Delta}^0_{Z,k} = \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}}{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}}$$
$$+ \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2},$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2}$$

$$\widehat{\Delta}^0_{DY} = \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}}{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}},$$

$$\widehat{\Delta}^0_{DZ,k} = \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right)}{\left[\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\right]^2}$$
$$- \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}}{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\right) - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}},$$

and $\widehat{V}$ is a 5×5 matrix defined by

$$\widehat{V}_{k\ell} = \frac{1}{N_1^2N_2^2}\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi_{ijk}\Xi'_{ij'\ell} + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\Xi_{ijk}\Xi'_{i'j\ell} + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Xi_{ijk}\Xi'_{ij\ell}\right.$$
$$\left. + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi_{ijk}D_{ij'}(1-\widehat{\pi})\Xi'_{i'j'\ell} + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi_{ijk}D_{i'j}(1-\widehat{\pi})\Xi'_{i'j'\ell}\right]$$

with the short-hand notation

$$\Xi_{i'j'k} = \begin{pmatrix} D_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij} \\ X_{i'j'k}\widehat{e}_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij} \\ Z_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij} \\ D_{i'j'}X_{i'j'k}\widehat{e}_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij} \\ D_{i'j'}Z_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij} \end{pmatrix}$$

for each (i', j'). With the above definition of $\widehat{\Omega}$, the covariance matrix of $\widehat{\beta}_{\text{debiased}}$ in (2.3.16) can be estimated by

$$\widehat{V}_{\widehat{\beta}_{\text{debiased}}} = \widehat{E}_{\text{debiased}}[XX'|R=1]^{-1}\widehat{\Omega}\widehat{E}_{\text{debiased}}[XX'|R=1]^{-1},$$

where $\widehat{E}_{\text{debiased}}[XX'|R=1]$ is defined in (2.3.15).

B.2.3 Standard Errors for (2.4.6) Under Conditions (2.4.1) and (2.4.2)

Let $\mathbb{1}_{ij}^w = \mathbb{1}\{W_{ij} = w\}$ for a short-hand notation. Also, let

$$\begin{aligned}\mathcal{D}(w) = &\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}_{ij}^w\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}_{ij}^w\right) \\ &- \left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}_{ij}^w\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}_{ij}^w\right).\end{aligned}$$

Based on (B.1), the variance of $\widehat{E}_{\text{debiased}}[Y|R=1, W=w]$ in (2.4.6) can be estimated by

$$\begin{aligned}&\widehat{V}_{\widehat{E}_{\text{debiased}}[Y|R=1,W=w]} = \\ &(\widehat{\Delta}_{DW}^w, \widehat{\Delta}_{YW}^w, \widehat{\Delta}_{ZW}^w, \widehat{\Delta}_{DYW}^w, \widehat{\Delta}_{DZW}^w, \widehat{\Delta}_W^w)'\widehat{V}^w(\widehat{\Delta}_{DW}^w, \widehat{\Delta}_{YW}^w, \widehat{\Delta}_{ZW}^w, \widehat{\Delta}_{DYW}^w, \widehat{\Delta}_{DZW}^w, \widehat{\Delta}_W^w),\end{aligned}$$

where

$$\widehat{\Delta}^w_{DW} = \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)^2\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2},$$

$$\widehat{\Delta}^w_{YW} = \frac{-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}}{\mathcal{D}(w)},$$

$$\widehat{\Delta}^w_{ZW} = \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}\mathbb{1}^w_{ij}}{\mathcal{D}(w)}$$
$$+ \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2},$$

$$\widehat{\Delta}^w_{DYW} = \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}}{\mathcal{D}(w)},$$

$$\widehat{\Delta}^w_{DZW} = \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\mathbb{1}^w_{ij}}{\mathcal{D}(w)},$$

$$\widehat{\Delta}^w_{W} = \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)^2}{\mathcal{D}(w)^2},$$

and $\widehat{V}^w$ is a 6×6 matrix defined by

$$\widehat{V}^w = \frac{1}{N_1^2N_2^2}\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi^w_{ij}\Xi^{w\prime}_{ij'} + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\Xi^w_{ij}\Xi^{w\prime}_{i'j} + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Xi^w_{ij}\Xi^{w\prime}_{ij}\right.$$
$$\left. + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi^w_{ij}D_{ij'}(1-\widehat{\pi}(W_{ij'}))\Xi^{w\prime}_{i'j'} + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi^w_{ij}D_{i'j}(1-\widehat{\pi}(W_{i'j}))\Xi^{w\prime}_{i'j'}\right]$$

with the short-hand notation

$$\Xi^w_{i'j'} = \begin{pmatrix} D_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}^w_{ij} \\ Y_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Y_{ij}\mathbb{1}^w_{ij} \\ Z_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij} \\ D_{i'j'}Y_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Y_{ij}\mathbb{1}^w_{ij} \\ D_{i'j'}Z_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij} \\ \mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}^w_{ij} \end{pmatrix}$$

for each (i', j').

B.2.4 Standard Errors for (2.4.11) Under Conditions (2.4.1) and (2.4.2)

The variance of $\widehat{E}_{\text{debiased}}[Y|R=1]$ in (2.4.11) can be estimated by

$$\widehat{V}_{\widehat{E}_{\text{debiased}}[Y|R=1]} = \sum_w \widehat{V}_{\widehat{E}_{\text{debiased}}[Y|R=1,W=w]}\hat{\varpi}(w)^2,$$

where $\widehat{V}_{\widehat{E}_{\text{debiased}}[Y|R=1,W=w]}$ is defined in Section B.2.3.

B.2.5 Standard Errors for (2.4.14) Under Conditions (2.4.2) and (2.4.12)

We continue to use the short-hand notations $\mathbb{1}^w_{ij} = \mathbb{1}\{W_{ij} = w\}$ and

$$\begin{aligned}\mathcal{D}(w) = &\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right) \\ &- \left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}^w_{ij}\right)\end{aligned}$$

from Appendix B.2.3.

Let $\widehat{e}_{ij} = Y_{ij} - X'_{ij}\widehat{\beta}_{\text{debiased}}$ denote the OLS residual, where $\widehat{\beta}_{\text{debiased}}$ is defined in (2.4.14). The k-th coordinate of X_{ij} will be denoted by X_{ijk}. In other words, $X_{ij} = (X_{ij1}, \ldots, X_{ijK})'$ where K denotes the number of coordinates. Let $\widehat{\Omega}^w$ denote the $K \times K$ matrix whose (k, ℓ)-th entry is defined by

$$\widehat{\Omega}^w_{k\ell} = (\widehat{\Delta}^w_{DW,k}, \widehat{\Delta}^w_{YW}, \widehat{\Delta}^w_{ZW,k}, \widehat{\Delta}^w_{DYW}, \widehat{\Delta}^w_{DZW,k}, \widehat{\Delta}^w_{W,k})'\widehat{V}^w_{k\ell}(\widehat{\Delta}^w_{DW,\ell}, \widehat{\Delta}^w_{YW}, \widehat{\Delta}^w_{ZW,\ell}, \widehat{\Delta}^w_{DYW}, \widehat{\Delta}^w_{DZW,\ell}, \widehat{\Delta}^w_{W,\ell}),$$

where

$$\widehat{\Delta}^w_{DW,k} = \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)^2\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2},$$

$$\widehat{\Delta}^w_{YW} = \frac{-\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}}{\mathcal{D}(w)},$$

$$\widehat{\Delta}^w_{ZW,k} = \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}}{\mathcal{D}(w)}$$
$$+ \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2},$$

$$\widehat{\Delta}^w_{DYW} = \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}}{\mathcal{D}(w)},$$

$$\widehat{\Delta}^w_{DZW,k} = \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}}{\mathcal{D}(w)},$$

$$\widehat{\Delta}^w_{W,k} = \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)}{\mathcal{D}(w)^2}$$
$$- \frac{\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij}\right)\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij}\right)^2}{\mathcal{D}(w)^2},$$

and $\widehat{V}^w_{k\ell}$ is a 6×6 matrix defined by

$$\widehat{V}^w_{k\ell} = \frac{1}{N_1^2N_2^2}\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi^w_{ijk}\Xi^{w\prime}_{ij'\ell} + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\Xi^w_{ijk}\Xi^{w\prime}_{i'j\ell} + \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Xi^w_{ijk}\Xi^{w\prime}_{ij\ell}\right.$$
$$\left. + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi^w_{ijk}D_{ij'}(1-\widehat{\pi}(W_{ij'}))\Xi^{w\prime}_{i'j'\ell} + \sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi^w_{ijk}D_{i'j}(1-\widehat{\pi}(W_{i'j}))\Xi^{w\prime}_{i'j'\ell}\right]$$

with the short-hand notation

$$\Xi^w_{i'j'k} = \begin{pmatrix} D_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}\mathbb{1}^w_{ij} \\ X_{i'j'k}\widehat{e}_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij} \\ Z_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} Z_{ij}\mathbb{1}^w_{ij} \\ D_{i'j'}X_{i'j'k}\widehat{e}_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}X_{ijk}\widehat{e}_{ij}\mathbb{1}^w_{ij} \\ D_{i'j'}Z_{i'j'}\mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} D_{ij}Z_{ij}\mathbb{1}^w_{ij} \\ \mathbb{1}^w_{i'j'} - \frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \mathbb{1}^w_{ij} \end{pmatrix}$$

for each (i', j').

With the above definition of $\widehat{\Omega}^w$, the covariance matrix of $\widehat{\beta}_{\text{debiased}}$ in (2.4.14) can be estimated by

$$\widehat{V}_{\widehat{\beta}_{\text{debiased}}} = \widehat{E}_{\text{debiased}}[XX'|R=1]^{-1}\left(\sum_w \widehat{\Omega}^w \hat{\varpi}(w)^2\right)\widehat{E}_{\text{debiased}}[XX'|R=1]^{-1},$$

where $\widehat{E}_{\text{debiased}}[XX'|R=1]$ is defined in (2.4.13).

C Analytical computations for estimation with large datasets

This appendix presents the analytical simplifications of our estimates of the false match rate, the corrected parameter of interest, and the standard errors, which are necessary in order to make the computation of the estimators tractable. These simplifications are specific to the cases that we study—computation of a population parameter based on a binary based on linkage (as in estimation of the migration rate), estimation of a regression of a second-period outcome on first-period regressors (as in intergenerational mobility), and estimation of a linear probability model of a binary outcome based on linkage on first-period regressors (as in the migration selection example). We also assume unique matching, so that each individual i in the first census can match to at most one individual j in the second census, and vice versa. Other cases that researchers might encounter can be simplified similarly. Throughout this appendix, we omit subscripts unless they are essential for clarity. The simplifications made in this appendix do not account for blocking of possible matches; in such a case, summations across i or j must simply be limited to observations within the same block as the observation in question.

C.1 Migration with unconfounded matching

The simplest case of the simplification is for migration with unconfounded matching. This case provides the potential for simplification because it is not necessary to actually compare each observation in census i to each observation in census j to determine if that match would result in an observed move; instead, it is possible to simply compute the number of potential matches for each individual i that would result in a move with knowledge of the total number of observations in census 2 and the number of observations in census 2 that are in the same location as individual i.

The first set of quantities can be simplified as follows. First, both $E[Z|D=1]$ and $E[Y|D=1]$ are straightforward to compute from the linked sample. The next set of simplifications are useful for the subsequent ones:

$$
\begin{aligned}
E[Z] &= \frac{1}{N_1 N_2} \sum_i \sum_j Z_{ij} = \frac{\sum_i (N_2 - N_i^{Z_0})}{N_1 N_2} \\
E[Y] &= \frac{1}{N_1 N_2} \sum_i \sum_j Y_{ij} = \frac{\sum_i (N_2 - N_i^{Y_0})}{N_1 N_2} \\
E[D] &= \frac{1}{N_1 N_2} \sum_i \sum_j D_{ij} = \frac{N_\ell}{N_1 N_2} \\
E[DZ] &= \frac{1}{N_1 N_2} \sum_i \sum_j D_{ij} Z_{ij} = \frac{N_\ell^{Z_1}}{N_1 N_2} \\
E[DY] &= \frac{1}{N_1 N_2} \sum_i \sum_j D_{ij} Y_{ij} = \frac{N_\ell^{Y_1}}{N_1 N_2},
\end{aligned}
$$

where N_1 is the number of observations in the first census, N_2 is the number of observations in the second census, $N_i^{Z_0}$ is the number of observations j in the second census with $\zeta_j = \zeta_i$ for observation i in the first census (that is, the number of observations in the second census that would not appear as a discrepancy if linked with observation i), $N_i^{Y_0}$ is the number of observations in the second

census that are in the same location as observation i in the first census, N_ℓ is the number of links, $N_\ell^{Z_1}$ is the number of links in which $Z=1$, and $N_\ell^{Y_1}$ is the number of links in which $Y=1$. Notice that these values can be computed either from the linked sample ($E[D]$, $E[DZ]$, and $E[DY]$) or from the first census alone, without needing to "cross" it with the second census ($E[Z]$ and $E[Y]$). Define also $N_i^{Z_0Y_1}$ as the number of observations in the second census with the same value of Z as individual i in the first census but in a different location, and define $N_i^{Z_0Y_0}$, $N_i^{Z_1Y_0}$, and $N_i^{Z_1Y_1}$ analogously.

Now, we can simplify separately each set of summations in the definition of $\hat{V}$. Note that we use simplified notation to refer to the components of Ξ_{ij}. In particular, define

$$\Xi_{ij} = \begin{pmatrix} \overline{D}_{ij} \\ \overline{Y}_{ij} \\ \overline{Z}_{ij} \\ \overline{DY}_{ij} \\ \overline{DZ}_{ij} \end{pmatrix}$$

We begin with $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Xi_{ij}'\Xi_{ij}$. The idea of the simplification is to express analytically $\sum_{j=1}^{N_2}\Xi_{ij}'\Xi_{ij}$ for each i so that the number of observations that must be considered in computing the double summation is N_1 instead of N_1N_2. The matrix $\Xi_{ij}'\Xi_{ij}$ has 25 elements, and we approach each separately, noting that the matrix is symmetric, meaning that only 15 distinct components must be computed. The simplifications are as follows.

$$\sum_j \overline{D}_{ij}^2 = \begin{cases} N_2E[D]^2 & i \text{ unlinked} \\ (1-E[D])^2+(N_2-1)E[D]^2 & i \text{ linked.} \end{cases}$$

$$\sum_j \overline{D}_{ij}\overline{Y}_{ij} = \begin{cases} N_i^{Y_0}E[D]E[Y]-(N_2-N_i^{Y_0})E[D](1-E[Y]) & i \text{ unlinked} \\ (1-E[D])(1-E[Y])+N_i^{Y_0}E[D]E[Y]-(N_2-N_i^{Z_0}-1)E[D](1-E[Y]) & i \text{ linked mover} \\ -(1-E[D])E[Y]+(N_i^{Y_0}-1)E[D]E[Y]-(N_2-N_i^{Z_0})E[D](1-E[Y]) & i \text{ linked stayer} \end{cases}$$

$$\sum_j \overline{D}_{ij}\overline{Z}_{ij} = \begin{cases} N_i^{Z_0}E[D]E[Z]-(N_2-N_i^{Z_0})E[D](1-E[Z]) & i \text{ unlinked} \\ (1-E[D])(1-E[Z])+N_i^{Z_0}E[D]E[Z]-(N_2-N_i^{Z_0}-1)E[D](1-E[Z]) & i \text{ linked discrepancy} \\ -(1-E[D])E[Z]+(N_i^{Z_0}-1)E[D]E[Z]-(N_2-N_i^{Z_0})E[D](1-E[Z]) & i \text{ linked non-discrepancy} \end{cases}$$

$$\sum_j \overline{D}_{ij}\overline{DY}_{ij} = \begin{cases} N_2E[D]E[DY] & i \text{ unlinked} \\ (1-E[D])(1-E[DY])+(N_2-1)E[D]E[DY] & i \text{ linked mover} \\ -(1-E[D])E[DY]+(N_2-1)E[D]E[DY] & i \text{ linked stayer} \end{cases}$$

$$\sum_j \overline{D}_{ij}\overline{DZ}_{ij} = \begin{cases} N_2E[D]E[DZ] & i \text{ unlinked} \\ (1-E[D])(1-E[DZ])+(N_2-1)E[D]E[DZ] & i \text{ linked discrepancy} \\ -(1-E[D])E[DZ]+(N_2-1)E[D]E[DZ] & i \text{ linked non-discrepancy} \end{cases}$$

$$\sum_j \overline{Y}_{ij}^2 = N_i^{Y_0}E[Y]^2+(N_2-N_i^{Y_0})(1-E[Y])^2$$

$$\sum_j \overline{Y}_{ij}\overline{Z}_{ij} = N_i^{Y_0Z_0}E[Y]E[Z]-N_i^{Y_0Z_1}E[Y](1-E[Z])-N_i^{Y_1Z_0}(1-E[Y])E[Z]+N_i^{Y_1Z_1}(1-E[Y])(1-E[Z])$$

$$\sum_j \overline{Y}_{ij}\overline{DY}_{ij} = \begin{cases} N_i^{Y_0}E[DY]E[Y]-(N_2-N_i^{Y_0})E[DY](1-E[Y]) & i \text{ unlinked or linked stayer} \\ (1-E[DY])(1-E[Y])+N_i^{Y_0}E[DY]E[Y]-(N_2-N_i^{Y_0}-1)E[DY](1-E[Y]) & i \text{ linked mover} \end{cases}$$

$$\sum_j \overline{Y}_{ij}\overline{DZ}_{ij} = \begin{cases} N_i^{Y_0}E[DZ]D[Y]-(N_2-N_i^{Y_0})E[DZ](1-E[Y]) & i \text{ unlinked or linked non-discrepancy} \\ (1-E[Y])(1-E[DZ])+N_i^{Y_0}E[Y]E[DZ]-(N_2-N_i^{Y_0}-1)(1-E[Y])E[DZ] & i \text{ linked mover discrepancy} \\ -E[Y](1-E[DZ])+(N_i^{Y_0}-1)E[Y]E[DZ]-(N_2-N_i^{Y_0})(1-E[Y])E[DZ] & i \text{ linked stayer discrepancy} \\ -(1-E[Y])E[DZ]+N_i^{Y_0}E[Y]E[DZ]-(N_2-N_i^{Y_0}-1)(1-E[Y])E[DZ] & i \text{ linked mover non-discrepancy} \\ E[Y]E[DZ]+(N_i^{Y_0}-1)E[Y]E[DZ]-(N_2-N_i^{Y_0})(1-E[Y])E[DZ] & i \text{ linked stayer non-discrepancy} \end{cases}$$

$$\sum_j \overline{Z}_{ij}^2 = N_i^{Z_0}E[Z]^2+(N_2-N_i^{Z_0})(1-E[Z])^2$$

$$\sum_j \overline{Z}_{ij}\overline{DY}_{ij}=\begin{cases} N_i^{Z_0}E[DY]D[Z]-(N_2-N_i^{Z_0})E[DY](1-E[Z]) & i \text{ unlinked or linked stayer} \\ (1-E[Z])(1-E[DY])+N_i^{Z_0}E[Z]E[DY]-(N_2-N_i^{Z_0}-1)(1-E[Z])E[DY] & i \text{ linked mover discrepancy} \\ -E[Z](1-E[DY])+(N_i^{Z_0}-1)E[Z]E[DY]-(N_2-N_i^{Z_0})(1-E[Z])E[DY] & i \text{ linked mover non-discrepancy} \\ -(1-E[Z])E[DY]+N_i^{Z_0}E[Z]E[DY]-(N_2-N_i^{Z_0}-1)(1-E[Z])E[DY] & i \text{ linked stayer discrepancy} \\ E[Z]E[DY]+(N_i^{Z_0}-1)E[Z]E[DY]-(N_2-N_i^{Z_0})(1-E[Z])E[DY] & i \text{ linked stayer non-discrepancy} \end{cases}$$

$$\sum_j \overline{Z}_{ij}\overline{DZ}_{ij}=\begin{cases} N_i^{Z_0}E[Z]E[DZ]-(N_2-N_i^{Z_0})(1-E[Z])E[DZ] & i \text{ unlinked or linked non-discrepancy} \\ (1-E[DZ])(1-E[Z])+N_i^{Z_0}E[DZ]E[Z]-(N_2-N_i^{Z_0}-1)E[DZ](1-E[DZ]) & i \text{ linked discrepancy} \end{cases}$$

$$\sum_j \overline{DY}_{ij}^2=\begin{cases} N_2E[DY]^2 & i \text{ unlinked or linked stayer} \\ (1-E[DY])^2+(N_1-1)E[DY]^2 & i \text{ linked mover} \end{cases}$$

$$\sum_j \overline{DY}_{ij}\overline{DZ}_{ij}=\begin{cases} N_2E[DY]E[DZ] & i \text{ unlinked or linked stayer non-discrepancy} \\ (1-E[DY])(1-E[DZ])+(N_2-1)E[DY]E[DZ] & i \text{ linked mover discrepancy} \\ -E[DY](1-E[DZ])+(N_2-1)E[DY]E[DZ] & i \text{ linked stayer discrepancy} \\ -(1-E[DY])E[DZ]+(N_2-1)E[DY]E[DZ] & i \text{ linked mover non-discrepancy} \end{cases}$$

$$\sum_j \overline{DZ}_{ij}^2=\begin{cases} N_2E[DZ]^2 & i \text{ unlinked or linked non-discrepancy} \\ (1-E[DZ])^2(N_2-1)E[DZ]^2 & i \text{ linked discrepancy} \end{cases}$$

Next, we consider $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi'_{ij}\Xi_{ij'}$. The idea of the simplification is to express analytically $\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi'_{ij}\Xi_{ij'}$ for each i so that the number of observations that must be considered in computing the double summation is N_1 instead of N_1N_2. The matrix $\Xi'_{ij}\Xi_{ij'}$ has 25 elements, and we approach each separately. The simplifications are as follows.

$$\sum_j\sum_{j'\neq j}\overline{D}_{ij}\overline{D}_{ij'}=\begin{cases} N_2(N_2-1)E[D]^2 & i \text{ unlinked} \\ -(1-E[D])(N_2-1)E[D]-(N_2-1)E[D]\{-(N_2-2)E[D]+(1-E[D])\} & i \text{ linked} \end{cases}$$

$$\sum_j\sum_{j'\neq j}\overline{D}_{ij}\overline{Y}_{ij'}=\begin{cases} -E[D]N_i^{Y_0}\{-(N_i^{Y_0}-1)E[Y]+(N_2-N_i^{Y_0})(1-E[Y])\} \\ \quad -E[D](N_2-N_i^{Y_0})\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} & i \text{ unlinked} \\ (1-E[D])\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} \\ \quad -E[D]N_i^{Y_0}\{-(N_i^{Y_0}-1)E[Y]+(N_2-N_i^{Y_0})(1-E[Y])\} \\ \quad -E[D](N_2-N_i^{Y_0}-1)\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} & i \text{ linked mover} \\ (1-E[D])\{-(N_i^{Y_0}-1)E[Y]+(N_2-N_i^{Y_0})(1-E[Y])\} \\ \quad -E[D](N_i^{Y_0}-1)\{-(N_i^{Y_0}-1)E[Y]+(N_2-N_i^{Y_0})(1-E[Y])\} \\ \quad -E[D](N_2-N_i^{Y_0})\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} & i \text{ linked stayer} \end{cases}$$

$$\sum_j\sum_{j'\neq j}\overline{D}_{ij}\overline{Z}_{ij'}=\begin{cases} -E[D]N_i^{Z_0}\{-(N_i^{Z_0}-1)E[Z]+(N_2-N_i^{Z_0})(1-E[Z])\} \\ \quad -E[D](N_2-N_i^{Z_0})\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} & i \text{ unlinked} \\ (1-E[D])\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} \\ \quad -E[D]N_i^{Z_0}\{-(N_i^{Z_0}-1)E[Z]+(N_2-N_i^{Z_0})(1-E[Z])\} \\ \quad -E[D](N_2-N_i^{Z_0}-1)\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} & i \text{ linked discrepancy} \\ (1-E[D])\{-(N_i^{Z_0}-1)E[Z]+(N_2-N_i^{Z_0})(1-E[Z])\} \\ \quad -E[D](N_i^{Z_0}-1)\{-(N_i^{Z_0}-1)E[Z]+(N_2-N_i^{Z_0})(1-E[Z])\} \\ \quad -E[D](N_2-N_i^{Z_0})\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} & i \text{ linked non-discrepancy} \end{cases}$$

$$\sum_j\sum_{j'\neq j}\overline{D}_{ij}\overline{DY}_{ij'}=\begin{cases} N_2(N_2-1)E[D]E[DY] & i \text{ unlinked} \\ -(1-E[D])(N_2-1)E[DY]-E[D](N_2-1)\{-(N_2-2)E[DY]+(1-E[DY])\} & i \text{ linked mover} \\ -(1-E[D])(N_2-1)E[DY]+E[D](N_2-1)^2E[DY] & i \text{ linked stayer} \end{cases}$$

$$\sum_j\sum_{j'\neq j}\overline{D}_{ij}\overline{DZ}_{ij'}=\begin{cases} N_2(N_2-1)E[D]E[DZ] & i \text{ unlinked} \\ -(1-E[D])(N_2-1)E[DZ]-E[D](N_2-1)\{-(N_2-2)E[DZ]+(1-E[DZ])\} & i \text{ linked discrepancy} \\ -(1-E[D])(N_2-1)E[DZ]+E[D](N_2-1)^2E[DZ] & i \text{ linked non-discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Y}_{ij} \overline{D}_{ij'} = \begin{cases} N_i^{Y_0} E[Y](N_2-1)E[D] - (N_2 - N_i^{Y_0})(1-E[Y])(N_2-1)E[D] & i \text{ unlinked} \\ -(1-E[Y])(N_2-1)E[D] - N_i^{Y_0} E[Y]\{-(N_2-2)E[D]+(1-E[D])\} + (N_2 - N_i^{Y_0}-1)(1-E[Y])\} & \\ \quad -(N_2-2)E[D]+(1-E[D])\} & i \text{ linked mover} \\ E[Y](N_2-1)E[D] - (N_i^{Y_0}-1)E[Y]\{-(N_2-2)E[D]+(1-E[D])\} + (N_2 - N_i^{Y_0})(1-E[Y])\} & \\ \quad -(N_2-2)E[D]+(1-E[D])\} & i \text{ linked stayer} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Y}_{ij} \overline{Y}_{ij'} = -N_i^{Y_0} E[Y]\{-(N_i^{Y_0}-1)E[Y] + (N_2 - N_i^{Y_0})(1-E[Y])\}$$
$$+ (N_2 - N_i^{Y_0}(1-E[Y])\{-N_i^{Y_0} E[Y] + (N_2 - N_i^{Y_0}-1)(1-E[Y])\}$$

$$\sum_j \sum_{j' \neq j} \overline{Y}_{ij} \overline{Z}_{ij'} = -N_i^{Y_0 Z_0} E[Y]\{-(N_i^{Z_0}-1)E[Z] + (N_2 - N_i^{Z_0})(1-E[Z])\} - N_i^{Y_0 Z_1} E[Y]\{-N_i^{Z_0} E[Z] + (N_2 - N_i^{Z_0}-1)(1-E[Z])\}$$
$$+ N_i^{Y_1 Z_0}(1-E[Y])\{-(N_i^{Z_0}-1)E[Z] + (N_2 - N_i^{Z_0})(1-E[Z])\} + N_i^{Y_1 Z_1}(1-E[Y])\{-N_i^{Z_0} E[Z] + (N_2 - N_i^{Z_0}-1)(1-E[Z])\}$$

$$\sum_j \sum_{j' \neq j} \overline{Y}_{ij} \overline{DY}_{ij'} = \begin{cases} N_i^{Y_0} E[Y](N_2-1)E[DY] - (N_2 - N_i^{Y_0}(1-E[Y])(N_2-1)E[DY] & i \text{ unlinked or linked stayer} \\ -(1-E[Y])(N_2-1)E[DY] - N_i^{Y_0} E[Y]\{-(N_2-2)E[DY]+(1-E[DY])\} & \\ \quad +(N_2 - N_i^{Y_0}-1)(1-E[Y])\{-(N_2-2)E[DY]+(1-E[DY])\} & i \text{ linked mover} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Y}_{ij} \overline{DZ}_{ij'} = \begin{cases} N_i^{Y_0} E[Y](N_2-1)E[DZ] - (N_2 - N_i^{Y_0}(1-E[Y])(N_2-1)E[DZ] & i \text{ unlinked or linked non-discrepancy} \\ -(1-E[Y[)(N_2-1)E[DZ] - N_i^{Y_0} E[Y]\{-(N_2-2)E[DZ]+(1-E[DZ])\} & \\ \quad +(N_2 - N_i^{Y_0}-1)(1-E[Y])\{-(N_2-2)E[DZ]+(1-E[DZ])\} & i \text{ linked mover discrepancy} \\ E[Y[(N_2-1)E[DZ] - (N_i^{Y_0}-1)E[Y]\{-(N_2-2)E[DZ]+(1-E[DZ])\} & \\ \quad +(N_2 - N_i^{Y_0})(1-E[Y])\{-(N_2-2)E[DZ]+(1-E[DZ])\} & i \text{ linked stayer discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Z}_{ij} \overline{D}_{ij'} = \begin{cases} N_i^{Z_0} E[Z](N_2-1)E[D] - (N_2 - N_i^{Z_0})(1-E[Z])(N_2-1)E[D] & i \text{ unlinked} \\ -(1-E[Z])(N_2-1)E[D] - N_i^{Z_0} E[Z]\{-(N_2-2)E[D]+(1-E[D])\} + (N_2 - N_i^{Z_0}-1)(1-E[Z])\} & \\ \quad -(N_2-2)E[D]+(1-E[D])\} & i \text{ linked discrepancy} \\ E[Z](N_2-1)E[D] - (N_i^{Z_0}-1)E[Z]\{-(N_2-2)E[D]+(1-E[D])\} + (N_2 - N_i^{Z_0})(1-E[Z])\} & \\ \quad -(N_2-2)E[D]+(1-E[D])\} & i \text{ linked non-discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Z}_{ij} \overline{Y}_{ij'} = -N_i^{Z_0 Y_0} E[Z]\{-(N_i^{Y_0}-1)E[Y] + (N_2 - N_i^{Y_0})(1-E[Y])\} - N_i^{Z_0 Y_1} E[Z]\{-N_i^{Y_0} E[Y] + (N_2 - N_i^{Y_0}-1)(1-E[Y])\}$$
$$+ N_i^{Z_1 Y_0}(1-E[Z])\{-(N_i^{Y_0}-1)E[Y] + (N_2 - N_i^{Y_0})(1-E[Y])\} + N_i^{Z_1 Y_1}(1-E[Z])\{-N_i^{Y_0} E[Y] + (N_2 - N_i^{Y_0}-1)(1-E[Y])\}$$

$$\sum_j \sum_{j' \neq j} \overline{Z}_{ij} \overline{Z}_{ij'} = -N_i^{Z_0} E[Z]\{-(N_i^{Z_0}-1)E[Z] + (N_2 - N_i^{Z_0})(1-E[Z])\}$$
$$+ (N_2 - N_i^{Z_0}(1-E[Z])\{-N_i^{Z_0} E[Z] + (N_2 - N_i^{Z_0}-1)(1-E[Z])\}$$

$$\sum_j \sum_{j' \neq j} \overline{Z}_{ij} \overline{DY}_{ij'} = \begin{cases} N_i^{Z_0} E[Z](N_2-1)E[DY] - (N_2 - N_i^{Z_0}(1-E[Z])(N_2-1)E[DY] & i \text{ unlinked or linked stayer} \\ -(1-E[Z[)(N_2-1)E[DY] - N_i^{Z_0} E[Z]\{-(N_2-2)E[DY]+(1-E[DY])\} & \\ \quad +(N_2 - N_i^{Z_0}-1)(1-E[Z])\{-(N_2-2)E[DY]+(1-E[DY])\} & i \text{ linked mover discrepancy} \\ E[Z[(N_2-1)E[DY] - (N_i^{Z_0}-1)E[Z]\{-(N_2-2)E[DY]+(1-E[DY])\} & \\ \quad +(N_2 - N_i^{Z_0})(1-E[Z])\{-(N_2-2)E[DY]+(1-E[DY])\} & i \text{ linked mover non-discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Z}_{ij} \overline{DZ}_{ij'} = \begin{cases} N_i^{Z_0} E[Z](N_2-1)E[DZ] - (N_2 - N_i^{Z_0}(1-E[Z])(N_2-1)E[DZ] & i \text{ unlinked or linked non-discrepancy} \\ -(1-E[Z])(N_2-1)E[DZ] - N_i^{Z_0} E[Z]\{-(N_2-2)E[DZ]+(1-E[DZ])\} & \\ \quad +(N_2 - N_i^{Z_0}-1)(1-E[Z])\{-(N_2-2)E[DZ]+(1-E[DZ])\} & i \text{ linked discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DY}_{ij} \overline{D}_{ij'} = \begin{cases} N_2 E[DY](N_2-1)E[D] & i \text{ unlinked} \\ -(1-E[DY])(N_2-1)E[D] - (N_2-1)E[DY]\{-(N_2-2)E[D]+(1-E[D])\} & i \text{ linked mover} \\ E[DY](N_2-1)E[D] - (N_2-1)E[DY]\{-(N_2-2)E[D]+(1-E[D])\} & i \text{ linked stayer} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DY}_{ij} \overline{Y}_{ij'} = \begin{cases} -N_i^{Y_0} E[DY]\{-(N_i^{Y_0}-1)E[Y]+(N_2-N_i^{Y_0})(1-E[Y])\} \\ \quad -(N_2-N_i^{Y_0})E[DY]\{-N_i^{Y_0}E[Y]+(N)2-N_i^{Y_0}-1)(1-E[Y])\} & i \text{ unlinked or linked stayer} \\ (1-E[DY])\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} \\ \quad -N_i^{Y_0}E[DY]\{-(N_i^{Y_0}-1)E[Y]+(N_2-N_i^{Y_0})(1-E[Y])\} \\ \quad -E[DY](N_2-N_i^{Y_0}-1)\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} & i \text{ linked mover} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DY}_{ij} \overline{Z}_{ij'} = \begin{cases} -N_i^{Z_0} E[DY]\{-(N_i^{Z_0}-1)E[Z]+(N_2-N_i^{Z_0})(1-E[Z])\} \\ \quad -(N_2-N_i^{Z_0})E[DY]\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} & i \text{ unlinked or linked stayer} \\ (1-E[DY])\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} \\ \quad -N_i^{Z_0}E[DY]\{-(N_i^{Z_0}-1)E[Z]+(N)2-N_i^{Z_0})(1-E[Z])\} \\ \quad -(N_2-N_i^{Z_0}-1)E[DY]\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} & i \text{ linked mover discrepancy} \\ (1-E[DY])\{-(N_i^{Z_0}-1)E[Z]+(N)2-N_i^{Z_0})(1-E[Z])\} \\ \quad -(N_i^{Z_0}-1)E[DY]\{-(N_i^{Z_0}-1)E[Z]+(N)2-N_i^{Z_0})(1-E[Z])\} \\ \quad -(N_2-N_i^{Z_0})E[DY]\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} & i \text{ linked mover non-discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DY}_{ij} \overline{DY}_{ij'} = \begin{cases} N_2(N_2-1)E[DY]^2 & i \text{ unlinked or linked stayer} \\ -(1-E[Dy])(N_2-1)E[DY]-(N_2-1)E[DY]\{-(N_2-2)E[DY]+(1-E[DY])\} & i \text{ linked mover} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DY}_{ij} \overline{DZ}_{ij'} = \begin{cases} N_2E[DY](N_2-1)E[DZ] & i \text{ unlinked or linked stayer non-discrepancy} \\ -(1-E[DY])(N_2-1)E[DZ]+(N_2-1)^2E[DY]E[DZ] & i \text{ linked mover non-discrepancy} \\ -(1-E[DY])(N_2-1)E[DZ]-(N_2-1)E[DY]\{-(N_2-2)E[DZ]+(1-E[DZ])\} & i \text{ linked mover discrepancy} \\ E[DY](N_2-1)E[DZ]-(N_2-1)E[DY]\{-(N_2-2)E[DZ]+(1-E[DZ])\} & i \text{ linked stayer discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZ}_{ij} \overline{D}_{ij'} = \begin{cases} N_2E[DZ](N_2-1)E[D] & i \text{ unlinked} \\ -(1-E[DZ])(N_2-1)E[D]-(N_2-1)E[DZ]\{-(N_2-2)E[D]+(1-E[D])\} & i \text{ linked discrepancy} \\ E[DZ](N_2-1)E[D]-(N_2-1)E[DZ]\{-(N_2-2)E[D]+(1-E[D])\} & i \text{ linked non-discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZ}_{ij} \overline{Y}_{ij'} = \begin{cases} -N_i^{Y_0} E[DZ]\{-(N_i^{Y_0}-1)E[Y]+(N_2-N_i^{Y_0})(1-E[Y])\} \\ \quad -(N_2-N_i^{Y_0})E[DZ]\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} & i \text{ unlinked or linked non-discrepancy} \\ (1-E[DZ])\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} \\ \quad -N_i^{Y_0}E[DZ]\{-(N_i^{Y_0}-1)E[Y]+(N)2-N_i^{Y_0})(1-E[Y])\} \\ \quad -(N_2-N_i^{Y_0}-1)E[DZ]\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} & i \text{ linked mover discrepancy} \\ (1-E[DZ])\{-(N_i^{Y_0}-1)E[Y]+(N)2-N_i^{Y_0})(1-E[Y])\} \\ \quad -(N_i^{Y_0}-1)E[DZ]\{-(N_i^{Y_0}-1)E[Y]+(N)2-N_i^{Y_0})(1-E[Y])\} \\ \quad -(N_2-N_i^{Y_0})E[DZ]\{-N_i^{Y_0}E[Y]+(N_2-N_i^{Y_0}-1)(1-E[Y])\} & i \text{ linked stayer discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZ}_{ij} \overline{Z}_{ij'} = \begin{cases} -N_i^{Z_0} E[DZ]\{-(N_i^{Z_0}-1)E[Z]+(N_2-N_i^{Z_0})(1-E[Z])\} \\ \quad -(N_2-N_i^{Z_0})E[DZ]\{-N_i^{Z_0}E[Z]+(N)2-N_i^{Z_0}-1)(1-E[Z])\} & i \text{ unlinked or linked non-discrepancy} \\ (1-E[DZ])\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} \\ \quad -N_i^{Z_0}E[DZ]\{-(N_i^{Z_0}-1)E[Z]+(N_2-N_i^{Z_0})(1-E[Z])\} \\ \quad -E[DZ](N_2-N_i^{Z_0}-1)\{-N_i^{Z_0}E[Z]+(N_2-N_i^{Z_0}-1)(1-E[Z])\} & i \text{ linked discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZ}_{ij} \overline{DY}_{ij'} = \begin{cases} N_2E[DZ](N_2-1)E[DY] & i \text{ unlinked or linked stayer non-discrepancy} \\ -(1-E[DZ])(N_2-1)E[DY]+(N_2-1)^2E[DZ]E[DY] & i \text{ linked stayer discrepancy} \\ -(1-E[DZ])(N_2-1)E[DY]-(N_2-1)E[DZ]\{-(N_2-2)E[DY]+(1-E[DY])\} & i \text{ linked mover discrepancy} \\ E[DZ](N_2-1)E[DY]-(N_2-1)E[DZ]\{-(N_2-2)E[DY]+(1-E[DY])\} & i \text{ linked mover non-discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZ}_{ij} \overline{DZ}_{ij'} = \begin{cases} N_2(N_2-1)E[DZ]^2 & i \text{ unlinked or linked non-discrepancy} \\ -(1-E[DZ])(N_2-1)E[DZ]-(N_2-1)E[DZ]\{-(N_2-2)E[DZ]+(1-E[DZ])\} & i \text{ linked discrepancy} \end{cases}$$

The other triple sum, $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{i' \neq i} \Xi'_{ij}\Xi_{i'j}$ is simplified analogously, but instead of deriving $\sum_{j=1}^{N_2}\sum_{j' \neq j} \Xi'_{ij}\Xi_{ij'}$ analytically for each i, we derive $\sum_{i=1}^{N_1}\sum_{i' \neq i} \Xi'_{ij}\Xi_{i'j}$ analytically for each j, meaning that the total sum can be computed in the second census. It is simply necessary to replace N_2 with N_1, $N_i^{Y_0}$ with $N_j^{Y_0}$, and so on.

Finally, we consider $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{i' \neq i}\sum_{j' \neq j} \Xi'_{ij}D_{i'j}(1-\hat{\pi})\Xi_{i'j'}$. Since we can factor the $(1-\hat{\pi})$ from the quadruple summation, we ignore it in our simplifications here. The idea of the simplification is to express analytically $\sum_{i=1}^{N_1}\sum_{i' \neq i}\sum_{j' \neq j} \Xi'_{ij}D_{i'j}\Xi_{i'j'}$ for each j so that the number of observations that must be considered in computing the double summation is N_2 instead of N_1N_2 and it can be

computed from the second census. The matrix $\Xi_{ij}\Xi'_{ij'}$ has 25 elements, and we approach each separately. Let $N_{i^*}^{Y_0}$ denote the value of $N_i^{Y_0}$ for the observation i in census 1 to which individual j is linked, and define other terms analogously. The simplifications are as follows.

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{D}_{ij} D_{i'j} \overline{D}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[D]^2 & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{D}_{ij} D_{i'j} \overline{Y}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -E[D](N_1-1)\{-N_{i^*}^{Y_0}E[Y]+(N_2-N_{i^*}^{Y_0}-1)(1-E[Y])\} & j \text{ linked mover} \\ -E[D](N_1-1)\{-(N_{i^*}^{Y_0}-1)E[Y]+(N_2-N_{i^*}^{Y_0})(1-E[Y])\} & j \text{ linked stayer} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{D}_{ij} D_{i'j} \overline{Z}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -E[D](N_1-1)\{-N_{i^*}^{Z_0}E[Z]+(N_2-N_{i^*}^{Z_0}-1)(1-E[Z])\} & j \text{ linked discrepancy} \\ -E[D](N_1-1)\{-(N_{i^*}^{Z_0}-1)E[Z]+(N_2-N_{i^*}^{Z_0})(1-E[Z])\} & j \text{ linked non-discrepancy} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{D}_{ij} D_{i'j} \overline{DY}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ E[D](N_1-1)(N_2-1)E[DY] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{D}_{ij} D_{i'j} \overline{DZ}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ E[D](N_1-1)(N_2-1)E[DZ] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Y}_{ij} D_{i'j} \overline{D}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ N_j^{Y_0}(N_2-1)E[D]E[Y]-(N_1-N_j^{Y_0}-1)(1-E[Y])(N_2-1)E[D] & j \text{ linked mover} \\ (N_j^{Y_0}-1)(N_2-1)E[D]E[Y]-(N_1-N_j^{Y_0})(1-E[Y])(N_2-1)E[D] & j \text{ linked stayer} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Y}_{ij} D_{i'j} \overline{Y}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -N_j^{Y_0}E[Y]\{N_{i^*}^{Y_0}E[Y]+(N_2-N_{i^*}^{Y_0}-1)(1-E[Y])\} & \\ \quad +(N_1-N_j^{Y_0}-1)(1-E[Y])\{-N_{i^*}^{Y_0}E[Y]+(N_2-N_{i^*}^{Y_0}-1)(1-E[Y])\} & j \text{ linked mover} \\ -(N_j^{Y_0}-1)E[Y]\{N_{i^*}^{Y_0}E[Y]+(N_2-N_{i^*}^{Y_0}-1)(1-E[Y])\} & \\ \quad +(N_1-N_j^{Y_0})(1-E[Y])\{-N_{i^*}^{Y_0}E[Y]+(N_2-N_{i^*}^{Y_0}-1)(1-E[Y])\} & j \text{ linked stayer} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Y}_{ij} D_{i'j} \overline{Z}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -N_j^{Y_0}E[Y]\{N_{i^*}^{Z_0}E[Z]+(N_2-N_{i^*}^{Z_0}-1)(1-E[Z])\} & \\ \quad +(N_1-N_j^{Y_0}-1)(1-E[Y])\{-N_{i^*}^{Z_0}E[Z]+(N_2-N_{i^*}^{Z_0}-1)(1-E[Z])\} & j \text{ linked mover discrepancy} \\ -(N_j^{Y_0}-1)E[Y]\{N_{i^*}^{Z_0}E[Z]+(N_2-N_{i^*}^{Z_0}-1)(1-E[Z])\} & \\ \quad +(N_1-N_j^{Y_0})(1-E[Y])\{-N_{i^*}^{Z_0}E[Z]+(N_2-N_{i^*}^{Z_0}-1)(1-E[Z])\} & j \text{ linked stayer discrepancy} \\ -N_j^{Y_0}E[Y]\{(N_{i^*}^{Z_0}-1)E[Z]+(N_2-N_{i^*}^{Z_0})(1-E[Z])\} & \\ \quad +(N_1-N_j^{Y_0}-1)(1-E[Y])\{-(N_{i^*}^{Z_0}-1)E[Z]+(N_2-N_{i^*}^{Z_0})(1-E[Z])\} & j \text{ linked mover non-discrepancy} \\ -(N_j^{Y_0}-1)E[Y]\{(N_{i^*}^{Z_0}-1)E[Z]+(N_2-N_{i^*}^{Z_0})(1-E[Z])\} & \\ \quad +(N_1-N_j^{Y_0})(1-E[Y])\{-(N_{i^*}^{Z_0}-1)E[Z]+(N_2-N_{i^*}^{Z_0})(1-E[Z])\} & j \text{ linked stayer non-discrepancy} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Y}_{ij} D_{i'j} \overline{DY}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -\{-N_j^{Y_0}E[Y]+(N_1-N_j^{Y_0}-1)(1-E[Y])\}(N_2-1)E[DY] & j \text{ linked mover} \\ -\{-(N_j^{Y_0}-1)E[Y]+(N_1-N_j^{Y_0})(1-E[Y])\}(N_2-1)E[DY] & j \text{ linked stayer} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Y}_{ij} D_{i'j} \overline{DZ}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -\{-N_j^{Y_0}E[Y]+(N_1-N_j^{Y_0}-1)(1-E[Y])\}(N_2-1)E[DZ] & j \text{ linked mover} \\ -\{-(N_j^{Y_0}-1)E[Y]+(N_1-N_j^{Y_0})(1-E[Y])\}(N_2-1)E[DZ] & j \text{ linked stayer} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Z}_{ij} D_{i'j} \overline{D}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ N_j^{Z_0}(N_2-1)E[D]E[Z]-(N_1-N_j^{Z_0}-1)(1-E[Z])(N_2-1)E[D] & j \text{ linked discrepancy} \\ (N_j^{Z_0}-1)(N_2-1)E[D]E[Z]-(N_1-N_j^{Z_0})(1-E[Z])(N_2-1)E[D] & j \text{ linked non-discrepancy} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Z}_{ij} D_{i'j} \overline{Y}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -N_j^{Z_0} E[Z]\{N_{i^*}^{Y_0} E[Y] + (N_2 - N_{i^*}^{Y_0} - 1)(1-E[Y])\} & \\ \quad + (N_1 - N_j^{Z_0} - 1)(1-E[Z])\{-N_{i^*}^{Y_0} E[Y] + (N_2 - N_{i^*}^{Y_0} - 1)(1-E[Y])\} & j \text{ linked mover discrepancy} \\ -(N_j^{Z_0} - 1) E[Z]\{N_{i^*}^{Y_0} E[Y] + (N_2 - N_{i^*}^{Y_0} - 1)(1-E[Y])\} & \\ \quad + (N_1 - N_j^{Z_0})(1-E[Z])\{-N_{i^*}^{Y_0} E[Y] + (N_2 - N_{i^*}^{Y_0} - 1)(1-E[Y])\} & j \text{ linked mover non-discrepancy} \\ -N_j^{Z_0} E[Z]\{(N_{i^*}^{Y_0} - 1) E[Y] + (N_2 - N_{i^*}^{Y_0})(1-E[Y])\} & \\ \quad + (N_1 - N_j^{Z_0} - 1)(1-E[Z])\{-(N_{i^*}^{Y_0} - 1) E[Y] + (N_2 - N_{i^*}^{Y_0})(1-E[Y])\} & j \text{ linked stayer discrepancy} \\ -(N_j^{Z_0} - 1) E[Z]\{(N_{i^*}^{Y_0} - 1) E[Y] + (N_2 - N_{i^*}^{Y_0})(1-E[Y])\} & \\ \quad + (N_1 - N_j^{Z_0})(1-E[Z])\{-(N_{i^*}^{Y_0} - 1) E[Y] + (N_2 - N_{i^*}^{Y_0})(1-E[Y])\} & j \text{ linked stayer non-discrepancy} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Z}_{ij} D_{i'j} \overline{Z}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -N_j^{Z_0} E[Z]\{N_{i^*}^{Z_0} E[Z] + (N_2 - N_{i^*}^{Z_0} - 1)(1-E[Z])\} & \\ \quad + (N_1 - N_j^{Z_0} - 1)(1-E[Z])\{-N_{i^*}^{Z_0} E[Z] + (N_2 - N_{i^*}^{Z_0} - 1)(1-E[Z])\} & j \text{ linked discrepancy} \\ -(N_j^{Z_0} - 1) E[Z]\{N_{i^*}^{Z_0} E[Z] + (N_2 - N_{i^*}^{Z_0} - 1)(1-E[Z])\} & \\ \quad + (N_1 - N_j^{Z_0})(1-E[Z])\{-N_{i^*}^{Z_0} E[Z] + (N_2 - N_{i^*}^{Z_0} - 1)(1-E[Z])\} & j \text{ linked non-discrepancy} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Z}_{ij} D_{i'j} \overline{DY}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -\{-N_j^{Z_0} E[Z] + (N_1 - N_j^{Z_0} - 1)(1-E[Z])\}(N_2 - 1) E[DY] & j \text{ linked discrepancy} \\ -\{-(N_j^{Z_0} - 1) E[Z] + (N_1 - N_j^{Z_0})(1-E[Z])\}(N_2 - 1) E[DY] & j \text{ linked non-discrepancy} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Z}_{ij} D_{i'j} \overline{DZ}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -\{-N_j^{Z_0} E[Z] + (N_1 - N_j^{Z_0} - 1)(1-E[Z])\}(N_2 - 1) E[DZ] & j \text{ linked discrepancy} \\ -\{-(N_j^{Z_0} - 1) E[Z] + (N_1 - N_j^{Z_0})(1-E[Z])\}(N_2 - 1) E[DZ] & j \text{ linked non-discrepancy} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DY}_{ij} D_{i'j} \overline{D}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1 - 1)(N_2 - 1) E[D] E[DY] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DY}_{ij} D_{i'j} \overline{Y}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -E[DY](N_1 - 1)\{-N_{i^*}^{Y_0} E[Y] + (N_2 - N_{i^*}^{Y_0} - 1)(1-E[Y])\} & j \text{ linked mover} \\ -E[DY](N_1 - 1)\{-(N_{i^*}^{Y_0} - 1) E[Y] + (N_2 - N_{i^*}^{Y_0})(1-E[Y])\} & j \text{ linked stayer} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DY}_{ij} D_{i'j} \overline{Z}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -E[DY](N_1 - 1)\{-N_{i^*}^{Z_0} E[Z] + (N_2 - N_{i^*}^{Z_0} - 1)(1-E[Z])\} & j \text{ linked discrepancy} \\ -E[DY](N_1 - 1)\{-(N_{i^*}^{Z_0} - 1) E[Z] + (N_2 - N_{i^*}^{Z_0})(1-E[Z])\} & j \text{ linked non-discrepancy} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DY}_{ij} D_{i'j} \overline{DY}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1 - 1)(N_2 - 1) E[DY]^2 & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DY}_{ij} D_{i'j} \overline{DZ}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1 - 1)(N_2 - 1) E[DY] E[DZ] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DZ}_{ij} D_{i'j} \overline{D}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1 - 1)(N_2 - 1) E[D] E[DZ] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DZ}_{ij} D_{i'j} \overline{Y}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -E[DZ](N_1 - 1)\{-N_{i^*}^{Y_0} E[Y] + (N_2 - N_{i^*}^{Y_0} - 1)(1-E[Y])\} & j \text{ linked mover} \\ -E[DZ](N_1 - 1)\{-(N_{i^*}^{Y_0} - 1) E[Y] + (N_2 - N_{i^*}^{Y_0})(1-E[Y])\} & j \text{ linked stayer} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DZ}_{ij} D_{i'j} \overline{Z}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -E[DZ](N_1 - 1)\{-N_{i^*}^{Z_0} E[Z] + (N_2 - N_{i^*}^{Z_0} - 1)(1-E[Z])\} & j \text{ linked discrepancy} \\ -E[DZ](N_1 - 1)\{-(N_{i^*}^{Z_0} - 1) E[Z] + (N_2 - N_{i^*}^{Z_0})(1-E[Z])\} & j \text{ linked non-discrepancy} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DZ}_{ij} D_{i'j} \overline{DY}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1 - 1)(N_2 - 1) E[DZ] E[DY] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DZ}_{ij} D_{i'j} \overline{DZ}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1 - 1)(N_2 - 1) E[DZ]^2 & j \text{ linked} \end{cases}$$

The other quadruple sum, $\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{i'\neq i} \sum_{j'\neq j} \Xi'_{ij} D_{ij'} (1-\hat{\pi}) \Xi_{i'j'}$ can be simplified anal-

ogously, but instead of deriving $\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j'\neq j}\Xi'_{ij}D_{i'j}Xi_{i'j'}$ analytically for each j, we derive $\sum_{j=1}^{N_2}\sum_{i'\neq i}\sum_{j'\neq j}\Xi'_{ij}D_{ij'}\Xi_{i'j'}$ for each i, meaning that the total sum can be computed in the first census. It is simply necessary to replace N_1 with N_2, $N_j^{Y_0}$ with $N_i^{Y_0}$, and so on.

C.2 Migration with confounded matching

The case of migration with confounded matching can be simplified in a manner similar to the simplification for migration with unconfounded matching. We make these simplifications under the assumption that $W_{ij} = W_i$; that is, that matching is confounded based on variables observed in the first census. There are several added complications. First, each quantity must be defined for each value of W rather than for the sample as a whole. Define the variable w_i as an indicator equal to one if observation i's value of W_i is equal to w. The quantities $E[Z|D=1,W=w]$ and $E[Y|D=1,W=w]$ for each w are straightforward to compute from the linked sample. We can also simplify

$$\frac{\sum_i\sum_j Z_{ij}w_i}{\sum_i\sum_j w_i} = \frac{\sum_i w_i(N_2 - N_i^{Z_0})}{N_2\sum_i w_i}$$
$$\frac{\sum_i\sum_j Y_{ij}w_i}{\sum_i\sum_j w_i} = \frac{\sum_i w_i(N_2 - N_i^{Y_0})}{N_2\sum_i w_i}$$

Next, we define analogs to the values $E[Z]$, $E[Y]$, etc. from above to incorporate w_i. In particular,

$$E[Zw] = \frac{1}{N_1N_2}\sum_i\sum_j Z_{ij}w_i = \frac{\sum_i w_i(N_2 - N_i^{Z_0})}{N_1N_2}$$
$$E[Yw] = \frac{1}{N_1N_2}\sum_i\sum_j Y_{ij}w_i = \frac{\sum_i w_i(N_2 - N_i^{Y_0})}{N_1N_2}$$
$$E[Dw] = \frac{1}{N_1N_2}\sum_i\sum_j D_{ij}w_i = \frac{N_\ell^w}{N_1N_2}$$
$$E[DZw] = \frac{1}{N_1N_2}\sum_i\sum_j D_{ij}Z_{ij}w_i = \frac{N_\ell^{wZ_1}}{N_1N_2}$$
$$E[DYw] = \frac{1}{N_1N_2}\sum_i\sum_j D_{ij}Y_{ij}w_i = \frac{N_\ell^{wY_1}}{N_1N_2}$$
$$E[w] = \frac{1}{N_1N_2}\sum_i\sum_j w_i = \frac{\sum_i w_i}{N_1},$$

where N_ℓ^w are the number of links with $W = w$, $N_e ll^{wZ_1}$ is the number of links with $W = w$ and $Z_{ij} = 1$, and $N_e ll^{wY_1}$ is the number of links with $W = w$ and $Y_{ij} = 1$. Define also

$$\Xi_{ij}^w = \begin{pmatrix} \overline{Dw}_{ij} \\ \overline{Yw}_{ij} \\ \overline{Zw}_{ij} \\ \overline{DYw}_{ij} \\ \overline{DZw}_{ij} \\ \overline{w}_i \end{pmatrix}$$

We begin with $\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \Xi_{ij}'^w \Xi_{ij}^w$. The idea of the simplification, as above, is to express analytically $\sum_{j=1}^{N_2} \Xi_{ij}'^w \Xi_{ij}^w$ for each i so that the number of observations that must be considered in computing the double summation is N_1 instead of $N_1 N_2$. When $w_i = 1$ (i.e., $W_i = w$), the simplification is analogous to that for unconfounded matching. So, for instance, when $w_i = 1$ we can simplify

$$\sum_j \overline{Dw}_{ij}^2 = \begin{cases} N_2 E[Dw]^2 & i \text{ unlinked and } w_i = 1 \\ (1 - E[Dw])^2 + (N_2 - 1)E[Dw]^2 & i \text{ linked and } w_i = 1 \end{cases}.$$

In what follows, we present the simplifications when $w_i = 0$ or when the new term $\overline{w}_i$ is involved.

$$\sum_j \overline{Dw}_{ij}^2 = \begin{cases} N_2 E[Dw]^2 & w_i = 0 \end{cases}$$

$$\sum_j \overline{Dw}_{ij} \overline{Yw}_{ij} = \begin{cases} N_2 E[Dw]E[Yw] & w_i = 0 \end{cases}$$

$$\sum_j \overline{Dw}_{ij} \overline{Zw}_{ij} = \begin{cases} N_2 E[Dw]E[Zw] & w_i = 0 \end{cases}$$

$$\sum_j \overline{Dw}_{ij} \overline{DYw}_{ij} = \begin{cases} N_2 E[Dw]E[DYw] & w_i = 0 \end{cases}$$

$$\sum_j \overline{Dw}_{ij} \overline{DZw}_{ij} = \begin{cases} N_2 E[Dw]E[DZw] & w_i = 0 \end{cases}$$

$$\sum_j \overline{Dw}_{ij} \overline{w}_i = \begin{cases} N_2 E[Dw]E[w] & w_i = 0 \\ -N_2 E[Dw](1 - E[w]) & w_i = 1 \text{ and } i \text{ unlinked} \\ (1 - E[Dw])(1 - E[w]) - (N_2 - 1)E[Dw](1 - E[w]) & w_i = 1 \text{ and } i \text{ linked} \end{cases}$$

$$\sum_j \overline{Yw}_{ij}^2 = \begin{cases} N_2 E[Yw]^2 & w_i = 0 \end{cases}$$

$$\sum_j \overline{Yw}_{ij} \overline{Zw}_{ij} = \begin{cases} N_2 E[Yw]E[Zw] & w_i = 0 \end{cases}$$

$$\sum_j \overline{Yw}_{ij} \overline{DYw}_{ij} = \begin{cases} N_2 E[Yw]E[DYw] & w_i = 0 \end{cases}$$

$$\sum_j \overline{Yw}_{ij} \overline{DZw}_{ij} = \begin{cases} N_2 E[Yw]E[DZw] & w_i = 0 \end{cases}$$

$$\sum_j \overline{Yw}_{ij} \overline{w}_i = \begin{cases} N_2 E[Yw]E[w] & w_i = 0 \\ -N_i^{Y_0} E[Yw](1 - E[w]) + (N_2 - N_i^{Y_0})(1 - E[Yw])(1 - E[w]) & w_i = 1 \end{cases}$$

$$\sum_j \overline{Zw}_{ij}^2 = \begin{cases} N_2 E[Zw]^2 & w_i = 0 \end{cases}$$

$$\sum_j \overline{Zw}_{ij} \overline{DYw}_{ij} = \begin{cases} N_2 E[Zw]E[DYw] & w_i = 0 \end{cases}$$

$$\sum_j \overline{Zw}_{ij} \overline{DZw}_{ij} = \begin{cases} N_2 E[Zw]E[DZw] & w_i = 0 \end{cases}$$

$$\sum_j \overline{Zw}_{ij} \overline{w}_i = \begin{cases} N_2 E[Zw]E[w] & w_i = 0 \\ -N_i^{Z_0} E[Zw](1 - E[w]) + (N_2 - N_i^{Z_0})(1 - E[Zw])(1 - E[w]) & w_i = 1 \end{cases}$$

$$\sum_j \overline{DYw}_{ij}^2 = \begin{cases} N_2 E[DYw]^2 & w_i=0 \end{cases}$$

$$\sum_j \overline{DYw}_{ij}\overline{DZw}_{ij} = \begin{cases} N_2 E[DYw]E[DZw] & w_i=0 \end{cases}$$

$$\sum_j \overline{DYw}_{ij}\overline{w}_i = \begin{cases} N_2 E[DYw]E[w] & w_i=0 \\ -N_2(1-E[w])E[DYw] & w_i=1 \text{ and } i \text{ unlinked or linked stayer} \\ (1-E[w])(1-E[DYw])-(N_2-1)(1-E[w])E[DYw] & w_i=1 \text{ and } i \text{ linked mover} \end{cases}$$

$$\sum_j \overline{DZw}_{ij}^2 = \begin{cases} N_2 E[DZw]^2 & w_i=0 \end{cases}$$

$$\sum_j \overline{DZw}_{ij}\overline{w}_i = \begin{cases} N_2 E[DZw]E[w] & w_i=0 \\ -N_2(1-E[w])E[DZw] & w_i=1 \text{ and } i \text{ unlinked or linked non-discrepancy} \\ (1-E[w])(1-E[DZw])-(N_2-1)(1-E[w])E[DZw] & w_i=1 \text{ and } i \text{ linked discrepancy} \end{cases}$$

$$\sum_j \overline{w}_i^2 = \begin{cases} N_2 E[w]^2 & w_i=0 \\ N_2(1-E[w])^2 & w_i=1 \end{cases}$$

Next, we consider $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi_{ij}'^{w}\Xi_{ij'}^{w}$. As above, the idea of the simplification is to express analytically $\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi_{ij}'^{w}\Xi_{ij'}^{w}$ for each i so that the number of observations that must be considered in computing the double summation is N_1 instead of N_1N_2. When $w_i = 1$ (i.e., $W_i = w$), the simplification is analogous to that for unconfounded matching. In what follows, we present the simplifications when $w_i = 0$ or when the new term $\overline{w}_i$ is involved.

$$\sum_j \sum_{j'\neq j} \overline{Dw}_{ij}\overline{Dw}_{ij'} = \begin{cases} N_2(N_2-1)E[Dw]^2 & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Dw}_{ij}\overline{Yw}_{ij'} = \begin{cases} N_2(N_2-1)E[Dw]E[Yw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Dw}_{ij}\overline{Zw}_{ij'} = \begin{cases} N_2(N_2-1)E[Dw]E[Zw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Dw}_{ij}\overline{DYw}_{ij'} = \begin{cases} N_2(N_2-1)E[Dw]E[DYw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Dw}_{ij}\overline{DZw}_{ij'} = \begin{cases} N_2(N_2-1)E[Dw]E[DZw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Dw}_{ij}\overline{w}_i = \begin{cases} N_2(N_2-1)E[Dw]E[w] & w_i=0 \\ -N_2(N_2-1)(1-E[w])E[Dw] & w_i=1 \text{ and } i \text{ unlinked} \\ (1-E[Dw])(1-E[w])(N_2-1)-(N_2-1)^2(1-E[w])E[Dw] & w_i=1 \text{ and } i \text{ linked} \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Yw}_{ij}\overline{Dw}_{ij'} = \begin{cases} N_2(N_2-1)E[Yw]E[Dw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Yw}_{ij}\overline{Yw}_{ij'} = \begin{cases} N_2(N_2-1)E[Yw]^2 & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Yw}_{ij}\overline{Zw}_{ij'} = \begin{cases} N_2(N_2-1)E[Yw]E[Zw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Yw}_{ij}\overline{DYw}_{ij'} = \begin{cases} N_2(N_2-1)E[Yw]E[DYw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Yw}_{ij}\overline{DZw}_{ij'} = \begin{cases} N_2(N_2-1)E[Yw]E[DZw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Yw}_{ij}\overline{w}_i = \begin{cases} N_2(N_2-1)E[Yw]E[w] & w_i=0 \\ (N_2-N_i^{Y0})(N_2-1)(1-E[w])(1-E[Yw])-N_i^{Y0}(N_2-1)(1-E[w])E[Yw] & w_i=1 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Zw}_{ij}\overline{Dw}_{ij'} = \begin{cases} N_2(N_2-1)E[Zw]E[Dw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j'\neq j} \overline{Zw}_{ij}\overline{Yw}_{ij'} = \begin{cases} N_2(N_2-1)E[Zw]E[Yw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Zw}_{ij} \overline{Zw}_{ij'} = \begin{cases} N_2(N_2-1)E[Zw]^2 & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Zw}_{ij} \overline{DYw}_{ij'} = \begin{cases} N_2(N_2-1)E[Zw]E[DYw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Zw}_{ij} \overline{DZw}_{ij'} = \begin{cases} N_2(N_2-1)E[Zw]E[DZw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{Zw}_{ij} \overline{w}_i = \begin{cases} N_2(N_2-1)E[Zw]E[w] & w_i=0 \\ (N_2-N_i^{Z0})(N_2-1)(1-E[w])(1-E[Zw]) - N_i^{Z0}(N_2-1)(1-E[w])E[Zw] & w_i=1 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DYw}_{ij} \overline{Dw}_{ij'} = \begin{cases} N_2(N_2-1)E[DYw]E[Dw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DYw}_{ij} \overline{Yw}_{ij'} = \begin{cases} N_2(N_2-1)E[DYw]E[Yw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DYw}_{ij} \overline{Zw}_{ij'} = \begin{cases} N_2(N_2-1)E[DYw]E[Zw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DYw}_{ij} \overline{DYw}_{ij'} = \begin{cases} N_2(N_2-1)E[DYw]^2 & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DYw}_{ij} \overline{DZw}_{ij'} = \begin{cases} N_2(N_2-1)E[DYw]E[DZw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DYw}_{ij} \overline{w}_i = \begin{cases} N_2(N_2-1)E[DYw]E[w] & w_i=0 \\ -N_2(N_2-1)(1-E[w])E[DYw] & w_i=1 \text{ and } i \text{ unlinked or linked stayer} \\ (1-E[DYw])(1-E[w])(N_2-1) - (N_2-1)^2(1-E[w])E[DYw] & w_i=1 \text{ and } i \text{ linked mover} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZw}_{ij} \overline{Dw}_{ij'} = \begin{cases} N_2(N_2-1)E[DZw]E[Dw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZw}_{ij} \overline{Yw}_{ij'} = \begin{cases} N_2(N_2-1)E[DZw]E[Yw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZw}_{ij} \overline{Zw}_{ij'} = \begin{cases} N_2(N_2-1)E[DZw]E[Zw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZw}_{ij} \overline{DYw}_{ij'} = \begin{cases} N_2(N_2-1)E[DZw]E[DYw] & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZw}_{ij} \overline{DZw}_{ij'} = \begin{cases} N_2(N_2-1)E[DZw]^2 & w_i=0 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{DZw}_{ij} \overline{w}_i = \begin{cases} N_2(N_2-1)E[DZw]E[w] & w_i=0 \\ -N_2(N_2-1)(1-E[w])E[DZw] & w_i=1 \text{ and } i \text{ unlinked or linked non-discrepancy} \\ (1-E[DZw])(1-E[w])(N_2-1) - (N_2-1)^2(1-E[w])E[DZw] & w_i=1 \text{ and } i \text{ linked discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{w}_i \overline{Dw}_{ij'} = \begin{cases} N_2(N_2-1)E[w]E[Dw] & w_i=0 \\ -N_2(N_2-1)E[Dw](1-E[w]) & w_i=1 \text{ and } i \text{ unlinked} \\ -(N_2-1)E[Dw](1-E[w]) + (N_2-1)(1-E[w])\{-(N_2-2)E[Dw] + (1-E[Dw)\} & w_i=1 \text{ and } i \text{ linked} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{w}_i \overline{Yw}_{ij'} = \begin{cases} N_2(N_2-1)E[w]E[Yw] & w_i=0 \\ (1-E[w])\{(N_2-N_i^{Y0})[-N_i^{Y0}E[Yw] + (N_2-N_i^{Y0}-1)(1-E[Yw])] \\ \quad + N_i^{Y0}[-(N_i^{Y0}-1)E[Yw] + (N_2-N+i^{Y0})(1-E[Yw])]\} & w_i=1 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{w}_i \overline{Zw}_{ij'} = \begin{cases} N_2(N_2-1)E[w]E[Zw] & w_i=0 \\ (1-E[w])\{(N_2-N_i^{Z0})[-N_i^{Z0}E[Zw] + (N_2-N_i^{Z0}-1)(1-E[Zw])] \\ \quad + N_i^{Z0}[-(N_i^{Z0}-1)E[Zw] + (N_2-N+i^{Z0})(1-E[Zw])]\} & w_i=1 \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{w}_i \overline{DYw}_{ij'} = \begin{cases} N_2(N_2-1)E[w]E[Dw] & w_i=0 \\ -N_2(N_2-1)E[DYw](1-E[w]) & w_i=1 \text{ and } i \text{ unlinked or linked stayer} \\ -(N_2-1)E[DYw](1-E[w]) + (N_2-1)(1-E[w])\{-(N_2-2)E[DYw] + (1-E[DYw)\} & w_i=1 \text{ and } i \text{ linked mover} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{w}_i \overline{DZw}_{ij'} = \begin{cases} N_2(N_2-1)E[w]E[Dw] & w_i=0 \\ -N_2(N_2-1)E[DZw](1-E[w]) & w_i=1 \text{ and } i \text{ unlinked or linked non-discrepancy} \\ -(N_2-1)E[DZw](1-E[w]) \\ \quad + (N_2-1)(1-E[w])\{-(N_2-2)E[DZw] + (1-E[DZw)\} & w_i=1 \text{ and } i \text{ linked discrepancy} \end{cases}$$

$$\sum_j \sum_{j' \neq j} \overline{w}_i \overline{w}_i = \begin{cases} N_2(N_2-1)E[w]^2 & w_i=0 \\ N_2(N_2-1)(1-E[w])^2 & w_i=1 \end{cases}$$

Unlike in the unconfounded case, the other triple sum, $\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{i' \neq i} \Xi'^2_{ij} \Xi^2_{i'j}$ cannot be simplified analogously. This is because the data available in each census are no longer symmetric, since w_i is observed only in the first census. Thus, we simplify it separately here. Analogous to the prior cases, the idea is to derive $\sum_{i=1}^{N_1} \sum_{i' \neq i} \Xi'_{ij} \Xi_{i'j}$ analytically for each j, meaning that the total sum can be computed in the second census. Define N_1^w as the number of observations in census 1 with $W_i = w$, w_i^* as the value of w_i for the observation i in census 1 to which an observation j in census 2 is linked, $N_j^{Y_0 w_1}$ as the number of observations in census 1 with the same location as observation j and $W = w$, and $N_j^{Z_0 w_1}$, etc. analogously. Define also Y_{i^*j} as the value of Y_{ij} for observation j in census 2 and the observation i^* to which it is linked in census 1, and define Z_{i^*j} analogously. The simplifications are as follows.

$$\sum_i \sum_{i' \neq i} \overline{Dw}_{ij} \overline{Dw}_{i'j} = \begin{cases} N_1(N_1-1)E[Dw]^2 & j \text{ unlinked} \\ -(N_1-1)E[Dw](w_{i^*}-E[Dw]) - (N_1-1)E[Dw]\{w_{i^*}-(N_1-1)E[Dw]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Dw}_{ij} \overline{Yw}_{i'j} = \begin{cases} -E[Dw]\{N_1[N_1^w - N_j^{Y_0 w_1} - (N_1-1)E[Yw]] - (N_1^w - N_j^{Y_0 w_1})\} & j \text{ unlinked} \\ (w_{i^*}-E[Dw])\{N_1^w - N_j^{Y_0 w_1} - (N_1-1)E[Yw] - w_{i^*} Y_{i^*j}\} & \\ \quad -E[Dw]\{(N_1-1)[N_1^w - N_j^{Y_0 w_1} - (N_1-1)E[Yw]] - (N_1^w - N_j^{Y_0 w_1} - w_{i^*} Y_{i^*j})\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Dw}_{ij} \overline{Zw}_{i'j} = \begin{cases} -E[Dw]\{N_1[N_1^w - N_j^{Z_0 w_1} - (N_1-1)E[Zw]] - (N_1^w - N_j^{Z_0 w_1})\} & j \text{ unlinked} \\ (w_{i^*}-E[Dw])\{N_1^w - N_j^{Z_0 w_1} - (N_1-1)E[Zw] - w_{i^*} Z_{i^*j}\} & \\ \quad -E[Dw]\{(N_1-1)[N_1^w - N_j^{Z_0 w_1} - (N_1-1)E[Zw]] - (N_1^w - N_j^{Z_0 w_1} - w_{i^*} Z_{i^*j})\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Dw}_{ij} \overline{DYw}_{i'j} = \begin{cases} N_1(N_1-1)E[Dw]E[DYw] & j \text{ unlinked} \\ -(N_1-1)E[DYw](w_{i^*}-E[Dw]) - (N_1-1)E[Dw]\{w_{i^*} Y_{i^*j} - (N_1-1)E[DYw]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Dw}_{ij} \overline{DZw}_{i'j} = \begin{cases} N_1(N_1-1)E[Dw]E[DZw] & j \text{ unlinked} \\ -(N_1-1)E[DZw](w_{i^*}-E[Dw]) - (N_1-1)E[Dw]\{w_{i^*} Z_{i^*j} - (N_1-1)E[DZw]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Dw}_{ij} \overline{w}_{i'} = \begin{cases} -N_1 E[Dw](N_1^w - (N_1-1)E[w]) + N_1^w E[Dw] & j \text{ unlinked} \\ (w_{i^*}-E[Dw])(N_1^w - (N_1-1)E[w] - w_{i^*}) - (N_1-1)E[Dw](N_1^w - (N_1-1)E[w]) & \\ \quad +(N_1^w - w_{i^*})E[Dw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Yw}_{ij} \overline{Dw}_{i'j} = \begin{cases} -(N_1^w - N_j^{Y_0 w_1})(N_1-1)E[Dw] + N_1 E[Yw](N_1-1)E[Dw] & j \text{ unlinked} \\ -(w_{i^*} Y_{i^*j} - E[Yw](N_1-1)E[Dw] + (N_1^w - N_j^{Y_0 w_1} - w_{i^*} Y_{i^*j})[w_{i^*} - (N_1-1)E[Dw]] & \\ \quad -(N_1-1)[w_{i^*} - (N_1-1)E[Dw]]E[Yw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Yw}_{ij} \overline{Yw}_{i'j} = (N_1^w - N_j^{Y_0 w_1})(N_1^w - N_j^{Y_0 w_1} - (N_1-1)E[Yw]) - (N_1^w - N_j^{Y_0 w_1})$$
$$-N_1 E[Yw](N_1^w - N_j^{Y_0 w_1} - (N_1-1)E[Yw] + (N_1^w - N_j^{Y_0 w_1})E[Yw]$$

$$\sum_i \sum_{i' \neq i} \overline{Yw}_{ij} \overline{Zw}_{i'j} = (N_1^w - N_j^{Y_0 w_1})(N_1^w - N_j^{Z_0 w_1} - (N_1-1)E[Zw]) - N_j^{Y_1 Z_1 w_1}$$
$$-N_1 E[Yw](N_1^w - N_j^{Z_0 w_1} - (N_1-1)E[Zw]) + (N_1^w - N_j^{Z_0 w_1})E[Yw]$$

$$\sum_i \sum_{i' \neq i} \overline{Yw}_{ij} \overline{DYw}_{i'j} = \begin{cases} -(N_1^w - N_j^{Y_0 w_1})(N_1-1)E[DYw] + N_1 E[Yw](N_1-1)E[DYw] & j \text{ unlinked} \\ -(w_{i^*} Y_{i^*j} - E[Yw])(N_1-1)E[DYw] + (N_1^w - N_j^{Y_0 w_1} - w_{i^*} Y_{i^*j})[w_{i^*} Y_{i^*j} - (N_1-1)E[DYw]] & \\ \quad -N_1-1)[w_{i^*} Y_{i^*j} - (N_1-1)E[DYw]]E[Yw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Yw}_{ij} \overline{DZw}_{i'j} = \begin{cases} -(N_1^w - N_j^{Y_0 w_1})(N_1-1)E[DZw] + N_1 E[Yw](N_1-1)E[DZw] & j \text{ unlinked} \\ -(w_{i^*} Y_{i^*j} - E[Yw])(N_1-1)E[DZw] + (N_1^w - N_j^{Y_0 w_1} - w_{i^*} Y_{i^*j})[w_{i^*} Y_{i^*j} - (N_1-1)E[DZw]] & \\ \quad -N_1-1)[w_{i^*} Y_{i^*j} - (N_1-1)E[DZw]]E[Yw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Yw}_{ij} \overline{w}_{i'} = (N_1^w - N_j^{Y_0 w_1})(N_1^w - (N_1-1)E[w]) - (N_1^w - N_j^{Y_0 w_1}) - N_1 E[Yw](N_1^w - (N_1-1)E[w]) + N_1^w E[Yw]$$

$$\sum_i \sum_{i' \neq i} \overline{Zw}_{ij} \overline{Dw}_{i'j} = \begin{cases} -(N_1^w - N_j^{Z_0 w_1})(N_1 - 1)E[Dw] + N_1 E[Zw](N_1 - 1)E[Dw] & j \text{ unlinked} \\ -(w_{i*} Z_{i*j} - E[Zw](N_1 - 1)E[Dw] + (N_1^w - N_j^{Z_0 w_1} - w_{i*} Z_{i*j})[w_{i*} - (N_1 - 1)E[Dw]] \\ \quad -(N_1 - 1)[w_{i*} - (N_1 - 1)E[Dw]]E[Zw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Zw}_{ij} \overline{Yw}_{i'j} = (N_1^w - N_j^{Z_0 w_1})(N_1^w - N_j^{Y_0 w_1} - (N_1 - 1)E[Yw]) - N_j^{Z_1 Y_1 w_1}$$

$$- N_1 E[Zw](N_1^w - N_j^{Y_0 w_1} - (N_1 - 1)E[Yw]) + (N_1^w - N_j^{Y_0 w_1})E[Zw]$$

$$\sum_i \sum_{i' \neq i} \overline{Zw}_{ij} \overline{Zw}_{i'j} = (N_1^w - N_j^{Z_0 w_1})(N_1^w - N_j^{Z_0 w_1} - (N_1 - 1)E[Zw]) - (N_1^w - N_j^{Z_0 w_1})$$

$$- N_1 E[Zw](N_1^w - N_j^{Z_0 w_1} - (N_1 - 1)E[Zw] + (N_1^w - N_j^{Z_0 w_1})E[Zw]$$

$$\sum_i \sum_{i' \neq i} \overline{Zw}_{ij} \overline{DYw}_{i'j} = \begin{cases} -(N_1^w - N_j^{Z_0 w_1})(N_1 - 1)E[DYw] + N_1 E[Zw](N_1 - 1)E[DYw] & j \text{ unlinked} \\ -(w_{i*} Z_{i*j} - E[Zw])(N_1 - 1)E[DYw] + (N_1^w - N_j^{Z_0 w_1} - w_{i*} Z_{i*j})[w_{i*} Z_{i*j} - (N_1 - 1)E[DYw]] \\ \quad -N_1 - 1)[w_{i*} Z_{i*j} - (N_1 - 1)E[DYw]]E[Zw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Zw}_{ij} \overline{DZw}_{i'j} = \begin{cases} -(N_1^w - N_j^{Z_0 w_1})(N_1 - 1)E[DZw] + N_1 E[Zw](N_1 - 1)E[DZw] & j \text{ unlinked} \\ -(w_{i*} Z_{i*j} - E[Zw])(N_1 - 1)E[DZw] + (N_1^w - N_j^{Z_0 w_1} - w_{i*} Z_{i*j})[w_{i*} Z_{i*j} - (N_1 - 1)E[DZw]] \\ \quad -N_1 - 1)[w_{i*} Z_{i*j} - (N_1 - 1)E[DZw]]E[Zw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{Zw}_{ij} \overline{w}_{i'} = (N_1^w - N_j^{Z_0 w_1})(N_1^w - (N_1 - 1)E[w]) - (N_1^w - N_j^{Z_0 w_1}) - N_1 E[Zw](N_1^w - (N_1 - 1)E[w]) + N_1^w E[Zw]$$

$$\sum_i \sum_{i' \neq i} \overline{DYw}_{ij} \overline{Dw}_{i'j} = \begin{cases} N_1(N_1 - 1)E[Dw]E[DYw] & j \text{ unlinked} \\ -(N_1 - 1)E[Dw](w_{i*} Y_{i*j} - E[DYw]) - (N_1 - 1)E[DYw]\{w_{i*} - (N_1 - 1)E[Dw]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DYw}_{ij} \overline{Yw}_{i'j} = \begin{cases} -E[DYw]\{N_1[N_1^w - N_j^{Y_0 w_1} - (N_1 - 1)E[Yw]] - (N_1^w - N_j^{Y_0 w_1})\} & j \text{ unlinked} \\ (w_{i*} Y_{i*j} - E[DYw])\{N_1^w - N_j^{Y_0 w_1} - (N_1 - 1)E[Yw] - w_{i*} Y_{i*j}\} \\ \quad -E[DYw]\{(N_1 - 1)[N_1^w - N_j^{Y_0 w_1} - (N_1 - 1)E[Yw]] - (N_1^w - N_j^{Y_0 w_1} - w_{i*} Y_{i*j})\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DYw}_{ij} \overline{Zw}_{i'j} = \begin{cases} -E[DYw]\{N_1[N_1^w - N_j^{Z_0 w_1} - (N_1 - 1)E[Zw]] - (N_1^w - N_j^{Z_0 w_1})\} & j \text{ unlinked} \\ (w_{i*} Y_{i*j} - E[DYw])\{N_1^w - N_j^{Z_0 w_1} - (N_1 - 1)E[Zw] - w_{i*} Z_{i*j}\} \\ \quad -E[DYw]\{(N_1 - 1)[N_1^w - N_j^{Z_0 w_1} - (N_1 - 1)E[Zw]] - (N_1^w - N_j^{Z_0 w_1} - w_{i*} Z_{i*j})\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DYw}_{ij} \overline{DYw}_{i'j} = \begin{cases} N_1(N_1 - 1)E[DYw]E[DYw] & j \text{ unlinked} \\ -(N_1 - 1)E[DYw](w_{i*} Y_{i*j} - E[DYw]) - (N_1 - 1)E[DYw]\{w_{i*} Y_{i*j} - (N_1 - 1)E[DYw]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DYw}_{ij} \overline{DZw}_{i'j} = \begin{cases} N_1(N_1 - 1)E[DYw]E[DZw] & j \text{ unlinked} \\ -(N_1 - 1)E[DZw](w_{i*} Y_{i*j} - E[DYw]) - (N_1 - 1)E[DYw]\{w_{i*} Z_{i*j} - (N_1 - 1)E[DZw]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DYw}_{ij} \overline{w}_{i'} = \begin{cases} -N_1 E[DYw](N_1^w - (N_1 - 1)E[w]) + N_1^w E[DYw] & j \text{ unlinked} \\ (w_{i*} Y_{i*j} - E[DYw])(N_1^w - (N_1 - 1)E[w] - w_{i*}) - (N_1 - 1)E[DYw](N_1^w - (N_1 - 1)E[w]) \\ \quad +(N_1^w - w_{i*})E[DYw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DZw}_{ij} \overline{Dw}_{i'j} = \begin{cases} N_1(N_1 - 1)E[Dw]E[DZw] & j \text{ unlinked} \\ -(N_1 - 1)E[Dw](w_{i*} Z_{i*j} - E[DZw]) - (N_1 - 1)E[DZw]\{w_{i*} - (N_1 - 1)E[Dw]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DZw}_{ij} \overline{Yw}_{i'j} = \begin{cases} -E[DZw]\{N_1[N_1^w - N_j^{Y_0 w_1} - (N_1 - 1)E[Yw]] - (N_1^w - N_j^{Y_0 w_1})\} & j \text{ unlinked} \\ (w_{i*} Z_{i*j} - E[DZw])\{N_1^w - N_j^{Y_0 w_1} - (N_1 - 1)E[Yw] - w_{i*} Y_{i*j}\} \\ \quad -E[DZw]\{(N_1 - 1)[N_1^w - N_j^{Y_0 w_1} - (N_1 - 1)E[Yw]] - (N_1^w - N_j^{Y_0 w_1} - w_{i*} Y_{i*j})\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DZw}_{ij} \overline{Zw}_{i'j} = \begin{cases} -E[DZw]\{N_1[N_1^w - N_j^{Z_0 w_1} - (N_1 - 1)E[Zw]] - (N_1^w - N_j^{Z_0 w_1})\} & j \text{ unlinked} \\ (w_{i*} Z_{i*j} - E[DZw])\{N_1^w - N_j^{Z_0 w_1} - (N_1 - 1)E[Zw] - w_{i*} Z_{i*j}\} \\ \quad -E[DZw]\{(N_1 - 1)[N_1^w - N_j^{Z_0 w_1} - (N_1 - 1)E[Zw]] - (N_1^w - N_j^{Z_0 w_1} - w_{i*} Z_{i*j})\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DZw}_{ij} \overline{DYw}_{i'j} = \begin{cases} N_1(N_1 - 1)E[DZw]E[DYw] & j \text{ unlinked} \\ -(N_1 - 1)E[DYw](w_{i*} Z_{i*j} - E[DZw]) - (N_1 - 1)E[DZw]\{w_{i*} Y_{i*j} - (N_1 - 1)E[DYw]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DZw}_{ij} \overline{DZw}_{i'j} = \begin{cases} N_1(N_1 - 1)E[DZw]E[DZw] & j \text{ unlinked} \\ -(N_1 - 1)E[DZw](w_{i*} Z_{i*j} - E[DZw]) - (N_1 - 1)E[DZw]\{w_{i*} Z_{i*j} - (N_1 - 1)E[DZw]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{DZw}_{ij} \overline{w}_{i'} = \begin{cases} -N_1 E[DZw](N_1^w - (N_1-1)E[w]) + N_1^w E[DZw] & j \text{ unlinked} \\ (w_{i^*} Z_{i^* j} - E[DZw])(N_1^w - (N_1-1)E[w] - w_{i^*}) - (N_1-1)E[DZw](N_1^w - (N_1-1)E[w]) & \\ \quad + (N_1^w - w_{i^*})E[DZw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{w}_i \overline{Dw}_{i'j} = \begin{cases} -(N_1^w - N_1 E[w])(N_1-1)E[Dw] & j \text{ unlinked} \\ -(w_{i^*} - E[w])(N_1-1)E[Dw] + (N_1^w - w_{i^*} - (N_1-1)E[w])[w_{i^*} - (N_1-1)E[Dw]] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{w}_i \overline{Yw}_{i'j} = N_1^w (N_1^w - N_y^{Y_0 w_1} - (N_1-1)E[Yw]) - (N_1^w - N_j^{Y_0 w_1}) - N_1 E[w](N_1^w - N_j^{Y_0 w_1} - (N_1-1)E[Yw]) + (N_1^w - N_j^{Y_0 w_1})E[w]$$

$$\sum_i \sum_{i' \neq i} \overline{w}_i \overline{Zw}_{i'j} = N_1^w (N_1^w - N_Z^{Z_0 w_1} - (N_1-1)E[Zw]) - (N_1^w - N_j^{Z_0 w_1}) - N_1 E[w](N_1^w - N_j^{Z_0 w_1} - (N_1-1)E[Zw]) + (N_1^w - N_j^{Z_0 w_1})E[w]$$

$$\sum_i \sum_{i' \neq i} \overline{w}_i \overline{DYw}_{i'j} = \begin{cases} -(N_1^w - N_1 E[w])(N_1-1)E[DYw] & j \text{ unlinked} \\ -(w_{i^*} - E[w])(N_1-1)E[DYw] + (N_1^w - w_{i^*} - (N_1-1)E[w])[w_{i^*} Y_{i^* j} - (N_1-1)E[DYw]] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{w}_i \overline{DZw}_{i'j} = \begin{cases} -(N_1^w - N_1 E[w])(N_1-1)E[DZw] & j \text{ unlinked} \\ -(w_{i^*} - E[w])(N_1-1)E[DZw] + (N_1^w - w_{i^*} - (N_1-1)E[w])[w_{i^*} Z_{i^* j} - (N_1-1)E[DZw]] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \overline{w}_i \overline{w}_{i'} = N_1^w (N_1^w - (N-1-1)E[w]) - N_1^w - N_1 E[w](N_1^w - (N_1-1)E[w]) + N_1^w E[w]$$

Next, we consider $\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{i' \neq i} \sum_{j' \neq j} \Xi_{ij}^{\prime w} D_{i'j} (1 - \hat{\pi}(w)) \Xi_{i'j'}^{w}$. Since we can factor the $(1 - \hat{\pi}(w))$ from the quadruple summation, we ignore it in our simplifications here. The idea of the simplification is to express analytically $\sum_{i=1}^{N_1} \sum_{i' \neq i} \sum_{j' \neq j} \Xi_{ij}^{\prime w} D_{i'j} \Xi_{i'j'}^{w}$ for each j so that the number of observations that must be considered in computing the double summation is N_2 instead of $N_1 N_2$ and it can be computed from the second census. The simplifications are as follows.

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{Dw}_{ij} D_{i'j} \overline{Dw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[Dw]^2 & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{Dw}_{ij} D_{i'j} \overline{Yw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)E[Dw](N_2-1)E[Yw] & j \text{ linked and } w_{i^*}=0 \\ -E[Dw](N_1-1)\{-N_{i^*}^{Y_0} E[Yw] + (N_2 - N_{i^*}^{Y_0} - 1)(1 - E[Yw])\} & j \text{ linked mover and } w_{i^*}=1 \\ -E[Dw](N_1-1)\{-(N_{i^*}^{Y_0} - 1)E[Yw] + (N_2 - N_{i^*}^{Y_0})(1 - E[Yw])\} & j \text{ linked stayer and } w_{i^*}=1 \end{cases}$$

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{Dw}_{ij} D_{i'j} \overline{Zw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)E[Dw](N_2-1)E[Zw] & j \text{ linked and } w_{i^*}=0 \\ -E[Dw](N_1-1)\{-N_{i^*}^{Z_0} E[Zw] + (N_2 - N_{i^*}^{Z_0} - 1)(1 - E[Zw])\} & j \text{ linked discrepancy and } w_{i^*}=1 \\ -E[Dw](N_1-1)\{-(N_{i^*}^{Z_0} - 1)E[Zw] + (N_2 - N_{i^*}^{Z_0})(1 - E[Zw])\} & j \text{ linked non-discrepancy and } w_{i^*}=1 \end{cases}$$

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{Dw}_{ij} D_{i'j} \overline{DYw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ E[Dw](N_1-1)(N_2-1)E[DYw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{Dw}_{ij} D_{i'j} \overline{DZw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ E[Dw](N_1-1)(N_2-1)E[DZw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{Dw}_{ij} D_{i'j} \overline{w}_{i'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[Dw]E[w] & j \text{ linked and } w_{i^*}=0 \\ -(N_1-1)(N_2-1)E[Dw](1-E[w]) & j \text{ linked and } w_{i^*}=1 \end{cases}$$

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{Yw}_{ij} D_{i'j} \overline{Dw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_1^w - N_j^{Y_0 w_1} - Y_{i^* j} w_{i^*})(1 - E[Yw])(N_2-1)E[Dw] & \\ \quad + (N_1 - N_1^w + N_j^{Y_0 w_1} - (1 - Y_{i^* j} w_{i^*}))E[Yw](N_2-1)E[Dw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{Yw}_{ij} D_{i'j} \overline{Yw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_1^w - N_j^{Y_0 w_1})(1 - E[Yw])(N_2-1)E[Yw] + (N_1 - N_1^w + N_j^{Y_0 w_1} - 1)E[YW]^2(N_2-1) & j \text{ linked and } w_{i^*}=0 \\ [(N_1^w - N_j^{Y_0 w_1} - 1)(1 - E[Yw]) - E[Yw](N_1 - N_1^w + N_j^{Y_0 w_1})] & \\ \quad \times \{-N_{i^*}^{Y_0} E[Yw] + (N_2 - N_{i^*}^{Y_0} - 1)(1 - E[Yw])\} & j \text{ linked mover and } w_{i^*}=1 \\ [(N_1^w - N_j^{Y_0 w_1})(1 - E[Yw]) - E[Yw](N_1 - N_1^w + N_j^{Y_0 w_1} - 1)] & \\ \quad \times \{-(N_{i^*}^{Y_0} - 1)E[Yw] + (N_2 - N_{i^*}^{Y_0})(1 - E[Yw])\} & j \text{ linked stayer and } w_{i^*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Yw}_{ij} D_{i'j} \overline{Zw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -[(N_1^w - N_j^{Y_0 w_1} Y_{i^* j} w_{i^*})(1-E[Yw]) & \\ \quad -(N_1 - N_1^w + N_j^{Y_0 w_1} - (1-Y_{i^* j} w_{i^*}))E[Yw]](N_2-1)E[Zw] & j \text{ linked and } w_{i^*}=0 \\ [(N_1^w - N_j^{Y_0 w_1} Y_{i^* j} w_{i^*})(1-E[Yw]) & \\ \quad -(N_1 - N_1^w + N_j^{Y_0 w_1} - (1-Y_{i^* j} w_{i^*}))E[Yw]] & \\ \quad \times\{-N_{i^*}^{Z_0} E[Zw] + (N_2 - N_{i^*}^{Z_0} - 1)(1-E[Zw])\} & j \text{ linked discrepancy and } w_{i^*}=1 \\ [(N_1^w - N_j^{Y_0 w_1} Y_{i^* j} w_{i^*})(1-E[Yw]) & \\ \quad -(N_1 - N_1^w + N_j^{Y_0 w_1} - (1-Y_{i^* j} w_{i^*}))E[Yw]] & \\ \quad \times\{-(N_{i^*}^{Z_0} - 1)E[Zw] + (N_2 - N_{i^*}^{Z_0})(1-E[Zw])\} & j \text{ linked non-discrepancy and } w_{i^*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Yw}_{ij} D_{i'j} \overline{DYw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -[(N_1^w - N_j^{Y_0 w_1} - Y_{i^* j} w_{i^*})(1-E[Yw]) - (N_1 - N_1^w + N_j^{Y_0 w_1} - (1-Y_{i^* j} w_{i^*}))E[Yw]] & \\ \quad \times (N_2-1)E[DYw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Yw}_{ij} D_{i'j} \overline{DZw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -[(N_1^w - N_j^{Y_0 w_1} - Y_{i^* j} w_{i^*})(1-E[Yw]) - (N_1 - N_1^w + N_j^{Y_0 w_1} - (1-Y_{i^* j} w_{i^*}))E[Yw]] & \\ \quad \times (N_2-1)E[DZw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Yw}_{ij} D_{i'j} \overline{w}_{i'} = \begin{cases} 0 & j \text{ unlinked} \\ -[(N_1^w - N_j^{Y_0 w_1} - Y_{i^* j} w_{i^*})(1-E[Yw]) - (N_1 - N_1^w + N_j^{Y_0 w_1} - (1-Y_{i^* j} w_{i^*}))E[Yw]] & \\ \quad \times (N_2-1)(w_{i^*} - E[w]) & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Zw}_{ij} D_{i'j} \overline{Dw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_1^w - N_j^{Z_0 w_1} - Z_{i^* j} w_{i^*})(1-E[Zw])(N_2-1)E[Dw] & \\ \quad +(N_1 - N_1^w + N_j^{Z_0 w_1} - (1-Z_{i^* j} w_{i^*}))E[Zw](N_2-1)E[Dw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Zw}_{ij} D_{i'j} \overline{Yw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -[(N_1^w - N_j^{Z_0 w_1} Z_{i^* j} w_{i^*})(1-E[Zw]) & \\ \quad -(N_1 - N_1^w + N_j^{Z_0 w_1} - (1-Z_{i^* j} w_{i^*}))E[Zw]](N_2-1)E[Yw] & j \text{ linked and } w_{i^*}=0 \\ [(N_1^w - N_j^{Z_0 w_1} Z_{i^* j} w_{i^*})(1-E[Zw]) & \\ \quad -(N_1 - N_1^w + N_j^{Z_0 w_1} - (1-Z_{i^* j} w_{i^*}))E[Zw]] & \\ \quad \times\{-N_{i^*}^{Y_0} E[Yw] + (N_2 - N_{i^*}^{Y_0} - 1)(1-E[Yw])\} & j \text{ linked mover and } w_{i^*}=1 \\ [(N_1^w - N_j^{Z_0 w_1} Z_{i^* j} w_{i^*})(1-E[Zw]) & \\ \quad -(N_1 - N_1^w + N_j^{Z_0 w_1} - (1-Z_{i^* j} w_{i^*}))E[Zw]] & \\ \quad \times\{-(N_{i^*}^{Y_0} - 1)E[Yw] + (N_2 - N_{i^*}^{Y_0})(1-E[Yw])\} & j \text{ linked stayer and } w_{i^*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Zw}_{ij} D_{i'j} \overline{Zw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_1^w - N_j^{Z_0 w_1})(1-E[Zw])(N_2-1)E[Zw] + (N_1 - N_1^w + N_j^{Z_0 w_1} - 1)E[ZW]^2(N_2-1) & j \text{ linked and } w_{i^*}=0 \\ [(N_1^w - N_j^{Z_0 w_1} - 1)(1-E[Zw]) - E[Zw](N_1 - N_1^w + N_j^{Z_0 w_1})] & \\ \quad \times\{-N_{i^*}^{Z_0} E[Zw] + (N_2 - N_{i^*}^{Z_0} - 1)(1-E[Zw])\} & j \text{ linked discrepancy and } w_{i^*}=1 \\ [(N_1^w - N_j^{Z_0 w_1})(1-E[Zw]) - E[Zw](N_1 - N_1^w + N_j^{Z_0 w_1} - 1)] & \\ \quad \times\{-(N_{i^*}^{Z_0} - 1)E[Zw] + (N_2 - N_{i^*}^{Z_0})(1-E[Zw])\} & j \text{ linked non-discrepancy and } w_{i^*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Zw}_{ij} D_{i'j} \overline{DYw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -[(N_1^w - N_j^{Z_0 w_1} - Z_{i^* j} w_{i^*})(1-E[Zw]) - (N_1 - N_1^w + N_j^{Z_0 w_1} - (1-Z_{i^* j} w_{i^*}))E[Zw]] & \\ \quad \times (N_2-1)E[DYw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Zw}_{ij} D_{i'j} \overline{DZw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -[(N_1^w - N_j^{Z_0 w_1} - Z_{i^* j} w_{i^*})(1-E[Zw]) - (N_1 - N_1^w + N_j^{Z_0 w_1} - (1-Z_{i^* j} w_{i^*}))E[Zw]] & \\ \quad \times (N_2-1)E[DZw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Zw}_{ij} D_{i'j} \overline{w}_{i'} = \begin{cases} 0 & j \text{ unlinked} \\ -[(N_1^w - N_j^{Z_0 w_1} - Z_{i^* j} w_{i^*})(1-E[Zw]) - (N_1 - N_1^w + N_j^{Z_0 w_1} - (1-Z_{i^* j} w_{i^*}))E[Zw]] & \\ \quad \times (N_2-1)(w_{i^*} - E[w]) & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DYw}_{ij} D_{i'j} \overline{Dw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[Dw]E[DYw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DYw}_{ij} D_{i'j} \overline{Yw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DYw]E[Yw] & j \text{ linked and } w_{i*}=0 \\ -E[DYw](N_1-1)\{-N_{i*}^{Y_0}E[Yw]+(N_2-N_{i*}^{Y_0}-1)(1-E[Yw])\} & j \text{ linked mover and } w_{i*}=1 \\ -E[DYw](N_1-1)\{-(N_{i*}^{Y_0}-1)E[Yw]+(N_2-N_{i*}^{Y_0})(1-E[Yw])\} & j \text{ linked stayer and } w_{i*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DYw}_{ij} D_{i'j} \overline{Zw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DYw]E[Zw] & j \text{ linked and } w_{i*}=0 \\ -E[DYw](N_1-1)\{-N_{i*}^{Z_0}E[Zw]+(N_2-N_{i*}^{Z_0}-1)(1-E[Zw])\} & j \text{ linked discrepancy and } w_{i*}=1 \\ -E[DYw](N_1-1)\{-(N_{i*}^{Z_0}-1)E[Zw]+(N_2-N_{i*}^{Z_0})(1-E[Zw])\} & j \text{ linked non-discrepancy and } w_{i*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DYw}_{ij} D_{i'j} \overline{DYw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DYw]^2 & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DYw}_{ij} D_{i'j} \overline{DZw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DYw]E[DZw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DYw}_{ij} D_{i'j} \overline{w}_{i'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DYw]E[w] & j \text{ linked and } w_{i*}=0 \\ -(N_1-1)(N_2-1)E[DYw](1-E[w]) & j \text{ linked and } w_{i*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DZw}_{ij} D_{i'j} \overline{Dw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[Dw]E[DZw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DZw}_{ij} D_{i'j} \overline{Yw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DZw]E[Yw] & j \text{ linked and } w_{i*}=0 \\ -E[DZw](N_1-1)\{-N_{i*}^{Y_0}E[Yw]+(N_2-N_{i*}^{Y_0}-1)(1-E[Yw])\} & j \text{ linked mover and } w_{i*}=1 \\ -E[DZw](N_1-1)\{-(N_{i*}^{Y_0}-1)E[Yw]+(N_2-N_{i*}^{Y_0})(1-E[Yw])\} & j \text{ linked stayer and } w_{i*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DZw}_{ij} D_{i'j} \overline{Zw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DZw]E[Zw] & j \text{ linked and } w_{i*}=0 \\ -E[DZw](N_1-1)\{-N_{i*}^{Z_0}E[Zw]+(N_2-N_{i*}^{Z_0}-1)(1-E[Zw])\} & j \text{ linked discrepancy and } w_{i*}=1 \\ -E[DZw](N_1-1)\{-(N_{i*}^{Z_0}-1)E[Zw]+(N_2-N_{i*}^{Z_0})(1-E[Zw])\} & j \text{ linked non-discrepancy and } w_{i*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DZw}_{ij} D_{i'j} \overline{DYw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DZw]E[DYw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DZw}_{ij} D_{i'j} \overline{DZw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DZw]^2 & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{DZw}_{ij} D_{i'j} \overline{w}_{i'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)(N_2-1)E[DZw]E[w] & j \text{ linked and } w_{i*}=0 \\ -(N_1-1)(N_2-1)E[DZw](1-E[w]) & j \text{ linked and } w_{i*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{w}_i D_{i'j} \overline{Dw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_2-1)E[Dw]\{(N_1^w-w_{i*})(1-E[w])-(N_1-N_1^w-(1-w_{i*}))E[w]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{w}_i D_{i'j} \overline{Yw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_2-1)E[Yw]\{(N_1^w-w_{i*})(1-E[w])-(N_1-N_1^w-(1-w_{i*}))E[w]\} & j \text{ linked, } w_{i*}=0 \\ \{-(N_{i*}^{Y_0}-(1-Y_{i*j}))E[Yw]+(N_2-N_{i*}^{Y_0}-Y_{i*j})(1-E[Yw])\} & \\ \quad\times\{(N_1^w-w_{i*})(1-E[w])-(N_1-N_1^w-(1-w_{i*}))E[w]\} & j \text{ linked, } w_{i*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{w}_i D_{i'j} \overline{Zw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_2-1)E[Zw]\{(N_1^w-w_{i*})(1-E[w])-(N_1-N_1^w-(1-w_{i*}))E[w]\} & j \text{ linked, } w_{i*}=0 \\ \{-(N_{i*}^{Z_0}-(1-Z_{i*j}))E[Zw]+(N_2-N_{i*}^{Z_0}-Z_{i*j})(1-E[Zw])\} & \\ \quad\times\{(N_1^w-w_{i*})(1-E[w])-(N_1-N_1^w-(1-w_{i*}))E[w]\} & j \text{ linked, } w_{i*}=1 \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{w}_i D_{i'j} \overline{DYw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_2-1)E[DYw]\{(N_1^w-w_{i*})(1-E[w])-(N_1-N_1^w-(1-w_{i*}))E[w]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{w}_i D_{i'j} \overline{DZw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_2-1)E[DZw]\{(N_1^w-w_{i*})(1-E[w])-(N_1-N_1^w-(1-w_{i*}))E[w]\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \overline{w}_i D_{i'j} \overline{w}_{i'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_2-1)(w_{i*}-E[w])\{(N_1^w-w_{i*})(1-E[w])-(N_1-N_1^w-(1-w_{i*}))E[w]\} & j \text{ linked} \end{cases}$$

Finally, we consider $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{i'\neq i}\sum_{j'\neq j}\Xi'^{w}_{ij}D_{ij'}(1-\hat{\pi}(w))\Xi^{w}_{i'j'}$. Since we can factor the $(1-\hat{\pi}(w))$ from the quadruple summation, we ignore it in our simplifications here. The idea of the simplification is to express analytically $\sum_{i=j}^{N_2}\sum_{i'\neq i}\sum_{j'\neq j}\Xi'^{w}_{ij}D_{ij'}\Xi^{w}_{i'j'}$ for each i so that the number of observations that must be considered in computing the double summation is N_1 instead of N_1N_2 and it can be computed from the first census. The simplifications are as follows.

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Dw}_{ij}D_{ij'}\overline{Dw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ (N_1-1)(N_2-1)E[Dw]^2 & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Dw}_{ij}D_{ij'}\overline{Yw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ -(N_2-1)\{N_1^{w}-N_{j^*}^{Y_0w_1}-(N_1-1)E[Yw]-w_iY_{ij^*}\}E[Dw] & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Dw}_{ij}D_{ij'}\overline{Zw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ -(N_2-1)\{N_1^{w}-N_{j^*}^{Z_0w_1}-(N_1-1)E[Zw]-w_iZ_{ij^*}\}E[Dw] & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Dw}_{ij}D_{ij'}\overline{DYw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ E[Dw](N_1-1)(N_2-1)E[DYw] & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Dw}_{ij}D_{ij'}\overline{DZw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ E[Dw](N_1-1)(N_2-1)E[DZw] & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Dw}_{ij}D_{ij'}\overline{w}_{i'}=\begin{cases}0 & i \text{ unlinked}\\ -(N_2-1)E[Dw]\{N_1^{w}-(N_1-1)E[w]-w_i\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Yw}_{ij}D_{ij'}\overline{Dw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ -(N_1-1)E[Dw]\{(N_2-N_i^{Y_0}-Y_{ij^*})(w_i-E[Yw])-(N_i^{Y_0}-(1-Y_{ij^*}))E[Yw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Yw}_{ij}D_{ij'}\overline{Yw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ \{N_1^{w}-N_{j^*}^{Y_0w_1}-(N_1-1)E[Yw]-w_iY_{ij^*}\}\\ \quad\times\{(N_2-N_i^{Y_0}-Y_{ij^*})(w_i-E[Yw])-(N_i^{Y_0}-(1-Y_{ij^*}))E[Yw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Yw}_{ij}D_{ij'}\overline{Zw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ \{N_1^{w}-N_{j^*}^{Z_0w_1}-(N_1-1)E[Zw]-w_iZ_{ij^*}\}\\ \quad\times\{(N_2-N_i^{Y_0}-Y_{ij^*})(w_i-E[Yw])-(N_i^{Y_0}-(1-Y_{ij^*}))E[Yw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Yw}_{ij}D_{ij'}\overline{DYw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ -(N_1-1)E[DYw]\{(N_2-N_i^{Y_0}-Y_{ij^*})(w_i-E[Yw])-(N_i^{Y_0}-(1-Y_{ij^*}))E[Yw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Yw}_{ij}D_{ij'}\overline{DZw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ -(N_1-1)E[DZw]\{(N_2-N_i^{Y_0}-Y_{ij^*})(w_i-E[Yw])-(N_i^{Y_0}-(1-Y_{ij^*}))E[Yw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Yw}_{ij}D_{ij'}\overline{w}_{i'}=\begin{cases}0 & i \text{ unlinked}\\ \{N_1^{w}-(N_1-1)E[w]-w_i\}\{(N_2-N_i^{Y_0}-Y_{ij^*})(w_i-E[Yw])-(N_i^{Y_0}-(1-Y_{ij^*}))E[Yw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Zw}_{ij}D_{ij'}\overline{Dw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ -(N_1-1)E[Dw]\{(N_2-N_i^{Z_0}-Z_{ij^*})(w_i-E[Zw])-(N_i^{Z_0}-(1-Z_{ij^*}))E[Zw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Zw}_{ij}D_{ij'}\overline{Yw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ \{N_1^{w}-N_{j^*}^{Y_0w_1}-(N_1-1)E[Yw]-w_iY_{ij^*}\}\\ \quad\times\{(N_2-N_i^{Z_0}-Z_{ij^*})(w_i-E[Zw])-(N_i^{Z_0}-(1-Z_{ij^*}))E[Zw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Zw}_{ij}D_{ij'}\overline{Zw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ \{N_1^{w}-N_{j^*}^{Z_0w_1}-(N_1-1)E[Zw]-w_iZ_{ij^*}\}\\ \quad\times\{(N_2-N_i^{Z_0}-Z_{ij^*})(w_i-E[Zw])-(N_i^{Z_0}-(1-Z_{ij^*}))E[Zw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Zw}_{ij}D_{ij'}\overline{DYw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ -(N_1-1)E[DYw]\{(N_2-N_i^{Z_0}-Z_{ij^*})(w_i-E[Zw])-(N_i^{Z_0}-(1-Z_{ij^*}))E[Zw]\} & i \text{ linked}\end{cases}$$

$$\sum_j\sum_{i'\neq ij'\neq j}\overline{Zw}_{ij}D_{ij'}\overline{DZw}_{i'j'}=\begin{cases}0 & i \text{ unlinked}\\ -(N_1-1)E[DZw]\{(N_2-N_i^{Z_0}-Z_{ij^*})(w_i-E[Zw])-(N_i^{Z_0}-(1-Z_{ij^*}))E[Zw]\} & i \text{ linked}\end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{Zw}_{ij} D_{ij'} \overline{w}_{i'} = \begin{cases} 0 & i \text{ unlinked} \\ \{N_1^w - (N_1 - 1)E[w] - w_i\}\{(N_2 - N_i^{Z0} - Z_{ij*})(w_i - E[Zw]) - (N_i^{Z0} - (1 - Z_{ij*}))E[Zw]\} & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DYw}_{ij} D_{ij'} \overline{Dw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_1 - 1)(N_2 - 1)E[Dw]E[DYw] & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DYw}_{ij} D_{ij'} \overline{Yw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)E[DYw]\{N_1^w - N_{j*}^{Y_0 w_1} - (N - 1 - 1)E[Yw] - w_i Y_{ij*}\} & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DYw}_{ij} D_{ij'} \overline{Zw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)E[DYw]\{N_1^w - N_{j*}^{Z_0 w_1} - (N - 1 - 1)E[Zw] - w_i Z_{ij*}\} & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DYw}_{ij} D_{ij'} \overline{DYw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_1 - 1)(N_2 - 1)E[DYw]^2 & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DYw}_{ij} D_{ij'} \overline{DZw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_1 - 1)(N_2 - 1)E[DYw]E[DZw] & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DYw}_{ij} D_{ij'} \overline{w}_{i'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)E[DYw]\{N_1^w - (N_1 - 1)E[w] - w_i\} & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DZw}_{ij} D_{ij'} \overline{Dw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_1 - 1)(N_2 - 1)E[Dw]E[DZw] & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DZw}_{ij} D_{ij'} \overline{Yw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)E[DZw]\{N_1^w - N_{j*}^{Y_0 w_1} - (N - 1 - 1)E[Yw] - w_i Y_{ij*}\} & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DZw}_{ij} D_{ij'} \overline{Zw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)E[DZw]\{N_1^w - N_{j*}^{Z_0 w_1} - (N - 1 - 1)E[Zw] - w_i Z_{ij*}\} & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DZw}_{ij} D_{ij'} \overline{DYw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_1 - 1)(N_2 - 1)E[DZw]E[DYw] & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DZw}_{ij} D_{ij'} \overline{DZw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_1 - 1)(N_2 - 1)E[DZw]^2 & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{DZw}_{ij} D_{ij'} \overline{w}_{i'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)E[DZw]\{N_1^w - (N_1 - 1)E[w] - w_i\} & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{w}_i D_{ij'} \overline{Dw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)(w_i - E[w])(N_1 - 1)E[Dw] & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{w}_i D_{ij'} \overline{Yw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)(w_i - E[w])\{N_1^w - N_{j*}^{Y_0 w_1} - (N_1 - 1)E[Yw] - w_i Y_{ij*}\} & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{w}_i D_{ij'} \overline{Zw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)(w_i - E[w])\{N_1^w - N_{j*}^{Z_0 w_1} - (N_1 - 1)E[Zw] - w_i Z_{ij*}\} & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{w}_i D_{ij'} \overline{DYw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)(w_i - E[w])(N_1 - 1)E[DYw] & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{w}_i D_{ij'} \overline{DZw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2 - 1)(w_i - E[w])(N_1 - 1)E[DZw] & i \text{ linked} \end{cases}$$

$$\sum_j \sum_{i' \neq i} \sum_{j' \neq j} \overline{w}_i D_{ij'} \overline{w}_{i'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_2 - 1)(w_i - E[w])\{N_1^w - (N_1 - 1)E[w] - w_i\} & i \text{ linked} \end{cases}$$

C.3 Regression with confounded matching

The final case that we consider is regression with confounded matching (which subsumes as a special case regression with unconfounded matching). The components consisting only of $\overline{Dw}$, $\overline{Zw}$, $\overline{DZw}$,

and $\overline{w}$ are identical to the regression case specified above, so we provide only the simplifications that differ. The assumption necessary to implement these simplifications is that the regressors are from the first census and the outcome variable is either from the second census or a binary based on linkage (such as migration). This allows us to simplify terms that will arise below of the form $\sum_j y_{ij}$ to either $\sum_j y_j$ (which can be computed easily from the second census) or to $N_2 - N_i^{Y_0}$.

We begin by noting that $E[XX'|D=1,W=w]$ and $E[XY|D=1,W=w]$ can be easily observed from the linked sample. We can also simplify

$$E[XX'|W=w]_{k\ell} = \frac{\sum_i \sum_j x_{ik}x_{i\ell}w_i}{\sum_i \sum_j w_i} = \frac{\sum_i x_{ik}x_{i\ell}w_i}{\sum_i w_i}$$
$$E[XY|W=w]_{k} = \frac{\sum_i \sum_j x_{ik}y_{ij}w_i}{\sum_i \sum_j w_i} = \frac{\sum_i x_{ik}y_{ij}w_i}{\sum_i w_i}$$

Define also

$$E[XYw]_k = \frac{1}{N_1 N_2}\sum_i \sum_j x_{ik}y_{ij}w_i = \frac{\sum_i x_{ik}w_i \sum_j y_{ij}}{N_1 N_2}$$
$$E[XXw]_{k\ell} = \frac{1}{N_1 N_2}\sum_i \sum_j x_{ik}x_{i\ell}w_i = \frac{1}{N_1}\sum_i x_{ik}x_{i\ell}w_i$$
$$E[X\varepsilon w]_k = E[XYw]_k - \sum_\ell \beta_\ell E[XX'w]_{k\ell},$$

and define $E[DX\varepsilon w]_k$ as the sum of $w_i D_{ij} x_{ik}\varepsilon_{ij}$ in the linked sample.

Finally, define

$$\Xi^w_{ijk} = \begin{pmatrix} \overline{Dw}_{ij} \\ \overline{X\varepsilon w}_{ijk} \\ \overline{Zw}_{ij} \\ \overline{DX\varepsilon w}_{ijk} \\ \overline{DZw}_{ij} \\ \overline{w}_i \end{pmatrix}$$

We begin with $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\Xi'^w_{ijk}\Xi^w_{ij\ell}$. The idea of the simplification, as above, is to express analytically $\sum_{j=1}^{N_2}\Xi'^w_{ijk}\Xi^w_{ij\ell}$ for each i so that the number of observations that must be considered in computing the double summation is N_1 instead of $N_1 N_2$. As above, let i^* indicate the individual in census 1 to which a census-2 individual is matched and j^* indicate the individual in census 2 to which a census-1 individual is matched. The simplifications are as follows.

$$\sum_j \overline{Dw}_{ij}\overline{X\varepsilon w}_{ijk} = w_i x_{ik} y_{ij^*} D_i - D_i w_i \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right)$$
$$- E[Dw] w_i x_{ik} \sum_j y_{ij} + N_2 E[Dw] \left\{ w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right\}$$

$$\sum_j \overline{Dw}_{ij}\overline{DX\varepsilon w}_{ijk} = w_i x_{ik} y_{ij^*} D_i - D_i \left[w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + w_i E[DX\varepsilon w]_k - E[Dw] w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} \right]$$
$$- E[Dw] w_i x_{ik} y_{ij^*} D_i + N_2 E[Dw] E[DX\varepsilon w]_k$$

$$\sum_j \overline{X\varepsilon w}_{ijk}\overline{X\varepsilon w}_{ijm} = w_i x_{ik}x_{im}\sum_j y_{ij}^2 - w_i x_{ik}\left(w_i x_{im}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell}\right)\sum_j y_{ij}$$
$$- w_i x_{im}\left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)\sum_j y_{ij}$$
$$+ N_2\left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)\left(w_i x_{im}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell}\right)$$

$$\sum_j \overline{X\varepsilon w}_{ijk}\overline{Zw}_{ij} = w_i x_{ik}\sum_j y_{ij}Z_{ij} - w_i x_{ik}E[Zw]\sum_j y_{ij} - w_i\left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)\sum_j Z_{ij}$$
$$+ N_2 E[Zw]\left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)$$

$$\sum_j \overline{X\varepsilon w}_{ijk}\overline{DX\varepsilon w}_{ijm} = w_i x_{ik}x_{im}y_{ij^*}^2 D_i - w_i x_{ik}x_{im}y_{ij^*}D_i\sum_\ell \beta_\ell x_{i\ell} - w_i x_{ik}E[DX\varepsilon w]_m\sum_j y_{ij}$$
$$- \left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)w_i x_{im}y_{ij^*}D_i$$
$$+ \left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)\left(w_i x_{im}\sum_\ell \beta_\ell x_{i\ell}\right)D_i$$
$$+ N_2 E[DX\varepsilon w]_m\left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)$$

$$\sum_j \overline{X\varepsilon w}_{ijk}\overline{DZw}_{ij} = w_i x_{ik}Z_{ij^*}y_{ij^*}D_i - w_i x_{ik}E[DZw]\sum_j y_{ij} - \left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)w_i Z_{ij^*}D_i$$
$$+ N_2\left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)E[DZw]$$

$$\sum_j \overline{X\varepsilon w}_{ijk}\overline{w}_i = (w_i - E[w])w_i x_{ik}\sum_j y_{ij} - N_2(w_i - E[w])\left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)$$

$$\sum_j \overline{Zw}_{ij}\overline{DX\varepsilon w}_{ijk} = w_i x_{ik}Z_{ij^*}y_{ij^*}D_i - \left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell}\right)Z_{ij^*}D_i - E[DX\varepsilon w]_k w_i\sum_j Z_{ij} - E[Zw]w_i x_{ik}y_{ij^*}D_i$$
$$+ E[Zw]D_i w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + N_2 E[Zw]E[DX\varepsilon w]_k$$

$$\sum_j \overline{DX\varepsilon w}_{ijk}\overline{DX\varepsilon w}_{ijm} = w_i x_{ik}x_{im}D_i y_{ij^*}^2 - w_i x_{ik}x_{im}D_i y_{ij^*}\sum_\ell \beta_\ell x_{i\ell} - E[DX\varepsilon w]_m w_i x_{ik}y_{ij^*}D_i$$
$$- w_i x_{ik}x_{im}D_i y_{ij^*}\sum_\ell \beta_\ell x_{i\ell} + w_i x_{ik}w_{im}D_i\left(\sum_\ell \beta_\ell x_{i\ell}\right)^2$$
$$+ E[DX\varepsilon w]_m w_i x_{ik}D_i\sum_\ell \beta_\ell x_{i\ell} - w_i x_{im}E[DX\varepsilon w]_k D_i y_{ij^*}$$
$$+ E[DX\varepsilon w]_k w_i x_{im}D_i\sum_\ell \beta_\ell x_{i\ell} + N_2 E[DX\varepsilon w]_k E[DX\varepsilon w]_m$$

$$\sum_j \overline{DX\varepsilon w}_{ijk}\overline{DZw}_{ij} = w_i x_{ik}y_{ij^*}Z_{ij^*}D_i - E[DZw]w_i x_{ik}y_{ij^*}D_i - w_i x_{ik}Z_{ij^*}D_i\sum_\ell \beta_\ell x_{i\ell} + E[DZw]w_i x_{ik}D_i\sum_\ell \beta_\ell x_{i\ell}$$
$$- E[DX\varepsilon w]_k w_i Z_{ij^*}D_i + N_2 E[DX\varepsilon w]_k E[DZw]$$

$$\sum_j \overline{DX\varepsilon w}_{ijk}\overline{w}_i = (w_i - E[w])\left(w_i x_{ik}y_{ij^*}D_i - w_i x_{ik}D_i\sum_\ell \beta_\ell x_{i\ell} - N_2 E[DX\varepsilon w]_k\right)$$

Next, we consider $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi'^w_{ijk}\Xi^w_{ij'm}$. As above, the idea of the simplification is to express analytically $\sum_{j=1}^{N_2}\sum_{j'\neq j}\Xi'^w_{ijk}\Xi^w_{ij'm}$ for each i so that the number of observations that must be considered in computing the double summation is N_1 instead of N_1N_2.

$$\sum_j\sum_{j'\neq j}\overline{Dw}_{ij}\overline{X\varepsilon w}_{ij'k} = -w_i x_{ik}D_i y_{ij^*} + D_i w_i\left\{w_i x_{ik}\sum_j y_{ij} - (N_2-1)\left(w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)\right\}$$

$$+w_i x_{ik} E[Dw] \sum_j y_{ij} - N_2 E[Dw] \left\{ w_i x_{ik} \sum_j y_{ij} - (N_2-1) \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\}$$

$$\sum_j \sum_{j' \neq j} \overline{Dw}_{ij} \overline{DX\varepsilon w}_{ij'k} = -w_i x_{ik} y_{ij^*} D_i + w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} + w_i D_i \left\{ w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} - (N_2-1) E[DX\varepsilon w]_k \right\}$$

$$+E[Dw] w_i x_{ik} y_{ij^*} D_i - E[Dw] w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} - N_2 E[Dw] \left\{ w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} - (N_2-1) E[DX\varepsilon w]_k \right\}$$

$$\sum_j \sum_{j' \neq j} \overline{X\varepsilon w}_{ijk} \overline{Dw}_{ij'} = (D_i w_i - (N_2-1) E[Dw]) w_i x_{ik} \sum_j y_{ij} - w_i x_{ik} y_{ij^*} D_i$$

$$-N_2 \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) (D_i w_i - (N_2-1) E[Dw])$$

$$+D_i w_i \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right)$$

$$\sum_j \sum_{j' \neq j} \overline{X\varepsilon w}_{ijk} \overline{X\varepsilon w}_{ij'm} = -w_i x_{ik} x_{im} \sum_j y_{ij}^2 + w_i x_{ik} \left\{ w_i x_{im} \sum_j y_{ij} - (N_2-1) \left(w_i x_{im} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell} \right) \right\} \sum_j y_{ij}$$

$$+ \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) w_i x_{im} \sum_j y_{ij}$$

$$-N_2 \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right)$$

$$\times \left\{ w_i x_{im} \sum_j y_{ij} - (N_2-1) \left(w_i x_{im} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell} \right) \right\}$$

$$\sum_j \sum_{j' \neq j} \overline{X\varepsilon w}_{ijk} \overline{Zw}_{ij'} = -w_i x_{ik} \sum_j y_{ij} Z_{ij} + \left\{ w_i \sum_j Z_{ij} - (N_2-1) E[Zw] \right\} w_i x_{ik} \sum_j y_{ij}$$

$$+ \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) w_i \sum_j Z_{ij}$$

$$-N_2 \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \left[w_i \sum_j Z_{ij} - (N_2-1) E[Zw] \right]$$

$$\sum_j \sum_{j' \neq j} \overline{X\varepsilon w}_{ijk} \overline{Dx\varepsilon w}_{ij'm} = -w_i x_{ik} x_{im} D_i y_{ij^*}^2 + w_i x_{ik} x_{im} D_i y_{ij^*} \sum_\ell \beta_\ell x_{i\ell}$$

$$+ \left[w_i x_{im} y_{ij^*} D_i - w_i x_{im} D_i \sum_\ell \beta_\ell x_{i\ell} - (N_2-1) E[DX\varepsilon w]_m \right] w_i x_{ik} \sum_j y_{ij}$$

$$+ \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) w_i x_{im} y_{ij^*} D_i$$

$$- \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) w_i x_{im} D_i \sum_\ell \beta_\ell x_{i\ell}$$

$$-N_2 \left[w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right]$$

$$\times \left[w_i x_{im} y_{ij^*} D_i - w_i x_{im} D_i \sum_\ell \beta_\ell x_{i\ell} - (N_2-1) E[DX\varepsilon w]_m \right]$$

$$\sum_j \sum_{j' \neq j} \overline{X\varepsilon w}_{ijk} \overline{DZw}_{ij'} = -w_i x_{ik} y_{ij^*} D_i Z_{ij^*} + (w_i D_i Z_{ij^*} - (N_2-1) E[DZw]) w_i x_{ik} \sum_j y_{ij}$$

$$+ \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) w_i D_i Z_{ij^*}$$

$$- \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) (w_i D_i Z_{ij^*} - (N_2-1) E[DZw]) N_2$$

$$\sum_j \sum_{j' \neq j} \overline{X\varepsilon w}_{ijk} \overline{w}_i = (N_2-1)(w_i E[w]) \left[w_i x_{ik} \sum_j y_{ij} - N_2 \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right]$$

$$\sum_j \sum_{j' \neq j} \overline{Zw}_{ij} \overline{x\varepsilon w}_{ij'k} = -w_i x_{ik} \sum_j Z_{ij} y_{ij}$$
$$- E[Zw] \left\{ w_i x_{ik} \sum_j y_{ij} - (N_2 - 1) \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\} N_2$$
$$+ w_i x_{ik} E[Zw] \sum_j y_{ij} + w_i \left\{ w_i x_{im} \sum_j y_{ij} - (N_2 - 1) \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\} \sum_j Z_{ij}$$

$$\sum_j \sum_{j' \neq j} \overline{Zw}_{ij} \overline{Dx\varepsilon w}_{ij'k} = -w_i x_{ik} y_{ij^*} D_i Z_{ij^*} + w_i x_{ik} D_i Z_{ij^*} \sum_\ell \beta_\ell x_{i\ell}$$
$$+ w_i \left\{ w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} - (N_2 - 1) E[DX\varepsilon w]_k \right\} \sum_j Z_{ij}$$
$$+ E[Zw] w_i x_{ik} y_{ij^*} D_i - E[Zw] w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell}$$
$$- E[Zw] \left\{ w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} - (N_2 - 1) E[DX\varepsilon w]_k \right\} N_2$$

$$\sum_j \sum_{j' \neq j} \overline{DX\varepsilon w}_{ijk} \overline{Dw}_{ij'} = -w_i x_{ik} y_{ij^*} D_i + w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} + E[DX\varepsilon w]_k w_i D_i$$
$$+ (D_i w_i - (N_2 - 1) E[Dw]) \left(w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} \right)$$
$$- N_2 E[DX\varepsilon w]_k (D_i w_i - (N_2 - 1) E[Dw])$$

$$\sum_j \sum_{j' \neq j} \overline{DX\varepsilon w}_{ijk} \overline{X\varepsilon w}_{ij'm} = -w_i x_{ik} D_i x_{im} y_{ij^*}^2 + w_i x_{ik} y_{ij^*} D_i x_{im}$$
$$+ \left\{ w_i x_{im} \sum_j y_{ij} - (N_2 - 1) \left(w_i x_{im} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell} \right) \right\}$$
$$\times \left(w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} \right)$$
$$+ E[DX\varepsilon w]_k w_i x_{im} \sum_j y_{ij} - N_2 E[DX\varepsilon w]_k$$
$$\times \left\{ w_i x_{im} \sum_j y_{ij} - (N_2 - 1) \left(w_i x_{im} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell} \right) \right\}$$

$$\sum_j \sum_{j' \neq j} \overline{DX\varepsilon w}_{ijk} \overline{Zw}_{ij'} = -w_i x_{ik} y_{ij^*} D_i Z_{ij^*} + w_i x_{ik} D_i Z_{ij^*} \sum_\ell \beta_\ell x_{i\ell} + E[DX\varepsilon w]_k w_i \sum_j y_{ij}$$
$$+ \left\{ w_i \sum_j Z_{ij} - (N_2 - 1) E[Zw] \right\} \left\{ w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} - N_2 E[DX\varepsilon w]_k \right\}$$

$$\sum_j \sum_{j' \neq j} \overline{DX\varepsilon w}_{ijk} \overline{DX\varepsilon w}_{ij'm} = -w_i x_{ik} D_i x_{im} y_{ij^*}^2 + 2 w_i x_{ik} D_i x_{im} \sum_\ell \beta_\ell x_{i\ell} - w_i x_{ik} D_i x_{im} \left(\sum_\ell \beta_\ell x_{i\ell} \right)^2$$
$$+ \left(w_i x_{im} y_{ij^*} D_i - w_i x_{im} D_i \sum_\ell \beta_\ell x_{i\ell} - (N_2 - 1) E[DX\varepsilon w]_m \right) \left(w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} \right)$$
$$- E[DX\varepsilon w]_k \left(N_2 w_i x_{im} y_{ij^*} D_i - N_2 w_i x_{im} D_i \sum_\ell \beta_\ell x_{i\ell} - w_i x_{im} y_{ij^*} D_i \right.$$
$$\left. + w_i x_{im} D_i \sum_\ell \beta_\ell x_{i\ell} - N_2 (N_2 - 1) E[DX\varepsilon w]_m \right)$$

$$\sum_j \sum_{j' \neq j} \overline{DX\varepsilon w}_{ijk} \overline{DZw}_{ij'} = -w_i x_{ik} y_{ij^*} D_i Z_{ij^*} + w_i x_{ik} D_i Z_{ij^*} \sum_\ell \beta_\ell x_{i\ell}$$
$$- E[DX\varepsilon w]_k (N_2 w_i Z_{ij^*} D_i - D_i Z_{ij^*} w_i - N_2 (N_2 - 1) E[DZw])$$
$$+ (w_i Z_{ij^*} D_i - (N_2 - 1) E[DZw]) \left(w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} \right)$$

$$\sum_j \sum_{j' \neq j} \overline{DX\varepsilon w}_{ijk} \overline{w}_i = (w_i - E[w]) \left(w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} - N_2 E[DX\varepsilon w]_k \right) (N_2 - 1)$$

$$\sum_j \sum_{j' \neq j} \overline{DZw}_{ij} \overline{x\varepsilon w}_{ij'k} = -w_i x_{ik} y_{ij^*} D_i Z_{ij^*} + E[DZw] w_i x_{ik} \sum_j y_{ij}$$
$$+ \left\{ w_i x_{ik} \sum_j y_{ij} - (N_2 - 1) \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\}$$
$$\times [w_i D_i Z_{ij^*} - N_2 E[DZw]]$$

$$\sum_j \sum_{j' \neq j} \overline{DZw}_{ij} \overline{Dx\varepsilon w}_{ij'k} = -w_i x_{ik} y_{ij^*} D_i Z_{ij^*} + E[DZw] w_i x_{ik} y_{ij^*} D_i + w_i x_{ik} D_i Z_{ij^*} \sum_\ell \beta_\ell x_{i\ell}$$
$$- E[DZw] w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell}$$
$$\left\{ w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} - (N_2 - 1) E[DX\varepsilon w]_k \right\} (D_i w_i Z_{ij^*} - N_2 E[DZw])$$

$$\sum_j \sum_{j' \neq j} \overline{w}_i \overline{x\varepsilon w}_{ij'k} = (w_i - E[w]) \left\{ -w_i x_{ik} \sum_j y_{ij} \right.$$
$$\left. + N_2 \left[w_i x_{ik} \sum_j y_{ij} - (N_2 - 1) \left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right] \right\}$$

$$\sum_j \sum_{j' \neq j} \overline{w}_i \overline{Dx\varepsilon w}_{ij'k} = (w_i - E[w]) \left\{ -w_i x_{ik} y_{ij^*} D_i + w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i\ell} \right.$$
$$\left. + N_2 \left[w_i x_{ik} y_{ij^*} D_i - w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - (N_2 - 1) E[DX\varepsilon w]_k \right] \right\}$$

Next, we consider $\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{i' \neq i} \Xi'^w_{ijk} \Xi^w_{i'jm}$. As above, the idea of the simplification is to express analytically $\sum_{i=1}^{N_1} \sum_{i' \neq i} \Xi'^w_{ijk} \Xi^w_{i'jm}$ for each j so that the number of observations that must be considered in computing the double summation is N_2 instead of $N_1 N_2$.

$$\sum_i \sum_{i' \neq i} \overline{Dw}_{ij} \overline{X\varepsilon w}_{i'jk} = -w_{i^*} x_{i^*k} y_{ij^*} D_j + w_{i^*} x_{i^*k} D_j \sum_\ell \beta_\ell x_{i^*\ell} + E[Dw] \sum_i w_i x_{ik} y_{ij} - E[Dw] \sum_i w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell}$$
$$+ \left\{ \sum_i w_i x_{ik} y_{ij} - \sum_i w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - (N_1 - 1) \left(E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\}$$
$$\times (D_j w_{i^*} - N_1 E[Dw])$$

$$\sum_i \sum_{i' \neq i} \overline{Dw}_{ij} \overline{DX\varepsilon w}_{i'jk} = -w_{i^*} x_{i^*k} y_{ij^*} D_j + w_{i^*} x_{i^*k} D_j \sum_\ell \beta_\ell x_{i^*\ell}$$
$$+ D_j w_{i^*} \left\{ w_{i^*} x_{i^*k} y_{ij^*} D_j - w_{i^*} x_{i^*k} D_j \sum_\ell \beta_\ell x_{i^*\ell} - (N_1 - 1) E[DX\varepsilon w]_k \right\}$$
$$+ E[Dw] \left\{ w_{i^*} x_{i^*k} y_{ij^*} D_j - w_{i^*} x_{i^*k} D_j \sum_\ell \beta_\ell x_{i^*\ell} \right\}$$
$$- N_1 \left\{ w_{i^*} x_{i^*k} y_{ij^*} D_j - w_{i^*} x_{i^*k} D_j \sum_\ell \beta_\ell x_{i^*\ell} - (N_1 - 1) E[DX\varepsilon w]_k \right\}$$

$$\sum_i \sum_{i' \neq i} \overline{X\varepsilon w}_{ijk} \overline{Dw}_{i'j} = -w_{i^*} x_{i^*k} y_{ij^*} D_j + w_i x_{ik} D_i \sum_\ell \beta_\ell x_{i^*\ell}$$
$$+ (D_j w_{i^*} - (N_1 - 1) E[Dw]) \left(\sum_i w_i x_{ik} y_{ij} - \sum_i w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} \right)$$
$$+ \left(E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) D_j w_{i^*}$$
$$- N_1 \left(E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) (D_j w_{i^*} - (N_1 - 1) E[Dw])$$

$$\sum_i \sum_{i' \neq i} \overline{X\varepsilon w}_{ijk} \overline{X\varepsilon w}_{i'jm} = -\sum_i w_i x_{ik} x_{im} y_{ij}^2 + 2 \sum_i w_i x_{ik} y_{ij} x_{im} \sum_\ell \beta_\ell x_{i\ell} - \sum_i w_i x_{ik} x_{im} \left(\sum_\ell \beta_\ell x_{i\ell} \right)^2$$

$$+\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)\left\{\sum_i w_i x_{im}y_{ij}-\sum_i w_i x_{im}\sum_\ell\beta_\ell x_{i\ell}\right\}$$
$$+\left\{\sum_i w_i x_{im}y_{ij}-\sum_i w_i x_{im}\sum_\ell\beta_\ell x_{i\ell}-(N_1-1)\left(E[XYw]_m-\sum_\ell\beta_\ell E[XXw]_{m\ell}\right)\right\}$$
$$\times\left\{\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}\right\}-N_1\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)$$
$$\times\left\{\sum_i w_i x_{im}y_{ij}-\sum_i w_i x_{im}\sum_\ell\beta_\ell x_{i\ell}-(N_1-1)\left(E[XYw]_m-\sum_\ell\beta_\ell E[XXw]_{m\ell}\right)\right\}$$

$$\sum_i\sum_{i'\neq i}\overline{X\varepsilon w}_{ijk}\overline{Zw}_{i'j}=-\sum_i w_i x_{ik}y_{ij}Z_{ij}+\sum_i w_i x_{ik}Z_{ij}\sum_\ell\beta_\ell x_{i\ell}+\left\{\sum_i w_i Z_{ij}-(N_1-1)E[Zw]\right\}$$
$$\times\left(\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}\right)+\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)\sum_i Z_{ij}w_i$$
$$-N_1\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)\left\{\sum_i Z_{ij}w_i-(N_1-1)E[Zw]\right\}$$

$$\sum_i\sum_{i'\neq i}\overline{X\varepsilon w}_{ijk}\overline{DX\varepsilon w}_{i'jm}=-w_{i^*}x_{i^*k}D_jx_{i^*m}y_{i^*j}^2+2w_{i^*}x_{i^*k}y_{ij^*}D_jx_{i^*m}\sum_\ell\beta_\ell x_{i^*\ell}-w_{i^*}x_{i^*k}D_jx_{i^*m}\left(\sum_\ell\beta_\ell x_{i^*\ell}\right)^2$$
$$+\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)\left(w_{i^*}x_{i^*m}y_{ij^*}D_j-w_{i^*}x_{i^*m}D_j\sum_\ell\beta_\ell x_{i^*\ell}\right)$$
$$+\left(w_{i^*}x_{i^*m}y_{ij^*}D_j-w_{i^*}x_{i^*m}D_j\sum_\ell\beta_\ell x_{i^*\ell}-(N_1-1)E[DX\varepsilon w]_m\right)\left(\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}\right)$$
$$-N_1\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)\left(w_{i^*}x_{i^*m}y_{ij^*}D_j-w_{i^*}x_{i^*m}D_j\sum_\ell\beta_\ell x_{i^*\ell}-(N_1-1)E[DX\varepsilon w]_m\right)$$

$$\sum_i\sum_{i'\neq i}\overline{X\varepsilon w}_{ijk}\overline{DZw}_{i'j}=-w_{i^*}x_{i^*k}y_{ij^*}D_jZ_{i^*j}+w_{i^*}x_{i^*k}D_jZ_{i^*j}$$
$$+\left(D_jZ_{i^*j}w_{i^*}-(N_1-1)E[DZw]\right)\left(\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}\right)$$
$$+\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)N_1^w-N_1\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)(N_1^2-(N_1-1)E[w])$$

$$\sum_i\sum_{i'\neq i}\overline{X\varepsilon w}_{ijk}\overline{w}_{i'}=-\sum_i w_i x_{ik}y_{ij}+\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}+N_1^w-(N_1-1)E[w])\left(\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}\right)$$
$$+\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)N_1^w-N_1\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)(N_1^w-(N_1-1)E[w])$$

$$\sum_i\sum_{i'\neq i}\overline{Zw}_{ij}\overline{X\varepsilon w}_{i'jk}=-\sum_i w_i x_{ik}y_{ij}Z_{ij}+\sum_i w_i x_{ik}Z_{ij}\sum_\ell\beta_\ell x_{i\ell}+E[Zw]\left(\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}\right)$$
$$+\left\{\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}-(N_1-1)\left(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell}\right)\right\}\left(\sum_i Z_{ij}w_i-N_1E[Zw]\right)$$

$$\sum_i\sum_{i'\neq i}\overline{Zw}_{ij}\overline{DX\varepsilon w}_{i'jk}=-w_{i^*}x_{i^*k}y_{ij^*}D_jZ_{i^*j}+w_{i^*}x_{i^*k}D_jZ_{i^*j}\sum_\ell\beta_\ell x_{i^*\ell}+E[Zw]\left(w_{i^*}x_{i^*k}y_{ij^*}D_j-w_{i^*}x_{i^*k}D_j\sum_\ell\beta_\ell x_{i^*\ell}\right)$$
$$+\left\{w_{i^*}x_{i^*k}y_{ij^*}D_j-w_{i^*}x_{i^*k}D_j\sum_\ell\beta_\ell x_{i^*\ell}-(N_1-1)E[DX\varepsilon w]_k\right\}\left(\sum_i Z_{ij}w_i-N_1E[Zw]\right)$$

$$\sum_i\sum_{i'\neq i}\overline{DX\varepsilon w}_{ijk}\overline{Dw}_{i'j}=-w_{i^*}x_{i^*k}y_{ij^*}D_j+w_{i^*}x_{i^*k}D_j\sum_\ell\beta_\ell x_{i^*\ell}+(D_jw_{i^*}-(N_1-1)E[Dw])\left(w_{i^*}x_{i^*k}y_{ij^*}D_j-w_{i^*}x_{i^*k}D_j\sum_\ell\beta_\ell x_{i^*\ell}\right)$$
$$+E[DX\varepsilon w]_kD_jw_{i^*}-N_1E[DX\varepsilon w]_k(D_jw_{i^*}-(N_1-1)E[Dw])$$

$$\sum_i\sum_{i'\neq i}\overline{DX\varepsilon w}_{ijk}\overline{X\varepsilon w}_{i'jm}=-w_{i^*}x_{i^*k}D_jx_{i^*m}y_{i^*j}^2+2w_{i^*}x_{i^*k}y_{ij^*}D_jx_{i^*m}\sum_\ell\beta_\ell x_{i^*\ell}-w_{i^*}x_{i^*k}D_jx_{i^*m}\left(\sum_\ell\beta_\ell x_{i^*\ell}\right)^2$$

$$+E[DX\varepsilon w]_k\left(\sum_i w_i x_{im} y_{ij} - \sum_i w_i x_{im}\sum_\ell \beta_\ell x_{i\ell}\right)$$
$$+\left\{\sum_i w_i x_{im} y_{ij} - \sum_i w_i x_{im}\sum_\ell \beta_\ell x_{i\ell} - (N_1-1)\left(E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell}\right)\right\}$$
$$\times\left(w_{i^*}x_{i^*k}y_{ij^*}D_j - w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell} - N_1 E[DX\varepsilon w]_k\right)$$

$$\sum_i\sum_{i'\neq i}\overline{DX\varepsilon w}_{ijk}\overline{Zw}_{i'j} = -w_{i^*}x_{i^*k}y_{ij^*}D_j Z_{i^*j} + w_{i^*}x_{i^*k}D_j Z_{i^*j}\sum_\ell \beta_\ell x_{i^*\ell} + E[DX\varepsilon w]_k\sum_i Z_{ij}w_i$$
$$+\left(\sum_i Z_i w_i - (N-1-1)E[Zw]\right)\left(w_{i^*}x_{i^*k}y_{ij^*}D_j - w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell}\right)$$
$$-N_1 E[DX\varepsilon w]_k\left(\sum_i Z_{ij}w_i - (N_1-1)E[Zw]\right)$$

$$\sum_i\sum_{i'\neq i}\overline{DX\varepsilon w}_{ijk}\overline{DX\varepsilon w}_{i'jm} = -w_{i^*}x_{i^*k}D_j x_{i^*m}y_{i^*j}^2 + 2w_{i^*}x_{i^*k}y_{ij^*}D_j x_{i^*m}\sum_\ell \beta_\ell x_{i^*\ell} - w_{i^*}x_{i^*k}D_j x_{i^*m}\left(\sum_\ell \beta_\ell x_{i^*\ell}\right)^2$$
$$+E[DX\varepsilon w]_k\left(w_{i^*}x_{i^*m}y_{ij^*}D_j - w_{i^*}x_{i^*m}D_j\sum_\ell \beta_\ell x_{i^*\ell}\right)$$
$$+\left\{w_{i^*}x_{i^*m}y_{ij^*}D_j - w_{i^*}x_{i^*m}D_j\sum_\ell \beta_\ell x_{i^*\ell} - (N_1-1)E[DX\varepsilon w]_m\right\}$$
$$\times\left(w_{i^*}x_{i^*k}y_{ij^*}D_j - w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell} - N_1 E[DX\varepsilon w]_k\right)$$

$$\sum_i\sum_{i'\neq i}\overline{DX\varepsilon w}_{ijk}\overline{DZw}_{i'j} = -wxksydZ_{i^*j} + w_{i^*}x_{i^*k}D_j Z_{i^*j}\sum_\ell \beta_\ell x_{i^*\ell} + E[DX\varepsilon w]_k D_j w_{i^*}Z_{i^*j}$$
$$+\left(D_j Z_{i^*j}w_{i^*} - (N_1-1)E[DZw]\right)\left(w_{i^*}x_{i^*k}y_{ij^*}D_j - w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell}\right)$$
$$-N_1 E[DX\varepsilon w]_k(D_j Z_{i^*j}w_{i^*} - (N_1-1)E[DZw])$$

$$\sum_i\sum_{i'\neq i}\overline{DX\varepsilon w}_{ijk}\overline{w}_{i'} = -w_{i^*}x_{i^*k}y_{ij^*}D_j + w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell} + N_1^w E[DX\varepsilon w]_k$$
$$+(N_1^w - (N_1-1)E[w])\left(w_{i^*}x_{i^*k}y_{ij^*}D_j - w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell}\right) - N_1 E[DX\varepsilon w]_k(N_1^w - (N_1-1)E[w])$$

$$\sum_i\sum_{i'\neq i}\overline{DZw}_{ijk}\overline{X\varepsilon w}_{i'jk} = -w_{i^*}x_{i^*k}y_{ij^*}D_j Z_{i^*j} + w_{i^*}x_{i^*k}D_j Z_{i^*j}\sum_\ell \beta_\ell x_{i^*\ell} + E[DZw]\left(\sum_i w_i x_{ik}y_{ij} - \sum_i w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell}\right)$$
$$+\left\{\sum_i w_i x_{ik}y_{ij} - \sum_i w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} - (N_1-1)\left(E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)\right\}(D_j Z_{i^*j}w_{i^*} - N_1 E[DZw])$$

$$\sum_i\sum_{i'\neq i}\overline{DZw}_{ijk}\overline{DX\varepsilon w}_{i'jk} = -w_{i^*}x_{i^*k}y_{ij^*}D_j Z_{i^*j} + w_{i^*}x_{i^*k}D_j Z_{i^*j}\sum_\ell \beta_\ell x_{i^*\ell} + E[DZw]\left(w_{i^*}x_{i^*k}y_{ij^*}D_j - w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell}\right)$$
$$+\left\{w_{i^*}x_{i^*k}y_{ij^*}D_j - w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell} - (N_1-1)E[DX\varepsilon w]_k\right\}(D_j Z_{i^*j}w_{i^*} - N_1 E[DZw])$$

$$\sum_i\sum_{i'\neq i}\overline{w}_i\overline{X\varepsilon w}_{i'jk} = -\sum_i w_i x_{ik}y_{ij} + \sum_i w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} + E[w]\left(\sum_i w_i x_{ik}y_{ij} - \sum_i w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell}\right)$$
$$+\left\{\sum_i w_i x_{ik}y_{ij} - \sum_i w_i x_{ik}\sum_\ell \beta_\ell x_{i\ell} - (N-1-1)\left(E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\right)\right\}(N_1^w - N_1 E[w])$$

$$\sum_i\sum_{i'\neq i}\overline{w}_i\overline{DX\varepsilon w}_{i'jk} = -w_{i^*}x_{i^*k}y_{ij^*}D_j + w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell} + E[w]\left(w_{i^*}x_{i^*k}y_{ij^*}D_j - w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell}\right)$$
$$+\left\{w_{i^*}x_{i^*k}y_{ij^*}D_j - w_{i^*}x_{i^*k}D_j\sum_\ell \beta_\ell x_{i^*\ell} - (N_1-1)E[DX\varepsilon w]_k\right\}(N_1^w - N_1 E[w])$$

Next, we consider $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{i'\neq i}\sum_{j'\neq j}\Xi'^{w}_{ijk}D_{i'j}(1-\hat{\pi}(w))\Xi^{w}_{i'j'm}$. Since we can factor the $(1-\hat{\pi}(w))$ from the quadruple summation, we ignore it in our simplifications here. The idea of the simplification is to express analytically $\sum_{i=1}^{N_1}\sum_{i'\neq i}\sum_{j'\neq j}\Xi'^{w}_{ijk}D_{i'j}\Xi^{w}_{i'j'm}$ for each j so that the number of observations that must be considered in computing the double summation is N_2 instead of N_1N_2 and it can be computed from the second census. The simplifications are as follows.

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{Dw}_{ij}D_{i'j}\overline{X\varepsilon w}_{i'j'k}=\begin{cases}0 & j \text{ unlinked}\\ -(N_1-1)E[Dw] & \\ \quad\times\left\{w_{i^*x_{i^*k}}\left(-y_{i^*j}+\sum_j y_{i^*j}\right)\right. & \\ \quad\left.-(N_2-1)(w_{i^*x_{i^*k}}\sum_\ell\beta_\ell x_{i^*\ell}+E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell})\right\} & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{Dw}_{ij}D_{i'j}\overline{DX\varepsilon w}_{i'j'k}=\begin{cases}0 & j \text{ unlinked}\\ (N_1-1)E[Dw](N_2-1)E[DX\varepsilon w]_k & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{X\varepsilon w}_{ijk}D_{i'j}\overline{Dw}_{i'j'}=\begin{cases}0 & j \text{ unlinked}\\ -(N_2-1)E[Dw] & \\ \quad\times\left\{\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}-w_{i^*x_{i^*k}}y_{i^*j}+w_{i^*x_{i^*k}}\sum_\ell\beta_\ell x_{i^*\ell}\right. & \\ \quad\left.-(N_1-1)(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell})\right\} & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{X\varepsilon w}_{ijk}D_{i'j}\overline{X\varepsilon w}_{i'j'm}=\begin{cases}0 & j \text{ unlinked}\\ \left\{w_{i^*x_{i^*m}}\left(-y_{i^*j}+\sum_j y_{i^*j}\right)\right. & \\ \quad\left.-(N_2-1)(w_{i^*x_{i^*m}}\sum_\ell\beta_\ell x_{i^*\ell}+E[XYw]_m-\sum_\ell\beta_\ell E[XXw]_{m\ell})\right\} & \\ \quad\times\left\{\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}-w_{i^*x_{i^*k}}y_{i^*j}+w_{i^*x_{i^*k}}\sum_\ell\beta_\ell x_{i^*\ell}\right. & \\ \quad\left.-(N_1-1)(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell})\right\} & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{X\varepsilon w}_{ijk}D_{i'j}\overline{Zw}_{i'j'}=\begin{cases}0 & j \text{ unlinked}\\ \left\{w_{i^*}\left(-Z_{i^*j}-\sum_j Z_{i^*j}\right)-(N_2-1)E[Zw]\right\} & \\ \quad\times\left\{\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}-w_{i^*x_{i^*k}}y_{i^*j}+w_{i^*x_{i^*k}}\sum_\ell\beta_\ell x_{i^*\ell}\right. & \\ \quad\left.-(N_1-1)(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell})\right\} & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{X\varepsilon w}_{ijk}D_{i'j}\overline{DX\varepsilon w}_{i'j'm}=\begin{cases}0 & j \text{ unlinked}\\ -(N_2-1)E[DX\varepsilon w]_m & \\ \quad\times\left\{\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}-w_{i^*x_{i^*k}}y_{i^*j}+w_{i^*x_{i^*k}}\sum_\ell\beta_\ell x_{i^*\ell}\right. & \\ \quad\left.-(N_1-1)(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell})\right\} & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{X\varepsilon w}_{ijk}D_{i'j}\overline{DZw}_{i'j'}=\begin{cases}0 & j \text{ unlinked}\\ -(N_2-1)E[DZw] & \\ \quad\times\left\{\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}-w_{i^*x_{i^*k}}y_{i^*j}+w_{i^*x_{i^*k}}\sum_\ell\beta_\ell x_{i^*\ell}\right. & \\ \quad\left.-(N_1-1)(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell})\right\} & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{X\varepsilon w}_{ijk}D_{i'j}\overline{w}_{i'}=\begin{cases}0 & j \text{ unlinked}\\ (N_2-1)(w_i-E[w]) & \\ \quad\times\left\{\sum_i w_i x_{ik}y_{ij}-\sum_i w_i x_{ik}\sum_\ell\beta_\ell x_{i\ell}-w_{i^*x_{i^*k}}y_{i^*j}+w_{i^*x_{i^*k}}\sum_\ell\beta_\ell x_{i^*\ell}\right. & \\ \quad\left.-(N_1-1)(E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell})\right\} & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{Zw}_{ij}D_{i'j}\overline{X\varepsilon w}_{i'j'k}=\begin{cases}0 & j \text{ unlinked}\\ \left\{\sum_i Z_{ij}w_i-Z_{i^*j}w_{i^*}-(N_1-1)E[Zw]\right\} & \\ \quad\times\left\{w_{i^*x_{i^*k}}\left(-y_{i^*j}+\sum_j y_{i^*j}\right)\right. & \\ \quad\left.-(N_2-1)(w_{i^*x_{i^*k}}\sum_\ell\beta_\ell x_{i^*\ell}+E[XYw]_k-\sum_\ell\beta_\ell E[XXw]_{k\ell})\right\} & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{Zw}_{ij}D_{i'j}\overline{DX\varepsilon w}_{i'j'k}=\begin{cases}0 & j \text{ unlinked}\\ -\left\{\sum_i Z_{ij}w_i-Z_{i^*j}w_{i^*}-(N_1-1)E[Zw]\right\}(N_2-1)E[DX\varepsilon w]_k & j \text{ linked}\end{cases}$$

$$\sum_i\sum_{i'\neq i}\sum_{j'\neq j}\overline{DX\varepsilon w}_{ijk}D_{i'j}\overline{Dw}_{i'j'}=\begin{cases}0 & j \text{ unlinked}\\ (N_1-1)E[DX\varepsilon w]_k(N_2-1)E[Dw] & j \text{ linked}\end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DX\varepsilon w}_{ijk} D_{i'j} \overline{X\varepsilon w}_{i'j'm} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_1-1)E[DX\varepsilon w]_k \\ \quad \times \left\{ w_{i^*x_{i^*m}} \left(-y_{i^*j} + \sum_j y_{i^*j} \right) \right. \\ \quad \left. -(N_2-1)\left(w_{i^*x_{i^*m}} \sum_\ell \beta_\ell x_{i^*\ell} + E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell} \right) \right\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DX\varepsilon w}_{ijk} D_{i'j} \overline{Zw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_1-1)E[DX\varepsilon w]_k \left\{ w_{i^*} \left(-Z_{i^*j} - \sum_j Z_{i^*j} \right) - (N_2-1)E[Zw] \right\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DX\varepsilon w}_{ijk} D_{i'j} \overline{DX\varepsilon w}_{i'j'm} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)E[DX\varepsilon w]_k (N_2-1)E[DX\varepsilon w]_m & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DX\varepsilon w}_{ijk} D_{i'j} \overline{DZw}_{i'j'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)E[DX\varepsilon w]_k (N_2-1)E[DZw] & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DX\varepsilon w}_{ijk} D_{i'j} \overline{w}_{i'} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)E[DX\varepsilon w]_k (N_2-1)(w_i - E[w]) & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DZw}_{ij} D_{i'j} \overline{X\varepsilon w}_{i'j'k} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_1-1)E[DZw] \\ \quad \times \left\{ w_{i^*x_{i^*k}} \left(-y_{i^*j} + \sum_j y_{i^*j} \right) \right. \\ \quad \left. -(N_2-1)\left(w_{i^*x_{i^*k}} \sum_\ell \beta_\ell x_{i^*\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{DZw}_{ij} D_{i'j} \overline{DX\varepsilon w}_{i'j'k} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1-1)E[DZw](N_2-1)E[DX\varepsilon w]_k & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{w}_i D_{i'j} \overline{X\varepsilon w}_{i'j'k} = \begin{cases} 0 & j \text{ unlinked} \\ (N_1^w - w_i - (N_1-1)E[w]) \\ \quad \times \left\{ w_{i^*x_{i^*k}} \left(-y_{i^*j} + \sum_j y_{i^*j} \right) \right. \\ \quad \left. -(N_2-1)\left(w_{i^*x_{i^*k}} \sum_\ell \beta_\ell x_{i^*\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\} & j \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{w}_i D_{i'j} \overline{DX\varepsilon w}_{i'j'k} = \begin{cases} 0 & j \text{ unlinked} \\ -(N_1^w - w_i - (N_1-1)E[w])(N_2-1)E[DX\varepsilon w]_k & j \text{ linked} \end{cases}$$

Finally, we consider $\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{i'\neq i} \sum_{j'\neq j} \Xi'^w_{ijk} D_{ij'} (1-\hat{\pi}(w)) \Xi^w_{i'j'm}$. Since we can factor the $(1-\hat{\pi}(w))$ from the quadruple summation, we ignore it in our simplifications here. The idea of the simplification is to express analytically $\sum_{j=1}^{N_2} \sum_{i'\neq i} \sum_{j'\neq j} \Xi'^w_{ijk} D_{ij'} \Xi^w_{i'j'm}$ for each j so that the number of observations that must be considered in computing the double summation is N_1 instead of $N_1 N_2$ and it can be computed from the second census. The simplifications are as follows.

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Dw}_{ij} D_{ij'} \overline{X\varepsilon w}_{i'j'k} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2-1)E[Dw] \\ \quad \times \left\{ \sum_i w_i x_{ik} y_{ij^*} - \sum_i w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - w_i x_{ik} y_{ij^*} \right. \\ \quad \left. + w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - (N_1-1)\left(E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{Dw}_{ij} D_{ij'} \overline{DX\varepsilon w}_{i'j'k} = \begin{cases} 0 & i \text{ unlinked} \\ (N_2-1)E[Dw](N_1-1)E[DX\varepsilon w]_k & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{X\varepsilon w}_{ijk} D_{ij'} \overline{Dw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_1-1)E[Dw] \\ \quad \times \left\{ w_i x_{ik} \left(-y_{ij^*} + \sum_j y_{ij} \right) \right. \\ \quad \left. -(N_2-1)\left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq i} \sum_{j'\neq j} \overline{X\varepsilon w}_{ijk} D_{ij'} \overline{X\varepsilon w}_{i'j'm} = \begin{cases} 0 & i \text{ unlinked} \\ \left\{ \sum_i w_i x_{im} y_{ij^*} - \sum_i w_i x_{im} \sum_\ell \beta_\ell x_{i\ell} - w_i x_{im} y_{ij^*} \right. \\ \quad \left. + w_i x_{im} \sum_\ell \beta_\ell x_{i\ell} - (N_1-1)\left(E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell} \right) \right\} \\ \quad \times \left\{ w_i x_{ik} \left(-y_{ij^*} + \sum_j y_{ij} \right) \right. \\ \quad \left. -(N_2-1)\left(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell} \right) \right\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{X\varepsilon w}_{ijk} D_{ij'} \overline{Zw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ \Big(\sum_i w_i Z_{ij*} - Z_{ij*} w_i - (N_1-1)E[Zw]\Big) & \\ \quad \times \Big\{ w_i x_{ik} \Big(-y_{ij*} + \sum_j y_{ij}\Big) & \\ \quad -(N_2-1)\big(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\big)\Big\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{X\varepsilon w}_{ijk} D_{ij'} \overline{DX\varepsilon w}_{i'j'm} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_1-1)E[DX\varepsilon w]_m & \\ \quad \times \Big\{ w_i x_{ik} \Big(-y_{ij*} + \sum_j y_{ij}\Big) & \\ \quad -(N_2-1)\big(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\big)\Big\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{X\varepsilon w}_{ijk} D_{ij'} \overline{DZw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_1-1)E[DZw] & \\ \quad \times \Big\{ w_i x_{ik} \Big(-y_{ij*} + \sum_j y_{ij}\Big) & \\ \quad -(N_2-1)\big(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\big)\Big\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{X\varepsilon w}_{ijk} D_{ij'} \overline{w}_{i'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_1-1)(w_i - E[w]) & \\ \quad \times \Big\{ w_i x_{ik} \Big(-y_{ij*} + \sum_j y_{ij}\Big) & \\ \quad -(N_2-1)\big(w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} + E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\big)\Big\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{Zw}_{ij} D_{ij'} \overline{X\varepsilon w}_{i'j'k} = \begin{cases} 0 & i \text{ unlinked} \\ \Big(w_i \Big(-Z_{ij*} + \sum_j Z_{ij}\Big) - (N_2-1)E[Zw]\Big) & \\ \quad \times \Big\{ \sum_i w_i x_{ik} y_{ij*} - \sum_i w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - w_i x_{ik} y_{ij*} & \\ \quad + w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - (N_1-1)\big(E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\big)\Big\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{Zw}_{ij} D_{ij'} \overline{DX\varepsilon w}_{i'j'k} = \begin{cases} 0 & i \text{ unlinked} \\ -\Big(w_i \Big(-Z_{ij*} + \sum_j Z_{ij}\Big) - (N_2-1)E[Zw]\Big)(N_1-1)E[DX\varepsilon w]_k & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{DX\varepsilon w}_{ijk} D_{ij'} \overline{Dw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_2-1)E[DX\varepsilon w]_k (N_1-1)E[Dw] & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{DX\varepsilon w}_{ijk} D_{ij'} \overline{X\varepsilon w}_{i'j'm} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2-1)E[DX\varepsilon w]_k & \\ \quad \times \Big\{ \sum_i w_i x_{im} y_{ij*} - \sum_i w_i x_{im} \sum_\ell \beta_\ell x_{i\ell} - w_i x_{im} y_{ij*} & \\ \quad + w_i x_{im} \sum_\ell \beta_\ell x_{i\ell} - (N_1-1)\big(E[XYw]_m - \sum_\ell \beta_\ell E[XXw]_{m\ell}\big)\Big\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{DX\varepsilon w}_{ijk} D_{ij'} \overline{Zw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2-1)E[DX\varepsilon w]_k \Big(\sum_i w_i Z_{ij*} - Z_{ij*} w_i - (N_1-1)E[Zw]\Big) & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{DX\varepsilon w}_{ijk} D_{ij'} \overline{DX\varepsilon w}_{i'j'm} = \begin{cases} 0 & i \text{ unlinked} \\ (N_2-1)E[DX\varepsilon w]_k (N_1-1)E[DX\varepsilon w]_m & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{DX\varepsilon w}_{ijk} D_{ij'} \overline{DZw}_{i'j'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_2-1)E[DX\varepsilon w]_k (N_1-1)E[DZw] & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{DX\varepsilon w}_{ijk} D_{ij'} \overline{w}_{i'} = \begin{cases} 0 & i \text{ unlinked} \\ (N_2-1)E[DX\varepsilon w]_k (N_1-1)(w_i - E[w]) & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{DZw}_{ij} D_{ij'} \overline{X\varepsilon w}_{i'j'k} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2-1)E[DZw] & \\ \quad \times \Big\{ \sum_i w_i x_{ik} y_{ij*} - \sum_i w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - w_i x_{ik} y_{ij*} & \\ \quad + w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - (N_1-1)\big(E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\big)\Big\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i'\neq ij'\neq j} \sum \overline{DZw}_{ij} D_{ij'} \overline{DX\varepsilon w}_{i'j'k} = \begin{cases} 0 & i \text{ unlinked} \\ (N_2-1)E[DZw](N_1-1)E[DX\varepsilon w]_k & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{w}_i D_{ij'} \overline{X\varepsilon w}_{i'j'k} = \begin{cases} 0 & i \text{ unlinked} \\ (N_2-1)(w_i - E[w]) & \\ \quad \times \Big\{ \sum_i w_i x_{ik} y_{ij*} - \sum_i w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - w_i x_{ik} y_{ij*} & \\ \quad + w_i x_{ik} \sum_\ell \beta_\ell x_{i\ell} - (N_1-1)\big(E[XYw]_k - \sum_\ell \beta_\ell E[XXw]_{k\ell}\big) \Big\} & i \text{ linked} \end{cases}$$

$$\sum_i \sum_{i' \neq i} \sum_{j' \neq j} \overline{w}_i D_{ij'} \overline{DX\varepsilon w}_{i'j'k} = \begin{cases} 0 & i \text{ unlinked} \\ -(N_2-1)(w_i - E[w])(N_1-1)E[DX\varepsilon w]_k & i \text{ linked} \end{cases}$$