# Scale Economies, Input-Output Loops, and Trade*

# Online Appendix

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## 1 An Alternative Proof of Theorem 1(i) in Allen *et al.* (2024, AAL)

The system of equations (1) in AAL is

$$x_{ih} = \sum_{j=1}^{N} f_{ijh}\left(\boldsymbol{x}_{j\cdot}\right),$$

where $\boldsymbol{x}_{j\cdot} \equiv \left(x_{j1}, \ldots, x_{jH}\right)^T$. Taking logarithms, we obtain

$$\ln x_{ih} = \ln \sum_{j=1}^{N} f_{ijh}\left(\boldsymbol{x}_{j\cdot}\right).$$

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Define the following notation: $y_{ih} \equiv \ln x_{ih}$, $\boldsymbol{y}_{i\cdot} \equiv (y_{i1}, \ldots, y_{iH})^T$, $\boldsymbol{y}_{\cdot h} \equiv (y_{1h}, \ldots, y_{Nh})^T$, $\boldsymbol{y} \equiv (\boldsymbol{y}_{\cdot 1}, \ldots, \boldsymbol{y}_{\cdot H})^T$, and $g_{ih}(\boldsymbol{y}) \equiv \ln \sum_{j=1}^{N} f_{ijh}(\boldsymbol{x}_{j\cdot})$. The Jacobian entries are given by

$$J_{ih,jh'}(\boldsymbol{y}) \equiv \left| \frac{\partial g_{ih}(\boldsymbol{y})}{\partial y_{jh'}} \right| = \frac{f_{ijh}(\boldsymbol{x}_{j\cdot})}{\sum_{j'=1}^{N} f_{ij'h}\left(\boldsymbol{x}_{j'\cdot}\right)} \cdot \left| \frac{\partial \ln f_{ijh}(\boldsymbol{x}_{j\cdot})}{\partial \ln x_{jh'}} \right|.$$

By assumption, there exists an $H \times H$ matrix $\mathbf{A}$ with entries $A_{hh'}$ such that

$$\left| \frac{\partial \ln f_{ijh}(\boldsymbol{x}_{j\cdot})}{\partial \ln x_{jh'}} \right| \leq A_{hh'} \quad \text{for all } i, j, h, h' \text{ and all } \boldsymbol{x}_{j\cdot};$$

and $\rho(\mathbf{A}) < 1$.

Let $\boldsymbol{g}_{\cdot h} \equiv (g_{1h}, \ldots, g_{Nh})^T$ and $\boldsymbol{g} \equiv (\boldsymbol{g}_{\cdot 1}, \ldots, \boldsymbol{g}_{\cdot H})^T$. By the mean-value theorem, for any $\boldsymbol{y}$ and $\boldsymbol{y}'$:

$$\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{y}') = J(\tilde{\boldsymbol{y}}) \cdot (\boldsymbol{y} - \boldsymbol{y}'),$$

where $\tilde{\boldsymbol{y}}$ is some point between $\boldsymbol{y}$ and $\boldsymbol{y}'$. We have

$$\begin{aligned} \left| g_{ih}(\boldsymbol{y}) - g_{ih}(\boldsymbol{y}') \right| &\leq \sum_{h'=1}^{H} \sum_{j=1}^{N} J_{ih,jh'}(\tilde{\boldsymbol{y}}) \left| y_{jh'} - y'_{jh'} \right| \\ &= \sum_{h'=1}^{H} \sum_{j=1}^{N} \frac{f_{ijh}(\tilde{\boldsymbol{x}}_{j\cdot})}{\sum_{j'=1}^{N} f_{ij'h}\left(\tilde{\boldsymbol{x}}_{j'\cdot}\right)} \cdot \left| \frac{\partial \ln f_{ijh}(\tilde{\boldsymbol{x}}_{j\cdot})}{\partial \ln x_{jh'}} \right| \cdot \left| y_{jh'} - y'_{jh'} \right| \\ &\leq \sum_{h'=1}^{H} A_{hh'} \max_j \left| y_{jh'} - y'_{jh'} \right|. \end{aligned} \tag{1}$$

Up until now, we just repeated the setup and preliminary argument from AAL. From here, AAL uses the Perov Fixed Point Theorem to argue that there exists a unique fixed point of $\boldsymbol{g}$. Instead, we are going to prove that $\boldsymbol{g}$ is a global contraction mapping with respect to an appropriately chosen norm.

Let $\tilde{\rho} \in (\rho(\mathbf{A}), 1)$. Define $\boldsymbol{v}_{(H)} \equiv (\tilde{\rho} I - \mathbf{A})^{-1} \iota$, where $I$ is the $H \times H$ identity matrix and $\iota = (1, \ldots, 1)^T$. Since $\tilde{\rho} > \rho(A)$, the matrix $\tilde{\rho} I - \mathbf{A}$ is an M-matrix, and therefore $(\tilde{\rho} I - \mathbf{A})^{-1}$ is a nonnegative matrix with positive diagonal entries. Hence, $\boldsymbol{v}_{(H)}$ is a positive vector. Also, $(\tilde{\rho} I - \mathbf{A}) \boldsymbol{v}_{(H)} = \iota$, and so $\mathbf{A}\boldsymbol{v}_{(H)} = \tilde{\rho}\boldsymbol{v}_{(H)} - \iota < \tilde{\rho}\boldsymbol{v}_{(H)}$. Denoting the entries of $\boldsymbol{v}_{(H)}$ by $v_1, \ldots, v_H$, the $h$-th component of the vector inequality $\mathbf{A}\boldsymbol{v}_{(H)} < \tilde{\rho}\boldsymbol{v}_{(H)}$ is given by

$$\sum_{h'} A_{hh'} v_{h'} < \tilde{\rho} v_h. \tag{2}$$

For any vector $\boldsymbol{y} \in \mathbb{R}^{HN}$, define the vector norm $\|\boldsymbol{y}\|_v \equiv \max_h \left\{ \frac{1}{v_h} \max_i |y_{hi}| \right\}$. We can also write this norm as $\|\boldsymbol{y}\|_v \equiv \left\| D_v^{-1} \boldsymbol{y} \right\|_\infty$, where $D_v$ is a diagonal matrix with vector $\boldsymbol{v} = (v_1, \ldots, v_1, \ldots, v_H, \ldots, v_H)^T$ on the diagonal. We have

$$
\begin{aligned}
\|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{y}')\|_v &= \max_h \left\{ \frac{1}{v_h} \max_i |g_{ih}(\boldsymbol{y}) - g_{ih}(\boldsymbol{y}')| \right\} \\
&\leq \max_h \left\{ \frac{1}{v_h} \max_i \left| \sum_{h'=1}^{H} A_{hh'} \max_j \left| y_{jh'} - y'_{jh'} \right| \right| \right\} \\
&= \max_h \left\{ \sum_{h'=1}^{H} \left( \frac{1}{v_h} A_{hh'} v_{h'} \right) \cdot \left( \frac{1}{v_{h'}} \max_j \left| y_{jh'} - y'_{jh'} \right| \right) \right\} \\
&\leq \max_h \left\{ \sum_{h'=1}^{H} \frac{1}{v_h} A_{hh'} v_{h'} \right\} \cdot \max_{h'} \left\{ \frac{1}{v_{h'}} \max_j \left| y_{jh'} - y'_{jh'} \right| \right\}.
\end{aligned}
$$

The first inequality above uses (1). The last inequality uses the property that for any matrix $B$ and vector $\boldsymbol{b}$, we have $\|B\boldsymbol{b}\|_\infty \leq |||B|||_\infty \cdot \|\boldsymbol{b}\|_\infty$, where $|||\cdot|||_\infty$ is the matrix norm induced by the sup vector norm. Using inequality (2), we obtain

$$
\|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{y}')\|_v \leq \max_h \left\{ \frac{1}{v_h} v_h \tilde{\rho} \right\} \cdot \left\| y_{jh'} - y'_{jh'} \right\|_v = \tilde{\rho} \cdot \left\| y_{jh'} - y'_{jh'} \right\|_v .
$$

Since $\tilde{\rho} < 1$, we conclude that $\boldsymbol{g}$ is a global contraction mapping with respect to the vector norm $\|\cdot\|_v$.

## 2 Counterexample to the Application of the AAL Approach to the $\ln \boldsymbol{F}(\boldsymbol{L})$ Mapping

Consider the following parameterization of the economy from the main text: $K = 2$, $\bar{L} = 1$,

$$
A = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}, \quad \mathcal{L} \equiv (I - A)^{-1} = \begin{pmatrix} 1.125 & 0.125 \\ 0.125 & 1.125 \end{pmatrix};
$$

and $\alpha_k = 0.8$, $\varepsilon_k = 1.5$, $\theta_k = 0.5$, $\overline{T}_k = 1$, $e_k = 0$, $E_k = 1$, and $\left[p_k^F\right]^{-\varepsilon_1} = 1$ for $k = 1, 2$. Let us check that the condition (UC) holds. We have for $k = 1, 2$:

$$
\varepsilon_k (\theta_1 \ell_{1k} + \theta_2 \ell_{2k}) = 0.9375 < 1.
$$

Set $w = 1$ and consider two vectors of labor allocations $\boldsymbol{L}^{(1)} = (0.1, 10)^T$ and $\boldsymbol{L}^{(2)} =$

$(10, 0.1)^T$. We have from the first step:

$$\boldsymbol{p}^{(1)} \approx \begin{pmatrix} 2.715 \\ 0.271 \end{pmatrix}, \; \boldsymbol{\lambda}^{(1)} \approx \begin{pmatrix} 0.183 \\ 0.876 \end{pmatrix}; \quad \boldsymbol{p}^{(2)} \approx \begin{pmatrix} 0.271 \\ 2.715 \end{pmatrix}, \; \boldsymbol{\lambda}^{(2)} \approx \begin{pmatrix} 0.876 \\ 0.183 \end{pmatrix}.$$

The Jacobian matrices are given by

$$J\left(\boldsymbol{L}^{(1)}\right) \approx \begin{pmatrix} 0.730 & 0.393 \\ 0.019 & 0.832 \end{pmatrix} \quad \text{and} \quad J\left(\boldsymbol{L}^{(2)}\right) \approx \begin{pmatrix} 0.832 & 0.019 \\ 0.393 & 0.720 \end{pmatrix},$$

with $\rho\left(J\left(\boldsymbol{L}^{(1)}\right)\right) = \rho\left(J\left(\boldsymbol{L}^{(2)}\right)\right) \approx 0.88 < 1$. The bound matrix $\overline{J}$ with elements $\overline{J}_{ks} \equiv \max\left\{J_{ks}\left(\boldsymbol{L}^{(1)}\right), J_{ks}\left(\boldsymbol{L}^{(2)}\right)\right\}$ for all $k, s$, is

$$\overline{J} \approx \begin{pmatrix} 0.832 & 0.393 \\ 0.393 & 0.832 \end{pmatrix}.$$

And we have $\rho\left(\overline{J}\right) = 1.225 > 1$.

# References


ALLEN, T., ARKOLAKIS, C. and LI, X. (2024). On the Equilibrium Properties of Spatial Models. *American Economic Review: Insights*, **6** (4), 472–89.