Technical Appendix

A. Additional Literature Review

The paper is related to several well-developed literatures, the literature on Secular Stagnation, the dynamic business cycle literature with search and matching frictions cum Nash wage bargaining, the dynamic asset pricing literature and the literature on consumption.

Beginning with Summers (2013) the Secular Stagnation literature is large. A highly readable introduction to the issues involved may be found in the volume edited by Teulings and Baldwin (2017). Most of the literature views the central mechanism generating Secular Stagnation as the inability of the interest rate to decline enough to raise investment to the level of savings in a full employment environment, resulting in subpar economic growth, the problem of the so called "Zero Lower Bound." Within this environment, the literature studies possible forces that reduce the propensity to save or increase the propensity to invest. Eggertsson and Mehrotra (2014) and Eggertsson et al. (2019b) provide important theoretical support. Eggertsson et al. (2019a) considers the consequences of ageing on the savings/investment balance. While we do not emphasize demographic forces, in the principal version of the model we study, growing wealth inequality is identified with a reduction in the measure of the stockholder class and thus a decline in the entire population. Eggertsson et al. (2019b) demonstrates that increasing income inequality can lower the natural rate, a feature of the present where it is a purely business cycle phenomenon arising largely from non-homothetic savings behavior on the part of stockholders. Rannenberg (2019) considers the effect of rising income inequality on aggregate savings and the full employment interest rate. For the present paper, it is the influence of the default free bond market on the labor market that drives our results. In this sense the model is directly in the spirit of Phelps (1994), although the modeling perspective is different. Daly (2016) argues that prior to the year 2000 interest rate reductions were driven by large increases in saving while post 2000 risk premium shocks have driven both an increase in the equity premium and a reduction in the safe rate. While the present model is not focused on replicating financial quantities, it does generate the phenomenon documented in Daly (2016) as a consequence of increasing wealth inequality.

Benigno and Fornaro (2017) provide a Keynesian growth story where pessimistic expectations lead to low growth which the central bank cannot restore because of the ZLB: low growth generates low profits which depresses investment in innovation. Fagereng et al. (2019) provide recent evidence that wealthy households save a larger share of their income when capital gains are accorded to savings, an observation that lends support to rising income inequality as a driving force in reducing real interest rates. Secular Stagnation is also related to phenomena referred to as the "Safe Asset Shortage" (Caballero and Fahri (2018) and Caballero (2008, 2015)). This literature attributes the secular decline in real interest rates to a combination of the 'global savings glut,' mandated regulatory changes that require financial institutions to hold more safe assets, etc. While this literature considers factors leading to the secular rate decline, it does not consider the mechanism detailed in the present model.

The literature arguing for macroprudential policies is related to the present model as it is based on the presence of pecuniary externalities derived from market incompleteness. These policies often involve financial market interventions as supplements when monetary policy is no longer effective. Fahri and Werning (2016) provide a framework for evaluating these policies and an excellent overview off the literature review. As an illustration, we point to Korinek and Simsek (2016) who evaluate the usefulness of ex-ante debt limits in anticipation of deleveraging shocks when the ZLB is possible since individual households decisions to deleverage entail aggregate demand externalities. Our perspective is different as we look to the labor market and the form of the present value wage contract to (partially) internalize the externality, and then explore its implications for the characteristics of the business cycle and asset pricing.

Because of the model's structure, the paper is directly related to the extensive dynamic macro literature with search and matching frictions; viz, Andolfatto (1996), Merz (1995), Shimer (2005), Hagedorn and Manovskii (2008), Christoffel and Kuester (2008) and Gertler and Trigari (2009). An excellent literature review of search and matching in the dynamic macro context is Yashiv (2007). Hornstein et al. (2005) provide an excellent survey of the intuition and issues involved in introducing search and matching frictions into dynamic business cycle models.

Of added relevance is the pure dynamic asset pricing literature. Prominent contributions are too numerous to detail and are generally well known; specifically we mention only Lucas (1978), Mehra and Prescott (1985), Jermann (1998), Boldrin et al. (2001), Danthine and Donaldson (2002), Guvenen (2009), Kaltenbrunner and Lochstoer (2010), Kim (2011), Gabaix (2012), Gourio (2012), Favalukis and Lin (2015), Lansing (2015) and Petrosky-Nadeau et al (2017). Fahri and Gourio (2018), Kehoe et al. (2023) and Ai and Bhandari (2018) are more recent contributions. Fahri and Gourio (2016) can explain a wide class of financial statistics using and otherwise standard production-growth model into which they incorporate monopolistic competition and capital quality shocks. Unlike the emphasis of the present paper, they set aside labor market considerations by assuming inelastic labor supply. Ai and Bhandri (2018) introduce agency frictions into a model of long-term contracting where neither stockholders nor workers can commit to the contract. We discuss the issue of commitment to present value wage bargaining and offer a mechanism by which both stockholders and workers voluntarily participate in egalitarian wage bargaining. In Ai and Bhandri (2018) workers anticipate greater separations in bad times (less firm investment in maintaining the match) and demand a contract yielding a highly procyclical stockholder consumption share and high stockholder SDF volatility. In contrast, we maintain the standard assumption of a constant separation rate while focusing more on the consequences of endogenous countercyclical payoff-relevant bargaining power on the properties of the equilibrium labor contract. Kehoe et al. (2023) focus on the unemployment volatility puzzle in a model that is also consistent with wage cyclicality and observed risk free rate volatility. This is accomplished by postulating time varying risk over the business cycle via and exogenous habit process akin to the process feature in Campbell and Cochrane (1999). The present model has differential, fixed habit parameters as a device for motivating the observed distribution of wealth in the United States, and studies how its evolution jointly impacts the labor and bond markets. Nevertheless, the model generates more than realistic unemployment volatility. Again, matching labor market volatility statistics is a byproduct of our modeling perspective, and not a principal objective of the present study. Each of these papers assumes pure stockholder wealth maximization, and financial market completeness. Our central mechanism also relies on financial market incompleteness, however.

Saito (1996), Basak and Cuoco (1998), and Guvenen (2009) are models of limited stock market participation like the present model. Again, all three focus on explaining financial stylized facts. Basak and Cuoco (1998) and Saito (1995) have no labor income flow. A higher equity premium results from increasing financial leverage as non-stockholders have no other way to accumulate wealth than to buy bonds. Guvenen (2009) is a production business cycle model that also shares the same structure of financial market incompleteness as in the present paper. Non-stockholders have labor income but, again, his focus is on replicating financial market regularities whereas our focus is on the resolution of the investment externality through "reform" of the labor market. In addition, the labor market in Guvenen (2009) is competitive, whereas in the present paper it is governed by the egalitarian generalization of Nash wage bargaining. Nevertheless, the present model shares many of Guvenen's (2009) model feature: workers trade bonds to assist in consumption smoothing and the bonds traded in the economy are issued by the firms themselves. However, in the present model's stakeholder economy, the labor contract itself provides risk-mitigation services to workers. In this sense, consumption smoothing in the present model occurs both within and without the firm, with the tradeoff between these mechanisms driving many of our results.

Much related literature focuses purely on the impact of incomplete financial markets on asset pricing, with a particular focus on the magnitude of the equity premium. Basically, this literature can be divided into production vs. exchange models and models with uninsurable risks vs. those with aggregate risk or both, these categories not being mutually exclusive. Incompleteness may take the form of the uninsurable components of income risk or restrictions on the types of securities eligible to be traded, in the latter case typically only a stock and a one-period bond; limited participation incompleteness where only a bond is traded by all agents is also paradigmatic. Basak and Cuoco (1998), Krueger (1999), Kreuger and Lustig (2010), and Levine and Zame (2002) are examples of these paradigmatic exchange models.

In an exchange economy with a continuum of agents where, as in the present model, only a stock and bond are traded, Krueger and Lustig (2010) present conditions under which the introduction of idiosyncratic, uninsurable income risk has no effect on the premium for aggregate risk. Krueger (1999) explores whether endogenous debt constraints relative to pure bond trading can better explain the lack of perfect risk sharing evident in micro data. Under bond trading alone Basak and Cuoco (1998) consider how limited stock market participation allows an exchange model to better replicate a variety of financial data. This feature is also present in the production models of Gomes and Michaelides (2007) and Guvenen (2009), and in the model of this paper. Gomes and Michaelides (2007) find that limited participation has a negligible effect on the premium. In the present paper, the extent of bond trading both allows workers to manage unemployment risk and directly influences the form of the labor contract. In these other papers the bond is purely a security for risk sharing and has no direct connection to the labor market, unlike the present paper.

The model generates some ancillary implications which we note in passing. Hall (2017) emphasizes a countercyclical discount rate as crucial to resolving the vacancy and unemployment volatility puzzles emphasized in Shimer (2005). Confirming Hall's (2017) intuition, both unemployment and vacancy postings in the model are volatile and negatively correlated with one another at business cycle frequencies. Furthermore, distribution risk in the model may be interpreted as representing the role of a wage shock emphasized by Shimer (2005) as necessary to account for the cyclical movement of key labor market variables.

The consumption/savings literature allows us to interpret the savings patterns of both stockholders and non-stockholders particularly as illustrated in the case of extreme wealth inequality. In particular, we rely on the insights of Deaton (1991) and Carroll (1992, 2009). The non-homothetic savings behavior of the stockholders in the model when wealth inequality is extreme reflects the apparent savings pattern of the top 10% of the US wealth distribution, as persuasively argued by Main et al. (2021). For an overview of the vast consumption literature, the reader is referred to the excellent survey of Carroll (2001).

A.1. Supplementary References

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B. Derivations and Proofs

B.1. Derivations for Section 2.1.

Conditional upon her period t information set, Ω_t^s, the recursive formulation of the stockholder's problem may be represented as:

$$V^s(\Omega_t^s) = \max_{\{c_t^s, h_t^s, e_{t+1}^s, b_{t+1}^s\}} \left[\begin{array}{c} u^s\left(c_t^s - \chi^s \mathbf{c}_{t-1}^s, H(h_t^s)\right) \\ +\lambda_t^s[w_t^s h_t^s + (p_t^e + d_t)e_t^s + p_t^b b_t^s - c_t^s - p_t^e e_{t+1}^s - b_{t+1}^s] \\ +\beta \mathbb{E}(V^s(\Omega_{t+1}^s)|\Omega_t^s) \end{array} \right] \tag{30}$$

where λ_t^s is the Lagrange multiplier associated with budget constraint (2). The solution to problem (30) is thus characterized by three standard necessary and sufficient first-order conditions:

$$w_t^s = H_1(h_t^s) \tag{31}$$

$$p_t^e = \beta \mathbb{E}_t[\widetilde{\Lambda}_{t,t+1}^s(\widetilde{p}_{t+1}^e + \widetilde{d}_{t+1})] \tag{32}$$

$$p_t^b = \beta \mathbb{E}_t(\widetilde{\Lambda}_{t,t+1}^s | \Omega_t^s) \tag{33}$$

where $\Lambda_{t,t+1}^s = \lambda_{t+1}^s / \lambda_t^s = u_1^s(c_{t+1}^s - \chi^s \mathbf{c}_t^s - H(h_{t+1}^s)) / u_1^s(c_t^s - \chi^s \mathbf{c}_{t-1}^s - H(h_t^s))$ denotes the stockholder's intertemporal marginal rate of substitution (IMRS), to be determined in equilibrium. In what follows we denote $\left\{\beta \widetilde{\Lambda}_{t,t+1}^s\right\}$ by $\left\{\widetilde{M}_{t,t+1}\right\}$. It represents the economy-wide stochastic discount factor (SDF) for all valuation purposes.

B.2. Derivations for Section 2.2.

Conditional upon Ω_t^n, the recursive formulation of a worker's problem may be expressed as:

$$V^n(\Omega_t^n) = \max_{\{b_{t+1}^n, h_t^n\}} \begin{bmatrix} u^n(c_t^n - \chi^n \mathbf{c}_{t-1}^n - n_t L(h_t^n)) \\ +\lambda_t^n \left(b_t^n + w_t^n h_t^n n_t + b(1-n_t) - p_t^b b_{t+1}^n - c_t^n\right) \\ +\beta \mathbb{E} V^n(\Omega_{t+1}^n | \Omega_t^n) \end{bmatrix} \tag{34}$$

where $c_t^n = n_t c_t^{n,e} + (1-n_t) c_t^{n,u}$, and λ_t^n is the Lagrange multiplier associated with the worker's budget constraint (6).[44, 45] The solution to problem (34) is characterized by the three necessary and sufficient first-order conditions:

$$u_1^n(c_t^n - \chi^n \mathbf{c}_{t-1}^n - n_t L(h_t^n)) = \lambda_t^n \tag{35}$$

$$w_t^n = L_1(h_t^n) \tag{36}$$

$$p_t^b = \beta \mathbb{E}\left(\frac{u_1^n(c_{t+1}^n - \chi^n \mathbf{c}_t^n - n_{t+1} L(h_{t+1}^n))}{u_1^n(c_t^n - \chi^n \mathbf{c}_{t-1}^n - n_t L(h_t^n))} | \Omega_t^n\right). \tag{37}$$

Note that workers' hours are supplied under the condition that the (hourly) wage equals the marginal rate of substitution of consumption for leisure, and that $\mathbf{c}_t^n = \int c_t^n d\omega$ is average worker consumption in period t with ω the measure of workers.

B.3. Deriving Non-Stockholder-Worker Family Problem (34) from Problem (4) – (7)

By perfect risk sharing within the non-stockholder-worker family,

$$u_1^n(c_t^{n,e} - \chi^n \mathbf{c}_{t-1}^{n,e} - L(h_t^n)) = u_1^n(c_t^{n,u} - \chi^n \mathbf{c}_{t-1}^{n,u})$$

where we assume $L(0) = 0$. By the form of GHH utility,

$$c_t^{n,e} - \chi^n \mathbf{c}_{t-1}^{n,e} - L(h_t^n) = c_t^{n,u} - \chi^n \mathbf{c}_{t-1}^{n,u}. \tag{38}$$

[44] $\Omega_t^n = \{w_t^n, n_t, s_t, b_t^n, p_t^b, \mathbf{c}_{t-1}^n\}$.

[45] The transition from (4) – (7) to (34) is not obvious. It follows from optimal risk sharing within the family. See Part B of the Appendix.

Note that total consumption of the family is:

$$\begin{aligned} c_t^n &= n_t c_t^{n,e} + (1-n_t)\, c_t^{n,u} \\ &= n_t \left(c_t^{n,u} - \chi^n \mathbf{c}_{t-1}^{n,u} + \chi^n \mathbf{c}_{t-1}^{n,e} + L(h_t^n) \right) + (1-n_t) c_t^{n,u} \\ &= c_t^{n,u} + n_t \chi^n \left(\mathbf{c}_{t-1}^{n,e} - \mathbf{c}_{t-1}^{n,u} \right) + n_t L(h_t^n) \end{aligned}$$

Thus,

$$c_t^n - n_t L\left(h_t^n\right) = c_t^{n,u} + n_t \left(\chi^n \mathbf{c}_{t-1}^{n,e} - \chi^n \mathbf{c}_{t-1}^{n,u} \right).$$

To both sides of the equation immediately above, subtract $\left[n_t \chi^n \mathbf{c}_{t-1}^{n,e} + (1-n_t)\, \chi^n \mathbf{c}_{t-1}^{n,u} \right]$. This yields

$$c_t^n - \chi^n \left[n_t \mathbf{c}_{t-1}^{n,e} + (1-n_t)\, \mathbf{c}_{t-1}^{n,u} \right] - n_t L\left(h_t^n\right) = c_t^{n,u} - \chi^n \mathbf{c}_{t-1}^{n,u}$$

or,

$$\begin{aligned} c_t^n - \chi^n \mathbf{c}_{t-1}^n - n_t L\left(h_t^n\right) &= c_t^{n,u} - \chi^n \mathbf{c}_{t-1}^{n,u} \\ &= c_t^{n,e} - \chi^n \mathbf{c}_{t-1}^{n,e} - L\left(h_t^n\right) \quad \text{by (38)} \end{aligned}$$

Thus,

$$\begin{aligned} u^n \left(c_t^n - \chi^n \mathbf{c}_{t-1}^n - n_t L\left(h_t^n\right) \right) &= u^n \left(c_t^{n,u} - \chi^n \mathbf{c}_{t-1}^{n,u} \right) \\ &= u^n \left(c_t^{n,e} - \chi^n \mathbf{c}_{t-1}^{n,e} - L\left(h_t^n\right) \right). \end{aligned}$$

This implies the objective function in (4),

$$n_t u^n \left(c_t^{n,e} - \chi^n \mathbf{c}_{t-1}^{n,e} - L\left(h_t^n\right) \right) + (1-n_t)\, u^n \left(c_t^{n,u} - \chi^n \mathbf{c}_{t-1}^{n,u} \right) = u^n \left(c_t^n - \chi^n \mathbf{c}_{t-1}^n - n_t L\left(h_t^n\right) \right).$$

The constraint is similarly modified as indicated. In this sense problems (4) – (7) and problem (34) are equivalent.

Note also two implicit assumptions underlying the equivalence: (i) whether employed or unemployed, non-stockholders have the same habit parameter; (ii) the habit of the unemployed depends only on the past consumption of the unemployed and similarly for the employed.

B.4. Within-Family Risk Sharing: A Recursive Representation

We derive Equation (34) for our representative non-stockholder with *any general time -separable preferences* within the context of the present model. For example, the non-stockholder's decision problem can be written as the representative family's optimum problem:

$$\begin{aligned} \max_{\left\{h_t^n, c_t^{n,e}, c_t^{n,u}, b_{t+1}^n\right\}} \; & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [n_t u^n (c_t^{n,e}, 1 - h_t^n) + (1-n_t) u^n (c_t^{n,u}, 1-e)] \qquad (39) \\ & \text{subject to} \\ & n_t c_t^{n,e} + (1-n_t) c_t^{u,e} + p_t^b b_{t+1}^n \le w_t^n h_t^n n_t + b(1-n_t) + T_t + b_t^n \\ & n_{t+1} = (1-\rho) n_t + s_t (1-n_t) \end{aligned}$$

where $c_t^{n,e}$ and $c_t^{n,u}$ are the period t employed non-stockholder's consumption and the unemployed non-stockholder's consumption, respectively, h_t^n is labour hours supplied by those employed in period t and the *constant* e represents constant search efforts made by the unemployed. In what follows, the derivation is unchanged with the addition of a habit, preferences are that GHH or $L(0)$ replaces $1-e$ as in the specific model of the present paper. The corresponding recursive representation is given by:

$$V^n(\Omega_t^n) = \max_{\{h_t^n, c_t^{n,e}, c_t^{n,u}, b_{t+1}^n\}} n_t u^n(c_t^{n,e}, 1-h_t^n) + (1-n_t)u^n(c_t^{n,u}, 1-e) + \beta \mathbb{E}_t[\tilde{V}^n(\Omega_{t+1}^n)] \tag{40}$$

subject to

$$n_t c_t^{n,e} + (1-n_t)c_t^{u,e} + p_t^b b_{t+1}^n \leq w_t^n h_t^n n_t + b(1-n_t) + T_t + b_t^n$$
$$n_{t+1} = (1-\rho)n_t + s_t(1-n_t)$$

Its Bellman equation is written as:

$$V^n(n_t,\, b_t^n; \Omega_t^n) = \max_{\{h^n, c_t^{n,e}, c_t^{n,u}, b_{t+1}^n\}} \begin{bmatrix} n_t u^n(c_t^{n,e}, 1-h_t^n) + (1-n_t)u^n(c_t^{n,u}, 1-e) \\ +\lambda_t^n(w_t^n h_t^n n_t + b(1-n_t) + T_t + b_t^n - n_t c_t^{n,e} - (1-n_t)c_t^{u,e} - p_t^b b_{t+1}^n) \\ +\beta \mathbb{E}_t[\tilde{V}^n((1-\rho)n_t + s_t(1-n_t),\; b_{t+1}^n; \Omega_{t+1}^n)] \end{bmatrix}. \tag{41}$$

where Ω_t^n is the representative worker's information set.

The Beveniste-Schneikman theorem gives:

$$\frac{\partial V^n}{\partial n_t} = u^n(c_t^{n,e}, 1-h_t^n) - u^n(c_t^{n,u}, 1-e) + \lambda_t^n(w_t^n h_t^n - b - (c_t^{n,e} - c_t^{n,u})) + \beta \mathbb{E}_t[\frac{\partial \tilde{V}^n}{\partial n_{t+1}}(1-\rho-s_t)]. \tag{42}$$

The risk-sharing condition within the family implies that:

$$\lambda_t^n = u_1^n(c_t^{n,e}, 1-h_t^n) = u_1^n(c_t^{n,u}, 1-e). \tag{43}$$

B.5. Derivations for Section 2.4.

The necessary and sufficient first-order condition for the firm's optimal investment decision is characterized by the investment Euler equation:

$$1 = \mathbb{E}(\widetilde{M}_{t,t+1} G_1(\frac{i_t}{k_t})[f_1(k_{t+1}, \mu_s \widetilde{h}_{t+1}^s, n_{t+1}\widetilde{h}_{t+1}^n)\widetilde{z}_{t+1} + \frac{(1-\delta) + G(\frac{\widetilde{i}_{t+1}}{k_{t+1}})}{G_1(\frac{\widetilde{i}_{t+1}}{k_{t+1}})} - \frac{\widetilde{i}_{t+1}}{k_{t+1}}]|\Omega_t^f). \tag{44}$$

The first-order condition for the firm 's optimal hiring decision for stockholder hours is given by

$$h_t^s : w_t^s = f_2(k_t, \mu_s h_t^s, n_t h_t^n) z_t, \tag{45}$$

while the first-order condition for the firm's optimal hiring rate for workers is given by

$$x_t : \kappa x_t = \mathbb{E}_t \widetilde{M}_{t,t+1} \widetilde{J}_{t+1} \tag{46}$$

where $J_t \equiv \frac{\partial V^f(\Omega_t)}{\partial n_t}$ is the firm's shadow value of hiring one additional worker (to be characterized shortly).[46]

[46] Here and in what follows we economize on notation by using the symbol $\mathbb{E}_t[\,\cdot\,]$ to represent $\mathbb{E}_t[\,\cdot \mid \Omega_t^i]$, $i \in \{s,\; f,\; n\}$ when there is no ambiguity.

Defining $V^f(\Omega_t^f) \equiv d_t + p_t^e$, the recursive representation of the firm's problem may be written as:

$$V^f(\Omega_t^f) = d_t + \mathbb{E}_t(\tilde{M}_{t,t+1} V^f(\Omega_{t+1}^f)|\Omega_t^f).$$

The necessary and sufficient first-order condition for the firm's optimal investment decision is given by:

$$i_t : \ (-1) + \mathbb{E}(\tilde{M}_{t,t+1} V_{k_{t+1}}^f |\Omega_t^f) \frac{\partial k_{t+1}}{\partial i_t} = 0.$$

By the envelope theorem,

$$\frac{\partial V^f(\Omega_t^f)}{\partial k_t} = f_1(k_t, \mu_s h_t^s, n_t h_t^n) z_t + \mathbb{E}_t(\tilde{M}_{t,t+1} V_{k_{t+1}}^f |\Omega_t^f) \frac{\partial k_{t+1}}{\partial k_t}.$$

B.6. Proof of Propsition 2.1.

Proof. The egalitarian bargaining solution directly implies

$$(1-\eta)\lambda_t^n (EP_t - U_t) = \eta \lambda_t^s J_t \tag{47}$$

where we have substituted identifications (18) and (19) into (20). Letting $\phi_t \equiv \lambda_t^s / \lambda_t^n$, as before, equation (47) can be rewritten as

$$(1-\eta)\frac{1}{\phi_t}(EP_t - U_t) = \eta J_t \tag{48}$$

where we identify $\{\widetilde{\phi}_t\}$ as "distribution risk."

Define

$$\eta_t = \frac{\eta}{(1-\eta)\frac{1}{\phi_t} + \eta}.$$

Dividing both sides of equation (48) by $(1-\eta)(1/\phi_t) + \eta$ yields

$$(1-\eta_t)(EP_t - U_t) = \eta_t J_t, \ \text{ or}$$
$$\frac{EP_t - U_t}{\eta_t} = \frac{J_t}{(1-\eta_t)},$$

which is equivalent to the DMP sharing rule with time-varying shares. In either case the joint match value, S_t, is given by

$$S_t = EP_t - U_t + J_t.$$

■

B.7. Proof of Corollary 2.1.

To show that our wage-setting mechanism features present value bargaining, we solely focus on the firm's decision variables. It will suffice to show that the job value, J_t, on the firm side can be expressed by the present value of bargained wages, as capital owners' discounts would apply to the wage determination in equilibrium, as a result of our bilateral wage bargaining between two heterogeneous agents.

As regards hiring decisions, note first that in our decentralized economy, the (representative) firm faces the following Bellman equation vis-à -vis the *job value*, J_t, to determine its *optimal* hiring rate (equivalently job vacancies)—*optimal* in the sense that the firm optimizes its decisions at the individual level:

$$
\begin{aligned}
J_t &= \max_{\{x_t\}} \left[(1-\alpha)(1-\mu)\left(\frac{y_t}{n_t}\right) - w_t^n h_t^n - \frac{\kappa}{2} x_t^2 + (1-\rho+x_t)\,\mathbb{E}_t\left[\tilde{M}_{t+1} J_{t+1}\right]\right] \\
&= (1-\alpha)(1-\mu)\left(\frac{y_t}{n_t}\right) - w_t^n h_t^n - \frac{\kappa}{2}\left(x_t^*\right)^2 + (1-\rho+x_t^*)\,\mathbb{E}_t\left[\tilde{M}_{t+1} J_{t+1}\right] \\
&= (1-\alpha)(1-\mu)\left(\frac{y_t}{n_t}\right) - w_t^n h_t^n - \frac{\kappa}{2}\left(x_t^*\right)^2 + (1-\rho)\,\mathbb{E}_t\left[\tilde{M}_{t+1} J_{t+1}\right] + x_t^* \cdot \underbrace{\mathbb{E}_t\left[\tilde{M}_{t+1} J_{t+1}\right]}_{\kappa \cdot x_t^*}
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{ \begin{array}{l}
(1-\alpha)(1-\mu)\cdot \underbrace{\left[\left(\frac{y_t}{n_t}\right) + \mathbb{E}_t\left[\sum_{j=1}^{\infty}(1-\rho)^j \cdot \left(\prod_{k=1}^{j} \tilde{M}_{t+k}\right)\left(\frac{y_{t+j}}{n_{t+j}}\right)\right]\right]}_{X_t\text{: the present value of current and future productivity}} \\
-w_t^n h_t^n - \mathbb{E}_t\left[\sum_{j=1}^{\infty}\left(\prod_{k=1}^{j}(1-\rho)^j \cdot \tilde{M}_{t+k}\right) w_{t+j}^n h_{t+j}^n\right] \\
+\frac{\kappa}{2}\left(x_t^*\right)^2 + \mathbb{E}_t\left[\sum_{j=1}^{\infty}\left(\prod_{k=1}^{j}(1-\rho)^j \cdot \tilde{M}_{t+k}\right)\frac{\kappa}{2}\left(x_{t+j}^*\right)^2\right]
\end{array}\right\} \\
&= \left\{ \begin{array}{l}
(1-\alpha)(1-\mu)\cdot X_t - \mathbb{E}_t\left[\sum_{j=0}^{\infty}\left(\prod_{k=0}^{j}(1-\rho)^j \cdot \tilde{M}_{t+k}\right) w_{t+j}^n h_{t+j}^n\right] \\
+\frac{\kappa}{2}\cdot \mathbb{E}_t\left[\sum_{j=0}^{\infty}\left(\prod_{k=0}^{j}(1-\rho)^j \cdot \tilde{M}_{t+k}\right)\left(x_{t+j}^*\right)^2\right]
\end{array}\right\}
\end{aligned}
$$

where the job value, J_t, is derived from the envelope condition, $\partial V_t^f / \partial n_t$, of the firm value $V_t^f \equiv d_t + p_t^e$: i.e.

$$
\begin{aligned}
J_t = &\frac{\partial V_t^f}{\partial n_t} \\
&\text{subject to} \\
&n_{t+1} = (1-\rho)n_t + x_t n_t = (1-\rho+x_t)n_t.
\end{aligned}
$$

Note that

$$w_t^n h_t^n + \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^n h_{t+j}^n \right]$$

$$= \left\{ \begin{array}{l} w_t^n h_t^n + \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^n h_{t+j}^n \right] \\ + w_t^n \bar{h}^n - w_t^n \bar{h}^n + \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^n \bar{h}^n \right] \\ - \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^n \bar{h}^n \right] \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \bar{h}^n \cdot \left[w_t^n + \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^n \right] \right] \\ + w_t^n (h_t^n - \bar{h}^n) + \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^n (h_{t+j}^n - \bar{h}^n) \right] \end{array} \right\}$$

In equilibrium, we have

$$w_t^n = w_t^{\text{KRN}} \quad \text{for all } t.$$

Accordingly,

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^n h_{t+j}^n \right]$$

$$= \left\{ \begin{array}{l} \underbrace{\bar{h}^n}_{\substack{\text{the steady state} \\ \text{value of } h_t^n}} \cdot \underbrace{\left[w_t^{\text{KRN}} + \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^{\text{KRN}} \right] \right]}_{W_t^{\text{pv}}\text{: the present value of bargained wages}} \\ + w_t^{\text{KRN}} (h_t^n - \bar{h}^n) + \mathbb{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^{\text{KRN}} (h_{t+j}^n - \bar{h}^n) \right] \end{array} \right\}$$

$$= \bar{h}^n \cdot W_t^{\text{pv}} + \bar{h}^n \cdot \mathbb{E}_t [\sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) \cdot w_{t+j}^{\text{KRN}} \cdot \underbrace{\frac{(h_{t+j}^n - \bar{h}^n)}{\bar{h}^n}}_{\substack{\text{percentage deviations} \\ \text{from the steady state } \bar{h}^n \\ \text{(business cycle fluctuations)}}}]$$

In equilibrium or in the process of bargaining, the job value J_t the firm faces can also be rewritten as

$$J_t = \left\{ \begin{array}{l} (1-\alpha)(1-\mu) \cdot X_t - \bar{h}^n \cdot W_t^{\text{pv}} \\ + \bar{h}^n \cdot \mathbb{E}_t \left[\sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) w_{t+j}^{\text{KRN}} \cdot \frac{(h_{t+j}^n - \bar{h}^n)}{\bar{h}^n} \right] \\ + \frac{\kappa}{2} \cdot \mathbb{E}_t \left[\sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} (1-\rho)^j \cdot \tilde{M}_{t+k} \right) \left(x_{t+j}^* \right)^2 \right]. \end{array} \right\}$$

B.8. Proof of Proposition 3.2.

To emphasize generality, we suppress the arguments of the non-stockholders utility function. Non-stockholder's matching surplus in the numeraire terms can be expressed as:

$$
\begin{aligned}
W_t - U_t &\equiv \frac{1}{\lambda_t^n}\frac{\partial V^n}{\partial n_t} \\
&= \frac{1}{\lambda_t^n}(u_t^{n,e} - u_t^{n,u}) + (w_t^n h_t^n - b - (c_t^{n,e} - c_t^{n,u})) + \beta E_t[\frac{\tilde{\lambda}_{t+1}^n}{\lambda_t^n}(W_{t+1} - U_{t+1})(1-\rho-s_t)]
\end{aligned} \tag{49}
$$

while the firm's matching surplus is given by:

$$
J_t = \frac{\partial y_t}{\partial n_t} - w_t^n h_t^n - \frac{\kappa}{2}x_t^2 + (1-\rho)\beta E_t[\frac{\tilde{\lambda}_{t+1}^s}{\lambda_t^s}J_{t+1}]. \tag{50}
$$

where λ_t^s represents stockholder marginal utility. The egalitarian bargaining problem between stockholders and non-stockholders gives the following solution:

$$
\underbrace{(1-\eta)\,\lambda_t^n\,(W_t - U_t)}_{\equiv \frac{\partial V_t^n}{\partial n_t}} \quad = \quad \underbrace{\eta \lambda_t^s J_t}_{\equiv \frac{\partial V_t^s}{\partial n_t}} \tag{51}
$$

Note that the optimal condition for opening job vacancies reads as:

$$
\kappa x_t = \beta E_t \frac{\tilde{\lambda}_{t+1}^s}{\lambda_t^s} J_{t+1} \tag{52}
$$

and thus the firm's matching surplus can be rewritten as:

$$
J_t = \frac{\partial y_t}{\partial n_t} - w_t^n h_t^n + \frac{\kappa}{2}x_t^2 + (1-\rho)\kappa x_t. \tag{53}
$$

Using the optimal condition for opening job vacancies (52) and the first order condition for the Kalai-Rawls-Nash (KRN) bargaining problem (51), the non-stockholder's marginal lifetime utility value for employment can be rewritten as:

$$
\begin{aligned}
\frac{\partial V^n}{\partial n_t} &= u^n(c_t^{n,e}, 1-h_t^n) - u^n(c_t^{n,u}, 1-e) + \lambda_t^n(w_t^n h_t^n - b - (c_t^{n,e} - c_t^{n,u})) + \beta \mathbb{E}_t[\frac{\eta}{1-\eta}\tilde{\lambda}_{t+1}^s J_{t+1}(1-\rho-s_t)] \\
&= u^n(c_t^{n,e}, 1-h_t^n) - u^n(c_t^{n,u}, 1-e) + \lambda_t^n(w_t^n h_t^n - b - (c_t^{n,e} - c_t^{n,u})) + \frac{\eta}{1-\eta}(1-\rho-s_t)\kappa x_t \lambda_t^s
\end{aligned} \tag{54}
$$

and the non-stockholder's matching surplus can be rewritten as:

$$
W_t - U_t = \frac{1}{\lambda_t^n}(u_t^{n,e} - u_t^{n,u}) + (w_t^n h_t^n - b - (c_t^{n,e} - c_t^{n,u})) + \frac{\eta}{1-\eta}(1-\rho-s_t)\kappa x_t \frac{\lambda_t^s}{\lambda_t^n}. \tag{55}
$$

Then, by re-plugging, the sharing rule for KRN wage bargaining is given by:

$$
\begin{aligned}
&(1-\eta)\lambda_t^n(W_t-U_t) \\
&= (1-\eta)(u_t^{n,e}-u_t^{n,u}) + (1-\eta)\lambda_t^n(w_t^n h_t^n - b - (c_t^{n,e}-c_t^{n,u})) + \eta(1-\rho-s_t)\kappa x_t \lambda_t^s \\
&\underbrace{=}_{\text{sharing rule}} \quad \eta\lambda_t^s J_t \\
&= \eta\lambda_t^s[\frac{\partial y_t}{\partial n_t} - w_t^n h_t^n + \frac{\kappa}{2}x_t^2 + (1-\rho)\kappa x_t].
\end{aligned} \tag{56}
$$

Rearragement gives:

$$
\begin{aligned}
&(1-\eta)\lambda_t^n w_t^n h_t^n + \eta\lambda_t^s w_t^n h_t^n + (1-\eta)(u_t^{n,e}-u_t^{n,u}) \\
&= (1-\eta)\lambda_t^n(b+(c_t^{n,e}-c_t^{n,u})) + \eta\lambda_t^s[\frac{\partial y_t}{\partial n_t} + \frac{\kappa}{2}x_t^2] + \eta s_t\kappa x_t\lambda_t^s.
\end{aligned} \tag{57}
$$

Our distribution risk is defined as:

$$
\phi_t \equiv \frac{\lambda_t^s}{\lambda_t^n} \tag{58}
$$

and in terms of distribution risk the sharing rule is written as:

$$
\begin{aligned}
&[(1-\eta)\frac{1}{\phi_t} + \eta]w_t^n h_t^n + (1-\eta)\frac{(u_t^{n,e}-u_t^{n,u})}{\lambda_t^n}\frac{1}{\phi_t} \\
&= \eta\left[\frac{\partial y_t}{\partial n_t} + \frac{\kappa}{2}x_t^2 + \kappa s_t x_t\right] + (1-\eta)\frac{1}{\phi_t}(b+(c_t^{n,e}-c_t^{n,u}))\ .
\end{aligned} \tag{59}
$$

The egalitarian bargaining wage with distribution risk now can be stated as:

$$
w_t^n h_t^n = \eta_t[\frac{\partial y_t}{\partial n_t} + \frac{\kappa}{2}x_t^2 + \kappa s_t x_t] + (1-\eta_t)(b+(c_t^{n,e}-c_t^{n,u}) - \frac{(u_t^{n,e}-u_t^{n,u})}{\lambda_t^n}) \tag{60}
$$

where

$$
\eta_t = \frac{\eta}{(1-\eta)\frac{1}{\phi_t}+\eta} \tag{61}
$$

B.9. Alternative Form of Proposition 2.2.

Equation (24) in Proposition 3.2 may also be written as

$$
\begin{aligned}
w_t^n h_t^n = (1-\eta_t)&\left[b + (c_t^{n,e} - c_t^{n,u} - \chi^n(c_{t-1}^{n,e} - c_{t-1}^{n,u})) + \left(\frac{u^n(c_t^{n,e}-\chi^n c_{t-1}^{n,e} - L(h_t^n)) - u^n(c_t^{n,u}-\chi^n c_{t-1}^{n,u})}{\lambda_t^n}\right)\right] \\
&+\eta_t\left[h_t^n f_3(k_t,\mu_s h_t^s, n_t h_t^n) z_t - \frac{\kappa}{2}x_t^2 + \kappa x_t s_t\right]
\end{aligned} \tag{62}
$$

where $w_t^n,\ h_t^n,\ \lambda_t^n, c_t^{n,e},\ c_t^{n,u}$ and b are as previously defined. The first term in (62),

$$\zeta_t = \left[b + \left(c_t^{n,e} - c_t^{n,u} - \chi^n \left(\mathrm{c}_{t-1}^{n,e} - \mathrm{c}_{t-1}^{n,u}\right)\right) + \left(\frac{u^n \left(c_t^{n,u} - \chi^n \mathrm{c}_{t-1}^{n,u}\right) - u^n \left(c_t^{n,e} - \chi^n \mathrm{c}_{t-1}^{n,e} - L\left(h_t^n\right)\right)}{\lambda_t^n} \right) \right],$$

represents an employed worker's dynamic outside option (reservation) value (of being unemployed). It consists of three components: (i) b, the exogenously given unemployment benefit, (ii) $\left(c_t^{n,e} - c_t^{n,u}\right) - \chi^n \left(\mathrm{c}_{t-1}^{n,e} - \mathrm{c}_{t-1}^{n,u}\right)$, the difference in consumption when employed vs. unemployed, and (iii) the utility benefit of not supplying hours when not working. Under GHH preferences and the representative family model for non-stockholders, this third term is zero, with the non-stockholder's dynamic outside option, henceforth denoted ζ_t, correspondingly simplified to:

$$\zeta_t = b + \left(c_t^{n,e} - c_t^{n,u}\right) - \chi^n \left(\mathrm{c}_{t-1}^{n,e} - \mathrm{c}_{t-1}^{n,u}\right) = b + \left(L\left(h_t^n\right) - h\left(0\right)\right).$$

The expression $\pi_{n_t} = \left[h_t^n f_3\left(k_t, \mu_s h_t^s,\ n_t h_t^n\right) z_t - \frac{\kappa}{2} x_t^2 - \kappa x_t s_t\right]$ represents the match related benefit to the firm (stockholders) of one marginally added worker. Equation (62) can then be expressed as:

$$w_t^n h_t^n = \left(1 - \eta_t\right) \zeta_t + \eta_t \pi_{n_t} \equiv \eta_t^{firm} \zeta_t + \eta_t^{non-stockholder} \pi_{n_t}. \tag{63}$$

We will use (62) – (63) later on to make more explicit the effects of countercyclical distribution risk on the stability of the worker wage bill.

C. Steady State Equilibrium

C.1. Steady State Equations

The steady-state computation proceeds as follows: first, beginning with Shimer (2005), it is commonplace in the literature to fix $\bar{\theta} = \bar{v}/\bar{u} = 1$. Following the Andolfatto (1996) calibration, the quarterly match probability for the firm is fixed at $\bar{q} = .9$. Since, in the steady-state

$$\rho \bar{n} = \bar{m} = \bar{q}\bar{v} = \bar{q}\bar{u},$$

with $\rho = .10$, we employ the relationship $\bar{u} = 1 - \bar{n}$ to obtain $\bar{u} = .10 = \bar{v}$ and $\bar{n} = .90$. Accordingly, it follows that $\bar{m} = .09$, $\bar{x} = \bar{m}/\bar{n} = .10$, and $\bar{s} = \bar{m}/\bar{u} = .9$. Note that $\bar{u}$ is the steady state rate of unemployment in the worker group, not in the whole economy. The steady state rate of unemployment in the economy should be $\bar{u}/\left(1 + \mu^s\right)$, taking a value of .91 in the baseline version as an illustration. Together with the identification $\bar{p}^b = \beta$ there are fourteen equations in the fourteen unknowns $\bar{k}$, $\bar{h}^s$, $\bar{h}^n$, $\bar{w}^s$, $\bar{w}^n$, $\bar{n}$, $\bar{\tau}$, $\bar{c}^s$, $\bar{c}^n$, $\bar{d}$, $\bar{b}^n$, $\bar{b}^s$, $\bar{p}^e$, and $\bar{\pi}_n$. The equations defining the steady-state are:

(i) $1 = \beta\left[f_1\left(\bar{k},\ \mu_s \bar{h}^s,\ \bar{n}\bar{h}^n\right)\bar{z} + (1 - \delta)\right]$

(ii) $\bar{w}^s = f_2\left(\bar{k},\ \mu_s \bar{h}^s,\ \bar{n}\bar{h}^n\right)\bar{z}$

(iii) $\bar{w}^n \bar{n} = \bar{\eta}\bar{\pi} + (1 - \bar{\eta})\,\bar{\tau}$

(iv) $\bar{\tau} = b + B_n\left(\bar{h}^n\right)^{\psi}$

(v) $\bar{\phi} = \frac{\left(\bar{c}^s (1-\chi^s) - B_s\left(\bar{h}^s\right)^{\psi}\right)^{-\gamma}}{\left(\bar{c}^n (1-\chi^n) - \bar{n} B_n\left(\bar{h}^n\right)^{\psi}\right)^{-\gamma}}$

(vi) $\bar{\eta} = \frac{\eta}{(1-\eta)\frac{1}{\bar{\phi}} + \eta}$

(vii) $\bar{c}^s = \bar{d} + \bar{h}^s \bar{w}^s + \bar{b}\left(1 - \bar{p}^b\right)$

(viii) $\bar{c}^n = \bar{w}^n \bar{n} \bar{h}^n + b\left(1 - \bar{n}\right) + \bar{b}^n \left(1 - \bar{p}^b\right)$

(ix) $\bar{d} = f\left(\bar{k},\, \mu_s \bar{h}^s,\, \bar{n}\bar{h}^n\right)\bar{z} - \delta \bar{k} - \mu_s \bar{w}^s \bar{h}^s - \bar{w}^n \bar{h}^n \bar{n} - \left(\frac{\kappa}{2}\right)\bar{x}\,\bar{n} - \left(1 - \bar{p}^b\right)\phi \bar{k}$

(x) $\overline{\Omega}^{\mathrm{W}} = \frac{\mu_s \bar{b}^s + \bar{p}^e}{\mu_s \bar{b}^s + \bar{p}^e + \bar{b}^n}$

(xi) $\phi \bar{k} = \mu_s \bar{b}^s + \bar{b}^n$

(xii) $\bar{p}^e = \frac{\beta \bar{d}}{1 - \beta}$

(xiii) $\bar{J} = (1 - \alpha)(1 - \mu)\left[\frac{f\left(\bar{k}, \mu_s \bar{h}^s, \bar{n}\bar{h}^n\right)\bar{z}}{\bar{n}}\right] - \frac{\kappa}{2}\bar{x}^2 + \kappa \bar{x}\bar{s}$

(xiv) $\bar{y} = f\left(\bar{k},\, \mu_s \bar{h}^s,\, \bar{n}\bar{h}^n\right)\bar{z} = \mu_s \bar{c}^s + \bar{c}^n + \bar{i} + \frac{\kappa}{2}\bar{x}^2 \bar{n}$

C.2. "Proof" of The Musk-Tesla superstar Effect

1. Across all the cases, capital returns in steady state are identical:

$$(1 + r) = \beta^{-1}.$$

2. The Euler relation in steady state is expressed as

$$\beta^{-1} = \alpha \cdot \frac{y}{k} + 1 - \delta.$$

Thus the Euler relation above implies

$$\frac{y}{k} = \alpha^{-1} \cdot [\beta^{-1} - (1 - \delta)].$$

In other words, across all the cases, the steady-state output-capital ratio, y/k, is the same.

3. Our production technology takes the following form:

$$y = A \cdot k^{\alpha} (\underbrace{(\mu_s h^s)^{\mu} (h^n \cdot n)^{1-\mu}}_{H})^{1-\alpha} = A \cdot k^{\alpha} H^{1-\alpha}$$

where H represents aggregate labor input.
Reformulation of the production technology implies that

$$\frac{y}{k} = A \cdot \left(\frac{H}{k}\right)^{1-\alpha}. \qquad (64)$$

In formulation (64), the output-capital ratio, y/k, is fixed; A is a fixed production-scale parameter and α is invariant except for the technological change case. Thus, as H increases, the steady-state capital stock, k, should rise in tandem. Indeed, as μ decreases, H will increase, suggesting that the increasing concentration of capital ownership will occur with the long-term rate of interest remaining positive and constant.

4. The technological change follows suit: Given H/k, a decrease in $1-\alpha$ (an increase in α) implies that

$$\left(\frac{H}{k}\right)^{1-\alpha} < \left(\frac{H}{k}\right)^{1-\alpha^*} \tag{65}$$

where $\alpha = .36 < \alpha^*$. Thus, provided that the output-capital ratio y/k, and the aggregate labor input H are fixed, relation (65) implies that k should increase with less $1-\alpha^*$. In other words, given H, capital stock k should increase.

D. Deriving the Form of the Gini Coefficient of Wealth Inequality for the Two Agent Economy

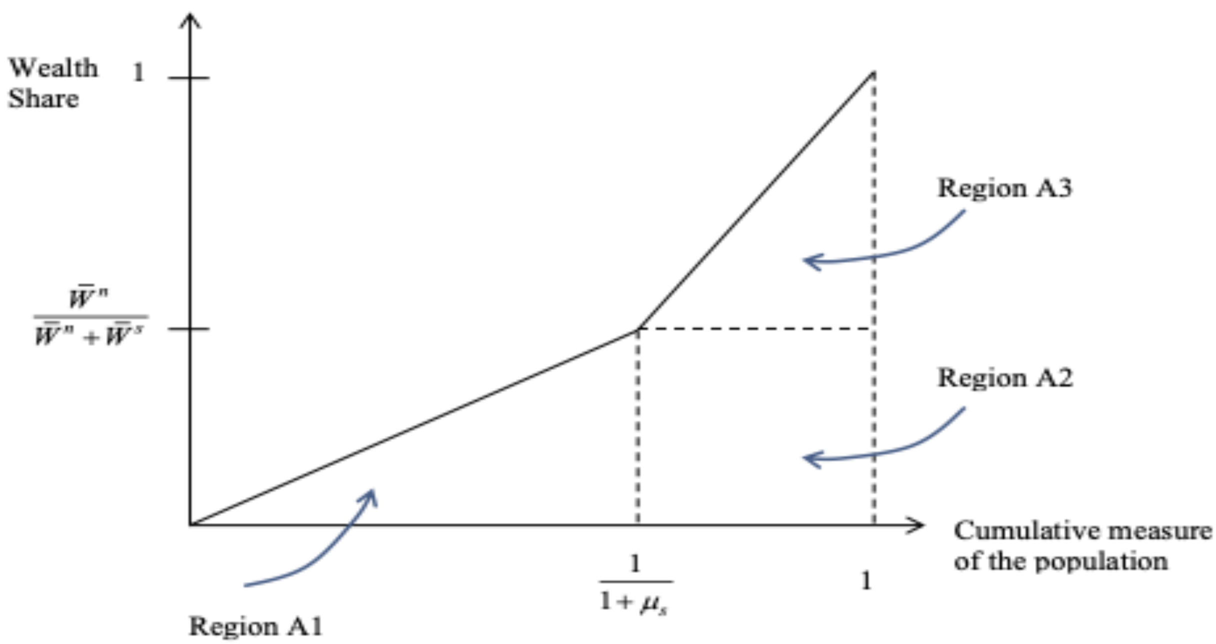

Figure 7: **Lorentz Curve**

The Gini coefficient is defined as

$$G = 1 - 2\left\{\begin{array}{c}\text{Area under the}\\ \text{Lorenz curve}\end{array}\right\}$$

We present the construction of $\overline{G}^W$. For the two agent model of the present paper, the Lorenz curve for steady state wealth has the following representation:

$$
\begin{aligned}
\overline{G}^{\text{w}} &= 1-2\left\{A1+A2+A3\right\} \\
&= 1-2\left\{\tfrac{1}{2}\left(\tfrac{1}{1+\mu_s}\right)\left(\tfrac{\bar{W}^n}{\bar{W}^n+\bar{W}^s}\right)+\tfrac{\mu_s}{1+\mu_s}\left(\tfrac{\bar{W}^n}{\bar{W}^n+\bar{W}^s}\right)+\tfrac{1}{2}\left(1-\left(\tfrac{\bar{W}^n}{\bar{W}^n+\bar{W}^s}\right)\right)\left(\tfrac{\mu_s}{1+\mu_s}\right)\right\} \\
&= 1-2\left\{\tfrac{1}{2}\left(\tfrac{\mu_s}{1+\mu_s}\right)+\left(\tfrac{\bar{W}^n}{\bar{W}^n+\bar{W}^s}\right)\left[\tfrac{1}{2}\left(\tfrac{1}{1+\mu_s}\right)+\tfrac{\mu_s}{1+\mu_s}-\tfrac{1}{2}\left(\tfrac{\mu_s}{1+\mu_s}\right)\right]\right\} \\
&= 1-2\left\{\tfrac{1}{2}\left(\tfrac{\mu_s}{1+\mu_s}\right)+\tfrac{1}{2}\left(\tfrac{\bar{W}^n}{\bar{W}^n+\bar{W}^s}\right)\right\} \\
&= 1-\left(\tfrac{\mu_s}{1+\mu_s}+\tfrac{\bar{W}^n}{\bar{W}^n+\bar{W}^s}\right)
\end{aligned}
$$

E. Model Solution: Asset Pricing Formulae

E.1. Introduction

In the present model, basic 'safe assets' are viewed as default-free (risk-free) bonds in financial markets, while thier risky counterparts being described as 'equity capital,' thereby suggesting basic asset pricing theory can be readily employed to characterize the resulting 'safe asset rates' and 'risk premia' both. Accordingly, Lucas's (1978) asset pricing equation will be exclusively exploited to determine the associated financial returns, including default-free rates and equity returns.

E.2. General Asset Pricing

Let $1 + R_{t+1}$ be the gross return on an asset held from period t to period $t + 1$. If the price and the cash flow of the asset in period t are denoted by P_t and F_t, respectively, then

$$1 + R_{t+1} = \frac{P_{t+1} + F_{t+1}}{P_t}. \tag{66}$$

The Arrow-Lucas-Rubinstein asset pricing equation requires that any asset with (66) must satisfy

$$1 = \mathbb{E}_t[\beta \frac{\tilde{\lambda}_{t+1}}{\lambda_t}(1 + \tilde{R}_{t+1})]. \tag{67}$$

Equivalently, we can rewrite (67) as follows:

$$0 = \log\beta + \log\mathbb{E}_t[\exp(\tilde{\hat{\lambda}}_{t+1} - \hat{\lambda}_t + \tilde{r}_{t+1})] \tag{68}$$

where $\hat{\lambda}_t$ is the log-deviation of marginal utility of consumption from its steady state value and $r_{t+1} \equiv \log(1 + R_{t+1})$. Assuming that λ_{t+1} and $1 + R_{t+1}$ are jointly lognormally distributed and using the standard formula for the expectation of lognormally distributed variables, equation (68) can be written as:

$$0 = \log\beta + \mathbb{E}_t[\tilde{\hat{\lambda}}_{t+1} - \hat{\lambda}_t] + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}[\sigma^2_{\lambda_t} + \sigma^2_{r_t} + 2\rho_{\lambda r_t}\sigma_{\lambda t}\sigma_{r_t}] \tag{69}$$

where $\sigma^2_{\lambda t} \equiv \text{Var}_t[\tilde{\hat{\lambda}}_{t+1} - \hat{\lambda}_t] = \mathbb{E}_t[(\tilde{\hat{\lambda}}_{t+1} - \mathbb{E}_t\hat{\lambda}_{t+1})^2]$, $\sigma^2_{rt} \equiv \text{Var}_t[\tilde{r}_{t+1}] = \mathbb{E}_t[(\tilde{r}_{t+1} - \mathbb{E}_t\tilde{r}_{t+1})^2]$, and $\rho_{\lambda r_t}$ is the conditional correlation, i.e. $\rho_{\lambda r_t}\sigma_{\lambda_t}\sigma_{r_t} \equiv \text{Cov}_t[(\tilde{\hat{\lambda}}_{t+1} - \hat{\lambda}_t), \tilde{r}_{t+1}]$.

E.3. Default Free Rate

A default-free asset (one quarter real bond) with the risk-free rate $r_t^b \equiv \log(1 + R_t^b)$ can be priced in much simpler way. Since $\sigma^2_{r_t^b} = \mathbb{E}_t[(\tilde{r}^b_{t+1} - \mathbb{E}_t r^b_{t+1})^2] = 0$, we have

$$r_t^b = -\log\beta - \mathbb{E}_t[\tilde{\hat{\lambda}}_{t+1} - \hat{\lambda}_t] - \frac{1}{2}\sigma^2_{\lambda_t}. \tag{70}$$

Then the simple risk-free rate is given by

$$1 + R_t^b = \exp r_t^b.$$

The unconditional moments of the simple risk-free rate can be calculated using the log-normal formula for the unconditional expectation:

$$\mathbb{E}[1 + \tilde{R}_t^b] = \exp(\mathbb{E}[\tilde{r}_t^b] + \frac{1}{2}\mathrm{Var}[\tilde{r}_t^b])$$

$$\mathrm{Var}[\tilde{R}_t^b] = \mathrm{Var}[1 + \tilde{R}_t^b] = \exp(2\mathbb{E}[\tilde{r}_t^b] + 2\mathrm{Var}[\tilde{r}_t^b]) - \exp(2\mathbb{E}[\tilde{r}_t^b] + \mathrm{Var}[\tilde{r}_t^b]).$$

The unconditional moments of r_t^b are given by

$$\mathbb{E}[\tilde{r}_t^b] = -\log\beta - \frac{1}{2}\mathrm{Var}[\tilde{\hat{\lambda}}_{t+1} - \mathbb{E}_t\hat{\lambda}_{t+1}]$$

$$\mathrm{Var}[\tilde{r}_{t,n}^b] = \mathrm{Var}[\mathbb{E}_t[\tilde{\hat{\lambda}}_{t+1} - \hat{\lambda}_t]].$$

E.4. Equity

To quantitatively assess 'equity capital risk' arising in the present model, we again resort to basing asset pricing theory: 'equity capital risk' will be measured by an equity risk premium, the difference between equity and risk-free bond returns, suggesting the determination of equity returns is a priority. Accordingly, to calculate the latter returns, we adopt a slightly different strategy, to follow, relative to the risk-free-rate case.

The Arrow-Lucas-Rubinstein asset pricing equation tells us that the period t equity price p_t^e must equal the present value of all future dividends discounted by the pricing kernel:

$$p_t^e = \mathbb{E}_t\left[\sum_{k=1}^{\infty}\beta^k\frac{\tilde{\lambda}_{t+k}}{\lambda_t}\tilde{d}_{t+k}\right] \tag{71}$$

where $\frac{\widetilde{\lambda}_{t+k}}{\lambda_t}$ is the stochastic discount factor of stockholders due to the presumed limited participation in the stock market.

Note that equivalently, the period t equity price p_t^e can be written as:

$$p_t^e = \sum_{k=1}^{\infty} \mathbb{E}_t \left[\beta^k \frac{\tilde{\lambda}_{t+k}}{\lambda_t} \tilde{d}_{t+k}\right] \tag{72}$$
$$= \sum_{k=1}^{\infty} \mathbb{E}_t \left[\beta^k \frac{\tilde{\lambda}_{t+k}}{\lambda_t} \frac{\tilde{d}_{t+k}}{d_t} d_t\right]$$
$$= \sum_{k=1}^{\infty} \mathbb{E}_t \left[\beta^k \exp(\tilde{\hat{\lambda}}_{t+k} - \hat{\lambda}_t + \tilde{\hat{d}}_{t+k} - \hat{d}_t) d_t\right]$$

where $\hat{d}_t$ is the log-deviation of dividend from its steady state value.
Applying the standard log-normal formula to the random variables $\{\hat{\lambda}_{t+k} - \hat{\lambda}_t + \hat{d}_{t+k} - \hat{d}_t\}_{k=1}^{\infty}$, each conditional expectation term in (72) can be written as:

$$\mathbb{E}_t[\exp(\tilde{\hat{\lambda}}_{t+k} - \hat{\lambda}_t + \tilde{\hat{d}}_{t+k} - \tilde{\hat{d}}_t)] \tag{73}$$
$$= \exp[\mathbb{E}_t[\tilde{\hat{\lambda}}_{t+k} - \hat{\lambda}_t + \tilde{\hat{d}}_{t+k} - \tilde{\hat{d}}_t] + \frac{1}{2}\text{Var}_t[\tilde{\hat{\lambda}}_{t+k} - \hat{\lambda}_t + \tilde{\hat{d}}_{t+k} - \tilde{\hat{d}}_t].$$

Both terms, $\mathbb{E}_t[\tilde{\hat{\lambda}}_{t+k} - \hat{\lambda}_t + \tilde{\hat{d}}_{t+k} - \tilde{\hat{d}}_t]$ and $\text{Var}_t[\tilde{\hat{\lambda}}_{t+k} - \hat{\lambda}_t + \tilde{\hat{d}}_{t+k} - \tilde{\hat{d}}_t]$, respectively, can be computed and then we approximate the equity price p_t^e in (72).

In the same fashion, the simple quarterly equity return is evaluated by

$$1 + R_{t,t+1}^e = \frac{p_{t+1}^e + d_{t+1}}{p_t^e}$$
$$= \frac{\left[\sum_{k=1}^{\infty} \mathbb{E}_{t+1}\left[\beta^k \exp(\tilde{\hat{\lambda}}_{t+1+k} - \hat{\lambda}_{t+1} + \tilde{\hat{d}}_{t+1+k} - \hat{d}_{t+1}) d_{t+1}\right]\right] + d_{t+1}}{\sum_{k=1}^{\infty} \mathbb{E}_t\left[\beta^k \exp(\tilde{\hat{\lambda}}_{t+k} - \hat{\lambda}_t + \tilde{\hat{d}}_{t+k} - \hat{d}_t) d_t\right]},$$

for sufficiently large number n.

E.5. Present-Value Variables

Inspired by Hall (2017), each of present-value variables, including $\widetilde{W}^{\text{pv}}$ and $\widetilde{X}$, pursued here represents a type of investment whose benefit flows have 'financial risk' comparable to corporate earnings, thereby being discounted by a fiancial discount implict in the stock market. According to standard asset pricing theory, this implict market discount identifies itself as the discount factor, $\beta^k \cdot \widetilde{\lambda}_{t+k}/\lambda_t$, present in equation (71), suggesting that these present-value variables, being treated as "financial assets," can be quantitatively evaluated using the joint log-normality assumption as in (73). To illustrate the point thus far, let Θ_t be a present-value variable with its benefit flows, $\{\widetilde{\vartheta}_{t+k}\}$. Along the prior line of reasoning, Θ_t is expressed as

$$\Theta_t = \vartheta_t + \mathbb{E}_t \left[\sum_{k=1}^{\infty} \beta^k \frac{\tilde{\lambda}_{t+k}}{\lambda_t} \widetilde{\vartheta}_{t+k}\right]. \tag{74}$$

Equation (74) can be approximated by the same procedure as in equation (71). First, "approximate" the right-hand side of (74) as $\exp(\hat{\vartheta}_t) + \sum_{k=1}^{\infty} \mathbb{E}_t \left[\beta^k \exp(\tilde{\hat{\lambda}}_{t+k} - \hat{\lambda}_t + \tilde{\hat{\vartheta}}_{t+k} - \hat{\vartheta}_t) \exp(\hat{\vartheta}_t)\right]$ provided that the log-linear solutions, $\{\hat{\lambda}_t\}$ and $\{\hat{\vartheta}_t\}$, respectively, are available. Second, using the log-normality assumption employed in (73), compute each conditional expectation term $\mathbb{E}_t[\exp(\tilde{\hat{\lambda}}_{t+k} - \hat{\lambda}_t + \tilde{\hat{\vartheta}}_{t+k} - \tilde{\hat{\vartheta}}_t)]$ and sum up the right-hand side of (74) for a sufficiently large number k.

E.6. Macro-cum-Asset-Pricing Formulae

This section further systematizes the above non-linear (log-normal) prcing formulae by combining an approximate (mostly log-linear) state-space system with the vector of state variables, often regarded as "exact" solutions to DSGE models.

The solution method pursued here has two basic steps: we first solve for the approximate dynamics of the model, while applying the log-linear pricing formulae discussed above as a second step. For the first step, we solve for the model's decision rules using any of the usual methods for solving linear rational expectations (notable references are Uhlig (1999), Klein (2000), and Sims (2002), to mention but a few). Once the model's aprroximate (log-linear) decision rules are found, they can be represented by the corresponding approximate state-space system. More specifically, the model's log-linear decision rules, to follow, can be represented in the matrix form:

$$\begin{aligned} \mathbf{y}_t &= \mathbf{R} \cdot \mathbf{x}_{t-1} + \mathbf{S} \cdot \mathbf{z}_t \\ \mathbf{x}_t &= \mathbf{P} \cdot \mathbf{x}_{t-1} + \mathbf{Q} \cdot \mathbf{z}_t \\ \mathbf{z}_{t+1} &= \mathbf{N} \cdot \mathbf{z}_t + \epsilon_{t+1} \end{aligned} \tag{75}$$

where $\mathbf{x}$ represents the $n_x \times 1$ vector of predetermined variables; $\mathbf{y}$ is the $n_y \times 1$ vector of non-predetermined (control) variables, while $\mathbf{z}$ denotes the $n_e \times 1$ vector of exogenous state variables.

We now conduct the state-space representation of the above recursive laws of motion (75) by introducing a new $(n_x + n_e) \times 1$vector, $\mathbf{s}_t$, of all state variables, whether predetermined or exogenous, which is defined by $[\mathbf{x}_{t-1}, \mathbf{z}_t]'$. The first row equation in (75) can be rewritten using the vector $\mathbf{s}_t$: that is, by rearrangement,

$$\begin{aligned} \underbrace{\mathbf{y}_t}_{n_y\times 1} &= \underbrace{\mathbf{R}}_{n_y\times(n_x-n_e)} \cdot \underbrace{\mathbf{x}_{t-1}}_{(n_x-n_e)\times 1} + \underbrace{\mathbf{S}}_{n_y\times n_e} \cdot \underbrace{\mathbf{z}_t}_{n_e\times 1} \\ &= \underbrace{\begin{bmatrix}\mathbf{R} & \mathbf{S}\end{bmatrix}}_{\equiv \mathbf{T}} \cdot \underbrace{\begin{bmatrix}\mathbf{x}_{t-1} \\ \mathbf{z}_t\end{bmatrix}}_{\equiv \mathbf{s}_t} \\ &= \mathbf{T} \cdot \mathbf{s}_t, \end{aligned}$$

which constitutes the *observation equation* of the the state-space system of interest. First note that the rest of two equations in (75) can be expressed as follows:

$$\begin{aligned} \mathbf{x}_t &= \underbrace{\begin{bmatrix}\mathbf{P} & \mathbf{Q}\end{bmatrix}}_{(n_x-n_e)\times n_x} \cdot \begin{bmatrix}\mathbf{x}_{t-1} \\ \mathbf{z}_t\end{bmatrix} + \underbrace{\begin{bmatrix}\mathbf{0}\end{bmatrix}}_{(n_x-n_e)\times n_e} \cdot \underbrace{\epsilon_{t+1}}_{n_e\times 1} \\ \mathbf{z}_{t+1} &= \begin{bmatrix}\underbrace{\mathbf{0}}_{n_e\times(n_x-n_e)} & \mathbf{N}\end{bmatrix} \cdot \begin{bmatrix}\mathbf{x}_{t-1} \\ \mathbf{z}_t\end{bmatrix} + \mathbf{I} \cdot \underbrace{\epsilon_{t+1}}_{n_e\times 1}. \end{aligned}$$

Accordingly, these two equations are simplified in the single compact form:

$$\underbrace{\begin{bmatrix} \mathbf{x}_t \\ \mathbf{z}_{t+1} \end{bmatrix}}_{\equiv \mathbf{s}_{t+1}} = \underbrace{\begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{0} & \mathbf{N} \end{bmatrix}}_{\equiv \mathbf{M}} \cdot \underbrace{\begin{bmatrix} \mathbf{x}_{t-1} \\ \mathbf{z}_t \end{bmatrix}}_{\equiv \mathbf{s}_t} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}}_{\equiv \mathbf{W}} \cdot \epsilon_{t+1},$$

which defines the *state equation* of the state-space system. More succinctly, the latter equation is expressed as:

$$\mathbf{s}_{t+1} = \mathbf{M} \cdot \mathbf{s}_t + \mathbf{W} \cdot \epsilon_{t+1}.$$

To summarize, the desired state-space system with the *observation and state equations* is:

$$\begin{aligned} \mathbf{y}_t &= \mathbf{T} \cdot \mathbf{s}_t \\ \mathbf{s}_{t+1} &= \mathbf{M} \cdot \mathbf{s}_t + \mathbf{W} \cdot \epsilon_{t+1}. \end{aligned} \tag{76}$$

The next step is to "combine" the state-space system (76) with the log-normal pricing formulae discussed in the previous section. To illustrate this point, it will suffice to present the price of equity as in discussed in section F.4.

We slightly rewrite the equity price (71) using the risk aversion parameter γ:

$$\begin{aligned} p_t^e &= \mathbb{E}_t \sum_{k=1}^{\infty} \beta^k \left(\frac{\Lambda_{t+k}}{\Lambda_t} \right)^{-\gamma} D_{t+k} \\ &= \sum_{k=1}^{\infty} \underbrace{\beta^k \mathbb{E}_t \left(\frac{\Lambda_{t+k}}{\Lambda_t} \right)^{-\gamma} D_{t+k}}_{\equiv V_t[D_{t+k}]} \\ &= \sum_{k=1}^{\infty} V_t \left[D_{t+k} \right]. \end{aligned} \tag{77}$$

In formulation (77), Λ_t denotes a "subutility aggregator" in the class of time-separable preferences with the risk aversion parameter γ; their "stand-in" felicity utility function is of the form $\Lambda_t^{1-\gamma}/(1-\gamma)$. By way of illustration, we have $\Lambda_t = c_t$ in the standard CRRA utility case, while having $\Lambda_t = c_t - \chi c_{t-1}$ for the habit-formation one. In the present model, Λ_t will take the form of $c_t - \chi c_{t-1} - G(h_t)$. $V_t [D_{t+k}]$ represents the valuation of any claim to a potentially random future payout D_{t+k}—in the case of "wage assets," this payout D_{t+k} is understood as a future flow wage w_{t+k}^n. In other words, $V_t [D_{t+k}]$, often referred to as 'strips' in the finance literature (e.g., Jermann (1998)), is a type of assets with a single period payout.

This 'strip' asset $V_t [D_{t+k}]$ is the key part of the log-normal pricing formulae, nesting the pricing of risk-free assets in section F.3. as a special case. In a nutshell, any long-term financial assets can be 'stripped down' into a sequence of 'strips,' each of whom reflects the joint log-normality of its payout growth and the market stochatic discount factor applied to its time horizon. More specifically, $V_t [D_{t+k}]$

can be expressed as

$$
\begin{aligned}
V_t\left[D_{t+k}\right] &= \beta^k \mathbb{E}_t \left(\frac{\Lambda_{t+k}}{\Lambda_t}\right)^{-\gamma} D_{t+k} \\
&= \beta^k \mathbb{E}_t \left(\frac{\Lambda_{t+k}}{\Lambda_t}\right)^{-\gamma} \frac{D_{t+k}}{D_t} D_t \\
&= \beta^k \mathbb{E}_t \exp\left[(-\gamma)(\lambda_{t+k}-\lambda_t)+d_{t+k}-d_t\right] \underbrace{\exp d_t}_{D_t} \\
&= \beta^k \exp d_t \left[\exp\left[\mathbb{E}_t\left((-\gamma)(\lambda_{t+k}-\lambda_t)+d_{t+k}-d_t\right)+\frac{1}{2}\mathrm{Var}_t\left((-\gamma)(\lambda_{t+k}-\lambda_t)+d_{t+k}-d_t\right)\right]\right],
\end{aligned}
\tag{78}
$$

which rationalizes our discussion. Furthermore, the last line of formulation (78) will be expressed in terms of the system (76), thereby completing the second step of our model solution.

From the state-space system (76), the log-deviations of Λ_t and D_t can be represented, respectively, as the corresponding row vectors of the observation equation:

$$
\begin{aligned}
\lambda_t &= \log \Lambda_t - \log \overline{\Lambda} \\
&= \mathbf{T}_\lambda \mathbf{s}_t \\
d_t &= \log D_t - \log \overline{D} \\
&= \mathbf{T}_d \mathbf{s}_t .
\end{aligned}
\tag{79}
$$

First note that the last line of formulation (78) can be decomposed into two main parts, the expectation and variance terms, respectively. Next using the expressions in relationship (79), the prior two are computed accordingly:

$$
\begin{aligned}
&\mathbb{E}_t\left[\lambda_{t+k}-\lambda_t+d_{t+k}-d_t\right] \\
&= \left[\mathbf{T}_\lambda+\mathbf{T}_d\right]\mathbf{M}^k\cdot\mathbf{s}_t-\left[\mathbf{T}_\lambda+\mathbf{T}_d\right]\cdot\mathbf{s}_t \\
&= \left[\mathbf{T}_\lambda+\mathbf{T}_d\right]\left[\mathbf{M}^k-\mathbf{I}\right]\cdot\mathbf{s}_t \\
&\mathrm{Var}_t\left[\lambda_{t+k}-\lambda_t+d_{t+k}-d_t\right] \\
&= \mathbb{E}_t\left[\lambda_{t+k}-\lambda_t+d_{t+k}-d_t-\mathbb{E}_t\left[\lambda_{t+k}-\lambda_t+d_{t+k}-d_t\right]\right]^2 \\
&= \mathbb{E}_t\left[\lambda_{t+k}+d_{t+k}-\mathbb{E}_t\left[\lambda_{t+k}+d_{t+k}\right]\right]^2 \\
&= \sum_{j=1}^{k-1}\left(\mathbf{T}_\lambda+\mathbf{T}_d\right)\mathbf{M}^j\mathbf{W}\mathbb{E}_t\left[\epsilon_{t+j}\right]^2\left[\left(\mathbf{T}_\lambda+\mathbf{T}_d\right)\mathbf{M}^j\mathbf{W}\right]'+\left(\mathbf{T}_\lambda+\mathbf{T}_d\right)\mathbf{W}\mathbb{E}_t\left[\epsilon_{t+k}\right]^2\left[\left(\mathbf{T}_\lambda+\mathbf{T}_d\right)\mathbf{W}\right]'.
\end{aligned}
\tag{80}
$$

In summary, any present-value variables are regarded as assets with a sequence of 'strips' and each 'strip' will be evaluated 'non-linearly' using formulations (78) and (80). Then for sufficiently large time horizon n, these 'strips' will be summed up.

E.7. Supplementary References

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F. Investment Wedge

F.1. Overview

The purpose of this exercise is to show that in our baseline economy, investment volatility can be enhanced by adding firm-specific investment shocks unrelated to productivity shocks, as in Fahri and Gourio (2018). Admittedly, our baseline model generates insufficient investment volatility relative to its real-life counterpart. For that purpose, we incoporate into our baseline model an *investment wedge*, introduced first by Chari et al. (2007) in the context of their "Business Cycle Accounting." The Business Cycle Accounting methodology rests upon the equivalence result that a large class of models, including models with varioustypes of frictions, is *observationally* equivalent to a prototype model with various types of time-varying wedges that distort the equilibrium decisions of agents operating in otherwise competitive markets. These wedges manifest themselves into time-varying productivity, labor income taxes, investment taxes, and government consumption. In their terms, these wedges, respectively, are labelled *efficiency wedges*, *labor wedges*, *investment wedges*, and *government consumption wedges*.

The Business Cycle Accounting (hereafter, BCA) analysis concludes that the efficiency and labor wedges play key parts in economic fluctuations of the US economy, particularly in episodes of crises, including the Great Depression, while the investment wedge plays a tertiary role. It is widely accepted that the efficiency wedge resembles TFP shocks in RBC models, while the labor wedge is equivalent to the result from a variety of labor market frictions, including DMP-type labor search and matching. The investment wedge can include a variety of investment-specific technology shocks (e.g. shocks on transformation technology between consumption and investment goods), credit and financial frictions that distort intertemporal investment Euler equations.

Building on an otherwise RBC economy with a single TFP shock and DMP-type labor market frictions, our model can be mapped into a proto-type model with the efficiency and labor wedges in the BCA context. Moreover, the model's equilibrium outcome coincides with what the BCA analysis claims: labor market frictions due to market incompleteness and TFP shocks are main drivers to account for episodes of economic crises such as Secular Stagnation or the Great Recession. In the same vein, the model's investment volatility, directly linked to any kind of *investment wedges*, will play a tertiary role in generating key labor and finanical statistics that feature conspicuously in long slumps.

Thus, we reason that the introduction of an investment wedge into the baseline model would matter only for increasing investment volatility without compromising its key mechanism. Basel upon this reasoning, we introduce an *investment wedge* in the form of a *investment-specific technology shock* employed widely in the business cycle literature, although the investment wedge here is intended to include a broad category of investment-specific shocks and fiancial frictions related with investment decisions.

As in the basline case, the (representative) firm's decision problem is to maximize its pre-dividend stock market value $d_t + p_t^e$ on a period-by-period basis given its information set Ω_t^f but with the investment wedge ν_t ("nu"):

$$\max_{\{i_t, k_{t+1}, x_t\}} d_t + p_t^e \equiv d_t + \mathbb{E}(\beta \Lambda_{t,t+1}^s (p_{t+1}^e + d_{t+1}) | \Omega_t^f) \tag{81}$$

sujject to

$$d_t \equiv f(k_t, \mu_s h_t^s, n_t h_t^n) z_t - \left(\frac{1}{\nu_t}\right) \cdot i_t - \mu_s w_t^s h_t^s - w_t^n h_t^n n_t - \frac{\kappa}{2} x_t^2 n_t - \varphi \bar{k} + p_t^b \varphi \bar{k}$$

$$k_{t+1} = (1-\delta) k_t + \Phi_i(i_t, k_t)$$

$$\Phi_i(i_t, k_t) \equiv G\left(\frac{i_t}{k_t}\right) \cdot k_t$$

$$n_{t+1} = (1-\rho) n_t + x_t n_t$$

$$\log \nu_{t+1} = \rho_\nu \log \nu_t + \epsilon_{t+1}^\nu.$$

ν_t is stochastically indepedent of z_t, i.e. TFP shocks.

The *original* implication of investment-specific technical change ν_t is that the number of consumption units that must be exchanged to acquire an efficiency unit of the investment good is $\frac{1}{\nu_t}$. Therefore, in the recursive stationary equilibrium of this economy, the real price of an investment good is $P \equiv \frac{1}{v}$; Any positive shock on ν_t implies a decline in the real price of investment, leading to investment booms. This mechanism materializes Keynes' view that shocks to the marginal efficiency of investment are important for business fluctuations (Greenwood et al, 1988). However, as we identify a series of shocks on ν_t with the investment wedge, any positive shock on ν_t implies that investment-related fiancial contraints (e.g. credit and collateral constraints) are relaxed in investment booms.

F.2. Derivation

1. **First-order Conditions**

   To facilitate derivation, we abstract from the firm's hiring decision x_t in the problem (81); this abstraction should be deemed innocuous, as the investment wedge only concerns investment decision, by construction. We then transform the firm's problem (81) into the following simpler but recursive formulation:

where $\mathbf{s}_t$ is a set of all state variables relevant to the firm's decision.

In the problem (**??**), λ_t and Q_t indicate the marginal utility of shareholders and Tobin's Q, respectively.

The (necessary and sufficient) first-order conditions for the firm's investment decision are given by:

$$\partial i_t: \quad -\left(\frac{1}{\nu_t}\right) + Q_t \cdot (\Phi_i(i_t, k_t)) = 0$$

$$\partial k_{t+1}: \quad -Q_t + \beta \cdot \mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t} \cdot \frac{\partial V^f}{\partial k_{t+1}}(\mathbf{s}_{t+1})\right] = 0$$

where $\Phi_i(i, \cdot)$ stands for $\frac{\partial}{\partial i}\Phi(i, \cdot)$.

By the envelope theorem,

$$\frac{\partial}{\partial k_t}V^f(\mathbf{s}_t) = \frac{\partial y_t}{\partial k_t} + Q_t \cdot ((1-\delta) + \Phi_k(i_t, k_t))$$

where $\Phi_k(\cdot, k)$ similarly stands for $\frac{\partial}{\partial k}\Phi(\cdot, k)$.
The investment Euler equation is thus represented as:

$$\begin{aligned}1 =& \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \cdot Q_t^{-1} \cdot \left(\frac{\partial y_{t+1}}{\partial k_{t+1}} + Q_{t+1} \cdot ((1-\delta) + \Phi_k(i_{t+1}, k_{t+1})) \right) \right] \\ =& \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \cdot R_{t+1} \right]\end{aligned}$$

where R_{t+1} indicates the firm's real return on investment:

$$R_{t+1} \equiv Q_t^{-1} \cdot \left(\frac{\partial y_{t+1}}{\partial k_{t+1}} + Q_{t+1} \cdot ((1-\delta) + \Phi_k(i_{t+1}, k_{t+1})) \right).$$

Tobin's Q is also determined by

$$Q_t = \frac{1}{\nu_t \cdot \Phi_i(i_t, k_t)}.$$

2. **Log-linearization**

   (a) Parameterization

   The adjustment cost function $\Phi(i_t, k_t)$ takes the form employed in Jermann (1998) and Kaltenbrunner and Lochstoer (2010):

   $$\Phi(i_t, k_t) = G(\frac{i_t}{k_t})k_t$$

   where $G(\cdot)$ is given by

   $$G(\frac{i_t}{k_t}) = \frac{a_1}{1-\frac{1}{\xi}}(\frac{i_t}{k_t})^{1-\frac{1}{\xi}} + a_2$$

   and α_1 and α_2 are chosen so that $G(\delta) = \delta$, and $G_1(\delta) = 1$.
   Accordingly, we determine the following terms:

   $$\Phi_i(i_t, k_t) \equiv G'\left(\frac{i_t}{k_t}\right) = a_1 \cdot \left(\frac{i_t}{k_t}\right)^{-\frac{1}{\xi}}$$

   $$Q_t = \frac{1}{\nu_t \cdot \Phi_i(i_t, k_t)} = a_1 \cdot \left(\frac{i_t}{k_t}\right)^{\frac{1}{\xi}} \cdot \nu_t^{-1}$$

   (b) Log-linearization

   It will suffice to log-linearize real returns on investment R_{t+1} in the investment euler equation:

$$\widehat{R_{t+1}} \approx -\widehat{Q_t} + \frac{\alpha \cdot \frac{\bar{y}}{\bar{k}}}{\alpha \cdot \frac{\bar{y}}{\bar{k}} + \bar{Q} \cdot (1-\delta)} \cdot \alpha \cdot \widehat{\frac{y_{t+1}}{k_{t+1}}} + \frac{\bar{Q} \cdot (1-\delta)}{\alpha \cdot \frac{\bar{y}}{\bar{k}} + \bar{Q} \cdot (1-\delta)} \cdot Q_{t+1} \widehat{\cdot (1} - \delta) \tag{82}$$

$$= -\widehat{Q_t} + \frac{\alpha \cdot \frac{\bar{y}}{\bar{k}}}{\alpha \cdot \frac{\bar{y}}{\bar{k}} + \bar{Q} \cdot (1-\delta)} \cdot \left(\widehat{y_{t+1}} - \widehat{k_{t+1}}\right) + \frac{\bar{Q} \cdot (1-\delta)}{\alpha \cdot \frac{\bar{y}}{\bar{k}} + \bar{Q} \cdot (1-\delta)} \cdot \widehat{Q_{t+1}}$$

The above parametric choice of $\Phi(i_t, k_t)$ makes it possible to log-linearize the terms related with Tobin's Q in the formula (82), as shown below:

$$\widehat{Q_t} = a_1 \cdot \widehat{\left(\frac{i_t}{k_t}\right)^{\frac{1}{\xi}}} \cdot \nu_t^{-1}$$
$$= \left(\frac{1}{\xi}\right) \cdot \left(\hat{i}_t - \hat{k}_t\right) - \hat{\nu}_t.$$

F.3. Calibration

The investment wedge ν_t evolves according to the law of motion:

$$\log \nu_{t+1} = \rho_\nu \cdot \log \nu_t + \epsilon_{t+1}^{\nu}$$

where the $\{\epsilon_t^\nu\}$ are distributed i.i.d. normal, with mean zero and stochatically independent of disturbances $\{\tilde{\varepsilon}_t\}$ on the productivity shock $\tilde{z}_t$.

In the above shock process, we have two *free* parameters; the autocorrelation coefficient ρ_ν and the standard deviation $\sigma_{\epsilon,\nu}$ of the $\{\epsilon_t^\nu\}$. We thus conduct a *hyperparameter search* on a pair of free parameters $(\rho_\nu, \sigma_{\epsilon,\nu})$ to target and deliver a standard deviation of risk-free rates of 2.31% (annualized) and a quarterly standard deviation of investment growth rates $\text{SD}(\Delta \log i_t)$ of 4.00% from the US data (1970 Q1-2015 Q4); for instance, we can minimize a criterion function, imposing a weight of 100 for the target moment of investment growth rates and a weight of one for that of risk-free rates on squared deviations. The overall minium is found at $\rho_\nu = 0.50$ and $\sigma_{\epsilon,\nu} = 3.23(\%)$. At the overall minimum, model simulations deliver $\text{SD}(\tilde{r}^b) = 2.59$ and $\text{SD}(\Delta \log i_t) = 3.60$.

F.4. Simulation Results

Table 4 in the main text reports standard business cycle statistics in the "investment wedge" case and make comparison with its baseline counterpart. As conjectured, the investment wedge acts just as *additional* shocks on the baseline model driven only by TFP disturbances.The standard deviations of all macro variables increase *absolutely*, but their relative counterparts do not change substantially except investment i. The investment wedge ν_t mostly concentrates on investment volatility; indeed $\text{SD}(\Delta \log i_t)$ increases from 1.27% (baseline) to 3.60%. One notable difference is that the mean risk free rate now enter the negative territory unambiguously, without changing market incompleteness substantially as in the Polarization Trap case. The negative mean risk free rate in the present scenario can emerge rather for a mechanical reason. To confirm that this is the case, rearrange and express the log-linearized real interest rate of the present economy:

$$\widehat{R}_{t+1}^{\text{invt. wedge}} = \widehat{R}_{t+1}^{\text{baseline}} + \hat{v}_t - \frac{\bar{Q}\cdot(1-\delta)}{\alpha\cdot\frac{\bar{y}}{\bar{k}} + \bar{Q}\cdot(1-\delta)}\cdot\hat{v}_{t+1}$$
$$= \widehat{R}_{t+1}^{\text{baseline}} + \hat{v}_t - \frac{\bar{Q}\cdot(1-\delta)}{\alpha\cdot\frac{\bar{y}}{\bar{k}} + \bar{Q}\cdot(1-\delta)}\cdot(\hat{v}_t + \epsilon_{t+1}^{\nu})$$
$$= \widehat{R}_{t+1}^{\text{baseline}} + (\frac{\alpha\cdot\frac{\bar{y}}{\bar{k}}}{\alpha\cdot\frac{\bar{y}}{\bar{k}} + \bar{Q}\cdot(1-\delta)})\cdot\hat{v}_t + \text{ white noise term.} \tag{83}$$

Accordingly, taking the expectations on both sides of formulation (83), we have the following relationship:

$$\mathbb{E}_t\widehat{R}_{t+1}^{\text{invt. wedge}} = \mathbb{E}_t\widehat{R}_{t+1}^{\text{baseline}} + (\frac{\alpha\cdot\frac{\bar{y}}{\bar{k}}}{\alpha\cdot\frac{\bar{y}}{\bar{k}} + \bar{Q}\cdot(1-\delta)})\cdot\hat{v}_t. \tag{84}$$

Relationship (84) indicates that the mean risk free rate will be *mechanically increased* by additional shocks $\tilde{v}_t$ along with TFP disturbances $\tilde{z}_t$ relative to the baseline case.

E.7. Supplementary References

1. Chari, V. V., Kehoe, P. J., McGrattan, E. R., 2007. "Business cycle accounting." *Econometrica* 75, 781–836.