Online Appendix to
The Effects of Parental Income and Family Structure on Intergenerational Mobility: A Trajectories-Based Approach

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1 Identification

In this section, we give a formal exposition of the identification of our model. Let (f_i, g_i, x_i) be i.i.d. observations as assumed in the text.

Assumption 1. We assume that
(a) (ε_i) is independent of (f_i, g_i, x_i).
(b) (f_i, g_i, x_i) has support $\mathcal{H} \times \mathcal{H} \times \mathcal{X}$.
(c) $\mathcal{T}$ is a set of continuous functions with zero infimum on $\mathcal{X}$.
(d) $\mathcal{P}$ is a set of strictly positive probability density functions on $\mathbb{R}$ with zero mean and unit variance.

Many of the conditions in Assumption 1 are directly comparable to those used in Matzkin (1992) and Yan (2024) for identification of binary choice models with no functional and operative arguments.

Theorem 1.1. *Under Assumption 1, θ is identified in Θ.*

Proof of Theorem 1.1. To establish identifiability of our model, we need to show that $\theta = \theta_0$ whenever $\pi_j(f, g, x; \theta) = \pi_j(f, g, x; \theta_0)$ for $j = 1, \ldots, m-1$ for all $(f, g, x) \in \mathcal{H} \times \mathcal{H} \times \mathcal{X}$, the support given by the condition (b) in Assumption 1. We will show that our model is identified for $m = 2$, from which it follows that our model is generally over-identified for $m > 2$. Subsequently, we let F and F_0 be the distribution functions of ρ and ρ_0, respectively. We break our proof into two parts.

Identification of α, β and Γ. Let $\pi_j(f, g, x; \theta) = \pi_j(f, g, x_i; \theta_0)$ for all $(f, g, x) \in \mathcal{H} \times \mathcal{H} \times \mathcal{X}$ and for any $j = 1, \ldots, m-1$ given. Assume $m = 2$, in which case μ becomes a scalar. It follows that

$$F\big(\mu-\langle\alpha,f\rangle-\langle\beta,g\rangle-\langle f,\Gamma g\rangle-\tau(x)\big)=F_0\big(\mu_0-\langle\alpha_0,f\rangle-\langle\beta_0,g\rangle-\langle f,\Gamma_0 g\rangle-\tau_0(x)\big)$$

for all $(f,g,x)\in\mathcal{H}\times\mathcal{H}\times\mathcal{X}$. Fix $x\in\mathcal{X}$ and define

$$G(z)=F\big(z+\mu-\tau(x)\big)\quad\text{and}\quad G_0(z)=F_0\big(z+\mu_0-\tau_0(x)\big).$$

It is easy to see that G and G_0 are also distribution functions, and

$$G(-\langle\alpha,f\rangle-\langle\beta,g\rangle-\langle f,\Gamma g\rangle)=G_0(-\langle\alpha_0,f\rangle-\langle\beta_0,g\rangle-\langle f,\Gamma_0 g\rangle)$$

for all $f,g\in\mathcal{H}$. Furthermore, by the condition (d) in Assumption 1, G and G_0 are strictly increasing and have unit variance.

Take $g=0$. If $\alpha_0=0$, we have $G(-\langle\alpha,f\rangle)=G_0(0)$ holds for all $f\in\mathcal{H}$, and therefore, $\alpha=0$. If $\alpha_0\neq 0$, fix an arbitrary f such that $\langle\alpha_0,f\rangle\neq 0$. Then for any $c\in\mathbb{R}$,

$$G\big(-\langle\alpha,cf\rangle\big)=G_0\big(-\langle\alpha_0,cf\rangle\big).$$

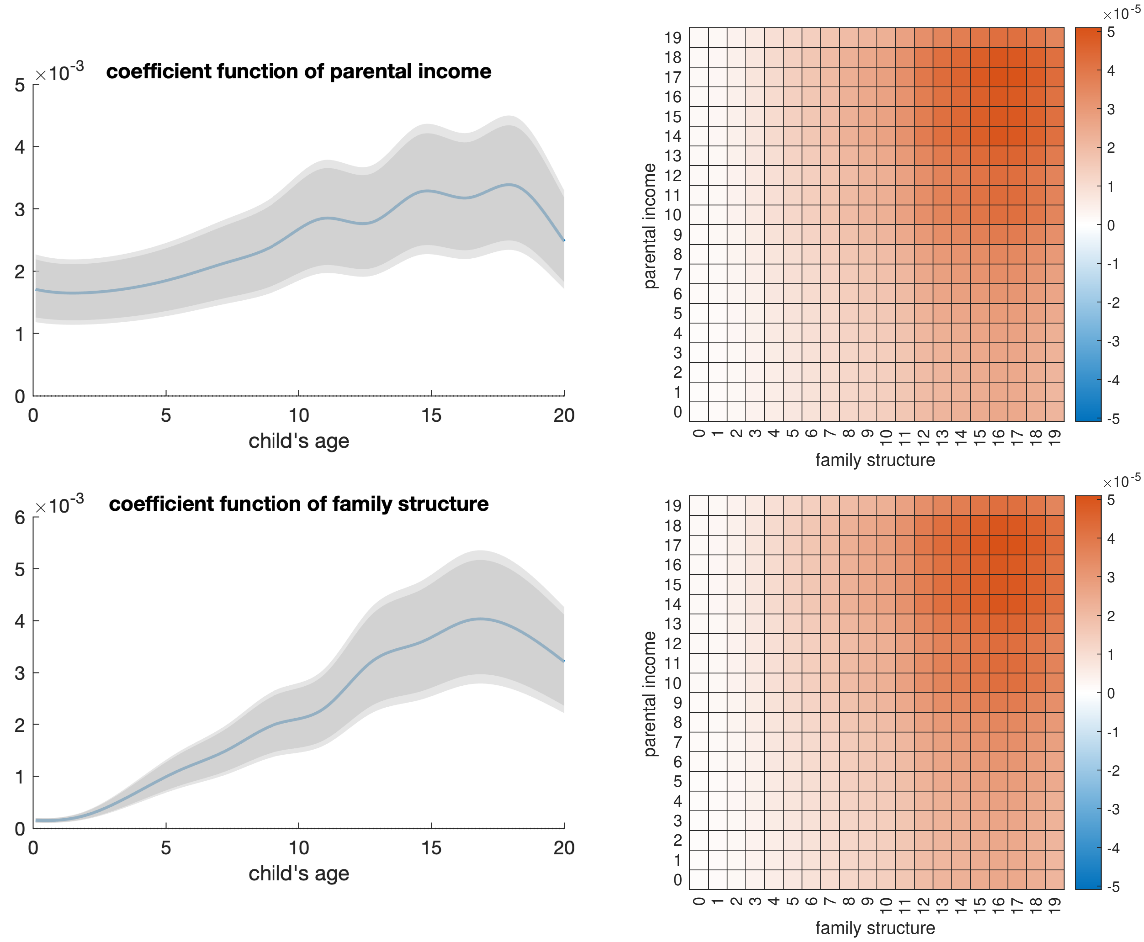
Since both G and G_0 have unit variance, this implies that $\langle\alpha,f\rangle=\langle\alpha_0,f\rangle$, and that $G=G_0$. Since f is arbitrary on $\mathcal{N}(\alpha_0)^\perp$, we conclude that $\alpha=\alpha_0$. We can similarly show that $\beta=\beta_0$ by taking $f=0$ instead. Now we have $G(-\langle\alpha_0,f\rangle-\langle\beta_0,g\rangle-\langle f,\Gamma g\rangle)=G(-\langle\alpha_0,f\rangle-\langle\beta_0,g\rangle-\langle f,\Gamma_0 g\rangle)$ for all $f,g\in\mathcal{H}$. Since G is strictly increasing, we have $\langle f,\Gamma g\rangle=\langle f,\Gamma_0 g\rangle$ for all $f,g\in\mathcal{H}$, which implies that $\Gamma=\Gamma_0$. This shows that α_0,β_0 and Γ_0 are identified.

Identification of μ,τ, and ρ. Once α_0,β_0 and Γ_0 are identified, the distribution of $\langle\alpha_0,f_i\rangle+\langle\beta_0,g_i\rangle+\langle f_i,\Gamma_0 g_i\rangle$ is identified. Also, this distribution has support $\mathbb{R}$. It then follows from Theorem 2.1 of Yan (2024) that τ_0 and ρ_0 are identified. Given that all other parameters are identified, the identification of μ_0 follows from the assumption that τ_0 has infimum 0 over its domain $\mathcal{X}$. □

2 Additional Results for Robustness Check

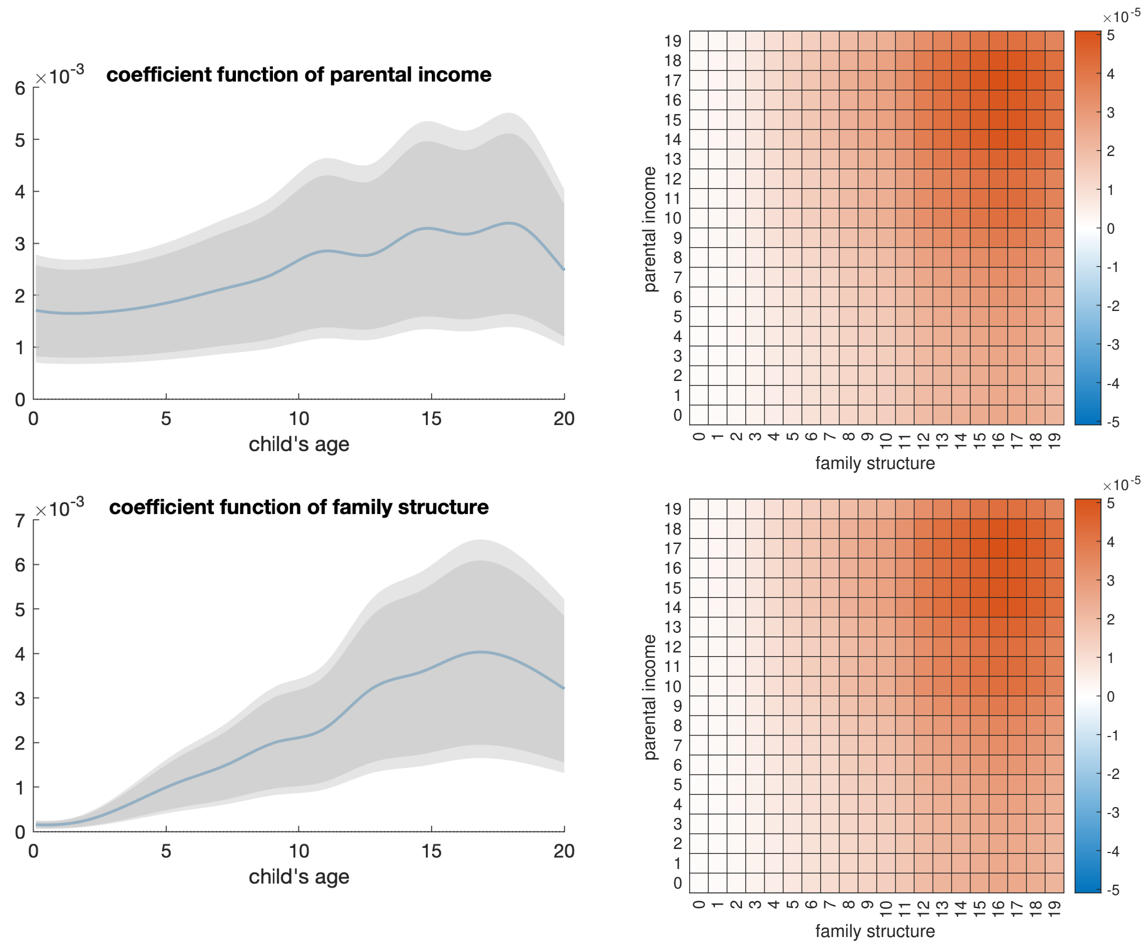
In this online appendix, we present additional estimation results for the model under alternative specifications of the error term distribution, different numbers of functional principal components used to represent family income and family structure trajectories, and the inclusion of survey weights.

Figure A1: Main and Interaction Effects of Parental Income and Family Structure (Probit)



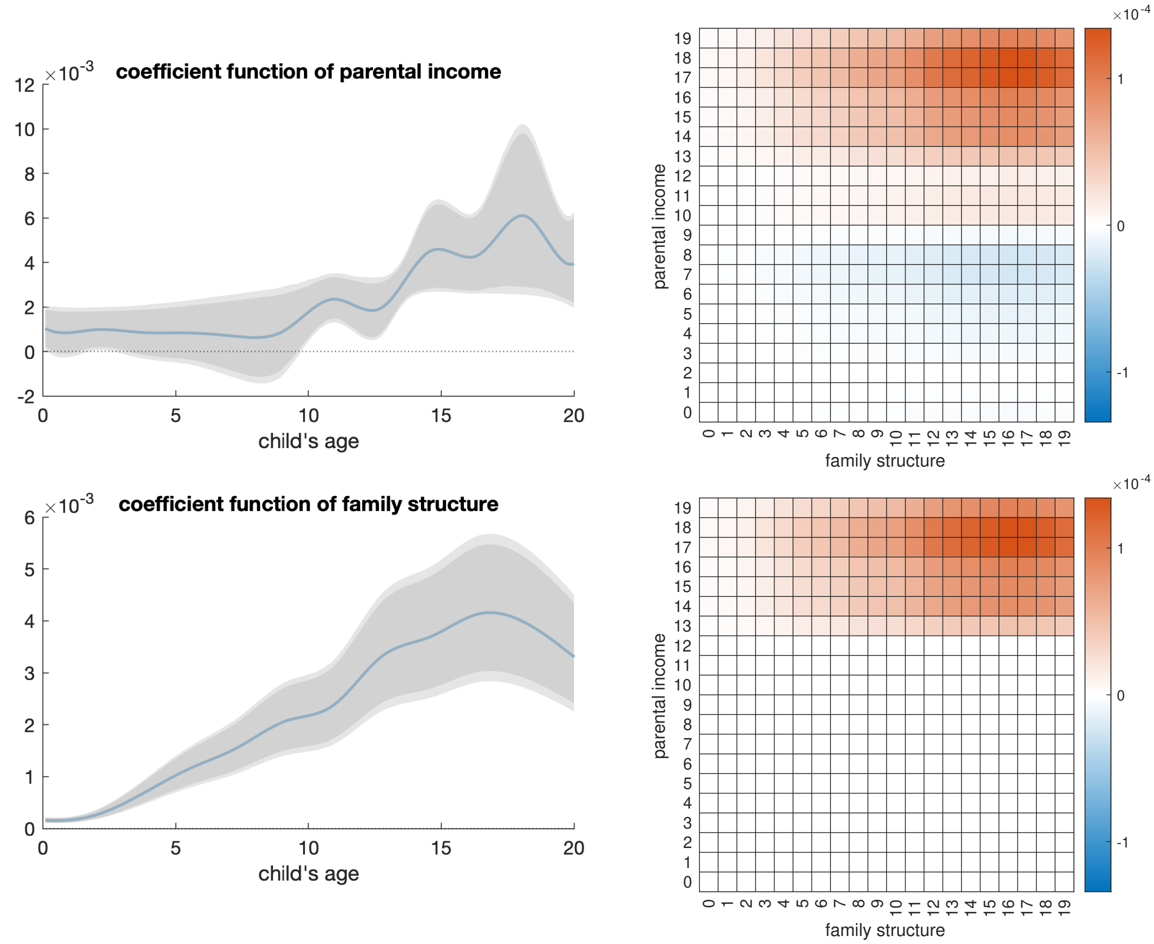
Notes: The upper-left panel shows the estimated functional coefficient $\widehat{\alpha}(t)$ for family income trajectories, while the lower-left panel shows the estimated functional coefficient $\widehat{\beta}(t)$ for family structure trajectories. The dark and light shaded areas represent the 90% and 95% confidence bands, respectively. The upper right panel displays the heat map of the estimated interaction function $\widehat{\Gamma}(t, s)$ between family income and family structure trajectories. The lower-right panel displays the corresponding heat map, retaining only the grid cells with values significantly different from zero at the 5% significance level.

Figure A2: Main and Interaction Effects of Parental Income and Family Structure (Logit)



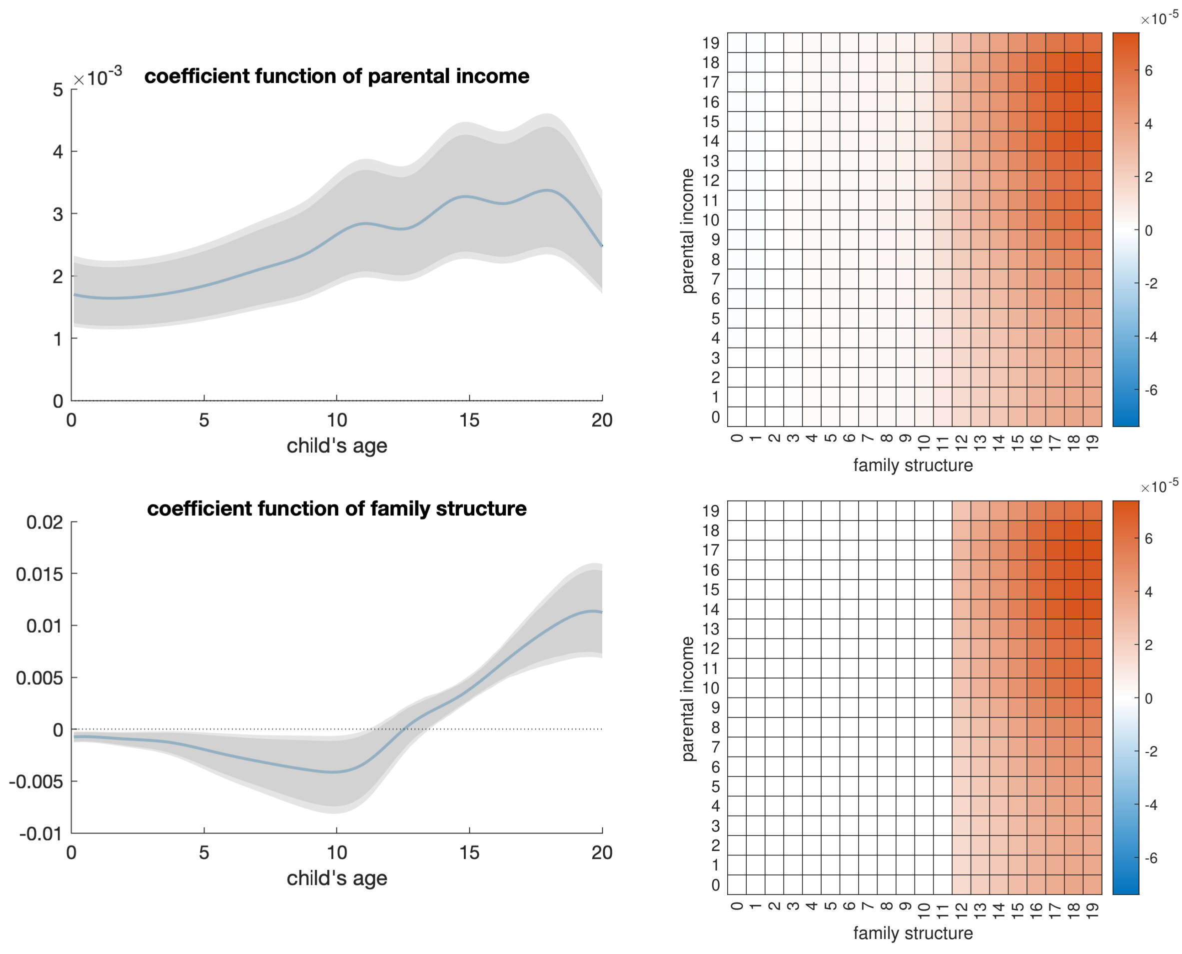
Notes: The upper-left panel shows the estimated functional coefficient $\widehat{\alpha}(t)$ for family income trajectories, while the lower-left panel shows the estimated functional coefficient $\widehat{\beta}(t)$ for family structure trajectories. The dark and light shaded areas represent the 90% and 95% confidence bands, respectively. The upper right panel displays the heat map of the estimated interaction function $\widehat{\Gamma}(t,s)$ between family income and family structure trajectories. The lower-right panel displays the corresponding heat map, retaining only the grid cells with values significantly different from zero at the 5% significance level.

Figure A3: Main and Interaction Effects of Parental Income and Family Structure (with $p_f = 2, p_g = 1$)



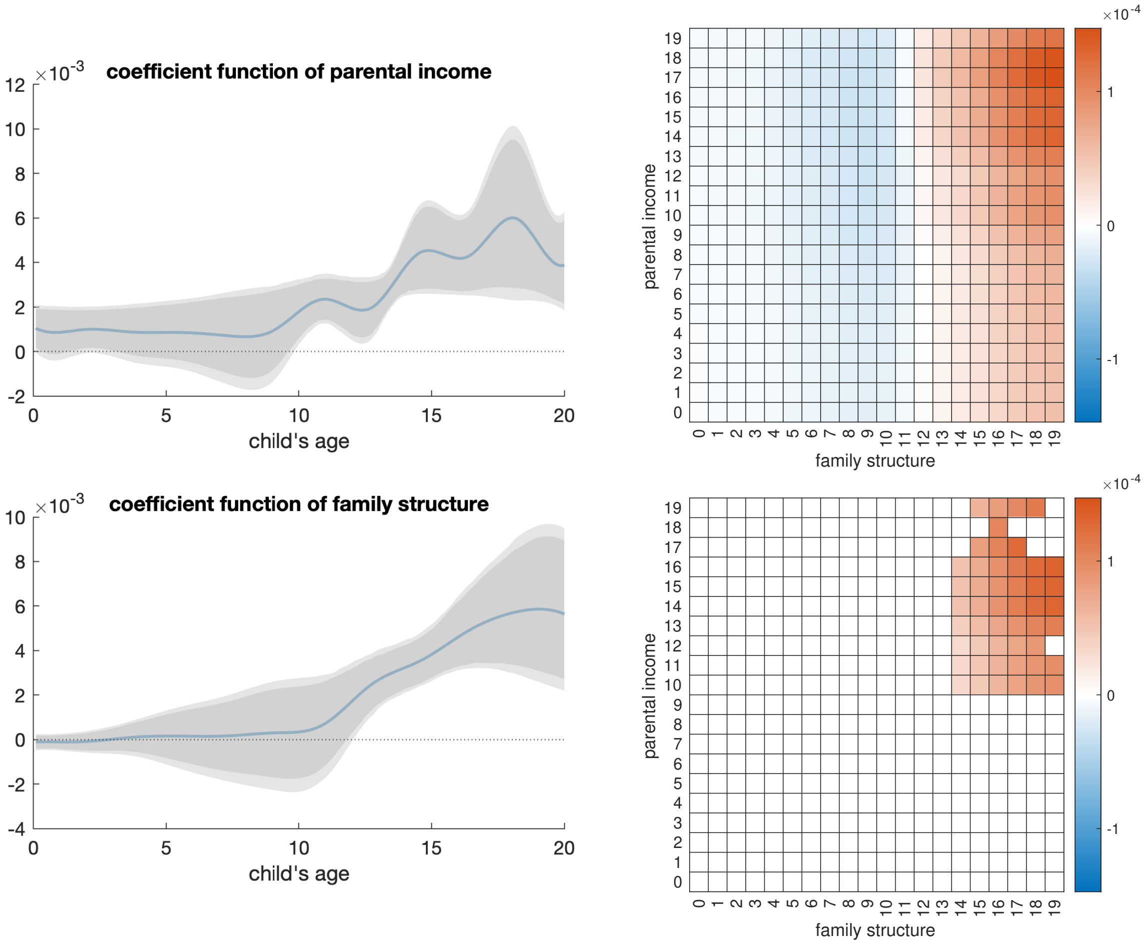
Notes: The upper-left panel shows the estimated functional coefficient $\widehat{\alpha}(t)$ for family income trajectories, while the lower-left panel shows the estimated functional coefficient $\widehat{\beta}(t)$ for family structure trajectories. The dark and light shaded areas represent the 90% and 95% confidence bands, respectively. The upper right panel displays the heat map of the estimated interaction function $\widehat{\Gamma}(t, s)$ between family income and family structure trajectories. The lower-right panel displays the corresponding heat map, retaining only the grid cells with values significantly different from zero at the 5% significance level.

Figure A4: Main and Interaction Effects of Parental Income and Family Structure (with $p_f = 1, p_g = 2$)



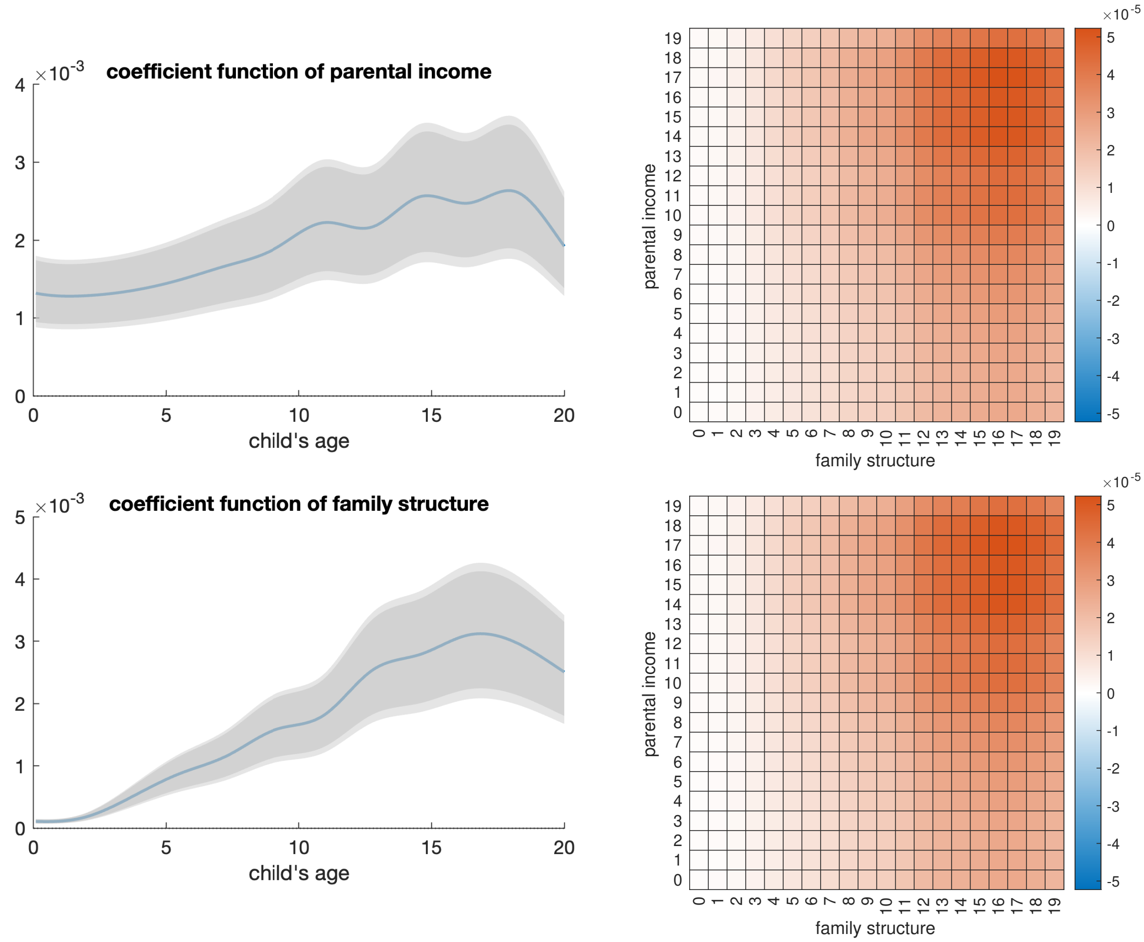
Notes: The upper-left panel shows the estimated functional coefficient $\widehat{\alpha}(t)$ for family income trajectories, while the lower-left panel shows the estimated functional coefficient $\widehat{\beta}(t)$ for family structure trajectories. The dark and light shaded areas represent the 90% and 95% confidence bands, respectively. The upper right panel displays the heat map of the estimated interaction function $\widehat{\Gamma}(t, s)$ between family income and family structure trajectories. The lower-right panel displays the corresponding heat map, retaining only the grid cells with values significantly different from zero at the 5% significance level.

Figure A5: Main and Interaction Effects of Parental Income and Family Structure (with $p_f = 2, p_g = 2$)



Notes: The upper-left panel shows the estimated functional coefficient $\widehat{\alpha}(t)$ for family income trajectories, while the lower-left panel shows the estimated functional coefficient $\widehat{\beta}(t)$ for family structure trajectories. The dark and light shaded areas represent the 90% and 95% confidence bands, respectively. The upper right panel displays the heat map of the estimated interaction function $\widehat{\Gamma}(t, s)$ between family income and family structure trajectories. The lower-right panel displays the corresponding heat map, retaining only the grid cells with values significantly different from zero at the 5% significance level.

Figure A6: Main and Interaction Effects of Parental Income and Family Structure (with Survey Weights)



Notes: The upper-left panel shows the estimated functional coefficient $\widehat{\alpha}(t)$ for family income trajectories, while the lower-left panel shows the estimated functional coefficient $\widehat{\beta}(t)$ for family structure trajectories. The dark and light shaded areas represent the 90% and 95% confidence bands, respectively. The upper right panel displays the heat map of the estimated interaction function $\widehat{\Gamma}(t,s)$ between family income and family structure trajectories. The lower-right panel displays the corresponding heat map, retaining only the grid cells with values significantly different from zero at the 5% significance level.

References

Matzkin, R. (1992). Nonparametric and distribution-free estimation of the binary threshold crossing and the binary choice models. *Econometrica*, 60(2):239–270.

Yan, G. (2024). A kernelization-based approach to nonparametric binary choice models. Manuscript.