Present Bias and Discount Rate Risk

Online Appendix

October 26, 2025

1 Survey evidence: Additional results and robustness tests

In this section we report additional results and robustness tests complementing the empirical evidence on time-varying present bias documented in Section 3. Table 1 repeats the unconditional average tests of the forecast errors, but uses raw survey data that are not filtered for errors and outliers. We can see that the results are qualitatively similar to those after applying the filters. The consumption growth forecast error both in the full sample and before Covid is again about 2% on average. The regression coefficients on education and income are also positive and significant, which indicates that less sophisticated agents make larger forecast errors about their own consumption growth (education and income are coded into three brackets 0, 0.5, and 1, where 1 stands for less sophisticated individuals). Even though the coefficients on the indicator of not saving generally are larger than in the original regression, they are insignificant which could be attributed to the noise caused by extreme observations in the raw data.

Since income could change unexpectedly and lead to a higher than planned consumption, we next show that our results are robust to including income when identifying present-biased agents. Table 2, Column (1) reports the baseline estimation of the composite bias variable as a linear combination of education, income (both classified in three buckets 0, 0.5, and 1) and an indicator of not saving in general, while Column (2) excludes income. In Table 3 we show that our results

documenting the time variation in the present bias are not affected by the inclusion of income in the composite bias variable. In particular, the main coefficient of interest β_3 which shows whether the degree of the bias changes with economic condition is still positive and significant after excluding income. Hence, our finding that present-biased agents' overconsumption increases in times of stress is not determined by changes in their income.

Another potential concern could be that our results are affected by financially constrained individuals who may have hand to mouth consumption. To alleviate this concern, we repeat the analysis presented in Table 3 after excluding financially constrained agents (see the definition of financial constraints in Section 3). First, we can see that the least sophisticated financially unconstrained individuals make significant consumption growth forecast errors of about 6% per year, similar to those in the full sample. Second, these forecast errors are time-varying and increase significantly in times of stress, based on the interaction term coefficient β_3. These results are also robust to excluding income from the composite bias cohort variable.

Finally, we test whether our findings are driven by overall pessimism about aggregate quantities. In particular, in Table 5 we regress the agents' average reported unemployment pessimism, defined as the probability that unemployment rate in the U.S. will increase in the next 12 months (see Section 3 for details), on the composite bias variables including and excluding income, and the demographic characteristics reported in the survey. We see that individuals who are less sophisticated and more susceptible to the present bias believe there is a lower chance that unemployment will increase in the future, i.e. they are more optimistic, rather than more pessimistic about aggregate quantities. This alleviates the potential concern that these individuals might have an overall pessimistic economic outlook that drives the pessimism about their own consumption growth.

Table 1: Raw data: Consumption growth forecast errors and demographics

The table reports the difference between individuals' realized and expected consumption growth based on the raw, unfiltered New York Fed's Survey of Consumer Expectations data. Column (1) presents the average consumption growth forecast error based on the full sample period (December 2014 until December 2022), while Column (2) focuses on the Pre-Covid (pre-2020) period. Columns (3) and (4) report the estimates of a regression with time and state fixed effects of consumption growth forecast errors on demographic characteristics: education, income, age, risk tolerance (each classified in three normalized categories 0, 0.5 and 1, consistent with the New York Fed's categorization), and an indicator of not saving in general. The realized and expected consumption growth are trimmed at 1% by time. Individuals between 25 and 80 years old are included in the final sample of 12,579 unique individuals. The t-statistics, reported in brackets below, are based on Driscoll-Kraay standard errors using 5 lags. Significance at 10%, 5% and 1% is denoted by *, **, and ***, respectively.

	(1)	(2)	(3)	(4)
Education			4.369***	4.518***
			(4.558)	(4.275)
Income			2.544***	3.110**
			(2.908)	(2.431)
Not save generally			0.530	0.527
			(0.888)	(0.689)
Age				0.154
				(0.174)
Risk tolerance				2.908
				(0.998)
Constant	2.287***	2.855***		
	(5.033)	(7.002)		
Time and State FE	N	N	Y	Y
Driscoll-Kraay SE	Y	Y	Y	Y
Pre-Covid period	N	Y	N	N
R-Squared	0.000	0.000	0.004	0.004
N	27,734	17,764	25,135	23,408

Table 2: Bias estimation

The table reports the estimation of a composite bias variable. In Column (1) it is based on education, income, (each classified in three buckets 0, 0.5, and 1, consistent with the survey categories) and an indicator of not saving in general cohorts, while in Column (2) it is based on education and indicator of not saving in general. The expected and realized consumption growth, that along with demographic characteristics are available in the New York Fed's Survey of Consumer Expectations data for the sample period from December 2014 until December 2022. The realized and expected consumption growth are trimmed at 1% by time. Individuals between 25 and 80 years old are included in the final sample of 11,928 unique individuals. The t-statistics, reported in brackets below, are based on standard errors clustered by state and cohort bucket. Significance at 10%, 5% and 1% is denoted by *, **, and ***, respectively.

	Bias 1	**Bias 2**
	(1)	(2)
Education	4.376***	4.449***
	(8.351)	(9.118)
Income	1.162***	
	(2.702)	
Not save generally	0.636*	0.576
	(1.774)	(1.612)
Time and State FE	Y	Y
Clustered SE	Y	Y
R-Squared	0.048	0.061
N	5,557	3,571

Table 3: Time variation in consumption growth forecast errors by bias cohort
The table reports the estimates of a regression with time and state fixed effects of average consumption growth forecast errors based on bias buckets, a stress indicator, and the interaction between bias and stress. In Columns (1) and (2) the stress indicator is a state-level economic conditions indicator (as in Baumeister, Leiva-León, and Sims, 2021) and in Columns (3) and (4) – a state-level change in unemployment compared to the year before (available in FRED). In Columns (1) and (3) the bias is estimated based on education, income and an indicator of not saving in general, while in Columns (2) and (4) the bias is estimated based on education and the indicator of not saving in general. The expected and realized consumption growth along with demographic characteristics used to estimate the bias buckets are available in the New York Fed's Survey of Consumer Expectations data for the sample period from December 2014 until December 2022. The realized and expected consumption growth are trimmed at 1% by time and individuals between 25 and 80 years old are included in the sample of 11,928 unique individuals. The t-statistics, reported in brackets below, are based on clustered by cohort and state standard errors. Significance at 10%, 5% and 1% is denoted by *, **, and ***, respectively.

	Change in unemployment		**Economic conditions**	
	Bias 1	Bias 2	Bias 1	Bias 2
	(1)	(2)	(3)	(4)
Bias	6.250***	5.084***	6.283***	5.084***
	(10.809)	(10.581)	(10.795)	(10.557)
Stress indicator	-0.348	-0.452*	-0.422	-0.496
	(-1.504)	(-1.918)	(-1.360)	(-1.455)
Bias x Stress indicator	0.775**	0.572*	0.896**	0.919***
	(2.134)	(1.942)	(2.482)	(2.870)
Time and State FE	Y	Y	Y	Y
Clustered SE	Y	Y	Y	Y
R-Squared	0.049	0.063	0.050	0.064
N	5,557	3,571	5,549	3,548

Table 4: Financially unconstrained sample: Time variation in consumption growth forecast errors

The table focuses on a subsample of financially unconstrained consumers. Detailed definition of the financially constrained consumers indicator is provided in Section 3. The table reports the estimates of a regression with time and state fixed effects of average consumption growth forecast errors based on bias buckets, a stress indicator, and the interaction between bias and stress. In Columns (1) and (2) the stress indicator is a state-level economic conditions indicator (as in Baumeister, Leiva-León, and Sims, 2021) and in Columns (3) and (4) – a state-level change in unemployment compared to the year before (available in FRED). In Columns (1) and (3) the bias is estimated based on education, income and an indicator of not saving in general, while in Columns (2) and (4) the bias is estimated based on education and the indicator of not saving in general. The expected and realized consumption growth along with demographic characteristics used to estimate the bias buckets are available in the New York Fed's Survey of Consumer Expectations data for the sample period from December 2014 until December 2022. The realized and expected consumption growth are trimmed at 1% by time and individuals between 25 and 80 years old are included in the final sample of 7,546 unique financially unconstrained individuals. The t-statistics, reported in brackets below, are based on clustered by cohort and state standard errors. Significance at 10%, 5% and 1% is denoted by *, **, and ***, respectively.

	Change in unemployment		**Economic conditions**	
	Bias 1	Bias 2	Bias 1	Bias 2
	(1)	(2)	(3)	(4)
Bias	6.206***	4.705***	6.294***	4.687***
	(8.070)	(6.320)	(8.124)	(6.373)
Stress indicator	-0.188	-0.240	0.099	-0.300
	(-0.565)	(-0.725)	(0.239)	(-0.681)
Bias × Stress indicator	1.428***	0.956**	1.306***	1.149***
	(2.980)	(2.382)	(2.769)	(2.976)
Time and State FE	Y	Y	Y	Y
Clustered SE	Y	Y	Y	Y
R-Squared	0.064	0.065	0.063	0.065
N	2,990	2,327	2,987	2,319

Table 5: Unemployment rate pessimism and demographics

The table reports the degree of unemployment rate pessimism based on the New York Fed's Survey of Consumer Expectations data. The dependent variable is the average reported unemployment pessimism, defined as the percentage chance that unemployment rate in the U.S. will be higher in the next 12 months based on the full sample period (December 2014 until December 2022). Column (1) reports the estimates of a regression with time and state fixed effects of unemployment pessimism on the composite bias variable based on education, income, (each classified in three buckets 0, 0.5, and 1, consistent with the New York Fed's categorization) and an indicator of not saving in general cohorts. Individuals with the lowest level of education and income are assigned the value of 1. In Column (2) the bias variable is based on education and an indicator of not saving in general cohorts. Columns (3) and (4) report the estimates of a regression with time and state fixed effects of unemployment pessimism on demographic characteristics: education, income, age, risk tolerance (each classified in three normalized categories 0, 0.5 and 1), and an indicator of not saving in general. The realized and expected consumption growth are trimmed at 1% by time. Individuals between 25 and 80 years old are included in the final sample of 11,928 unique individuals. The t-statistics, reported in brackets below, are based on Driscoll-Kraay standard errors using 5 lags. Significance at 10%, 5% and 1% is denoted by *, **, and ***, respectively.

	(1)	(2)	(3)	(4)
Bias 1	-4.777***			
	(-6.857)			
Bias 2		-4.906***		
		(-3.922)		
Education			-4.643***	-4.675***
			(-4.199)	(-4.248)
Income			0.386	0.463
			(0.495)	(0.542)
Not save generally			-0.355	-0.085
			(-0.896)	(-0.210)
Age				-1.809***
				(-2.591)
Risk tolerance				-0.490
				(-0.530)
Constant				
Time and state FE	Y	Y	Y	Y
Driscoll-Kraay SE	Y	Y	Y	Y
R-squared	0.026	0.027	0.027	0.028
N	23,386	23,586	23,386	22,521

2 Time-varying β: Log utility

2.1 General solution

2.1.1 TV value function

The problem of an agent with time-varying discounting β_t^{TV} is given by:

$$U_t^{TV}(W_t, \beta_t^{TV}) = \max_{C_t, \omega_t} \quad \log C_t + \beta_t^{TV} \mathbb{E}_t[U_{t+1}^{TV}(W_{t+1}, \beta_{t+1}^{TV})], \tag{1}$$

subject to $W_{t+1} = (W_t - C_t)(R_{f,t} + \omega_t^\top R_{t+1}^e)$. Let $C_{TV,t}$ denote the optimized consumption, $\phi_t^{TV} \equiv \frac{W_t}{C_{TV,t}}$ denote the optimal wealth-consumption ratio, and $R_{TV,t+1} \equiv R_{f,t} + \omega_t^{*\top} R_{t+1}^e$ denote the optimized portfolio return. Iterating (1) forward yields

$$\begin{aligned}
U_t^{TV}(W_t, \beta_t^{TV}) &= \max_{C_t, \omega_t} \quad \log C_t + \beta_t^{TV} \mathbb{E}_t \sum_{j=1}^{\infty} \prod_{i=1}^{j-1} \beta_{t+i}^{TV} \log C_{TV,t+j} \\
&= \log C_{TV,t} + \beta_t^{TV} \mathbb{E}_t \sum_{j=1}^{\infty} \prod_{i=1}^{j-1} \beta_{t+i}^{TV} \log \Big(\frac{W_{t+j}}{\phi_{t+j}^{TV}} \Big) \\
&= \log \Big(\frac{W_t}{\phi_t^{TV}} \Big) + \beta_t^{TV} \mathbb{E}_t \sum_{j=1}^{\infty} \Big(\prod_{i=1}^{j-1} \beta_{t+i}^{TV} \Big) \log \left(\frac{W_t \prod_{i=0}^{j-1} \frac{\phi_{t+i}^{TV} - 1}{\phi_{t+i}^{TV}} R_{TV,t+1+i}}{\phi_{t+j}^{TV}} \right) \\
&\equiv \Big[1 + \mathbb{E}_t \sum_{j=1}^{\infty} \prod_{i=0}^{j-1} \beta_{t+i}^{TV} \Big] \log W_t + A_t^{TV}.
\end{aligned} \tag{2}$$

Suppose β_t^{TV} follows a Markov process with states $\hat{\beta}^{TV}$ and transition probability matrix Π_β. Then $\mathbb{E}(\beta_{t+1}^{TV} | \beta_t^{TV} = \hat{\beta}^{TV}) = \Pi_\beta \hat{\beta}^{TV}$, $\mathbb{E}(\beta_t^{TV} \beta_{t+1}^{TV} | \beta_t^{TV} = \hat{\beta}^{TV}) = D(\hat{\beta}^{TV}) \Pi_\beta \hat{\beta}^{TV}$, and

$$\begin{aligned}
K^{TV}(\hat{\beta}^{TV}) &\equiv \mathbf{1} + \sum_{j=1}^{\infty} \mathbb{E}_t \Big[\prod_{i=0}^{j-1} \beta_{t+i}^{TV} \Big| \beta_t^{TV} = \hat{\beta}^{TV} \Big] = \mathbf{1} + \sum_{j=1}^{\infty} [D(\hat{\beta}^{TV}) \Pi_\beta]^{j-1} \hat{\beta}^{TV} \\
&= \mathbf{1} + \Big(I - D(\hat{\beta}^{TV}) \Pi_\beta \Big)^{-1} \hat{\beta}^{TV} = \Big(I - D(\hat{\beta}^{TV}) \Pi_\beta \Big)^{-1} \Big[\Big(I - D(\hat{\beta}^{TV}) \Pi_\beta \Big) \mathbf{1} + \hat{\beta}^{TV} \Big] \\
&= \Big(I - D(\hat{\beta}^{TV}) \Pi_\beta \Big)^{-1} \mathbf{1},
\end{aligned} \tag{3}$$

where the last equality used that $D(\hat{\beta}^{TV})\Pi_\beta \mathbf{1} = D(\hat{\beta}^{TV})\mathbf{1} = \hat{\beta}^{TV}$. Using (3) and (2) we get

$$U_t^{TV}(W_t, \beta_t^{TV}) \;=\; K^{TV}(\beta_t^{TV}) \log W_t + A_t^{TV}. \tag{4}$$

2.1.2 Optimal wealth-consumption ratios and SDF

Using (4) we can rewrite the TV agent's problem (1) as follows:

$$\begin{aligned} U_t^{TV}(W_t, \beta_t^{TV}) &= \max_{C_t, \omega_t} \quad \log C_t + \beta_t^{TV} \mathbb{E}_t[K^{TV}(\beta_{t+1}^{TV}) \log W_{t+1} + A_{t+1}^{TV}] \\ &= \max_{C_t, \omega_t} \quad \log C_t + \log(W_t - C_t) \beta_t^{TV} \mathbb{E}_t[K^{TV}(\beta_{t+1}^{TV})] \\ &+ \quad \beta_t^{TV} \mathbb{E}_t[K^{TV}(\beta_{t+1}^{TV}) \log(R_{f,t} + \omega_t^\top R_{t+1}^e) + A_{t+1}^{TV}]. \end{aligned} \tag{5}$$

The first-order condition for consumption is

$$\phi^{TV}(\beta_t^{TV}) \;\equiv\; \frac{W_t}{C_{TV,t}} = 1 + \beta_t^{TV} \mathbb{E}_t[K^{TV}(\beta_{t+1}^{TV})]. \tag{6}$$

Or, in vectorized form:

$$\begin{aligned} \phi^{TV}(\hat{\beta}^{TV}) &= \mathbf{1} + D(\hat{\beta}^{TV})\Pi_\beta K^{TV}(\hat{\beta}^{TV}) = \mathbf{1} + D(\hat{\beta}^{TV})\Pi_\beta \Big(I - D(\hat{\beta}^{TV})\Pi_\beta\Big)^{-1} \mathbf{1} \\ &= \Big(I - D(\hat{\beta}^{TV})\Pi_\beta\Big)\Big(I - D(\hat{\beta}^{TV})\Pi_\beta\Big)^{-1} \mathbf{1} + D(\hat{\beta}^{TV})\Pi_\beta \Big(I - D(\hat{\beta}^{TV})\Pi_\beta\Big)^{-1} \mathbf{1} \\ &= \Big[\Big(I - D(\hat{\beta}^{TV})\Pi_\beta\Big) + D(\hat{\beta}^{TV})\Pi_\beta\Big]\Big(I - D(\hat{\beta}^{TV})\Pi_\beta\Big)^{-1} \mathbf{1} \\ &= \Big(I - D(\hat{\beta}^{TV})\Pi_\beta\Big)^{-1} \mathbf{1} = K^{TV}(\hat{\beta}^{TV}). \end{aligned} \tag{7}$$

The first-order condition w.r.t. portfolio weights yields

$$0 \;=\; \mathbb{E}_t\Big[\phi^{TV}(\beta_{t+1}^{TV}) R_{TV,t+1}^{-1} R_{t+1}^e\Big]. \tag{8}$$

Using (6) and (8) gives us the following expression for the TV SDF:

$$M_{t+1}^{TV} \;=\; \beta_t^{TV} \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV}) - 1} R_{TV,t+1}^{-1}, \tag{9}$$

or, alternatively

$$\begin{aligned} M_{t+1}^{TV} &= \beta_t^{TV} \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})-1}\Big(\frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})-1}\frac{C_{TV,t+1}}{C_{TV,t}}\Big)^{-1} \\ &= \beta_t^{TV}\frac{C_{TV,t}}{C_{TV,t+1}}. \end{aligned} \tag{10}$$

2.2 Representative agent economy

In an economy populated by a unit mass continuum of TV agents, we would have $C_{TV,t} = C_t$, where the latter denotes aggregate consumption endowment. Thus, if aggregate consumption is constant, the SDF in this economy would be:

$$M_{t+1}^{TV} = \beta_t^{TV}\frac{C_t}{C_{t+1}} = \beta_t^{TV}. \tag{11}$$

In other words, we would have risk-neutral pricing in this economy.

It is clear from (7) that

$$\begin{aligned} \phi^{TI}(\hat{\theta}) &= \phi^{TV}(\hat{\beta}^{TV}) \Leftrightarrow \mathbf{1} = \Big(I - D(\hat{\beta}^{TV})\Pi_\beta\Big)\phi^{TI}(\hat{\theta}) = \phi^{TI}(\hat{\theta}) - D\left(\Pi_\beta\phi^{TI}(\hat{\theta})\right)\hat{\beta}^{TV} \Leftrightarrow \\ \hat{\beta}^{TV} &= D\left(\Pi_\beta\phi^{TI}(\hat{\theta})\right)^{-1}\Big(\phi^{TI}(\hat{\theta}) - \mathbf{1}\Big). \end{aligned} \tag{12}$$

If we choose $\Pi_\beta = \Pi$ in (12), we see that there exists a choice of state-vector $\hat{\beta}^{TV}$ such that the distribution of TV wealth-consumption ratios is identical to that of a TI agent. As is clear in the representative agent versions of the TI and TV economies: identically distributed wealth-consumption ratios does not result in identical asset pricing implications. In the TI economy we get a positive risk-premium on the consumption claim even when aggregate consumption is constant, whereas in the TV economy we get risk-neutral pricing.

2.3 Heterogeneous agents economy

Let $s_t \equiv \frac{W_{TV,t}}{W_t}$ denote the share of aggregate wealth W_t that belongs to TV agents and $1 - s_t$ be the share of aggregate wealth belonging to constant β agents, which we will still refer to as TC

for convenience. As in the previous section, the TC agent's pricing kernel is given by:

$$M_{t+1}^{TC} = \frac{1}{R_{TC,t+1}} = \beta \frac{C_t}{C_{t+1}} \frac{1-s_t}{1-s_{t+1}} \frac{\phi_t}{\phi_{t+1}}, \tag{13}$$

and from (9) we can write the TV agent's pricing kernel as

$$\begin{aligned} M_{t+1}^{TV} &= \beta_t^{TV} \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})-1} (R_{t+1}^{TV})^{-1} = \beta_t^{TV} \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})-1} \Big(\frac{s_{t+1}W_{t+1}}{\frac{\phi^{TV}(\beta_t^{TV})-1}{\phi^{TV}(\beta_t^{TV})} s_t W_t} \Big)^{-1} \\ &= \beta_t^{TV} \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})} \frac{s_t}{s_{t+1}} \frac{W_t}{W_{t+1}} = \beta_t^{TV} \frac{C_t}{C_{t+1}} \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})} \frac{s_t}{s_{t+1}} \frac{\phi_t}{\phi_{t+1}}. \end{aligned} \tag{14}$$

The wealth-share of TV agents evolves as follows:

$$s_{t+1} = s_t \frac{\phi^{TV}(\beta_t^{TV})-1}{\phi^{TV}(\beta_t^{TV})} \frac{\phi(s_t, \beta_t^{TV})}{\phi(s_t, \beta_t^{TV})-1} \frac{R_{TV,t+1}}{R_{C,t+1}}. \tag{15}$$

The aggregate wealth-consumption ratio is

$$\phi(s_t, \beta_t^{TV}) = \phi^{TV}(\beta_t^{TV}) \phi^{TC} \left(s_t \phi^{TC} + (1-s_t) \phi^{TV}(\beta_t^{TV}) \right)^{-1}. \tag{16}$$

2.3.1 Complete markets

In equilibrium with complete markets, the two SDFs are equal state-by-state. Hence,

$$\begin{aligned} M_{t+1}^{TC} &= M_{t+1}^{TV} \Leftrightarrow \beta \frac{1-s_t}{1-s_{t+1}} = \beta_t^{TV} \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})} \frac{s_t}{s_{t+1}} \Leftrightarrow \\ \beta(1-s_t)s_{t+1} &= \beta_t^{TV} s_t \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})} (1-s_{t+1}) \Leftrightarrow \\ s_{t+1} &= \frac{s_t \beta_t^{TV} \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})}}{(1-s_t)\beta + s_t \beta_t^{TV} \frac{\phi^{TV}(\beta_{t+1}^{TV})}{\phi^{TV}(\beta_t^{TV})}} \\ &= \frac{s_t \beta_t^{TV} \phi^{TV}(\beta_{t+1}^{TV})}{(1-s_t)\beta \phi^{TV}(\beta_t^{TV}) + s_t \beta_t^{TV} \phi^{TV}(\beta_{t+1}^{TV})}. \end{aligned} \tag{17}$$

Thus, the TC agents' wealth-share is given by:

$$1 - s_{t+1} = \frac{(1-s_t)\beta\phi^{TV}(\beta_t^{TV})}{(1-s_t)\beta\phi^{TV}(\beta_t^{TV}) + s_t\beta_t^{TV}\phi^{TV}(\beta_{t+1}^{TV})}. \tag{18}$$

Using (17) and (18) in (16) at $t+1$ we can express the aggregate wealth-consumption ratio next period as follows:

$$\begin{aligned}
\tilde{\phi}(\beta_{t+1}^{TV}; s_t, \beta_t^{TV}) &\equiv \phi(s_{t+1}, \beta_{t+1}^{TV}) = \phi^{TC}\phi^{TV}(\beta_{t+1}^{TV})\left(s_{t+1}\phi^{TC} + (1-s_{t+1})\phi^{TV}(\beta_{t+1}^{TV})\right)^{-1} \\
&= \phi^{TC}\phi^{TV}(\beta_{t+1}^{TV})\left(\frac{s_t\beta_t^{TV}\phi^{TV}(\beta_{t+1}^{TV})\phi^{TC} + (1-s_t)\beta\phi^{TV}(\beta_t^{TV})\phi^{TV}(\beta_{t+1}^{TV})}{(1-s_t)\beta\phi^{TV}(\beta_t^{TV}) + s_t\beta_t^{TV}\phi^{TV}(\beta_{t+1}^{TV})}\right)^{-1} \\
&= \phi^{TC}\frac{(1-s_t)\beta\phi^{TV}(\beta_t^{TV}) + s_t\beta_t^{TV}\phi^{TV}(\beta_{t+1}^{TV})}{(1-s_t)\beta\phi^{TV}(\beta_t^{TV}) + s_t\beta_t^{TV}\phi^{TC}}.
\end{aligned} \tag{19}$$

We immediately see that the aggregate wealth-consumption ratio next period is increasing in the TV agent's wealth-consumption ratio next period.

Substituting (16), (17), and (19) into (14), we get the following expression of the equilibrium stochastic discount factor:

$$M_{t+1} = \beta\frac{C_t}{C_{t+1}}\frac{\beta^{-1}\beta_t^{TV}s_t\phi^{TC} + (1-s_t)\phi^{TV}(\beta_t^{TV})}{s_t\phi^{TC} + (1-s_t)\phi^{TV}(\beta_t^{TV})}. \tag{20}$$

We therefore immediately see that the equilibrium stochastic discount factor does not depend on the wealth-consumption ratio $\phi^{TV}(\beta_{t+1}^{TV})$ of the TV agent at $t+1$. Hence, we get risk-neutral pricing if aggregate consumption is constant.

3 Robustness to alternative specifications: Time-varying δ

3.1 General solution

Suppose the problem of a TI agent instead takes the following form:

$$U_t^{TI}(W_t, \delta_t) = \max_{C_t, \omega_t} \log C_t + \beta \delta_t \mathbb{E}_t \Big[\theta U_{t+1}^{TI}(W_{t+1}, \delta_{t+1}) + (1-\theta) U_{t+1}^{TC}(W_{t+1}) \Big], \tag{21}$$

subject to $W_{t+1} = (W_t - C_t)(R_{f,t} + \omega_t^\top R_{t+1}^e)$. In other words, now we assume the subjective probability of remaining TI is constant, but instead the extra discounting δ is assumed to be time-varying. As before, the TC value function takes the form

$$U_t^{TC}(W_t) = \phi^{TC} \log W_t + A_t^{TC}, \tag{22}$$

where $\phi^{TC} \equiv \frac{1}{1-\beta}$ denotes the optimal wealth-consumption ratio of a TC agent.

Iterating (21) forward, letting ϕ_t^{TI} and $R_{TI,t+1}$ denote optimized wealth-consumption ratios and portfolio returns, we get:

$$\begin{aligned}
U_t^{TI}(W_t, \delta_t) &= \mathbb{E}_t \Big[\sum_{j=0}^{\infty} \Big(\prod_{i=0}^{j-1} \delta_{t+i} \Big) (\theta\beta)^j \log \frac{W_{t+j}}{\phi_{t+j}^{TI}} + \sum_{j=1}^{\infty} \Big(\prod_{i=0}^{j-1} \delta_{t+i} \Big) \beta^j \theta^{j-1} (1-\theta) U_{t+j}^{TC}(W_{t+j}) \Big] \\
&= \mathbb{E}_t \Big[\sum_{j=0}^{\infty} \Big(\prod_{i=0}^{j-1} \delta_{t+i} \Big) (\theta\beta)^j \log \frac{W_t \prod_{i=0}^{j-1} \frac{\phi_{t+i}^{TI}-1}{\phi_{t+i}^{TI}} R_{TI,t+1+i}}{\phi_{t+j}^{TI}} \\
&+ \sum_{j=1}^{\infty} \Big(\prod_{i=0}^{j-1} \delta_{t+i} \Big) \beta^j \theta^{j-1} (1-\theta) \Big(\phi^{TC} \log W_{t+j} + A_{t+j}^{TC} \Big) \Big] \\
&\equiv \mathbb{E}_t \Big[1 + \sum_{j=1}^{\infty} \Big(\prod_{i=0}^{j-1} \delta_{t+i} \Big) \Big((\theta\beta)^j + \beta^j \theta^{j-1} (1-\theta) \phi^{TC} \Big) \Big] \log W_t \\
&+ A_t^{TI}(\delta_t).
\end{aligned} \tag{23}$$

Assuming δ_t is a Markov process with K states $\hat{\delta}$ and a $K \times K$ transition probability matrix Π_δ, the conditional expectation $\mathbb{E}_t \prod_{i=0}^{j-1} \delta_{t+i}$ can take K values. In particular, $\mathbb{E}(\delta_{t+1} | \delta_t =$

$\hat{\delta}) = \Pi_\delta \hat{\delta}$ is the $K \times 1$ vector of conditional expectations. Furthermore, $\mathbb{E}(\delta_t \delta_{t+1} | \delta_t = \hat{\delta}) = D(\hat{\delta})\mathbb{E}(\delta_{t+1}|\delta_t = \hat{\delta}) = D(\hat{\delta})\Pi_\delta \hat{\delta}$ and $\mathbb{E}(\delta_t \delta_{t+1}\delta_{t+2}|\delta_t = \hat{\delta}) = D(\hat{\delta})\mathbb{E}(\mathbb{E}(\delta_{t+1}\delta_{t+2}|\delta_{t+1} = \hat{\delta})|\delta_t = \hat{\delta}) = D(\hat{\delta})\Pi_\delta \mathbb{E}(\delta_{t+1}\delta_{t+2}) = D(\hat{\delta})\Pi D(\hat{\delta})\Pi_\delta \hat{\delta}$. Thus,

$$\begin{aligned}
\mathbb{E}\Big(\prod_{i=0}^{j-1} \delta_{t+i} \Big| \delta_t = \hat{\delta}\Big) &= \Big(\prod_{i=1}^{j-1} D(\hat{\delta})\Pi_\delta\Big)\hat{\delta} = \Big(D(\hat{\delta})\Pi_\delta\Big)^{j-1}\hat{\delta} \qquad (24) \\
K^{TI}(\hat{\delta}) &\equiv \mathbf{1} + \mathbb{E}\Big[\sum_{j=1}^{\infty}\Big(\prod_{i=0}^{j-1}\delta_{t+i}\Big)(\beta\theta)^j\Big(1+\theta^{-1}(1-\theta)\phi^{TC}\Big)\Big|\delta_t = \hat{\delta}\Big] \\
&= \mathbf{1} + \beta\theta\Big(1+\theta^{-1}(1-\theta)\phi^{TC}\Big)\sum_{j=1}^{\infty}\Big(\beta\theta D(\hat{\delta})\Pi_\delta\Big)^{j-1}\hat{\delta} \\
&= \mathbf{1} + \beta\theta\Big(1+\theta^{-1}(1-\theta)\phi^{TC}\Big)\Big(I - \beta\theta D(\hat{\delta})\Pi_\delta\Big)^{-1}\hat{\delta} \\
&= \Big(I - \beta\theta D(\hat{\delta})\Pi_\delta\Big)^{-1}\Big[\Big(I - \beta\theta D(\hat{\delta})\Pi_\delta\Big)\mathbf{1} + \beta\theta\Big(1+\theta^{-1}(1-\theta)\phi^{TC}\Big)\hat{\delta}\Big] \\
&= \Big(I - \beta\theta D(\hat{\delta})\Pi_\delta\Big)^{-1}\Big[\mathbf{1} - \beta\theta\hat{\delta} + \beta\theta\Big(1+\theta^{-1}(1-\theta)\phi^{TC}\Big)\hat{\delta}\Big] \\
&= \Big(I - \theta\beta D(\hat{\delta})\Pi_\delta\Big)^{-1}\Big(\mathbf{1} + (1-\theta)\beta\hat{\delta}\phi^{TC}\Big). \qquad (25)
\end{aligned}$$

We can now re-write the problem in (21) as follows

$$\begin{aligned}
U_t^{TI}(W_t, \delta_t) &= \max_{C_t, \omega_t} \quad \log C_t + \log(W_t - C_t)\beta\delta_t\mathbb{E}_t\Big[\theta K^{TI}(\delta_{t+1}) + (1-\theta)\phi^{TC}\Big] \\
&+ \beta\delta_t\mathbb{E}_t\Big[\Big(\theta K^{TI}(\delta_{t+1}) + (1-\theta)\phi^{TC}\Big)\log(R_{f,t} + \omega_t^\top R_{t+1}^e) \\
&+ \theta A_{t+1}^{TI}(\delta_{t+1}) + (1-\theta)A_{t+1}^{TC}\Big]. \qquad (26)
\end{aligned}$$

The first-order condition w.r.t. consumption is

$$\phi^{TI}(\delta_t) \equiv \frac{W_t}{C_t^*} = 1 + \beta\delta_t\mathbb{E}_t\Big[\theta K^{TI}(\delta_{t+1}) + (1-\theta)\phi^{TC}\Big], \qquad (27)$$

or, equivalently in vectorized form:

$$\begin{aligned}
\phi^{TI}(\hat{\delta}) &= \mathbf{1} + \beta D(\hat{\delta})\Big[\theta\Pi_\delta K^{TI}(\hat{\delta}) + (1-\theta)\mathbf{1}\phi^{TC}\Big] \\
&= \mathbf{1} + \beta D(\hat{\delta})\Big[\theta\Pi_\delta\Big(I - \theta\beta D(\hat{\delta})\Pi_\delta\Big)^{-1}\Big(\mathbf{1} + (1-\theta)\beta\hat{\delta}\phi^{TC}\Big) + (1-\theta)\mathbf{1}\phi^{TC}\Big] \\
, &= \Big(\mathbf{1} + (1-\theta)\beta\hat{\delta}\phi^{TC}\Big) + \theta\beta D(\hat{\delta})\Pi_\delta\Big(I - \theta\beta D(\hat{\delta})\Pi_\delta\Big)^{-1}\Big(\mathbf{1} + (1-\theta)\beta\hat{\delta}\phi^{TC}\Big) \\
&= \Big[I + \theta\beta D(\hat{\delta})\Pi_\delta\Big(I - \theta\beta D(\hat{\delta})\Pi_\delta\Big)^{-1}\Big]\Big(\mathbf{1} + (1-\theta)\beta\hat{\delta}\phi^{TC}\Big) \\
&= \Big[\Big(I - \theta\beta D(\hat{\delta})\Pi_\delta\Big) + \theta\beta D(\hat{\delta})\Pi_\delta\Big]\Big(I - \theta\beta D(\hat{\delta})\Pi_\delta\Big)^{-1}\Big(\mathbf{1} + (1-\theta)\beta\hat{\delta}\phi^{TC}\Big) \\
&= \Big(I - \theta\beta D(\hat{\delta})\Pi_\delta\Big)^{-1}\Big(\mathbf{1} + (1-\theta)\beta\hat{\delta}\phi^{TC}\Big) = K^{TI}(\hat{\delta}). \qquad (28)
\end{aligned}$$

We can therefore express the value function as follows

$$U_t^{TI}(W_t, \delta_t) = \phi^{TI}(\delta_t)\log W_t + A_t^{TI}(\delta_t). \qquad (29)$$

The first-order conditions w.r.t. portfolio weights are

$$0 = \mathbb{E}_t\Big[\Big(\theta\phi^{TI}(\delta_{t+1}) + (1-\theta)\phi^{TC}\Big)R_{TI,t+1}^{-1}R_{t+1}^e\Big]. \qquad (30)$$

Combining (27) and (30) gives us the following expression for the TI SDF:

$$M_{t+1}^{TI} = \beta\delta_t\Big(\theta\frac{\phi^{TI}(\delta_{t+1})}{\phi^{TI}(\delta_t) - 1} + (1-\theta)\frac{\phi^{TC}}{\phi^{TI}(\delta_t) - 1}\Big)R_{TI,t+1}^{-1}. \qquad (31)$$

3.2 Continuum of infinitesimal representative TI agents

If we assume that the economy is populated by a unit mass continuum of identical TI agents, then aggregate wealth-consumption ratios must equal individual wealth-consumption ratios and individuals must find it optimal to simply hold the aggregate consumption claim. In other words:

$\phi_{t+1} = \phi^{TI}(\delta_{t+1})$ and $R_{C,t+1} = R_{TI,t+1}$. We can therefore express the SDF in (31) as follows:

$$\begin{aligned} M_{t+1}^{TI} &= \beta\delta_t\Big(\theta\frac{\phi^{TI}(\delta_{t+1})}{\phi^{TI}(\delta_t)-1} + (1-\theta)\frac{\phi^{TC}}{\phi^{TI}(\delta_t)-1}\Big)\Big(\frac{\phi_{t+1}C_{t+1}}{(\phi_t-1)C_t}\Big)^{-1} \\ &= \beta\delta_t\Big(\theta + (1-\theta)\frac{\phi^{TC}}{\phi^{TI}(\delta_{t+1})}\Big)\frac{C_t}{C_{t+1}}. \end{aligned} \tag{32}$$

With constant aggregate consumption, the SDF becomes

$$M_{t+1}^{TI} = \beta\delta_t\Big(\theta + (1-\theta)\frac{\phi^{TC}}{\phi^{TI}(\delta_{t+1})}\Big). \tag{33}$$

In other words, the SDF is low when wealth-consumption ratios are high, causing a positive risk-premium on the consumption claim even when aggregate consumption is constant. Note that there exists a choice of $\hat{\delta}$ that ensures the wealth-consumption ratios are identically distributed in the time-varying θ and time-varying δ economies:

$$\begin{aligned} \phi_\delta^{TI} = \phi_\theta^{TI} &\Leftrightarrow \Big(I - \theta\beta D(\hat{\delta})\Pi_\delta\Big)^{-1}\Big(\mathbf{1} + (1-\theta)\beta\hat{\delta}\phi^{TC}\Big) = \phi_\theta^{TI} \Leftrightarrow \\ \Big(\mathbf{1} + (1-\theta)\beta\hat{\delta}\phi^{TC}\Big) &= \Big(I - \theta\beta D(\hat{\delta})\Pi_\delta\Big)\phi_\theta^{TI} \Leftrightarrow \\ \phi_\theta^{TI} - \mathbf{1} &= D(\hat{\delta})\left(\mathbf{1}(1-\theta)\beta\phi^{TC} + \theta\beta\Pi_\delta\phi_\theta^{TI}\right) \Leftrightarrow \\ \hat{\delta} &= D\left(\mathbf{1}(1-\theta)\beta\phi^{TC} + \theta\beta\Pi_\delta\phi_\theta^{TI}\right)^{-1}\left(\phi_\theta^{TI} - \mathbf{1}\right). \end{aligned} \tag{34}$$

4 Partial market participation by TI agent

One concern might be that TI agents do not actively trade in the stock market. In this section we therefore consider the impact of assuming TI agents only participate in trading *some* rather than *all* assets in a very simple version of the model. Assume that the agents have log-utility. Furthermore, the TI agents can trade only in default free bonds in zero net supply whereas the TC agents are actively engaged in all markets. The portfolio returns of TI and TC agents can be written

$$\begin{aligned} R_{TI,t+1} &= R_{f,t} + \varphi_{TI,t}^{\top}(R_{B,t+1} - R_{f,t}) && (35) \\ R_{TI,t+1} &= R_{f,t} + \varphi_{TC,t}^{\top} R_{t+1}^{e}, && (36) \end{aligned}$$

where $R_{B,t+1}$ is the vector of returns on the bonds traded by the TI agent. Market clearing requires that: a) TI and TC consumption together make up aggregate consumption, b) TI and TC savings in total hold the aggregate consumption claim. We already know that the TC wealth-consumption ratio, ϕ^{TC}, is a constant, while the TI wealth-consumption ratio, $\phi^{TI}(\hat{\theta})$, depends solely on current θ. TI, and TC agents' savings are given by

$$\begin{aligned} W_{TI,t} - C_{TI,t} &= \frac{\phi^{TI}(\theta_t) - 1}{\phi^{TI}(\theta_t)} s_t W_t && (37) \\ W_{TC,t} - C_{TC,t} &= \frac{\phi^{TC} - 1}{\phi^{TC}} (1 - s_t) W_t, && (38) \end{aligned}$$

where s_t is the TI agents wealth share. By market clearing in the asset markets we have $(W_t - C_t)R_{C,t+1} = (W_{TI,t} - C_{TI,t})R_{TI,t+1} + (W_{TC,t} - C_{TC,t})R_{TC,t+1}$. Thus,

$$\begin{aligned} R_{TC,t+1} &= \frac{W_t - C_t}{W_{TC,t} - C_{TC,t}} R_{C,t+1} - \frac{W_{TI,t} - C_{TI,t}}{W_{TC,t} - C_{TC,t}} R_{TI,t+1} = R_{C,t+1} + \frac{W_{TI,t} - C_{TI,t}}{W_{TC,t} - C_{TC,t}} \Big(R_{C,t+1} - R_{TI,t+1} \Big) \\ &= R_{C,t+1} + \frac{\phi^{TC}}{\phi^{TC} - 1} \frac{\phi^{TI}(\theta_t) - 1}{\phi^{TI}(\theta_t)} \frac{s_t}{1 - s_t} (R_{C,t+1} - R_{TI,t+1}). \end{aligned} \quad (39)$$

In other words, the TC agents' portfolios uses a portfolio of bonds to lever up their position in the consumption claim. The amount of leverage is governed by $\frac{\phi^{TC}}{\phi^{TC}-1} \frac{\phi^{TI}(\theta_t)-1}{\phi^{TI}(\theta_t)} \frac{s_t}{1-s_t}$. Clearly, TC

leverage is high if either: 1) $\phi^{TI}(\theta_t)$ is high, i.e. θ_t is low, or 2) TI wealth-share s_t is high.

By market clearing for consumption we get

$$\begin{aligned} C_t &= C_{TI,t} + C_{TC,t} = \Big(\frac{s_t}{\phi^{TI}(\theta_t)} + \frac{1-s_t}{\phi^{TC}}\Big) W_t \Leftrightarrow \\ \phi(s_t,\theta_t) &\equiv \frac{W_t}{C_t} = \Big(\frac{s_t}{\phi^{TI}(\theta_t)} + \frac{1-s_t}{\phi^{TC}}\Big)^{-1}. \end{aligned} \tag{40}$$

In other words, the aggregate wealth-consumption ratio is increasing in the TI wealth-consumption ratio and therefore decreasing in θ_t. Since the return on the consumption claim is $R_{C,t+1} = \frac{\phi(s_{t+1},\theta_{t+1})}{\phi(s_t,\theta_t)-1}\frac{C_{t+1}}{C_t}$, we have that the consumption-claim return will be decreasing in θ_{t+1}. The pricing kernels are given by:

$$M_{TI,t+1} = \beta\delta \frac{\theta_t \phi^{TI}(\theta_{t+1}) + (1-\theta_t)\phi^{TC}}{\phi^{TI}(\theta_t) - 1} R_{TI,t+1}^{-1} \tag{41}$$

$$M_{TC,t+1} = R_{TC,t+1}^{-1} = \Big(R_{C,t+1} + \frac{\phi^{TC}}{\phi^{TC}-1}\frac{\phi^{TI}(\theta_t)-1}{\phi^{TI}(\theta_t)}\frac{s_t}{1-s_t}(R_{C,t+1} - R_{TI,t+1})\Big)^{-1}, \tag{42}$$

where the TC pricing kernel prices all assets and the TI pricing kernel prices the bonds: $0 = \mathbb{E}_t(M_{TC,t+1}R^e_{t+1})$ and $0 = \mathbb{E}_t(M_{TI,t+1}R^e_{B,t+1})$.

To illustrate the impact of partial participation by the TI agents, we consider a very simple version of the model. First, we solve the model at an annual frequency and assume aggregate consumption is i.i.d. and binomial: up 4.5% or down 0.5% with equal probability. Second, we let

$$\hat{\theta} = \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix}, \qquad \Pi = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{pmatrix}. \tag{43}$$

Third, $\beta = 0.99$ and $\delta = 0.9$. Finally, we assume the TI agents only have access to two bonds: a 1-period and a 2-period bond, while the TC agent has access to all assets.

Figure 1 plots the risk-premium on the consumption claim (y-axis) for various levels of current TI wealth-share as functions of current θ (x-axis). The solid lines are from the economy where

TI agents participate only in the bond market, whereas the dashed lines are from an otherwise identical economy where both TI and TC agents trade in a complete set of financial assets. There are two key takeaways from Figure 1. First, the risk-premium on the consumption claim is always higher in the economy where TI agents participate only in the bond market than when TI agents trade in a complete set of financial assets. Second, the overall pattern is very similar in the two versions of the economy: the risk-premium is hump-shaped in θ.

The result that TI agents might affect the risk-premium on the consumption claim and cause it to be time-varying even when they do not trade it might seem puzzling at first. However, the mechanism is straight-forward. If the TI agents hold all their wealth in bonds, that implies (by market clearing) that TC agents must hold a levered position in the consumption claim. In times when θ is low TI agents wish to save a lot and TC agents will be highly levered thereby driving up the consumption claim price. At the same time, by market clearing in consumption, this must be the time when TC consumption is also high. However, when θ increases, TI agents save less and consume more, thereby reducing the amount of leverage available to the TC agents, which in turn both lowers TC consumption and drives down the price on the consumption claim. Thus, both TC consumption and the price-dividend ratio on the consumption claim is decreasing in θ (through the impact on TC leverage) resulting in priced θ risk.

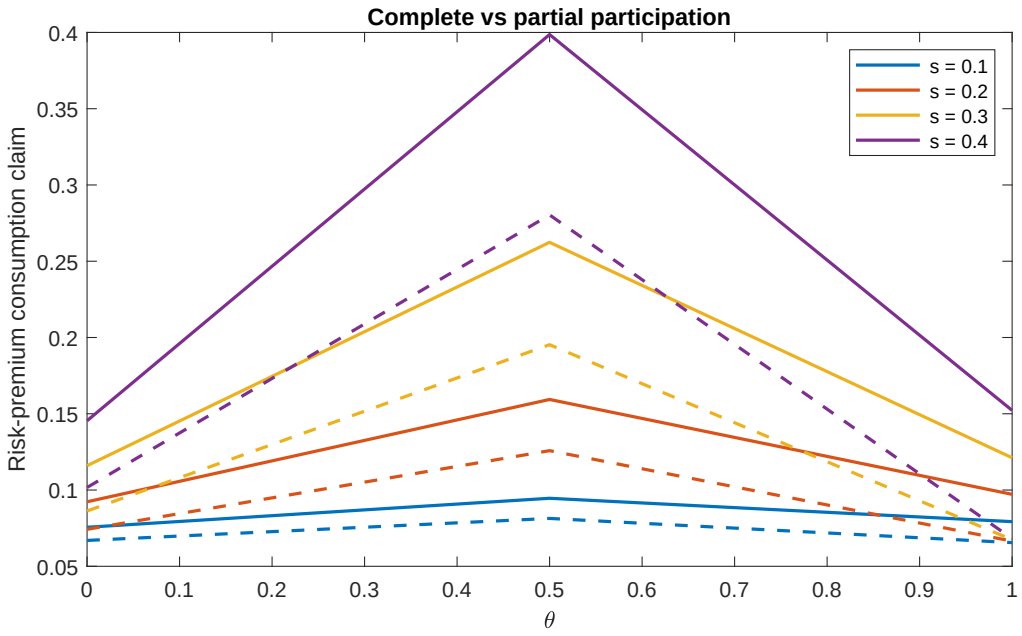

Figure 1: Partial vs complete participation risk-premium

This figure plots the risk premium on the consumption claim (y-axis) against current θ (x-axis) for various levels of current TI wealth-share s. The solid lines are from the economy where TI agents only trade in bonds, while the dashed lines are from the otherwise identical economy where both TI and TC agents trade in a complete set of financial assets.

5 Sophisticated vs Naive agents

In this section we show that our results do not depend qualitatively on the assumption that agents are naive about their true future type and unaware that they will remain time inconsistent. We first discuss the problems where the agents have a constant degree of present bias and are either sophisticated or fully naive about their type. Then we consider an alternative version of our model with time-varying present bias, where the agents are fully aware rather than naive about their type and have either expected utility or Epstein-Zin preferences.

Let $\phi_t \equiv \frac{W_t}{C_t}$ denote the wealth-consumption ratio and ω_t denote the portfolio an agent holds from period t to $t+1$. Throughout, all optimization problems are subject to the following budget constraint:

$$W_{t+1} \quad = \quad (W_t - C_t)(R_{f,t} + \omega_t^\top R_{t+1}^e) = \frac{\phi_t - 1}{\phi_t} W_t (R_{f,t} + \omega_t^\top R_{t+1}^e). \tag{44}$$

5.1 Constant degree of present bias

Pollak (1968) builds on the problem of time inconsistency in intertemporal choice developed by Strotz (1955) and distinguishes between naive and sophisticated present-biased agents. Naive agents do not anticipate the true consumption choices of their future selves and therefore repeatedly deviate from their plans, while sophisticated agents recognize their own bias and choose optimal strategies that are dynamically consistent given their foresight. Pollak (1968) shows that with log utility both naive and sophisticated agents make identical consumption-savings choices. Groneck, Ludwig, and Zimper (2024) extend this result to settings with CRRA and Epstein-Zin preferences and prove that sophisticated agents save more than naive agents when the risk aversion $\gamma > 1$ under CRRA utility and when the elasticity of intertemporal substitution $\psi \in (0, 1)$ under Epstein-Zin utility. In this section we show that the consumption-savings and portfolio choices of sophisticated and naive agents with time-varying present are identical with log utility but differ under CRRA and Epstein-Zin utility, consistent with the findings of Groneck, Ludwig, and Zimper (2024). Nevertheless, regardless of whether these agents are sophisticated or naive their presence still has strong effects on asset prices as long as their degree of present bias is

time-varying.

5.2 Time-varying present bias: Expected utility

The time t expected lifetime utility for a given sequence of wealth-consumption ratios and portfolio weights of a time-consistent (TC) and a time-inconsistent (TI) agent is

$$U_{TC,t}(\{\phi_j,\omega_j\}_{j=t}^{\infty};W_t) \equiv \mathbb{E}_t\sum_{j=t}^{\infty}\beta^{j-t}u_j\Big(\frac{W_j}{\phi_j}\Big) \tag{45}$$

$$\begin{aligned} U_{TI,t}(\{\phi_j,\omega_j\}_{j=t}^{\infty};W_t,\theta_t) &\equiv u_t\Big(\frac{W_t}{\phi_t}\Big)+\beta\delta\mathbb{E}_t\Big[\theta_t U_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1}) \\ &+ (1-\theta_t)U_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty};W_{t+1})\Big], \end{aligned} \tag{46}$$

where the notation $\tilde{\phi}_j,\tilde{\omega}_j$ is used to highlight that the sequence of wealth-consumption ratios and portfolio weights imagined by the TI agent might be different depending on whether she remains TI or becomes TC. In fact, what in effect separates a naive agent from a sophisticated agent is that a sophisticated agent understands that $\{\phi_j,\omega_j\}_{j=t+1}^{\infty}=\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty}$ – she does not believe that she will ever change type – whereas a naive agent believes that $\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty}=\{\phi_j^{TC},\omega_j^{TC}\}_{j=t+1}^{\infty}$, where the latter sequence is the *optimal* sequence of wealth-consumption ratios for a TC agent. In particular, we have $\{\phi_j^{TC},\omega_j^{TC}\}_{j=t+1}^{\infty}\neq\{\phi_j,\omega_j\}_{j=t+1}^{\infty}$ that she will actually end up following.

Thus, the objective of a fully sophisticated TI agent can be written

$$\begin{aligned} V_{TI,t}^{S}(W_t,\theta_t) &\equiv \max_{\phi_t,\omega_t} u_t\Big(\frac{W_t}{\phi_t}\Big)+\beta\delta\mathbb{E}_t\Big[\theta_t U_{TI,t+1}(\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1}) \\ &+ (1-\theta_t)U_{TC,t+1}(\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty};W_{t+1})\Big], \end{aligned} \tag{47}$$

where ϕ_t^S,ω_t^S is the wealth-consumption ratio and portfolio choice that maximizes (46) given that the agent knows she in the future will follow the sequence $\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty}$. Similarly, the objective of a naive TI agent can be written

$$\begin{aligned} V_{TI,t}^{N}(W_t,\theta_t) &\equiv \max_{\phi_t,\omega_t} u_t\Big(\frac{W_t}{\phi_t}\Big)+\beta\delta\mathbb{E}_t\Big[\theta_t U_{TI,t+1}(\{\phi_j^N,\omega_j^N\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1}) \\ &+ (1-\theta_t)U_{TC,t+1}(\{\phi_j^{TC},\omega_j^{TC}\}_{j=t+1}^{\infty};W_{t+1})\Big]. \end{aligned} \tag{48}$$

In everything that follows, we will assume the agent does not have access to a commitment device. Furthermore, we will assume that the sequences of wealth-consumption ratios and portfolio choices do not depend on the level of wealth. This latter assumption holds for instance if the future sequence is the result of optimizing expected utility with CRRA preferences. Thus, from the perspective of the time t self of the agent, the sequence of future wealth-consumption ratios and portfolio weights can be taken as given and will not depend on time t choice of wealth-consumption ratio or portfolio choice [1].

The first-order condition for consumption and portfolio choice can then be written as follows:

$$\begin{aligned} u_t'\Big(\frac{W_t}{\phi_t}\Big) &= \beta\delta\mathbb{E}_t\Big[\Big(\theta_t\frac{\partial U_{TI,t+1}(\{\phi_j,\omega_j,\theta_j\}_{j=t+1}^{\infty};W_{t+1})}{\partial W_{t+1}} \\ &+ (1-\theta_t)\frac{\partial U_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty};W_{t+1})}{\partial W_{t+1}}\Big)R_{p,t+1}\Big] \qquad (49) \\ 0 &= \beta\delta\mathbb{E}_t\Big[\Big(\theta_t\frac{\partial U_{TI,t+1}(\{\phi_j,\omega_j,\theta_j\}_{j=t+1}^{\infty};W_{t+1})}{\partial W_{t+1}} \\ &+ (1-\theta_t)\frac{\partial U_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty};W_{t+1})}{\partial W_{t+1}}\Big)R_{t+1}^e\Big], \qquad (50) \end{aligned}$$

where $R_{p,t+1}$ denotes the return on the agent's portfolio. Note that the optimality conditions for both a sophisticated and a naive agent are given by (49) and (50). The difference between the sophisticated and naive cases lies in the assumed future sequences of wealth-consumption ratios and portfolio choices.

5.2.1 CRRA-preferences

By definition,

$$W_{t+1} = (W_t - C_t)R_{p,t+1} = \frac{\phi_t - 1}{\phi_t}R_{p,t+1}W_t \qquad (51)$$

[1] If we had access to commitment devices, like e.g. illiquid assets, self t's portfolio choice would impact self $t+1$'s optimal wealth-consumption ratio and portfolio choice. Similarly, with non-constant relative risk aversion preferences, optimal portfolio next period will depend on wealth next period and thus on choice of portfolio today.

and as a result

$$W_{t+2} = \frac{\phi_{t+1}-1}{\phi_{t+1}} R_{p,t+2} W_{t+1} = \frac{\phi_{t+1}-1}{\phi_{t+1}} \frac{\phi_t - 1}{\phi_t} R_{p,t+1} R_{p,t+2} W_t \tag{52}$$

$$W_{t+n} = W_t \Big[\prod_{i=1}^{n} \frac{\phi_{t+n-i}-1}{\phi_{t+n-i}} R_{p,t+1+n-i} \Big]. \tag{53}$$

In the case of log-utility, the expected TC utility of the sequence of wealth-consumption ratios and portfolio weights $\{\phi_j, \omega_j\}_{j=t}^{\infty}$ is:

$$\begin{aligned} U_{TC,t}(\{\phi_j, \omega_j\}_{j=t}^{\infty}; W_t) &= \mathbb{E}_t \sum_{j=t}^{\infty} \beta^{j-t} \log\Big(\frac{W_j}{\phi_j}\Big) \\ &= \frac{W_t}{1-\beta} + \mathbb{E}_t \sum_{j=t}^{\infty} \beta^{j-t} \log\Big(\frac{\prod_{i=1}^{j-t} \frac{\phi_{j-i}-1}{\phi_{j-i}} R_{p,j+1-i}}{\phi_j}\Big) \\ &\equiv \phi^{TC} \log W_t + \eta_t^{TC}, \end{aligned} \tag{54}$$

where $\phi^{TC} \equiv \frac{1}{1-\beta}$ denotes the optimal wealth-consumption ratio of a TC agent. Note the independence of future wealth-consumption ratios and portfolio weights. The expected TI lifetime utility is:

$$\begin{aligned} U_{TI,t}(\{\phi_j, \omega_j\}_{j=t}^{\infty}; W_t, \theta_t) &= \log\Big(\frac{W_t}{\phi_t}\Big) + \sum_{l=t+1}^{\infty} (\beta\delta)^{l-t} \mathbb{E}_t \Big[\prod_{i=t}^{l-1} \theta_i \log\Big(\frac{W_l}{\phi_l}\Big) \\ &+ \prod_{i=t}^{l-2} \theta_i (1-\theta_{l-1})(\phi^{TC} \log W_l + \eta_l^{TC}) \Big] \\ &= \Big[1 + \mathbb{E}_t \sum_{l=t+1}^{\infty} (\beta\delta)^{l-t} \prod_{i=t}^{l-2} \theta_i \Big(\theta_{l-1} + (1-\theta_{l-1})\phi^{TC}\Big)\Big] \log W_t \\ &+ \mathbb{E}_t \sum_{l=t+1}^{\infty} \Big[\prod_{i=t}^{l-1} \theta_i (\beta\delta)^{l-t} \log\Big(\frac{\prod_{i=1}^{l-t} \frac{\phi_{l-i}-1}{\phi_{l-i}} R_{p,l+1-i}}{\phi_l}\Big) \\ &+ \prod_{i=t}^{l-2} \theta_i (1-\theta_{l-1})(\beta\delta)^{l-t} \Big(\phi^{TC} \log \prod_{i=1}^{l-t} \frac{\phi_{l-i}-1}{\phi_{l-i}} R_{p,l+1-i} + \eta_l^{TC}\Big)\Big] \\ &\equiv \phi^{TI}(\theta_t) \log W_t + \eta_t^{TI}. \end{aligned} \tag{55}$$

We see that if θ_t is a Markov process with state vector $\hat{\theta}$ and transition probability matrix Π, we

get the familiar closed form solution for $\phi^{TI}(\theta_t)$ in vectorized form:

$$\phi^{TI}(\hat{\theta}) \;=\; \Big(I - \beta\delta D(\hat{\theta})\Pi\Big)^{-1}\Big(\mathbf{1} + \beta\delta(\mathbf{1}-\hat{\theta})\phi^{TC}\Big). \tag{56}$$

We already know that (56) is the optimal TI wealth-consumption ratio for a naive TI agent. Note that in deriving (54) and (55) we did not say anything about the future sequence of wealth-consumption ratios and portfolio weights. However, we still get that the factors multiplying log wealth are the optimal TC and TI wealth-consumption ratios, respectively. From (49), it is then immediately clear that with log-utility, a sophisticated and a naive TI agent make the exact same consumption and portfolio decisions and will therefore have identical SDFs as well.

If $\gamma \neq 1$, we can then write (45) as follows

$$\begin{aligned} U_{TC,t}(\{\phi_j,\omega_j\}_{j=t}^{\infty};W_t) &= \mathbb{E}_t\sum_{j=t}^{\infty}\beta^{j-t}\frac{\left(\frac{W_j}{\phi_j}\right)^{1-\gamma}}{1-\gamma} \\ &= \mathbb{E}_t\sum_{j=t}^{\infty}\beta^{j-t}\Big[\frac{\prod_{i=1}^{j-t}\frac{\phi_{j-i}-1}{\phi_{j-i}}R_{p,j+1-i}}{\phi_j}\Big]^{1-\gamma}\frac{W_t^{1-\gamma}}{1-\gamma} \\ &\equiv K_t^{TC}(\{\phi_j,\omega_j\}_{j=t}^{\infty})\frac{W_t^{1-\gamma}}{1-\gamma}, \end{aligned} \tag{57}$$

where $K_t^{TC}(\{\phi_j,\omega_j\}_{j=t}^{\infty}) \equiv \mathbb{E}_t\sum_{j=t}^{\infty}\beta^{j-t}\left(\frac{\left[\prod_{i=1}^{j-t}\frac{\phi_{j-i}-1}{\phi_{j-i}}R_{p,j+1-i}\right]}{\phi_j}\right)^{1-\gamma}$. By forward induction, we

can write (46) as follows

$$
\begin{aligned}
U_{TI,t}(\{\phi_j,\omega_j\}_{j=t}^{\infty};W_t,\theta_t) &= u_t\Big(\frac{W_t}{\phi_t}\Big)+\sum_{l=t+1}^{\infty}(\beta\delta)^{l-t}\mathbb{E}_t\Bigg[\prod_{i=t}^{l-1}\theta_i u_l\Big(\frac{W_l}{\phi_l}\Big)\\
&+ \prod_{i=t}^{l-2}\theta_i(1-\theta_{l-1})U_{TC,l}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=l}^{\infty};W_l)\Bigg]\\
&= \frac{\Big(\frac{W_t}{\phi_t}\Big)^{1-\gamma}}{1-\gamma}+\sum_{l=t+1}^{\infty}(\beta\delta)^{l-t}\mathbb{E}_t\Bigg[\prod_{i=t}^{l-1}\theta_i\frac{\Big(\frac{W_l}{\phi_l}\Big)^{1-\gamma}}{1-\gamma}\\
&+ \prod_{i=t}^{l-2}\theta_i(1-\theta_{l-1})K_l^{TC}(\{\phi_j,\omega_j\}_{j=l}^{\infty})\frac{W_l^{1-\gamma}}{1-\gamma}\Bigg]\\
&= \Bigg\{\phi_t^{\gamma-1}+\sum_{l=t+1}^{\infty}(\beta\delta)^{l-t}\mathbb{E}_t\Bigg[\Big(\prod_{i=t}^{l-1}\theta_i\Big)\Big(\frac{\prod_{i=1}^{l-t}\frac{\phi_{l-i}-1}{\phi_{l-i}}R_{p,l+1-i}}{\phi_l}\Big)^{1-\gamma}\\
&+ \Big(\prod_{i=t}^{l-2}\theta_i\Big)(1-\theta_{l-1})K_l^{TC}(\{\phi_j,\omega_j\}_{j=l}^{\infty})\Big(\prod_{i=1}^{l-t}\frac{\phi_{l-i}-1}{\phi_{l-i}}R_{p,l+1-i}\Big)^{1-\gamma}\Bigg]\Bigg\}\\
&\times \frac{W_t^{1-\gamma}}{1-\gamma}\\
&\equiv K_t^{TI}(\{\phi_j,\omega_j\}_{j=t}^{\infty};\theta_t)\frac{W_t^{1-\gamma}}{1-\gamma}, \qquad (58)
\end{aligned}
$$

where

$$
\begin{aligned}
K_t^{TI}(\{\phi_j,\omega_j\}_{j=t}^{\infty};\theta_t) &\equiv \phi_t^{\gamma-1}+\sum_{l=t+1}^{\infty}(\beta\delta)^{l-t}\mathbb{E}_t\Bigg[\Big(\prod_{i=t}^{l-2}\theta_i\Big)\Big[\theta_{l-1}\phi_l^{\gamma-1}+(1-\theta_{l-1})K_l^{TC}(\{\phi_j,\omega_j\}_{j=l}^{\infty})\Big]\\
&\times \Big(\prod_{i=1}^{l-t}\frac{\phi_{l-i}-1}{\phi_{l-i}}R_{p,l+1-i}\Big)^{1-\gamma}\Bigg]. \qquad (59)
\end{aligned}
$$

With CRRA-preferences, it is easy to verify that the value function of a TC agent can be written

$$
V_{TC,t}(W_t)=\phi_{TC,t}^{\gamma}\frac{W_t^{1-\gamma}}{1-\gamma}, \qquad (60)
$$

where $\phi_{TC,t}\equiv\frac{W_t}{C_t}=1+\beta^{\frac{1}{\gamma}}\mathbb{E}_t[\phi_{TC,t+1}^{\gamma}R_{TC,t+1}^{1-\gamma}]^{\frac{1}{\gamma}}$ denotes the *optimal* wealth-consumption ratio for a TC agent.

We already know that for a naive TI agent, the value function can be written

$$V_{TC,t}(W_t) = \phi_{N,t}^{\gamma} \frac{W_t^{1-\gamma}}{1-\gamma}, \tag{61}$$

where $\phi_{N,t} \equiv \frac{W_t}{C_t} = 1 + (\beta\delta)^{\frac{1}{\gamma}} \mathbb{E}_t[(\theta_t \phi_{N,t+1}^{\gamma} + (1-\theta_t)\phi_{TC,t+1}^{\gamma}) R_{N,t+1}^{1-\gamma}]^{\frac{1}{\gamma}}$ denotes the *optimal* wealth-consumption ratio for a naive TI agent.

To find the value function for a sophisticated TI agent, we solve (47). Plugging into the first-order condition for optimal consumption (49) we get

$$\phi_{S,t} = 1 + (\beta\delta)^{\frac{1}{\gamma}} \mathbb{E}_t \Big[\Big(\theta_t K_{t+1}^{TI}(\{\phi_j^S, \omega_j^S\}_{j=t+1}^{\infty}; \theta_{t+1}) + (1-\theta_t) K_t^{TC}(\{\phi_j^S, \omega_j^S\}_{j=t+1}^{\infty}) \Big) R_{S,t+1}^{1-\gamma} \Big]^{\frac{1}{\gamma}}. \tag{62}$$

Plugging (62) into (47) gives us

$$\begin{aligned}
V_{TI,t}^S(W_t, \theta_t) &= \max_{\phi_t, \omega_t} u_t\Big(\frac{W_t}{\phi_t}\Big) + \beta\delta \mathbb{E}_t \Big[\theta_t U_{TI,t+1}(\{\phi_j^S, \omega_j^S\}_{j=t+1}^{\infty}; W_{t+1}, \theta_{t+1}) \\
&+ (1-\theta_t) U_{TC,t+1}(\{\phi_j^S, \omega_j^S\}_{j=t+1}^{\infty}; W_{t+1}) \Big] \\
&= \frac{\Big(\frac{W_t}{\phi_{S,t}}\Big)^{1-\gamma}}{1-\gamma} + \beta\delta \mathbb{E}_t \Big[\Big(\theta_t K_{t+1}^{TI}(\{\phi_j^S, \omega_j^S\}_{j=t+1}^{\infty}; \theta_{t+1}) + (1-\theta_t) K_{t+1}^{TC}(\{\phi_j^S, \omega_j^S\}_{j=t+1}^{\infty}) \Big) W_{t+1}^{1-\gamma} \Big] \\
&= \frac{W_t^{1-\gamma}}{1-\gamma} \Big\{ \phi_{S,t}^{\gamma-1} + \Big(\frac{\phi_{S,t}-1}{\phi_{S,t}} \Big)^{1-\gamma} \\
&\times \beta\delta \mathbb{E}_t \Big[\Big(\theta_t K_{t+1}^{TI}(\{\phi_j^S, \omega_j^S\}_{j=t+1}^{\infty}; \theta_{t+1}) + (1-\theta_t) K_{t+1}^{TC}(\{\phi_j^S, \omega_j^S\}_{j=t+1}^{\infty}) \Big) R_{S,t+1}^{1-\gamma} \Big] \Big\} \\
&= \frac{W_t^{1-\gamma}}{1-\gamma} \Big\{ \phi_{S,t}^{\gamma-1} + \Big(\frac{\phi_{S,t}-1}{\phi_{S,t}} \Big)^{1-\gamma} (\phi_{S,t}-1)^{\gamma} \Big\} \\
&= \Big\{ 1 + (\phi_{S,t}-1) \Big\} \phi_{S,t}^{\gamma-1} \frac{W_t^{1-\gamma}}{1-\gamma} \\
&= \phi_{S,t}^{\gamma} \frac{W_t^{1-\gamma}}{1-\gamma}.
\end{aligned} \tag{63}$$

Thus, we must have that

$$\phi_{S,t}^{\gamma} = K_t^{TI}(\{\phi_j^S, \omega_j^S\}_{j=t}^{\infty}; \theta_t). \tag{64}$$

Combining (49) and (50) we get that the sophisticated TI agent's SDF is given by

$$M_{S,t+1} = \beta\delta\Big[\theta_t\Big(\frac{\phi_{S,t+1}}{\phi_{S,t}-1}\Big)^{\gamma} + (1-\theta_t)\frac{K^{TC}_{t+1}(\{\phi^S_j,\omega^S_j\}^{\infty}_{j=t+1})}{(\phi_{S,t}-1)^{\gamma}}\Big]R^{-\gamma}_{S,t+1} \tag{65}$$

whereas the SDF or a naive TI agent is

$$M_{N,t+1} = \beta\delta\Big[\theta_t\Big(\frac{\phi_{N,t+1}}{\phi_{N,t}-1}\Big)^{\gamma} + (1-\theta_t)\Big(\frac{\phi_{TC,t+1}}{\phi_{N,t}-1}\Big)^{\gamma}\Big]R^{-\gamma}_{N,t+1}. \tag{66}$$

Note that the naive agent's expected life-time utility must be at least as large as that of the sophisticated agent, as the latter solves an equivalent problem with constraints. Since the resulting optimized expected life-time utilities can be written

$$V^S_{TI,t}(W_t,\theta_t) = \phi^{\gamma}_{S,t}\frac{W_t^{1-\gamma}}{1-\gamma} \tag{67}$$

$$V^N_{TI,t}(W_t,\theta_t) = \phi^{\gamma}_{N,t}\frac{W_t^{1-\gamma}}{1-\gamma} \tag{68}$$

it immediately follows that since $V^N_{TI,t}(W_t,\theta_t) \geq V^S_{TI,t}(W_t,\theta_t)$, $\phi_{N,t} \geq \phi_{S,t}$ if $\gamma < 1$ and $\phi_{N,t} \leq \phi_{S,t}$ if $\gamma > 1$. In other words, the sophisticated agent saves more (less) than the naive agent if $\gamma > 1$ ($\gamma < 1$). The two saves exactly the same with log-utility, i.e. $\gamma = 1$. This result confirms that the result in Groneck, Ludwig and Zimper (2024) that sophisticated agents save more than naive agents if and only if $\gamma > 1$ also extend to our setting.

5.3 Time-varying present bias: Epstein-Zin preferences

The life-time utilities of a given sequence of wealth-consumption ratios and portfolio weights $\{\phi_j,\omega_j\}^{\infty}_{j=t}$ are now given by

$$U_{TC,t}(\{\phi_j,\omega_j\}^{\infty}_{j=t};W_t) \equiv \Big\{\Big(\frac{W_t}{\phi_t}\Big)^{\rho} + \beta\mathbb{E}_t\Big[U_{TC,t+1}(\{\phi_j,\omega_j\}^{\infty}_{j=t+1};W_{t+1})^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}} \tag{69}$$

$$\begin{aligned} U_{TI,t}(\{\phi_j,\omega_j\}^{\infty}_{j=t};W_t,\theta_t) &\equiv \Big\{\Big(\frac{W_t}{\phi_t}\Big)^{\rho} + \beta\delta\mathbb{E}_t\Big[\theta_t U_{TI,t+1}(\{\phi_j,\omega_j\}^{\infty}_{j=t+1};W_{t+1},\theta_{t+1})^{\alpha} + \\ &+ (1-\theta_t)U_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}^{\infty}_{j=t+1};W_{t+1})^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}, \end{aligned} \tag{70}$$

where again the notation $\{\tilde{\phi}_j, \tilde{\omega}_j\}_{j=t+1}^{\infty}$ is used to indicate that the sequence of wealth-consumption ratios and portfolio weights in the mind of a TI agent *might* depend on whether she evaluates the TC life-time utility or the TI life-time utility. In particular, if the agent is fully sophisticated, she understands that she is always TI, thus she will use the same sequence to evaluate both the TC and TI continuation life-time utilities. A naive TI agent on the other hand, thinks that she might actually change type and become TC, in which case she will choose a different sequence than if she remains TI.

The value-functions of sophisticated and naive TI agents are therefore:

$$\begin{aligned} V^S_{TI,t}(W_t,\theta_t) &\equiv \max_{\phi_t,\omega_t}\Big\{\Big(\frac{W_t}{\phi_t}\Big)^{\rho} + \beta\delta\mathbb{E}_t\Big[\theta_t U_{TI,t+1}(\{\phi^S_j,\omega^S_j\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1})^{\alpha} + \\ &+ (1-\theta_t)U_{TC,t+1}(\{\phi^S_j,\omega^S_j\}_{j=t+1}^{\infty};W_{t+1})^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}} \qquad (71) \\ V^N_{TI,t}(W_t,\theta_t) &\equiv \max_{\phi_t,\omega_t}\Big\{\Big(\frac{W_t}{\phi_t}\Big)^{\rho} + \beta\delta\mathbb{E}_t\Big[\theta_t U_{TI,t+1}(\{\phi^N_j,\omega^N_j\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1})^{\alpha} + \\ &+ (1-\theta_t)U_{TC,t+1}(\{\phi^{TC}_j,\omega^{TC}_j\}_{j=t+1}^{\infty};W_{t+1})^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}, \qquad (72) \end{aligned}$$

where $\phi^S_t,\ \omega^S_t$ and $\phi^N_t,\ \omega^N_t$ denotes the wealth-consumption ratio and portfolio weights that solve (71) and (72) at time t, respectively. Similarly, $\phi^{TC}_t,\ \omega^{TC}_t$ denote the optimal wealth-consumption ratio and portfolio weights of a TC agent at time t.

The first-order conditions w.r.t. consumption and portfolio choice gives us the Epstein-Zin versions of (49) and (50):

$$\begin{aligned} \Big(\frac{W_t}{\phi_t}\Big)^{\rho-1} &= \beta\delta\mathbb{E}_t\Big[\theta_t U_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1})^{\alpha} + \\ &+ (1-\theta_t)U_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty};W_{t+1})^{\alpha}\Big]^{\frac{\rho}{\alpha}-1} \\ &\times \mathbb{E}_t\Big[\Big(\theta_t\frac{U_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1})^{\alpha}}{\partial W_{t+1}} + \\ &+ (1-\theta_t)\frac{U_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty};W_{t+1})^{\alpha}}{\partial W_{t+1}}\Big)R_{p,t+1}\Big] \qquad (73) \\ 0 &= \mathbb{E}_t\Big[\Big(\theta_t\frac{U_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1})^{\alpha}}{\partial W_{t+1}} + \\ &+ (1-\theta_t)\frac{U_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty};W_{t+1})^{\alpha}}{\partial W_{t+1}}\Big)R^e_{t+1}\Big]. \qquad (74) \end{aligned}$$

The first-order conditions take the same form for both the sophisticated and naive agent, the difference being in what future sequence of wealth-consumption ratios and portfolio weights are used to evaluate the expressions.

We next show that the life-time utilities can be written in the form $U_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1}) = K_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty},\theta_{t+1})W_{t+1}$ and $U_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty};W_{t+1}) = K_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty})W_{t+1}$. The first-order conditions therefore become:

$$\begin{aligned}\Big(\frac{W_t}{\phi_t}\Big)^{\rho-1} &= \beta\delta\Big(\frac{\phi_t-1}{\phi_t}\Big)^{\rho-1}W_t^{\rho-1}\mathbb{E}_t\Big[\Big(\theta_t K_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty},\theta_{t+1})^{\alpha}+\\ &+ (1-\theta_t)K_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty})^{\alpha}\Big)R_{p,t+1}^{\alpha}\Big]^{\frac{\rho}{\alpha}} \qquad (75)\\ 0 &= \mathbb{E}_t\Big[\Big(\theta_t K_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty},\theta_{t+1})^{\alpha}+\\ &+ (1-\theta_t)K_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty})^{\alpha}\Big)R_{p,t+1}^{\alpha-1}R_{t+1}^{e}\Big]. \qquad (76)\end{aligned}$$

Combining (75) and (76) allows us to write the SDF as follows:

$$\begin{aligned}M_{t+1} &= (\beta\delta)^{\frac{\alpha}{\rho}}\mathbb{E}_t\Big[\Big(\theta_t\frac{K_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty},\theta_{t+1})^{\alpha}}{(\phi_t-1)^{\frac{(1-\rho)\alpha}{\rho}}}+\\ &+ (1-\theta_t)\frac{K_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty})^{\alpha}}{(\phi_t-1)^{\frac{(1-\rho)\alpha}{\rho}}}\Big)R_{p,t+1}^{\alpha}\Big]. \qquad (77)\end{aligned}$$

5.3.1 Time-consistent preferences

With EZ-preferences, the value function of a TC agent can be written

$$V_{TC,t}(W_t) = \phi_{TC,t}^{\frac{1-\rho}{\rho}}W_t. \qquad (78)$$

Now, consider the life-time utility of a TC agent for an arbitrary (not necessarily optimal) sequence of wealth-consumption ratios and portfolio weights. To arrive at the necessary recursions, assume that the sequence $\{\phi_j,\omega_j\}_{j=t}^{\infty}$ is arbitrary from period t until period $T-1$ and then identically equal to the optimal sequence of wealth-consumption ratios and portfolio weights from period T onward. Then let $T\to\infty$.

We then have

$$
\begin{aligned}
U_{TC,T-1}(\{\phi_j,\omega_j\}_{j=T-1}^{\infty};W_{T-1}) &= \Big\{C_{T-1}^{\rho}+\beta\mathbb{E}_{T-1}\Big[V_{TC,T}(W_T)^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\\
&= \Big\{\Big(\frac{W_{T-1}}{\phi_{T-1}}\Big)^{\rho}+\beta\mathbb{E}_{T-1}\Big[\phi_{TC,T}^{\frac{(1-\rho)\alpha}{\rho}}W_T^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\\
&= \Big\{\Big(\frac{W_{T-1}}{\phi_{T-1}}\Big)^{\rho}+(W_{T-1}-C_{T-1})^{\rho}\beta\mathbb{E}_{T-1}\Big[\phi_{TC,T}^{\frac{(1-\rho)\alpha}{\rho}}R_{p,T}^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\\
&= \Big\{\Big(\frac{W_{T-1}}{\phi_{T-1}}\Big)^{\rho}+(\phi_{T-1}-1)^{\rho}\Big(\frac{W_{T-1}}{\phi_{T-1}}\Big)^{\rho}\beta\mathbb{E}_{T-1}\Big[\phi_{TC,T}^{\frac{(1-\rho)\alpha}{\rho}}R_{p,T}^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\\
&= \Big\{1+(\phi_{T-1}-1)^{\rho}\beta\mathbb{E}_{T-1}\Big[\phi_{TC,T}^{\frac{(1-\rho)\alpha}{\rho}}R_{p,T}^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\frac{W_{T-1}}{\phi_{T-1}}\\
&\equiv K_{TC,T-1}(\{\phi_j,\omega_j\}_{j=T-1}^{\infty})W_{T-1}, \qquad (79)
\end{aligned}
$$

where $K_{TC,T-1}(\{\phi_j,\omega_j\}_{j=T-1}^{\infty}) \equiv \Big\{1+(\phi_{T-1}-1)^{\rho}\beta\mathbb{E}_{T-1}\Big[\phi_{TC,T}^{\frac{(1-\rho)\alpha}{\rho}}R_{p,T}^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\phi_{T-1}^{-1}$. At $T-2$ we get

$$
\begin{aligned}
U_{TC,T-2}(\{\phi_j,\omega_j\}_{j=T-2}^{\infty};W_{T-2}) &= \Big\{C_{T-2}^{\rho}+\beta\mathbb{E}_{T-2}\Big[V_{TC,T-1}(W_{T-1})^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\\
&= \Big\{\Big(\frac{W_{T-2}}{\phi_{T-2}}\Big)^{\rho}+\beta\mathbb{E}_{T-2}\Big[K_{TC,T-1}(\{\phi_j,\omega_j\}_{j=T-1}^{\infty})^{\alpha}W_{T-1}^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\\
&= \Big\{1+(\phi_{T-2}-1)^{\rho}\beta\mathbb{E}_{T-1}\Big[K_{TC,T-1}(\{\phi_j,\omega_j\}_{j=T-1}^{\infty})^{\alpha}R_{p,T-1}^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\frac{W_{T-2}}{\phi_{T-2}}\\
&\equiv K_{TC,T-2}(\{\phi_j,\omega_j\}_{j=T-2}^{\infty})W_{T-2}. \qquad (80)
\end{aligned}
$$

It is easy to guess that the general recursion is as follows

$$
\begin{aligned}
U_{TC,t}(\{\phi_j,\omega_j\}_{j=t}^{\infty};W_t) &= K_{TC,t}(\{\phi_j,\omega_j\}_{j=t}^{\infty})W_t \qquad (81)\\
K_{TC,t}(\{\phi_j,\omega_j\}_{j=t}^{\infty}) &\equiv \Big\{1+(\phi_t-1)^{\rho}\beta\mathbb{E}_t\Big[K_{TC,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty})^{\alpha}R_{p,t+1}^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\phi_t^{-1}. \qquad (82)
\end{aligned}
$$

5.3.2 Time-inconsistent preferences

Similarly, guess that:

$$
\begin{aligned}
U_{TI,t+1}^{S}(\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty};W_{t+1},\theta_{t+1}) &= K_{TI,t+1}(\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty},\theta_{t+1})W_{t+1} \qquad (83)\\
K_{TI,t+1}(\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty},\theta_{t+1}) &\equiv (\phi_{t+1}^S)^{\frac{1-\rho}{\rho}} \qquad (84)
\end{aligned}
$$

then, by (75) we have that

$$\begin{aligned}(\phi_t^S-1)^{1-\rho} &= \beta\delta\mathbb{E}_t\Big[\Big(\theta_t K_{TI,t+1}(\{\phi_j,\omega_j\}_{j=t+1}^{\infty},\theta_{t+1})^{\alpha}+\\ &+ (1-\theta_t)K_{TC,t+1}(\{\tilde{\phi}_j,\tilde{\omega}_j\}_{j=t+1}^{\infty})^{\alpha}\Big)R_{p,t+1}^{\alpha}\Big]^{\frac{\rho}{\alpha}}. \qquad (85)\end{aligned}$$

Plugging (85) into (71) gives us:

$$\begin{aligned}V_{TI,t}^S(W_t,\theta_t) &= \Big\{\Big(\frac{W_t}{\phi_t^S}\Big)^{\rho}+\beta\delta\mathbb{E}_t\Big[\Big(\theta_t K_{TI,t+1}(\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty},\theta_{t+1})^{\alpha}\\ &+ (1-\theta_t)K_{TC,t+1}(\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty})^{\alpha}\Big)W_{t+1}^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\\ &= \Big\{\Big(\frac{W_t}{\phi_t^S}\Big)^{\rho}+\Big(\frac{\phi_t^S-1}{\phi_t^S}\Big)^{\rho}W_t^{\rho}\beta\delta\mathbb{E}_t\Big[\Big(\theta_t K_{TI,t+1}(\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty},\theta_{t+1})^{\alpha}\\ &+ (1-\theta_t)K_{TC,t+1}(\{\phi_j^S,\omega_j^S\}_{j=t+1}^{\infty})^{\alpha}\Big)R_{p,t+1}^{\alpha}\Big]^{\frac{\rho}{\alpha}}\Big\}^{\frac{1}{\rho}}\\ &= \Big\{\phi_t^{S-\rho}+\Big(\frac{\phi_t^S-1}{\phi_t^S}\Big)^{\rho}(\phi_t^S-1)^{1-\rho}\Big\}^{\frac{1}{\rho}}W_t\\ &= \Big\{1+(\phi_t^S-1)^{\rho}(\phi_t^S-1)^{1-\rho}\Big\}^{\frac{1}{\rho}}\phi_t^{S-1}W_t\\ &= (\phi_t^S)^{\frac{1-\rho}{\rho}}W_t \qquad (86)\end{aligned}$$

which verifies our guess.

We notice that the life-time utilities of the sophisticated and naive agents would be identical if the naive agent believed she was constrained to following the same sequence of wealth-consumption ratios and portfolio weights in the event she became a TC agent as when she remains TI. However, since she believes she is not constrained to follow this sequence, her continuation value function conditional on becoming TC is larger, thus $V_{TI,t}^N(W_t,\theta_t)\geq V_{TI,t}^S(W_t,\theta_t)$. We have:

$$V_{TI,t}^N(W_t,\theta_t)=\phi_t^{N\frac{1-\rho}{\rho}}W_t \qquad (87)$$

$$V_{TI,t}^S(W_t,\theta_t)=\phi_t^{S\frac{1-\rho}{\rho}}W_t. \qquad (88)$$

Since $V^N_{TI,t}(W_t,\theta_t) \geq V^S_{TI,t}(W_t,\theta_t) \Leftrightarrow {\phi_t^N}^{\frac{1-\rho}{\rho}} \geq {\phi_t^S}^{\frac{1-\rho}{\rho}}$, we get that

$$\phi_t^N \geq \phi_t^S, \qquad \text{if } \rho \in (0,1) \tag{89}$$

$$\phi_t^N \leq \phi_t^S, \qquad \text{if } \rho < 0 \tag{90}$$

In other words, we again confirm that the result in Groneck, Ludwig and Zimper (2024) hold in our setting with Epstein-Zin utility as well - the sophisticated agent saves more (less) than the naive agent if $\rho < 0$ ($\rho \in (0,1)$), i.e. $\psi \in (0,1)$ ($\psi > 1$).

Plugging into (77) we get the following SDFs for sophisticated and naive TI agents:

$$M_{S,t+1} = (\beta\delta)^{\frac{\alpha}{\rho}} \Big[\theta_t \Big(\frac{\phi^S_{t+1}}{\phi^S_t - 1}\Big)^{\frac{(1-\rho)\alpha}{\rho}} + (1-\theta_t)\Big(\frac{K_{TC,t+1}(\{\phi^S_j, \omega^S_j\}^{\infty}_{j=t+1})}{(\phi^S_t - 1)^{\frac{1-\rho}{\rho}}}\Big)^{\alpha}\Big] R^{\alpha-1}_{S,t+1} \tag{91}$$

$$M_{N,t+1} = (\beta\delta)^{\frac{\alpha}{\rho}} \Big[\theta_t \Big(\frac{\phi^N_{t+1}}{\phi^N_t - 1}\Big)^{\frac{(1-\rho)\alpha}{\rho}} + (1-\theta_t)\Big(\frac{\phi^{TC}_{t+1}}{\phi^N_t - 1}\Big)^{\frac{(1-\rho)\alpha}{\rho}}\Big] R^{\alpha-1}_{N,t+1}. \tag{92}$$

5.4 Numerical results - sophisticated vs naive TI agents

Table 6 compares the numerical results of the OLG model with Epstein-Zin preferences and fully sophisticated TI agents (column 1) to our baseline model with TI agents who are naive about their future type and consist of 50% or 80% of the total population (columns 2 and 3). Panels A and B give the conditional and unconditional moments, respectively.

The mean wealth share of the sophisticated TI agents (about 16% of the total wealth) is comparable to that of the naive TI agents, showing that even though they are fully aware of their future type, these agents still overconsume and save too little compared to TC agents. Similar to our baseline case, the presence of sophisticated TI agents strongly affects asset prices despite their low wealth share. The risk premium on the consumption and dividend claim are about 1.57% and 3.08%. Quantitatively, we notice that these values are slightly lower than those in the baseline case, partly due to the lower wealth share volatility and stock return volatility. The Sharpe ratios of the dividend claim in both specifications are comparable in magnitude (about 0.30). In addition, as we show in this section, sophisticated agents save less than naive agents under our calibration with EIS $\psi > 1$, pushing the risk-free rate to 3.38% – about 1%

above the baseline value. The average slope of the yield curve is still positive, but slightly lower (0.33) and less volatile.

Panel B shows that the model with sophisticated TI agents still yields time-varying discount rates due to the time-variation in the present bias. The volatility of the conditional risk premium on the consumption and the dividend claim of about 0.8 are, however, lower than those in the baseline specification. Finally, the price-dividend ratio is negatively correlated with conditional expected excess market returns, and the slope of the term structure is positively correlated with conditional expected excess bond returns, consistent with the data.

Table 6: Asset pricing moments: Sophisticated TI agent

The table reports asset pricing moments for the OLG model with fully sophisticated TI and TC agents and Epstein-Zin preferences. The calibration parameters are given in Table 4. $E(x)$ and $\sigma(x)$ denote the unconditional mean and variance of x, respectively, while $corr(x, y)$ denotes the correlation between x and y. SR stands for Sharpe ratio, s_t is the TI agent wealth share, $R_{C,t}$ is the return to the aggregate consumption claim, $R_{m,t}$ is the return to the dividend claim, $R_{f,t}$ is the one-period real risk-free rate, $R_t^{(10y)}$ is the return to a 10-year default-free, real, zero-coupon bond, $y_t^{(n)}$ is the yield of the n-maturity, default-free, real zero-coupon bond, and PD_t is the price-dividend ratio of the dividend claim. All moments, except for correlations, are annualized.

	Sophisticated	**Baseline**	**80% TI agents**
	(1)	(2)	(3)
	Panel A: Unconditional moments		
$E(s_t)$	16.24	15.53	17.20
$\sigma(s_t)$	3.17	6.79	8.82
$E(R_{C,t} - R_{f,t})$	1.57	2.18	3.83
$\sigma(R_{C,t} - R_{f,t})$	5.49	6.96	10.75
$E(R_{m,t} - R_{f,t})$	3.08	3.71	5.54
$\sigma(R_{m,t} - R_{f,t})$	10.80	11.79	14.81
$SR(R_{m,t} - R_{f,t})$	0.29	0.31	0.37
$E(R_{f,t})$	3.38	2.44	0.78
$\sigma(R_{f,t})$	0.34	1.70	3.02
$E\left(y_t^{(10y)} - y_t^{(1m)}\right)$	0.33	0.55	0.83
$\sigma\left(y_t^{(10y)} - y_t^{(1m)}\right)$	0.34	1.71	3.04
	Panel B: Conditional moments		
$\sigma(E_t(R_{C,t+1} - R_{f,t+1}))$	0.80	2.53	4.86
$\sigma(E_t(R_{m,t+1} - R_{f,t+1}))$	0.83	2.72	5.31
$\sigma\left(E_t\left(R_{t+1}^{(10y)} - R_{f,t+1}\right)\right)$	0.68	1.82	3.26
$Corr(PD_t, E_t(R_{m,t+1} - R_{f,t+1}))$	-0.85	-0.55	-0.50
$Corr\left(y_t^{(10y)} - y_t^{(1m)}, E_t\left(R_{t+1}^{(10y)} - R_{f,t+1}\right)\right)$	0.54	0.72	0.75

6 Further discussion about the equilibrium dynamics

In our calibration (i.e. $\gamma > 1$ and $\psi > 1$), we find that the TC wealth-consumption ratio at time $t+1$ is increasing in θ_{t+1}. Intuitively, the TC agent saves more as θ_{t+1} increases partly due to improved investment opportunities at $t+1$ (lower aggregate wealth-consumption ratio) and partly due to the expectation of improved investment opportunities in the future after $t+1$. Furthermore, the wealth-share at time $t+1$ of TI agents (TC agents) who were alive at time t, is weakly increasing (decreasing) in θ_{t+1}. In particular, if $\theta_t = 0$, TI and TC agents hold identical portfolios and the wealth-share at $t+1$ is therefore constant. When θ_t is larger, the TI (TC) agents buy portfolios that pay off more (less) in the states where θ_{t+1} increases. We therefore see that the growth in TC wealth-consumption ratios and growth in TC wealth-shares in the log SDF decomposition will partially offset each other.

The last endogenous term, the growth rate in aggregate wealth-consumption ratios, is more complex. The aggregate wealth-consumption ratio is a weighted average of TC and TI wealth-consumption ratios. TI agents' consumption-savings decision today is affected by their beliefs about how much they will consume if they become TC next period (i.e. next period TC wealth-consumption ratio) as well as how much they will consume if the remain TI (i.e. next period TI wealth-consumption ratio). When θ_t is very low, the TI agent is almost sure she will become a TC agent next period. Therefore the distribution of next period TC wealth-consumption ratios is very important for determining current $\phi_{TI,t}$. Since the distribution of $\phi_{TC,t+1}$ shifts rightwards when θ_t increases, this channel will give rise to a positive relation between TI wealth-consumption ratios at time t and θ_t. However, there is also an opposing effect, namely that higher θ_t makes it more likely that the agent remains TI next period, and since the TI wealth-consumption ratio is lower than that of a TC agent, this channel will give rise to a negative relation between $\phi_{TI,t}$ and θ_t.[2] In our calibration, we find that the first effect dominates for low values of θ_t while the second effect dominates for higher values of θ_t. The overall effect is therefore that $\phi_{TI,t}$ has a hump-shape in θ_t – initially increasing for low values of θ_t, then decreasing for higher values

[2]Note that in the case of log-utility, the TC wealth-consumption ratio is constant, thus only the second channel is at play, giving rise to a monotonically negative relationship between θ_t and $\phi_{TI,t}$

of θ_t. The aggregate wealth-consumption ratio inherits this hump-shaped pattern from the TI wealth-consumption ratio.

Figure 3 plots the shocks to log SDF and the decomposition for current TI wealth share equal to its mean and two values of current θ_t: 20th and 80th percentile. When θ_t is close to 0 (the first plot in Figure 3), the TI and TC agents hold essentially the same portfolio. Thus, there is almost no contribution from the shocks to wealth-share as seen from the flat red dashed line. Furthermore, the TC agents' wealth-consumption ratio is increasing in θ_{t+1} due to an expectation of better investment opportunities and higher risk going forward, which gives rise to the falling red solid line. Finally, the aggregate wealth-consumption ratio is also increasing in θ_{t+1} giving rise to the falling dotted red line. The increasing aggregate wealth-consumption ratio ϕ_{t+1} is due to the TC wealth-consumption ratio $\phi_{TC,t+1}$ being increasing in θ_{t+1}, which for low values of θ_t also causes the TI wealth-consumption ratio to be increasing in θ_{t+1}. Thus, for low values of θ_t, positive shocks to θ_{t+1} results in lower marginal utility, i.e. good states.

However, for higher values of θ_t (see the second plot in Figure 3), TI and TC agents find it optimal to hold very different portfolios. In particular, TI agents are averse to high θ_{t+1} states and therefore buy claims that pay off more in such states resulting in positive shocks to θ_{t+1} being associated with positive shocks to TI wealth-share, which in turn gives rise to the increasing dashed red line. The TC wealth-consumption ratio is still increasing in shocks to θ_{t+1}, which results in the declining solid red line. Finally, the aggregate wealth-consumption ratio at $t+1$ is now decreasing in shocks to θ_{t+1} due to the TI wealth-consumption ratio being decreasing in θ_{t+1}, resulting in the increasing red-dotted line. In this case, the positive contributions to marginal utility from the TI wealth-share and the aggregate wealth-consumption ratio more than offsets the negative impact on marginal utility of higher TC wealth-consumption ratio when shocks to θ_{t+1} are large and positive, while the opposite is the case when shocks to θ_{t+1} are negative. Thus, the impact of shocks to θ_{t+1} is not symmetric when θ_t is high - positive shocks have large positive impacts on marginal utility whereas negative shocks have small positive impacts on marginal utility, which gives rise to a convex shape of the pricing kernel. The convexity in the pricing kernel gives rise to a variance risk premium.

Figure 2 similarly plots the SDF in levels as functions of shocks to θ_{t+1} for three different levels of TI wealth share at time t: 30th percentile (solid blue lines), the 50th percentile (dashed red lines), and the 70th percentile (dashed yellow lines). In the two first panels, we plot the SDF at $t+1$ conditional on θ_t being at the 25th and 75th percentiles, respectively. In the last panel we first calculate the SDF as a function of shocks to θ_{t+1} for each possible value of θ_t then we use the unconditional distribution of θ_t to condition out the dependence on θ_t – i.e. we essentially get the "average" functional form of the SDF for a given TI wealth-share at time t. The first key takeaway from Figure 2 is that θ_t determines whether marginal utility is decreasing (increasing) in shocks to θ_{t+1}, which will be the case for low (high) θ_t. Second, higher TI wealth share s_t magnifies the existing pattern, but does not seem to qualitatively change it. Third, on "average", the SDF is convex and asymmetric in shocks to θ_{t+1} and large positive shocks are bad states, which explains the existence of an unconditional variance risk premium. Fourth, the degree of convexity varies both with θ_t and TI wealth-share, which gives rise to a time-varying variance risk premium.

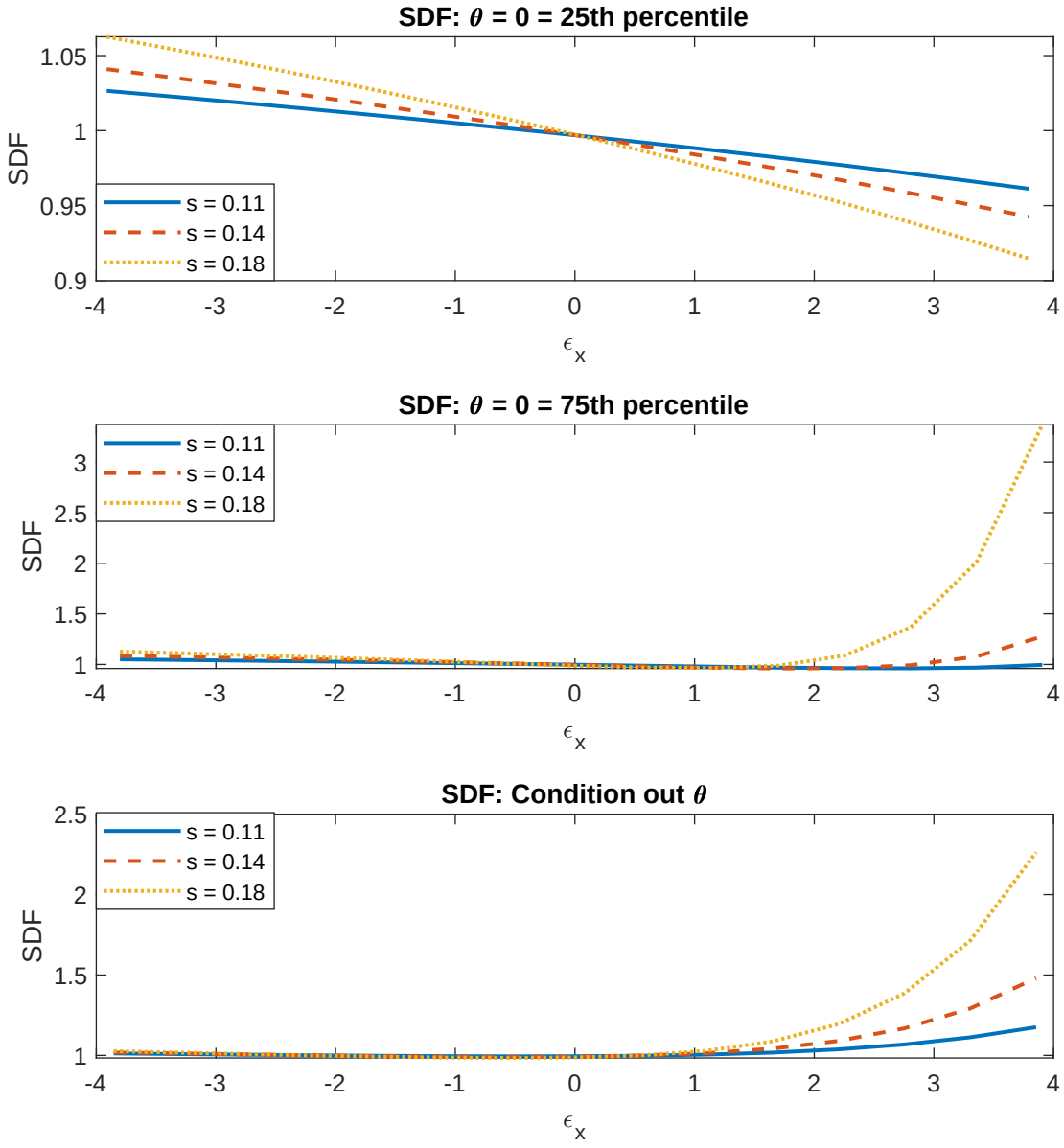

Figure 2: Stochastic discount factor

This figure plots the stochastic discount factor (y-axis) as a function of the shock (ϵ_x) to x_t, where the present bias parameter is given by $\theta_t = \left(\frac{e^{x_t}}{1+e^{x_t}}\right)^{\varphi}$. The SDF is presented for different wealth shares of the time-inconsistent agent s and for different values of θ (25th and 75th percentile and conditioning out θ). In this setting both the TI and TC agents have Epstein-Zin preferences and the calibration parameters are given in Table 4.

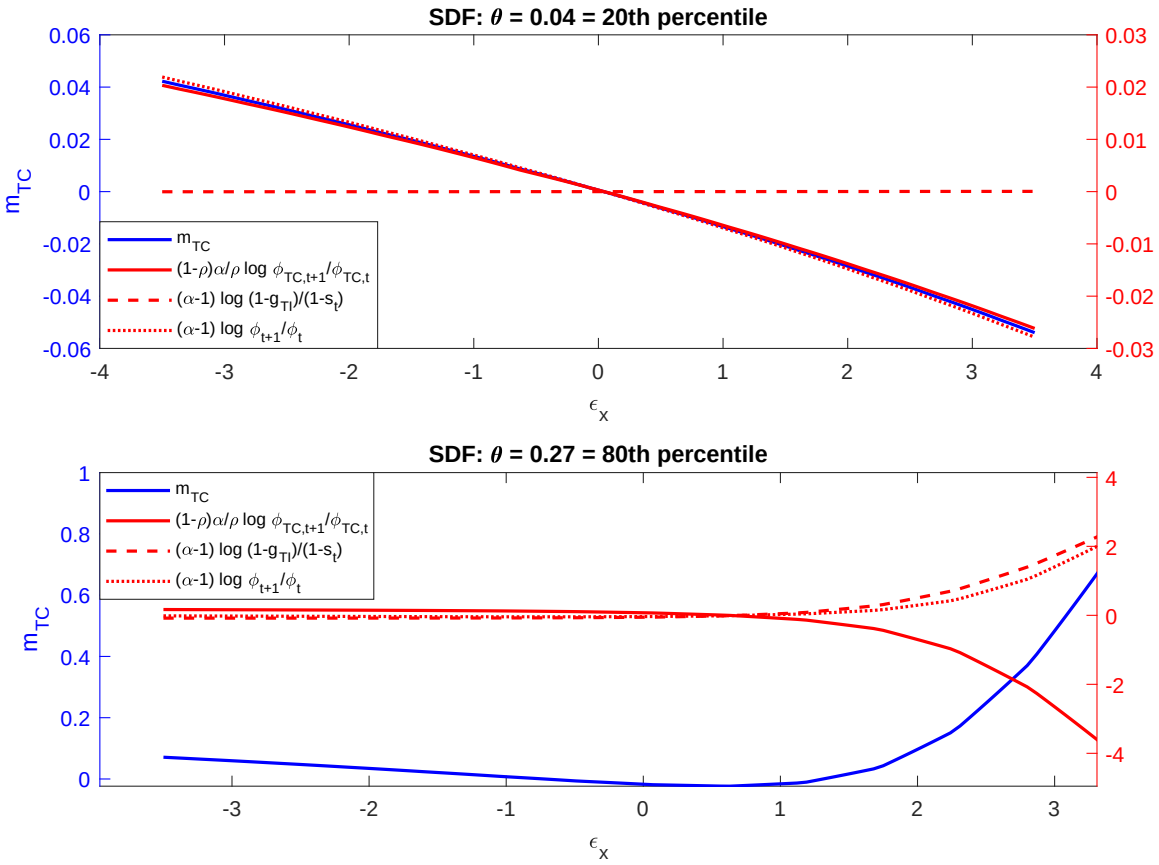

Figure 3: Stochastic discount factor decomposition

This figure plots the shocks to the log stochastic discount factor (left y-axis) and the shocks to the log decomposition of the SDF given by $m_{t+1} = \frac{\alpha}{\rho}\log(\beta\lambda) + \frac{\alpha-\rho}{\rho}\log\frac{\phi_{TC,t}}{\phi_{TC,t}-1} + \frac{(1-\rho)\alpha}{\rho}\log\frac{\phi_{TC,t+1}}{\phi_{TC,t}} + (\alpha-1)\log\frac{1-g_{TI,t+1}}{1-s_t} + (\alpha-1)\log\frac{\phi_{t+1}}{\phi_t} + (\alpha-1)\log\frac{C_{t+1}}{C_t}$ (right y-axis) as a function of the shock (ϵ_x) to x_t, where the present bias parameter is given by $\theta_t = \left(\frac{e^{x_t}}{1+e^{x_t}}\right)^{\varphi}$. The SDF is presented for the mean wealth share of the time-inconsistent agent s and for different values of θ (20th and 80th percentile). In this setting both the TI and TC agents have Epstein-Zin preferences and the calibration parameters are given in Table 4.