Appendix A Aggregate Productivity with Place-Based Compensation

In the body of the paper, we measure aggregate productivity by the amount of factor endowments that can be saved after winners compensate losers using lump-sum transfers. In this appendix, we briefly discuss how to extend this definition of aggregate productivity to cases where individual-level lump-sum transfers are not available, but place-based redistributive policies are.

Suppose that instead of individual-specific transfers, location-level consumption taxes, denoted by τ_r, can be levied on households that choose location r. In this case, the budget constraint of agent h is now:

$$\sum_r (1+\tau_r) p_r c_h \mathbf{1}\left[l_h = r\right] = Z \sum_r w_r a_{hr} \mathbf{1}\left[l_h = r\right] + T,$$

where T is a uniform lump-sum rebate (the same for all households). Budget balance requires that

$$T = \int \sum_r \tau_r p_r c_{hr} \mathbf{1}\left[l_h = r\right] dh. \tag{34}$$

All other conditions defining equilibrium are the same as in Section 2. We can now define the feasible set of allocations that can be supported using these place-based tax instruments:

$$\mathcal{C}^{pb}(t, Z) \equiv \left\{\{\boldsymbol{c}_h, l_h\}_{h\in H} \text{ supported via equilibrium given } \boldsymbol{z}(t),\, Z, \text{ and some consumption taxes}\right\}.$$

Given this feasible set, the definition of aggregate productivity, given place-based compensations, is the same as before replacing $\mathcal{C}(t, Z)$ with $\mathcal{C}^{pb}(t, Z)$.

Definition 7. Aggregate productivity with place-based compensation at t is

$$A^{pb}(t) = \max\left\{Z^{-1} : \{c_h, l_h\}_{h\in H} \in \mathcal{C}^{pb}(t, Z) \text{ and } (c_h, l_h) \succeq_h (c_h(0), l_h(0)) \text{ for every } h\right\}. \tag{35}$$

where $c_h(0)$ and $l_h(0)$ are the consumption and location of h in the status quo.

To see how this works in practice, consider the following one-good economy example.

Example 11 (One-Good Economy with Place-Based Policies). Consider the one-good economy again: there is a single freely-traded consumption good produced linearly from labor and productivity in each location r is z_r. Assume all agents have homogeneous skills, as in Assumption 2. Hence, production in each location is

$$y_r = z_r(t)L_r = z_r(t)\int Z\mathbf{1}[l_h = r]dh.$$

The following shows that, in the one-good economy, $A^{pb}(t)$ is bounded above by $A(t)$ but that, to a first-order approximation, the two are the same.

Proposition 10 (Hulten's Theorem for One-Good Economy). *In the one-good economy with $a_{hr} = 1$ for every h and r, we have*

$$A^{pb}(t) \leq A(t) \quad \textit{for all } t.$$

Moreover, around the status quo $t = 0$, the two measures coincide to a first-order approximation and both obey Hulten's theorem:

$$\Delta \log A^{pb} \approx \Delta \log A \approx \sum_r \lambda_r(0)\, \Delta \log z_r.$$

To go beyond a linear approximation, rearrange the budget-balance condition, (34), as

$$Z^{-1} = \frac{\sum_r w_r L_r}{\sum_r p_r c_r L_r} = \frac{\sum_r z_r L_r}{\sum_r c_r L_r}, \tag{36}$$

where c_r denotes per-capita consumption in location r, and we use the fact that the real wage in location r is equal to $z_r(t)$ in this one-good economy. Using equation (35) and (36), we can write

$$A^{pb}(t) = \max_{\boldsymbol{c}} \left\{ \frac{\sum_r z_r(t) L_r(\boldsymbol{c})}{\sum_r c_r L_r(\boldsymbol{c})} : u_h(c_r, l_h(\boldsymbol{c})) \geq u_h^0 \text{ for every } h \right\},$$

where $L_r(\boldsymbol{c})$ is labor supply in location r as a function of the vector of location-level per capita consumption $\boldsymbol{c}$.

If, in the solution to the problem above, for every location $r \in R$, some agent $h \in H$ stays in the same location as in the status-quo and all locations are non-empty, then compensating the stayers requires ensuring that consumption per capita in each location is at least as high as in the status quo. Hence, we can replace the H inequalities above with just R inequalities:

$$A^{pb}(t) = \max_{\boldsymbol{c}} \left\{ \frac{\sum_r z_r(t) L_r(\boldsymbol{c})}{\sum_r c_r L_r(\boldsymbol{c})} : c_r \geq c_r^0 \text{ for every } r \right\}.$$

This is relatively simple constrained optimization maximization problem given the supply function.

To make it more concrete, suppose that $\boldsymbol{L}(\boldsymbol{c})$ is the iso-elastic labor supply function:

$$\frac{L_r(\boldsymbol{c})}{\sum_{r'} L_{r'}(\boldsymbol{c})} = \frac{c_r^{\theta}}{\sum_{r'} c_{r'}^{\theta}}.$$

Then, it is straightforward to show that the solution to $A^{pb}(t)$ satisfies:

$$c_r^{pb}(t) = z_r(0)\mathbf{1}\left[\frac{z_r(t)/z_r(0)}{A^{pb}(t)} \leq \frac{\theta+1}{\theta}\right] + \frac{\theta}{\theta+1}\frac{z_r(t)}{A^{pb}(t)}\mathbf{1}\left[\frac{z_r(t)/z_r(0)}{A^{pb}(t)} > \frac{\theta+1}{\theta}\right].$$

If the shocks are sufficiently small, $\frac{z_r(t)/z_r(0)}{A^{pb}(t)} \leq \frac{\theta+1}{\theta}$ for every r, then the solution sets consumption in each location equal to its status quo value $c_r^{pb}(t) = z_r(0)$. This is achieved by scaling aggregate factor productivity by $1/A^{pb}(t)$ and setting consumption taxes appropriately. Since consumption is equal to its status-quo value in every location, no agent switches locations, and hence every agent is exactly indifferent to the status-quo.

However, if region r experiences much faster productivity growth than average — in the sense that $\frac{z_r(t)/z_r(0)}{A^{pb}(t)} > \frac{\theta+1}{\theta}$ — then consumption in region r must exceed its status-quo level to incentivize additional workers to move there. In this case, stayers in region r are strictly better off than in the status quo. Given the optimal choices $\boldsymbol{c}^{pb}(t)$ and the implied labor supplies $L_r(\boldsymbol{c}^{pb}(t))$, the place-based aggregate productivity index $A^{pb}(t)$ is obtained from (36).

Appendix B Proofs

Proof of Proposition 1. Define $u_h(\boldsymbol{z}, Z)$ to be the utility of household h in a decentralized equilibrium given productivity vector $\boldsymbol{z}$ and aggregate factor augmenting productivity Z. Since in equilibrium all households have the same utility, irrespective of their chosen location, there are no lump-sum transfers in the compensated equilibrium. Hence, to calculate $A(t)$, we simply need to solve

$$u_h(\boldsymbol{z}(t), 1/A(t)) = u_h(\boldsymbol{z}(0), 1).$$

For any regions r such that $L_r(\boldsymbol{z}(t), 1/A(t)) \times L_r(\boldsymbol{z}(0), 1) > 0$ we have that

$$g(c_r(\boldsymbol{z}(t), 1/A(t))) + \epsilon_r = g(c_r(\boldsymbol{z}(0), 1)) + \epsilon_r.$$

The result follows from this equation. If, in addition, we know that g is logarithmic, then a proportional change in consumption in every location has no effect on the mass of households that choose a given location. That is, if $g(c) = \log(c)$, then labor supplied in location r is $L_r(\boldsymbol{z}, 1/A(t)) = L_r(\boldsymbol{z}, 1)/A(t)$. Hence, we know that

$$c_r(\boldsymbol{z}(t), 1/A(t)) = c_r(\boldsymbol{z}(t), 1)/A(t).$$

The second part of the proposition follows from this equation. □

Proof of Proposition 2. The expenditure function is

$$e_h(\boldsymbol{w}, \boldsymbol{p}, u_h^0) = \min_{T_h, c_h, l_h} \{T_h : u_h(c_h, l_h) \geq u_h^0, \text{ and } \sum_r p_r c_h \mathbf{1}[l_h = r] \leq \sum_r a_{hr} w_r \mathbf{1}[l_h = r] + T_h\}.$$

Since utility is increasing in consumption, the weak inequalities above have to be equalities. The first constraint implies that:

$$u_h(c_h, l_h) = g(\bar{c}_{hl_h}) + \epsilon_{hl_h} = u_h^0$$

Hence,

$$e_h(\boldsymbol{w}, \boldsymbol{p}, u_h^0) = \min_{T_h, l_h} \{T_h : \sum_r p_r \bar{c}_{hr} \mathbf{1}[l_h = r] \leq \sum_r a_{hr} w_r \mathbf{1}[l_h = r] + T_h\}.$$

The weak inequality above is strict at an optimum point, so substituting out for T_h yields

$$e_h(\boldsymbol{w}, \boldsymbol{p}, u_h^0) = -\max_{l_h}\{\sum_r a_{hr} w_r \mathbf{1}[l_h = r] - \sum_r p_r \bar{c}_{hl_h} \mathbf{1}[l_h = r]\}.$$

The maximizer of this problem is the compensated choice $l_h^{\text{comp}}(\boldsymbol{w}, \boldsymbol{p}, u_h^0)$. □

Proof of Theorem 1. Aggregate productivity solves

$$A(t) = \max_{\boldsymbol{T}, Z} \left\{ Z^{-1} : \{c_h, l_h\}_{h \in H} \in \mathcal{C}(t, Z) \text{ and } (c_h, l_h) \succeq_h (c_h(0), l_h(0)) \text{ for every } h \right\}.$$

Let $\boldsymbol{T}^*$ be a maximizer of the problem above. The indifference condition of each household implies that:

$$u_h(c_h(\boldsymbol{T}^*), l_h(\boldsymbol{T}^*)) = u_h^0,$$

or that

$$g(c_h(\boldsymbol{T}^*)) + \epsilon_{hl_h(\boldsymbol{T}^*)} = u_h^0$$

Hence,

$$c_h(\boldsymbol{T}^*) = g^{-1}\left(u_h^0 - \epsilon_{hl_h(\boldsymbol{T}^*)}\right) \equiv \bar{c}_{hl_h(\boldsymbol{T}^*)}.$$

$$\frac{\frac{a_{hl_h(\boldsymbol{T}^*)} w_{l_h(\boldsymbol{T}^*)}(\boldsymbol{T}^*)}{A} + T_h^*}{p_{l_h(\boldsymbol{T}^*)}(\boldsymbol{T}^*)} = \bar{c}_{hl_h(\boldsymbol{T}^*)}.$$

Rearrange this to get that the optimal compensating transfers satisfies:

$$T_h^* = p_{l_h(\boldsymbol{T}^*)}(\boldsymbol{T}^*) \bar{c}_{hl_h(\boldsymbol{T}^*)} - \frac{a_{hl_h(\boldsymbol{T}^*)} w_{l_h(\boldsymbol{T}^*)}(\boldsymbol{T}^*)}{A}$$

The sum of these transfers must add up to zero, which implies that

$$\sum_h \left[p_{l_h(\boldsymbol{T}^*)}(\boldsymbol{T}^*) \bar{c}_{hl_h(\boldsymbol{T}^*)} - \frac{a_{hl_h(\boldsymbol{T}^*)} w_{l_h(\boldsymbol{T}^*)}(\boldsymbol{T}^*)}{A} \right] = 0.$$

Rearranging this gives an equation that A has to satisfy given the optimal $\boldsymbol{T}^*$:

$$A = \frac{\sum_h a_{hl_h(\boldsymbol{T}^*)} w_{l_h(\boldsymbol{T}^*)}(\boldsymbol{T}^*)}{\sum_h p_{l_h(\boldsymbol{T}^*)}(\boldsymbol{T}^*) \bar{c}_{hl_h(\boldsymbol{T}^*)}}.$$

We know by definition that

$$l_h(\boldsymbol{T}^*) = l_h^{\text{comp}}\left(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), u_h^0\right)$$

and we know that

$$T_h^* = e_h(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), u_h^0).$$

Using the fact that

$$L_r^{\text{comp}}(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), \boldsymbol{u}^0) = \sum_h a_{hr} \mathbf{1}[l_h^{\text{comp}}(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), u_h^0) = r].$$

and the fact that

$$\begin{aligned}\sum_h p_{l_h(\boldsymbol{T}^*)}(\boldsymbol{T}^*)\bar{c}_{hl_h(\boldsymbol{T}^*)} &= \frac{w_r}{A} L_r^{\text{comp}}(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), \boldsymbol{u}^0) + \sum_h e_h(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), u_h^0) \mathbf{1}\left[l_h^{\text{comp}} = r\right] \\ &= E_r^{\text{comp}}(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), \boldsymbol{u}^0),\end{aligned}$$

we can write

$$A = \frac{\sum_r w_r(\boldsymbol{T}^*) L_r^{\text{comp}}(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), \boldsymbol{u}^0)}{\sum_r E_r^{\text{comp}}(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), \boldsymbol{u}^0)}$$

where $\boldsymbol{w}(\boldsymbol{T}^*)$ and $\boldsymbol{p}(\boldsymbol{T}^*)$ are prices and wages in the compensated equilibrium given compensated location choices $L_r^{\text{comp}}(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), \boldsymbol{u}^0)$ and compensated final demand $E_r^{\text{comp}}(\boldsymbol{w}(\boldsymbol{T}^*)/A, \boldsymbol{p}(\boldsymbol{T}^*), \boldsymbol{u}^0)$ and total factor productivity $1/A$. □

Proof of Theorem 2. Define the expenditure function to be

$$\begin{aligned} e_h(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h) &= \min_{T_h} \left\{ T_h : \max_r g\left(\frac{\frac{w_r a_{hr}}{A} + T_h}{p_r}\right) + \epsilon_{hr} \geq u_h^0 \right\} \\ &= T_h + \mu_h \left[\max_r \left\{ g\left(\frac{\frac{w_r a_{hr}}{A} + T_h}{p_r}\right) + \epsilon_{hr} \right\} - u_h^0 \right]. \end{aligned}$$

Hence,

$$\frac{\partial e_h}{\partial u_h^0} = -\mu_h.$$

Define the indirect utility function of the agent to be

$$v_h(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, T_h) = \max\left\{ u_h(c_h, l_h) : p_{l_h} c_h = \frac{w_{l_h} a_{hl_h}}{A} + T_h \right\}.$$

Then, by the envelope theorem, we have

$$\frac{\partial v_h}{\partial T_h} = g'\left(\frac{\frac{w_r a_{hl_h}}{A} + T_h}{p_{l_h}}\right) \frac{1}{p_{l_h}}.$$

For households that switch location, the derivative above needs to be evaluated from the appropriate direction, see Milgrom and Segal (2002). However, this issue will not play an important role in the proof, because the set of households that are exactly indifferent between two locations is measure zero. We also have the identity that

$$e_h(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, v_h(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, T_h)) = T_h.$$

Differentiating this identity gives

$$\frac{\partial e_h}{\partial u_h}\frac{\partial v_h}{\partial T_h} = 1.$$

Combining equations, we have that

$$-\mu_h = \frac{\partial e_h}{\partial u_h} = \left[\frac{\partial v_h}{\partial T_h}\right]^{-1} = \frac{p_{l_h}}{g'\left(\frac{\frac{a_{hl_h} w_{l_h}}{A} + T_h}{p_{l_h}}\right)}.$$

We know that A satisfies the feasibility condition:

$$\sum_h e_h(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h^0) = 0. \tag{37}$$

For households that are not on the margin of switching locations, the expenditure function is totally differentiable in t. We totally differentiate in t and write $\boldsymbol{w}$ and $\boldsymbol{p}$ for the wages and prices in the compensated equilibrium without explicitly writing the superscript comp on every variable. The reason is that in this proof, we never refer to the decentralized equilibrium — only to the compensated one. We have

$$\begin{aligned}
\frac{\partial e_h(\boldsymbol{w}/A, \boldsymbol{p}, u_h)}{\partial\left(w_i/A\right)} &= a_{hl_h}\mu_h g'(\frac{\frac{a_{hl_h} w_{l_h}}{A} + T_h}{p_{l_h}})\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h} w_{l_h}}{A} + T_h}{p_{l_h}}) = i\right]\frac{1}{p_{l_h}} \\
&= -a_{hl_h}\frac{p_{l_h}}{g'\left(\frac{\frac{a_{hl_h} w_{l_h}}{A} + T_h}{p_{l_h}}\right)}\frac{1}{p_{l_h}} g'(\frac{\frac{a_{hl_h} w_{l_h}}{A} + T_h}{p_{l_h}})\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h} w_{l_h}}{A} + T_h}{p_{l_h}}) = i\right] \\
&= -a_{hl_h}\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h} w_{l_h}}{A} + T_h}{p_{l_h}}) = i\right],
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial e_h(\boldsymbol{w}/A,\boldsymbol{p},u_h)}{\partial p_i} &= -\mu_h g'(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}})\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}})=i\right]\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}}\frac{1}{p_{l_h}} \\
&= \frac{p_{l_h}}{g'\left(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}}\right)}g'(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}})\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}})=i\right]\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}}\frac{1}{p_{l_h}} \\
&= \frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}}\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}})=i\right]
\end{aligned}
$$

Total differentiating (37),

$$
\sum_r\sum_h \frac{\partial e_h(\frac{\boldsymbol{w}}{A},\boldsymbol{p},u_h^0)}{\partial\left[w_r/A\right]}d\left[\frac{w_r}{A}\right]+\sum_r\sum_h\frac{\partial e_h(\frac{\boldsymbol{w}}{A},\boldsymbol{p},u_h^0)}{\partial p_r}dp_r=0.
$$

Substituting the expressions above,

$$
\begin{aligned}
\sum_r\sum_h a_{hl_h}\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}})=r\right]d\left[\frac{w_r}{A}\right] &= \sum_r\sum_h\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}}\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}})=r\right]dp_r \\
\sum_r L_r^{\text{comp}}d\left[\frac{w_r}{A}\right] &= \sum_r\sum_h\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}}\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}})=r\right]dp_r \\
\sum_r L_r^{\text{comp}}d\left[\frac{w_r}{A}\right] &= \sum_r d\log p_r\sum_h\left[\frac{a_{hl_h}w_{l_h}}{A}+T_h\right]\mathbf{1}\left[l_h(\frac{\frac{a_{hl_h}w_{l_h}}{A}+T_h}{p_{l_h}})=r\right] \\
\sum_r L_r^{\text{comp}}\left[\frac{w_r}{A}\right]d\log\left[\frac{w_r}{A}\right] &= \sum_r E_r^{\text{comp}}d\log p_r \qquad (38) \\
\sum_r L_r^{\text{comp}}\left[\frac{w_r}{A}\right]d\log\left[A\right] &= \sum_r L_r^{\text{comp}}\left[\frac{w_r}{A}\right]d\log\left[w_r\right]-\sum_r E_r^{\text{comp}}d\log p_r \\
d\log A &= \sum_r\frac{L_r^{\text{comp}}\left[\frac{w_r}{A}\right]}{\sum_r L_r^{\text{comp}}\left[\frac{w_r}{A}\right]}d\log\left[w_r\right]-\sum_r\frac{E_r^{\text{comp}}}{\sum_r L_r^{\text{comp}}\left[\frac{w_r}{A}\right]}d\log p_r \\
d\log A &= \sum_r\lambda_r d\log w_r-\sum_r\chi_r d\log p_r.
\end{aligned}
$$

Shephard's lemma for producer r implies that

$$
d\log p_i=-d\log z_i+\sum_j\Omega_{ij}d\log p_j+\sum_r\Omega_{ir}d\log w_r,
$$

where w_r is the wage per efficiency unit of labor of type r in the compensated equilibrium.

Inverting this system of equations yields

$$d\log p = -\Psi d\log z + \Psi^F d\log w,$$

where $\Psi = (I-\Omega)^{-1}$ is the Leontief inverse and Ψ^F is the matrix of factor contents. Substitute this into the expression for $d\log A$ to get

$$\begin{aligned} d\log A &= \sum_r \lambda_r d\log w_r - \sum_r \chi_r \left[-\Psi d\log z + \Psi^F d\log w\right] \\ &= \sum_r \lambda_r d\log w_r + \sum_r \chi_r \Psi d\log z - \sum_r \chi_r \Psi^F d\log w \\ &= \sum_r \lambda_r d\log w_r + \sum_{i\in N} \lambda_i d\log z_i - \sum_r \lambda_r d\log w_r \\ &= \sum_{i\in N} \lambda_i d\log z_i. \end{aligned}$$

By the fundamental theorem of calculus,

$$\log A = \int \sum_{i\in N} \lambda_i d\log z_i,$$

as needed.

□

Proof of Proposition 3. To get (18), log differentiate input demand for CES. To get (19) use Shephard's lemma, constant-returns-to-scale, and the fact that price equals marginal cost. To get (20) log differentiate the definition of λ_r for $r \in R$. To get (21), log differentiate (6) (the equation is the same regardless of whether i is a factor or a good). All these equations are equilibrium conditions that must hold in both the decentralized economy and the compensated equilibrium. □

Proof of Proposition 4. The first part follows from:

$$\begin{aligned} L_r^{\text{comp}}(\boldsymbol{w}/A, p^c, \boldsymbol{u}) &= \int a_{hr} \left[r \in \arg\max_i \left\{ a_{hi}\frac{w_i}{Ap^c} + \frac{e_h(\boldsymbol{w}/A, p^c, u_h)}{p^c} + \epsilon_{hi} \right\} \right] \\ &= \int a_{hr} \left[r \in \arg\max_i \left\{ a_{hi}\frac{w_i}{Ap^c} + \epsilon_{hi} \right\} \right] \\ &= L(\boldsymbol{w}/A, p^c) = L(\boldsymbol{w}/(Ap^c), 1), \end{aligned}$$

showing that the compensated and uncompensated supply systems have the same functional form.

Equation (22) is just the total derivative of L_r^{comp} with respect to t. This pins down the equilibrium in conjunction with equations (17) to (21). The only unknown terms are the ones involving χ^{comp} in (21). However, since $\Omega_{ri} = \Omega_{r'i}$ for every r and r' in R (common consumption good), the terms involving χ^{comp} drop out of (21). In particular,

$$\sum_r d\chi_r \Omega_{ri} + \sum_r \chi_r d\Omega_{ri} = d\Omega_{ri},$$

since we know that $\sum_r d\chi_r = 0$ and $\sum_r \chi_r = 1$. □

Proof of Proposition 5. For the uncompensated equilibrium, the result follows easily from log-differentiating the definition:

$$\chi_r = \frac{w_r L_r}{\sum_{r'} w_{r'} L_{r'}}.$$

Now consider the change in the spending shares by location, $d\chi^{\text{comp}}$,in the compensated equilibrium. To keep the notation more manageable, we do not include the superscript comp on every variable and simply note that everything is in the compensated equilibrium.

Let $T(\epsilon, t)$ be the compensating transfer at t received by an agent with tastes ϵ. Let $B_r(t)$ denote the set of ϵ that choose location r in the compensated equilibrium. Note that it is defined by

$$B_r(t) = \left\{ \epsilon : g\left(\frac{\frac{w_r(t)}{A(t)} + T(\epsilon,t)}{p_r(t)}\right) + \epsilon_r - \max\left\{ g\left(\frac{\frac{w_{r'}(t)}{A(t)} + T(\epsilon,t)}{p_{r'}(t)}\right) + \epsilon_{r'} \right\} \leq 0 \right\} = \left\{\epsilon : \phi_r(\epsilon, t) \leq 0\right\}.$$

Households on the margin of choosing r are defined by $\phi(\epsilon, t) = 0$. Then the change in share of expenditures in each location satisfies:

$$\begin{aligned} d\log\chi_r &= d\log\int [p_r \bar{c}_{hr}] \mathbf{1}[l(h) = r]\, dh - \underbrace{d\log\sum_{r'}\int [p_r \bar{c}_{hr}] \mathbf{1}\left[l(h) = r'\right] dh}_{\equiv X}, \\ &= \frac{1}{\lambda_r} d\int [p_r \bar{c}_{hr}] \mathbf{1}[l(h) = r]\, dh - X \\ &= \frac{1}{\lambda_r} d\left[\int_{B_r(t)} [p_r \bar{c}_r(\epsilon)]\, f(\epsilon) d\epsilon\right] - X, \end{aligned}$$

where $\bar{c}_r(\epsilon)$ is the consumption-equivalent and $f(\epsilon)$ is the density of households with tastes ϵ. Using the fact that $p_r \bar{c}_r(\epsilon) = \frac{w_r}{A} + T(\epsilon, t)$, we can write

$$\begin{aligned} d\log\chi_r &= \frac{1}{\lambda_r} d\left[\int_{B_r}\left[\frac{w_r(t)}{A(t)} + T(\epsilon,t)\right] f(\epsilon)d\epsilon\right] - X \\ &= \frac{1}{\lambda_r}\int_{B_r}\left[d\frac{w_r}{A} + \frac{\partial}{\partial t}T(\epsilon,t)\right] f(\epsilon)d\epsilon + \frac{1}{\lambda_r}\int\left[\frac{w_r(t)}{A(t)} + T(\epsilon,t)\right]\delta(\phi(z,\epsilon))f(\epsilon)\|\nabla_\epsilon\phi(z,\epsilon)\|d\epsilon - X \end{aligned}$$

where the second line uses Leibniz' rule. At the status quo, where we differentiate, we have $\frac{w_r(0)}{A(0)} + T(\epsilon, 0) = w_r(0)$. Dropping time subscripts, we can write

$$d\log\chi_r = \frac{1}{\lambda_r}\int_{B_r}\left[d\frac{w_r}{A} + \frac{\partial}{\partial t}T(\epsilon,t)\right] f(\epsilon)d\epsilon + \frac{1}{\lambda_r}\int w_r\delta(\phi(z,\epsilon))f(\epsilon)\|\nabla_\epsilon\phi(z,\epsilon)\|d\epsilon - X.$$

The first summand is the change in the compensating income of each household in location r and the second summand is the wage in location r times the mass of households that move to location r in the compensated equilibrium. Hence, we can write

$$\begin{aligned} d\log\chi_r &= \frac{1}{\lambda_r}\left[\int_{B_r}\left[d\frac{w_r}{A} + dT(\epsilon)\right] f(\epsilon)d\epsilon\right] + \frac{1}{\lambda_r}\left[w_r dL_r\right] - X \\ &= \frac{1}{\lambda_r}\left[\frac{w_r}{p_r}dp_r\int_{B_r} f(\epsilon)d\epsilon\right] + \frac{1}{\lambda_r}\left[w_r dL_r\right] - X \\ &= \frac{1}{\lambda_r}\left[\frac{w_r}{p_r}dp_r\right] L_r + \frac{1}{\lambda_r}\left[w_r dL_r\right] - X \\ &= \frac{1}{\lambda_r}\left[\frac{L_r w_r}{p_r}dp_r\right] + \frac{1}{\lambda_r}\left[w_r dL_r\right] - X \\ &= d\log p_r + \frac{1}{\lambda_r}dL_r w_r - X \\ &= d\log p_r + d\log L_r - X. \end{aligned}$$

By definition, we must have

$$\mathbb{E}_\chi\left[d\log\chi\right] = 0,$$

hence,

$$X = \mathbb{E}_\chi\left[d\log p_r + d\log L_r\right].$$

This implies that, in the compensated equilibrium, we have

$$d\log\chi_r = \left[d\log p_r + d\log L_r\right] - \mathbb{E}_\chi\left[d\log p_r + d\log L_r\right].$$

□

Proof of Proposition 6. We start with the uncompensated equilibrium and then consider the compensated equilibrium.

Uncompensated Equilibrium. Let $B_i(t)$ denote the set of ϵ that choose i:

$$B_i(t) = \left\{\epsilon : g(\frac{w_i(t)}{p_i(t)}) + \epsilon_i \geq g(\frac{w_k(t)}{p_k(t)}) + \epsilon_k, \forall\, k \neq i\right\}$$
$$= \{\epsilon : \phi_i(t,\epsilon) \leq 0\},$$

where

$$\phi_i(t,\epsilon) = \max_k\{g(\frac{w_k(t)}{p_k(t)}) + \epsilon_k - g(\frac{w_i(t)}{p_i(t)}) - \epsilon_i\}.$$

The set of movers from i to j is given by

$$\begin{aligned}
L_{i\to j} &= \int \mathbf{1}(\epsilon \in B_i(0))\mathbf{1}(\epsilon \in B_j(t))f(\epsilon)d\epsilon \\
&= \int_{B_j(t)} \mathbf{1}(\epsilon \in B_i(0))f(\epsilon)d\epsilon. \\
dL_{i\to j} &= d\left[\int_{B_j(t)} \mathbf{1}(\epsilon \in B_i(0))f(\epsilon)d\epsilon\right] \\
&= \left[\int_{dB_j(t)} \mathbf{1}(\epsilon \in B_i(0))f(\epsilon)d\epsilon\right] \\
&= \left[\int \mathbf{1}(\epsilon \in B_i(0))\delta(\phi_j(0,\epsilon))\|\nabla_\epsilon \phi_j(0,\epsilon)\| f(\epsilon)d\epsilon\right],
\end{aligned}$$

where δ is the Dirac delta function which means the integral is evaluated at the boundary of $B_j(0)$, that is where $\phi_j(0,\epsilon) = 0$ and $\mathbf{1}(\epsilon \in B_i(0))$. For any vector $\boldsymbol{x}$, let $h(\boldsymbol{x})$ be the density of households with $\epsilon_j - \epsilon_i = x_i - x_j$ and $\epsilon_j - \epsilon_k \geq x_k - x_j$ and $\epsilon_i - \epsilon_k \geq x_k - x_i$. This the density of households that are indifferent between i and j (and prefer i and j to all other options) given payoffs are $x_i = g(c_i)$. Define $\epsilon_{ij} = x_i - x_j$ to be the cut-off between i and j. Then we can write

$$\begin{aligned}
dL_{i\to j} &= -h(\boldsymbol{w}(0)/\boldsymbol{p}(0))d\epsilon_{ij}\mathbf{1}\left[d\epsilon_{ij} \leq 0\right] \\
&= -h(g(\boldsymbol{w}(0)/\boldsymbol{p}(0))\left[g'(\frac{w_i(0)}{p_i(0)})d\left[\frac{w_i}{p_i}\right] - g'(\frac{w_j(0)}{p_j(0)})d\left[\frac{w_j}{p_j}\right]\right]\mathbf{1}\left[d\epsilon_{ij} \leq 0\right]. \qquad (39)
\end{aligned}$$

We relate this equation to cross-derivatives of the supply function $\boldsymbol{L}(\boldsymbol{p},\boldsymbol{w},\boldsymbol{T})$. In particular, note that in the decentralized equilibrium, $\boldsymbol{T} = 0$, the supply function depends only on the vector of real wages $\boldsymbol{w}/\boldsymbol{p}$. Furthermore, consider a perturbation where only the real wage in location j changes (all other real wages are held constant), then from the equation above, we can write

$$\frac{\partial L_{i\to j}}{\partial(w_j/p_j)} = h(g(\boldsymbol{w}(0)/\boldsymbol{p}(0))g'(w_j(0)/p_j(0)).$$

Since

$$\begin{aligned}\frac{\partial L_i}{\partial(w_j/p_j)} &= \sum_{k\neq i}\frac{\partial L_{k\to i}}{\partial(w_j/p_j)} - \sum_{k\neq i}\frac{\partial L_{i\to k}}{\partial(w_j/p_j)}\\ &= -\frac{\partial L_{i\to j}}{\partial(w_j/p_j)}\\ &= -h(g(\boldsymbol{w}(0)/\boldsymbol{p}(0))g'(w_j(0)/p_j(0)),\end{aligned} \tag{40}$$

where the second line follows from equation (39). Using these equations, we can write

$$\begin{aligned}dL_{i\to j} &= -h(g(\boldsymbol{w}(0)/\boldsymbol{p}(0))\left[g'(w_i(0)/p_i(0))d\left[\frac{w_i}{p_i}\right] - g'(w_j(0)/p_j(0))d\left[\frac{w_j}{p_j}\right]\right]\mathbf{1}\left[d\epsilon_{ij}\leq 0\right]\\ &= \left[\frac{\partial L_j}{\partial(w_i/p_i)}d\left[\frac{w_i}{p_i}\right] - \frac{\partial L_i}{\partial(w_j/p_j)}d\left[\frac{w_j}{p_j}\right]\right]\mathbf{1}\left[d\epsilon_{ij}\leq 0\right]\\ &= \left[\frac{\partial L_j}{\partial(w_i/p_i)}d\left[\frac{w_i}{p_i}\right] - \frac{\partial L_i}{\partial(w_j/p_j)}d\left[\frac{w_j}{p_j}\right]\right]\mathbf{1}\left[h(g(\boldsymbol{w}(0)/\boldsymbol{p}(0))d\epsilon_{ij}\leq 0\right]\\ &= \left[\frac{\partial L_j}{\partial(w_i/p_i)}d\left[\frac{w_i}{p_i}\right] - \frac{\partial L_i}{\partial(w_j/p_j)}d\left[\frac{w_j}{p_j}\right]\right]\mathbf{1}\left[\left[\frac{\partial L_j}{\partial(w_i/p_i)}d\left[\frac{w_i}{p_i}\right] - \frac{\partial L_i}{\partial(w_j/p_j)}d\left[\frac{w_j}{p_j}\right]\right]\geq 0\right],\end{aligned}$$

which is the expression in the proposition (swapping i and j for r and r').

Compensated Equilibrium. We now consider the compensated equilibrium. To show how location choices change, to a first order, let $T(\epsilon,t)$ be the compensating transfer in the compensated equilibrium for households with preferences ϵ. The set of households that are marginal movers between region i and j in the compensated equilibrium is

$$F_{ij}(t) = \left\{\epsilon : g(\frac{w_i(t)+T(\epsilon,t)}{p_i(t)}) + \epsilon_i = g(\frac{w_j(t)+T(\epsilon,t)}{p_j(t)}) + \epsilon_j \geq g(\frac{w_k(t)+T(\epsilon,t)}{p_k(t)}) + \epsilon_k, \forall\, k\notin\{j,i\}\right\}.$$

For every $\epsilon \in F_{ij}(t)$, the following equation holds:

$$\epsilon_{ij}(t) = \epsilon_{hj} - \epsilon_{hi} = g(\frac{w_i(t)/A(t) + T(\epsilon, t)}{p_i(t)}) - g(\frac{w_j(t)/A + T(\epsilon, t)}{p_j(t)}).$$

We call $\epsilon_{ij}(t)$ the cut-off values for i and j (this is different to the cut-off values in the uncompensated decentralized equilibrium because of the compensating transfers). In the status quo, transfers are zero, so the cutoff between regions i and j is

$$\epsilon_{ij}^0 = g(\frac{w_i^0}{p_i^0}) - g(\frac{w_j^0}{p_j^0}).$$

Differentiating the expression for $\epsilon_{ij}(t)$ with respect to t and suppressing dependence on t gives:

$$d\epsilon_{ij} = g'(\frac{w_i/A + T(\epsilon)}{p_i})d\left[\frac{w_i/A + T(\epsilon)}{p_i}\right] - g'(\frac{w_j/A + T(\epsilon)}{p_j})d\left[\frac{w_j/A + T(\epsilon)}{p_j}\right].$$

At the status quo, since $A = 1$ and $T(\epsilon, 0) = 0$, we have

$$d\epsilon_{ij} = g'(\frac{w_i}{p_i})\left[d\left[w_i/Ap_i\right] + \frac{1}{p_i}dT(\epsilon)\right] - g'(\frac{w_j}{p_j})d\left[d\left[w_j/Ap_j\right] + \frac{1}{p_j}dT(\epsilon)\right].$$

Note that

$$T(\epsilon, t) = e(\frac{\boldsymbol{w}(t)}{A(t)}, \boldsymbol{p}(t), u_h^0(\epsilon), \epsilon),$$

where the fourth summand captures how the expenditure function varies as a function of tastes, holding wages, prices, and utility constant. Hence, at the status quo, totally differentiating with respect to t yields:

$$dT(\epsilon) = e_1(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h^0, \epsilon) \cdot d\left[\frac{\boldsymbol{w}}{A}\right] + e_2(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h^0, \epsilon) \cdot d\boldsymbol{p} + \left[e_3(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h^0, \epsilon)\frac{du_h^0(\epsilon)}{d\epsilon} + e_4(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h^0, \epsilon)\right] d\epsilon,$$

where $\epsilon \in F_{ij}(t)$ and $d\epsilon$ is evaluated for some path that remains in $F_{ij}(z, t)$ as t changes. However, in the status quo, $T(\epsilon, 0) = 0$. Hence, moving in the cross-section of households in the status quo, we have

$$e_3(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h^0, \epsilon)\frac{du_h^0}{d\epsilon} + e_4(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h^0(\epsilon), \epsilon) = 0,$$

which means we can ignore change in the transfers due to changes in the identity of the marginal household, and the expression above simplifes to just

$$dT(\epsilon) = e_1(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h^0(\epsilon), \epsilon) \cdot d\left[\frac{\boldsymbol{w}}{A}\right] + e_2(\frac{\boldsymbol{w}}{A}, \boldsymbol{p}, u_h^0(\epsilon), \epsilon) \cdot d\boldsymbol{p},$$

where ϵ is evaluated for some household that's marginal between i and j. Using the envelope theorem,

$$\begin{aligned} dT(\epsilon) &= \lim_{t\to 0+}\left[\sum_{i\in R} \mathbf{1}\left[l_h(\frac{\frac{w_r}{A}+T(\epsilon_{ij})}{p_r}) = i\right]\left(-d\left[\frac{w_i}{A}\right] + \frac{w_i}{p_i}dp_i\right)\right] \\ &= \lim_{t\to 0+}\left[\sum_{i\in R} \mathbf{1}\left[l_h(\frac{w_r}{p_r}) = i\right]\left(-d\left[\frac{w_i}{A}\right] + \frac{w_i}{p_i}dp_i\right)\right], \end{aligned}$$

where $\epsilon \in F_{ij}(t)$. In particular, at the status quo, the change in compensating transfers dT is the same for every household that is marginal between choosing i and j regardless of the other values ϵ_{hk}. If $d\epsilon_{ij} < 0$ then households move from i to j in the compensated equilibrium, else the reverse. Denote the change in compensating transfers for these marginal households by the following symbol:

$$dT_{ij} = \lim_{t\to 0+}\left[\sum_{i\in R} \mathbf{1}\left[l_h(\frac{w_r}{p_r}) = i\right]\left(-d\left[\frac{w_i}{A}\right] + \frac{w_i}{p_i}dp_i\right)\right]$$

Hence, we can write

$$dT_{ij} = \left(-d\left[\frac{w_i}{A}\right] + \frac{w_i}{p_i}dp_i\right)\mathbf{1}\left[d\epsilon_{ij} < 0\right] + \left(-d\left[\frac{w_j}{A}\right] + \frac{w_j}{p_j}dp_j\right)\mathbf{1}\left[d\epsilon_{ij} \geq 0\right]$$

Substitute this into the expression for $d\epsilon_{ij}$, in the case where $d\epsilon_{ij} < 0$, so households move from i to j, we get

$$\begin{aligned} d\epsilon_{ij} = g'(\frac{w_i}{p_i})\left[d\left[w_i/Ap_i\right] + \frac{1}{p_i}\left[-d\left[\frac{w_i}{A}\right] + \frac{w_i}{p_i}dp_i\right)\right] - \\ g'(\frac{w_j}{p_j})\left[d\left[w_j/Ap_j\right] + \frac{1}{p_j}\left(-d\left[\frac{w_i}{A}\right] + \frac{w_i}{p_i}dp_i\right)\right]. \end{aligned}$$

Use the fact that

$$d\left[w_i/\left(Ap_i\right)\right] + \left[-d\left[w_i/A\right] + \frac{w_i}{p_i}dp_i\right]/p_i = 0$$

to get

$$d\epsilon_{ij} = -g'(\frac{w_j}{p_j})\left[d\left[w_j/Ap_j\right] - \frac{p_i}{p_j}d\left[w_i/\left(Ap_i\right)\right]\right]$$

which requires that $p_j d\left[w_j/Ap_j\right] > p_i d\left[w_i/\left(Ap_i\right)\right]$. Similarly, if $d\epsilon_{ij} > 0$, we instead get

$$d\epsilon_{ij} = g'(\frac{w_i}{p_i})\left[d\left[w_j/Ap_j\right] - \frac{p_i}{p_j}d\left[w_i/\left(Ap_i\right)\right]\right]$$

The mass of movers from i to j is then given by

$$\begin{aligned}
L_{i\to j} &= \int \mathbf{1}\left[l_h(\boldsymbol{p}(0),\boldsymbol{w}(0),\mathbf{0}) = i\right]\mathbf{1}\left[l_h(\boldsymbol{p}(t),\boldsymbol{w}(t)/A(t),T_h(t)) = j\right]dh. \\
dL_{i\to j} &= -h(\epsilon_{ij}^0)d\epsilon_{ij}\mathbf{1}\left[d\epsilon_{ij} < 0\right] \\
&= h(\epsilon_{ij}^0)g'(\frac{w_j}{p_j})\left[d\left[w_j/Ap_j\right] - \frac{p_i}{p_j}d\left[w_i/\left(Ap_i\right)\right]\right]\mathbf{1}\left[p_j d\left[w_j/Ap_j\right] > p_i d\left[w_i/\left(Ap_i\right)\right]\right],
\end{aligned}$$

where $h(\epsilon_{ij}^0)$ is the density of households for whom $\epsilon \in F_{ij}(0)$.

Note that in the uncompensated economy, in response to the perturbation $d\left[w_j/p_j\right] > 0$ — holding all other real wages fixed — (40) implies that

$$-\frac{\partial L_i}{\partial\left[w_j/p_j\right]}d\left[w_j/p_j\right] = h(\epsilon_{ij}^0)g'(w_j/p_j)d\left[w_j/p_j\right].$$

Hence, we can replace

$$-\frac{\partial L_i}{\partial\left[w_j/p_j\right]} = h(\epsilon_{ij}^0)g'(w_j/p_j). \tag{41}$$

Using this we can rewrite

$$\begin{aligned}
dL_{i\to j} &= h(\epsilon_{ij}^0)g'(\frac{w_j}{p_j})\left[d\left[w_j/Ap_j\right] - \frac{p_i}{p_j}d\left[w_i/\left(Ap_i\right)\right]\right]\mathbf{1}\left[p_j d\left[w_j/Ap_j\right] > p_i d\left[w_i/\left(Ap_i\right)\right]\right] \\
&= -\frac{\partial L_i}{\partial\left[w_j/p_j\right]}\frac{1}{p_j}\left[p_j d\left[w_j/Ap_j\right] - p_i d\left[w_i/\left(Ap_i\right)\right]\right]\mathbf{1}\left[p_j d\left[w_j/Ap_j\right] > p_i d\left[w_i/\left(Ap_i\right)\right]\right].
\end{aligned}$$

The change in the compensated labor supply of location i then follows from the equation that:

$$dL_i = \sum_{j\neq i} dL_{j\to i} - \sum_{j\neq i} dL_{i\to j}.$$

□

Proof of Proposition 7. By Hulten (1978), to first-order, we have:

$$\Delta \log Y \approx \sum_i \lambda_i(0) \Delta \log z_i + \sum_{r \in R} \lambda_r(0) \Delta \log L_r.$$

We then use the fact that to a first-order,

$$\lambda_r(0) \Delta \log L_r = \frac{w_r(0)}{\sum_{r'} w_{r'}(0) L_{r'}(0)} \Delta L_r.$$

□

Proof of Proposition 8. From Hulten's theorem, we know that

$$\log A^{\text{MFP}}(t) = \int_0^t \sum_i \lambda_i(s) \frac{d \log z_i}{ds} ds$$

whereas from Theorem 1, we know that

$$\log A(t) = \int_0^t \sum_i \lambda_i^{\text{comp}}(s) \frac{d \log z_i}{ds} ds.$$

Note that, since $\epsilon_{hr} = \bar{\epsilon}$ for every h and r, and there is a common consumption good with the law of one price, we have that

$$l_h^{\text{comp}}(\boldsymbol{w}, \boldsymbol{p}, u_h^0) = l_h(\boldsymbol{w}, \boldsymbol{p}).$$

Furthermore, since there is only a common consumption good, Domar weights do not depend on the distribution of spending (i.e. $\chi(t)$ and $\chi^{\text{comp}}(t)$ result in the same Domar weights). Hence

$$\lambda_i(s) = \lambda_i^{\text{comp}}(s),$$

which implies that

$$A(t) = A^{MFP}(t).$$

Next, since there is only one common consumption good, real output in this economy is just the total production/consumption of that good: $Y(t) = \sum_h c_h(t) / \sum_h c_h(0)$. Since households only value the consumption good, the first welfare theorem implies that the decentralized equilibrium at t maximizes the quantity of the consumption good produced at t. By Hulten's theorem, we know that

$$\frac{d\log Y}{dt} = \sum_{i\in N} \lambda_i(t)\frac{d\log z_i}{dt} + \sum_{r\in R} \lambda_r(t)\frac{d\log L_r}{dt},$$

where the second term is a reallocation effect. However, this term must equal zero, otherwise, the equilibrium at t would not be maximizing Y. Hence,

$$\frac{d\log Y(t)}{dt} = \frac{d\log A^{MFP}(t)}{dt}$$

at every t. The result follows by integrating both sides. □

Proof of Proposition 9. Without loss of generality, let the common consumption good price p^c be the numeraire. The social surplus function is

$$U(\boldsymbol{w}) = \mathbb{E}\left[\max_r \{a_{hi}w_r + \epsilon_{hr}\}\right],$$

where w_r is now the real wage in location r. Then,

$$\begin{aligned}
\frac{\partial U}{\partial w_r} &= \frac{\partial}{\partial w_r}\int \max_i \{a_{hi}w_i + \epsilon_{hi}\} f(\epsilon)d\epsilon \\
&= \int \frac{\partial}{\partial w_r}\left[\max_i \{a_{hi}w_i + \epsilon_{hi}\}\right] f(\epsilon)d\epsilon \\
&= \int a_{hr}\mathbf{1}\left\{a_{hr}w_r + \epsilon_{hr} \geq a_{hj}w_j + \epsilon_{hj}\forall j\right\} f(\epsilon)d\epsilon \\
&= L_r(\boldsymbol{w}).
\end{aligned}$$

Equation (38) in the proof of Theorem 2 states that with a common consumption good

$$\sum_r L_r^{\text{comp}}\left[\frac{w_r}{A}\right] d\log\left[\frac{w_r}{A}\right] = \sum_r L_r^{\text{comp}}\left[\frac{w_r}{A}\right] d\log p^c,$$

where the wages and prices are the ones in the compensated equilibrium. Since the consumer price index is the numeraire, this is the same as:

$$\sum_r L_r^{\text{comp}}\left[\frac{w_r}{A}\right] d\log\left[\frac{w_r}{A}\right] = 0.$$

Equivalently,

$$\sum_r L_r^{\text{comp}} d\left[\frac{w_r}{A}\right] = 0.$$

Integrating between 0 and t gives:

$$\int_{w(0)/p^c(0)}^{w(t)/(A(t))} \sum_r L_r^{\text{comp}}(\boldsymbol{x}) d\boldsymbol{x} = 0.$$

By Proposition 4, the compensated and uncompensated labor supply functions are the same. Furthermore, given productivities $\boldsymbol{z}(t)$ and aggregate factor-augmenting productivity $1/A(t)$, the compensated and uncompensated equilibrium real wages are the same. Hence, we can write

$$\int_{w(0)}^{w(t)/(A(t))} \sum_r L_r(\boldsymbol{x}) d\boldsymbol{x} = 0,$$

substituting in $\frac{\partial U}{\partial w_r}$ from above gives

$$\int_{w(0)}^{w(t)/(A(t))} \sum_r \frac{\partial U}{\partial w_r}(\boldsymbol{x}) d\boldsymbol{x} = 0,$$

which is equal to

$$U(w(t)/(A(t))) - U(w(0)) = 0,$$

by the fundamental theorem of calculus.

□

Lemma 1. *In the one-good economy, for any preferences $u_h(c_h, l) = f_h(g(c_h) + \epsilon_{hl})$, the following is true. The problem where aggregate productivity is calculated by choosing feasible locations agent by agent,*

$$\tilde{A} = \max_{\boldsymbol{l}} \frac{\sum_r \int z_r \mathbf{1}[l_h = r]\, dh}{\sum_r \int \bar{c}_{hr} \mathbf{1}[l_h = r]\, dh}, \tag{42}$$

yields the same answer as A with the same location choices. That is, $\tilde{A} = A$.

Proof of Lemma 1. Let $\tilde{A}$ be the value of the maximization problem in (42), and denote by $\tilde{\boldsymbol{l}}$ the corresponding optimal location profile. By Dinkelbach (1967), $\tilde{\boldsymbol{l}}$ also solves the alternative problem

$$\max_{\boldsymbol{l}} \left[\sum_r \int \frac{z_r}{\tilde{A}} \mathbf{1}[l_h = r]\, dh - \sum_r \int \bar{c}_{hr} \mathbf{1}[l_h = r]\, dh \right],$$

or equivalently

$$\max_{\boldsymbol{l}} \sum_r \int \left(\frac{z_r}{\tilde{A}} - \bar{c}_{hr} \right) \mathbf{1}[l_h = r] \; dh.$$

Given $\tilde{A}$, this problem is separable across h, so it can be solved household by household:

$$\tilde{l}_h \in \arg\max_{r \in R} \left\{ \frac{z_r}{\tilde{A}} - \bar{c}_{hr} \right\}.$$

By Proposition 2, the compensated location choice of household h when aggregate productivity is $1/\tilde{A}$ is

$$l_h^{\text{comp}} \in \arg\max_{r \in R} \left\{ \frac{z_r}{\tilde{A}} - \bar{c}_{hr} \right\},$$

so $\tilde{l}_h = l_h^{\text{comp}}$ for all h. By definition of A, the value of aggregate productivity that supports this compensated allocation is precisely $A = \tilde{A}$, as claimed. □

Proof of Proposition 10. The first part of the proposition is a consequence of Lemma 1. The lemma shows that, in a one-good economy, the value of aggregate productivity $A(t)$ obtained by maximizing over all feasible location profiles $\{l_h\}_{h \in H}$ coincides with $A(t)$ defined using individual-specific lump-sum transfers. Any allocation implementable with place-based policies corresponds to some location profile, and hence is feasible in the feasible agent-by-agent location problem. Therefore, place-based policies cannot achieve a strictly larger value of aggregate productivity than individual-specific transfers, and

$$A^{pb}(t) \leq A(t) \quad \text{for all } t.$$

For the second part, we construct an explicit feasible place-based policy that yields a lower bound for $A^{pb}(t)$ and then apply a squeezing argument. Let L_r^0 denote labor supply in location r in the status quo. Define

$$A^{lb}(t) \equiv \frac{\sum_r z_r(t) L_r^0}{\sum_r z_r(0) L_r^0}. \tag{43}$$

Consider the following policy. Scale productivities by $1/A^{lb}(t)$ and choose a vector of consumption tax rates $\{\tau_r^{lb}\}_{r \in R}$ and a common lump-sum rebate T^{lb} such that, in the resulting equilibrium,

$$\frac{z_r(t)/A^{lb}(t) + T^{lb}}{1 + \tau_r^{lb}} = c_r^0 = z_r(0) \quad \text{for every } r. \tag{44}$$

In words, after taxes and transfers, consumption per capita in every location is exactly equal to its status-quo level c_r^0. This implies that every household is exactly indifferent to the status quo and has no incentive to move. Hence, location choices remain at their status-quo values. Substituting $(1+\tau_r^{lb})c_r^0 = z_r(t)/A^{lb}(t) + T^{lb}$ into the budget-balance condition (34), one can verify that such a pair $(T^{lb}, \{\tau_r^{lb}\}_r)$ exists if $A^{lb}(t)$ is given by (43).

Since this policy keeps everyone indifferent to the status quo and is implementable with place-based consumption taxes, it is a feasible solution for the problem that defines $A^{pb}(t)$. Therefore,

$$A^{pb}(t) \geq A^{lb}(t) \quad \text{for all } t.$$

In the one-good economy with exogenous labor supplies $\{L_r^0\}$, the lower-bound index $A^{lb}(t)$ coincides with a standard aggregate productivity index: $A^{lb}(t) = \sum_r z_r(t)L_r^0 / \sum_r z_r(0)L_r^0$. By Hulten's theorem, its derivative at the status quo satisfies

$$\frac{d}{dt}\log A^{lb}(t)\Big|_{t=0} = \sum_r \lambda_r(0)\,\frac{d}{dt}\log z_r(t)\Big|_{t=0}, \tag{45}$$

where $\lambda_r(0) = z_r(0)L_r^0 / \sum_{r'} z_{r'}(0)L_{r'}^0$. On the other hand, the full-transfer index $A(t)$ also satisfies Hulten's theorem at the status quo (by Corollary 1). So,

$$\frac{d}{dt}\log A(t)\Big|_{t=0} = \sum_r \lambda_r(0)\,\frac{d}{dt}\log z_r(t)\Big|_{t=0}. \tag{46}$$

Combining $A^{lb}(t) \leq A^{pb}(t) \leq A(t)$ for all t with $A^{lb}(0) = A^{pb}(0) = A(0) = 1$, we have for t close to zero and $t \neq 0$,

$$\frac{\log A^{lb}(t) - \log A^{lb}(0)}{t} \leq \frac{\log A^{pb}(t) - \log A^{pb}(0)}{t} \leq \frac{\log A(t) - \log A(0)}{t}.$$

Using (45) and (46), the left and right terms converge to the same limit. By the squeeze theorem, the middle term must converge to the same limit. Hence $\log A^{pb}(t)$ is differentiable at $t = 0$ and

$$\frac{d}{dt}\log A^{pb}(t)\Big|_{t=0} = \sum_r \lambda_r(0)\,\frac{d}{dt}\log z_r(t)\Big|_{t=0}.$$

Rewriting this in terms of finite changes yields the first-order approximation in the statement of the proposition. □

Appendix C Implications of $g(\cdot)$ for behavior

Since utility is only pinned down up to monotone transformations, the function f_h has no observable implications. However, the shape of $g(c_h)$ has testable implications, since it controls the way income and substitution effects interact with each other. For example, if $g(c)$ is linear, then household choices are invariant to an additive increase in consumption in every location (regardless of the distribution of tastes). On the other hand, if $g(c)$ is log, then household choices are invariant to scaling consumption in every location (regardless of the distribution of tastes).

The following proposition shows that the conditional labor supply function $\boldsymbol{L}(\boldsymbol{w}|\boldsymbol{a})$, mapping vectors of real wages into labor supply in each location conditional on skills $\boldsymbol{a}$, uniquely pins down $g(c_h)$ up to an affine transformation.

Proposition 11 (Relation between L and g)**.** *Let the efficiency units of labor supplied by workers given real wages $\boldsymbol{w}$ conditional on skill type $\boldsymbol{a}$ to be:*

$$L_i(\boldsymbol{w}|\boldsymbol{a}) = \int a_i \mathbf{1}\left[g(w_i a_i) + \epsilon_{hi} \geq g(a_j w_j) + \epsilon_{hj} \quad \forall j\right] dh.$$

Given knowledge of $\boldsymbol{L}(\boldsymbol{w}|\boldsymbol{a})$, the function $g(c)$ is pinned down up to an affine transformation. A notable implication is the following. The function $\boldsymbol{L}(\boldsymbol{w}|\boldsymbol{a})$ has symmetric cross-derivatives in real wages if, and only if, $g(c)$ is affine.

That is, the functional form of $g(c_h)$ has testable implications. The linear case is noteworthy because, under this assumption, some of our calculations dramatically simplify. Intuitively, if $g(c_h)$ is linear, then a lump-sum transfer (in consumption units) to household h will not change household h's choice of location.[41]

Proof. Suppose that preferences take the form

$$u_h(c, r) = g(c) + \epsilon_{hr}.$$

Consider the set of $\epsilon's$ marginal between i and j that also do not strictly prefer any other region:

$$F_{ij}(\boldsymbol{w}|\boldsymbol{a}) = \{\epsilon : g(a_i w_i) - g(a_j w_j) = \epsilon_{hj} - \epsilon_{hi}, \, g(a_i w_i) - g(a_k w_k) \geq \epsilon_{hk} - \epsilon_{hi}, \, \forall k \neq i\}.$$

[41] As another example, the population share function without transfers $\boldsymbol{L}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{0})$ has symmetric cross semi-elasticities in real wages $\partial L_r / \partial \log w_{r'} / p_{r'} = \partial L_{r'} / \partial \log w_r / p_r$ if, and only if, $g(c_h)$ is a log function of c_h. The commonly used constant-elasticity Fréchet supply system is a special case and requires that $g(c_h)$ be log.

Define the set that prefer region i by $B_i(\boldsymbol{w}|\boldsymbol{a})$. We can write

$$L_i(\boldsymbol{w}|\boldsymbol{a}) = \int a_i \mathbf{1}\left[\epsilon \in B_i(\boldsymbol{w}|\boldsymbol{a})\right] f(\epsilon) d\epsilon,$$

where $f(\epsilon)$ is the density of ϵ. A marginal change in w_j lowers the share in region i by:

$$\frac{\partial L_i(\boldsymbol{w}|\boldsymbol{a})}{\partial w_j} = -a_i f(g(a_i w_i) - g(a_j w_j)) a_j g'(a_j w_j).$$

Similarly

$$\frac{\partial L_j(\boldsymbol{w}|\boldsymbol{a})}{\partial w_i} = -a_j f(g(a_i w_i) - g(a_j w_j)) a_i g'(a_i w_i).$$

Hence,

$$\frac{\frac{\partial L_i(\boldsymbol{w}|\boldsymbol{a})}{\partial w_j}}{\frac{\partial L_j(\boldsymbol{w}|\boldsymbol{a})}{\partial w_i}} = \frac{g'(a_j w_j)}{g'(a_i w_i)}.$$

Define the function

$$g_{ij}(w_i, w_j) = \frac{\partial L_i(\boldsymbol{w}|\boldsymbol{a})/\partial w_j}{\partial L_j(\boldsymbol{w}|\boldsymbol{a})/\partial w_i} = \frac{g'(a_j w_j)}{g'(a_i w_i)}.$$

Hence,

$$g'(a_j w_j) = g_{ij}(w_i, w_j) g'(a_i w_i).$$

By the fundamental theorem of calculus we have

$$g(w_j) - g(1) = g'(a_i w_i) \int_{1/a_j}^{w_j/a_j} g_{ij}(w_i, x) dx.$$

Specifically, at $w_i = 1/a_i$, we get

$$\begin{aligned} g(w_j) - g(1) &= g'(1) \int_{1/a_j}^{w_j/a_j} g_{ij}(1, x) dx \\ \frac{g(w_j) - g(1)}{g'(1)} &= \int_{1/a_j}^{w_j/a_j} g_{ij}(1, x) dx = G_{ij}(w_j/a_j) - G_{ij}(1/a_j), \end{aligned}$$

where $G_{ij}(x)$ is the antiderivative of the function $g_{ij}(1, x)$. Hence, the function g is pinned down up to an affine transformation. □

Appendix D Additional Examples and Derivations

D.1 Cobb-Douglas and Logit Example

Every good is produced using a Cobb-Douglas production technology, so (5) implies that the price of each good $i \in N$ can be written as

$$p_i = z_i^{-1} \prod_{j \in N} p_j^{\Omega_{ij}} \prod_{r \in R} w_r^{\Omega_{ir}},$$

where Ω_{ij} and Ω_{ir} are expenditure shares of i on j and r respectively (Cobb-Douglas implies that these expenditure shares are constant). We can solve out for λ_r in the market clearing condition, (6), to get that for every factor r:

$$\lambda_r = \sum_{r'} \chi_r \Psi_{c(r')r},$$

where $\Psi = (I - \Omega)^{-1}$ is the Leontief inverse (which, again, is constant due to the Cobb-Douglas assumption). This equation states that income of r must equal the dollar-weighted average factor content of final consumption $\sum_{r' \in R} \chi_{r'} \Psi_{c(r')r}$, where $\Psi_{c(r')r}$ is the total factor r content of consumption by agents in location r'.

Suppose that all consumers consume the same consumption good, so that the factor content Ψ_r is the same for every r'. The previous equation simplifies to

$$\lambda_r = \frac{w_r L_r}{\sum_{r''} w_{r''} L_{r''}} = \Psi_{0r}, \tag{47}$$

where 0 is the index for the common consumption good and the right-hand side is a constant, depending only on the Cobb-Douglas share parameters (Ψ_{0r} is the rth element of the 0th row of the constant Leontief inverse).

Suppose skills are homogeneous, $a_{hr} = 1$ and normalize $Z = 1$. Utility functions are given by $u_h(c_h, l_h) = f_h(c_h + \epsilon_{hr} \mathbf{1}[l_h = r])$ where ϵ_{hr} are drawn from type I extreme value distribution and f_h is any strictly increasing function. The labor supply function is

$$L_r(\boldsymbol{w}, \boldsymbol{p}, \boldsymbol{T}) = \frac{\exp(\theta w_r / p^c + B_r)}{\sum_{r'} \exp(\theta w_r' / p^c + B_r')}, \tag{48}$$

where p^c is the price of the final consumption good, and θ and B_r are parameters of the distribution of ϵ_{hr}. Given this functional form, equation (47) can be rewritten as

$$\frac{w_r \exp(\theta w_r / p^c + B_r)}{\sum_{r'} w_{r'} \exp(\theta w_r' / p^c + B_r')} = \Psi_{0r}$$

where the consumer price index satisfies

$$p^c = \prod_j z_j^{-\Psi_{0j}} \prod_r w_r^{\Psi_{0r}}.$$

These equations pin down all equilibrium prices, up to the choice of numeraire, which can then be used to pin down quantities.

D.2 Occupational Choice Example

Suppose that the common consumption good is a CES bundle of outputs from different industries:

$$y = \left(\sum_r \Omega_{0r}^{\frac{1}{\theta_0}} x_{0r}^{\frac{\theta_0 - 1}{\theta_0}} \right)^{\frac{\theta_0}{\theta_0 - 1}},$$

where the units of quantities are chosen so that Ω_{0r} are expenditures in the status quo. Industry r's output is

$$x_{0r} = z_r L_r.$$

Suppose that workers utility functions can be written as

$$u_h(c_h, r) = c_h + \epsilon_{hr},$$

where ϵ_{hr} is type I extreme value. In this case, $L(\boldsymbol{w}, \boldsymbol{p}, \boldsymbol{T})$ has the standard logit functional form. This implies that

$$\frac{\partial L_r}{\partial \left[w_{r'} / p^c \right]} = \theta L_r L_{r'}.$$

Substituting this into (22) yields

$$d \log L_i^{\text{comp}} = \theta \left(d \left[w_i / \left(A p^c \right) \right] - \mathbb{E}_{L^{\text{comp}}} \left[d \left[w_i / \left(A p^c \right) \right] \right] \right), \tag{49}$$

where the wages and prices are evaluated in the compensated equilibrium. We now apply Proposition 3. The sales share of industry r, in the compensated equilibrium, is given by its share of the wage bill:

$$d\log\lambda_l^{\text{comp}} = d\log w_l + d\log L_l^{\text{comp}} - \mathbb{E}_{\lambda^{\text{comp}}}\left[d\log w + d\log L^{\text{comp}}\right]. \tag{50}$$

At the same time, the sales of share industry r, in the compensated equilibrium, is also given by the share of household spending on industry r:

$$d\log\lambda_l^{\text{comp}} = (\theta_0 - 1)\left[d\log z_l - d\log w_l - \mathbb{E}_{\lambda^{\text{comp}}}\left[d\log z - d\log w\right]\right], \tag{51}$$

where we use the fact that $d\log p_r = d\log w_r - d\log z_r$. Shephard's lemma implies that the consumer price index in the compensated equilibrium is

$$d\log p^c = \sum_l \lambda_l^{\text{comp}}\left[d\log w_l - d\log z_l\right]. \tag{52}$$

Finally, from Theorem 2, the change in aggregate productivity is:

$$d\log A = \sum_i \lambda_i^{\text{comp}} d\log z_i. \tag{53}$$

Equations (49)-(53) form a system of ordinary differential equations that can be solved to obtain $A(t)$ without simulation methods. The boundary conditions are that at $t = 0$, expenditure and population shares coincide with the status quo, and $A(0) = 1$. Solving the system is simple: we discretize the productivity shocks, and iterate on the linear system, updating variables each time.

D.3 Additional Details on Example 9

In this example, there is a measure–1 continuum of agents, 2 locations, no differences in idiosyncratic skills, and a single good produced with labor in each locations. Agent h has utility

$$u_h(c) = \max_{r=1,2}\{c_r \varepsilon_{hr}\},$$

where $\varepsilon_{hr} > 0$ is i.i.d. Fréchet with shape parameter θ. Given a vector of consumption by location, $\{c_r\}$, the supply of labor in location r is

$$L_r = \frac{(c_r)^\theta}{\sum_{r'}(c_{r'})^\theta}.$$

The utilitarian social welfare function is

$$U = \int u_h(c)\, dh = \kappa \Big[\sum_r (c_r)^\theta \Big]^{1/\theta},$$

where $\kappa = \Gamma(1 - 1/\theta)$.

Maximizing social welfare function Consider the problem of picking $\{c_r^0\}$ to maximize U subject to individual rationality, which implies labor supply (D.3), and the resource constraint

$$\sum_r L_r^0 c_r^0 = \sum_r L_r^0 z_r. \tag{54}$$

Using the labor supply function, (54) becomes

$$\sum_r \frac{(c_r^0)^\theta}{\sum_{r'} (c_{r'}^0)^\theta} c_r^0 = \sum_r \frac{(c_r^0)^\theta}{\sum_{r'} (c_{r'}^0)^\theta} z_r,$$

or

$$\sum_r (c_r^0)^{\theta+1} = \sum_r (c_r^0)^\theta z_r.$$

Set the Lagrangian

$$\mathcal{L}(c, \lambda) = \sum_r c_r^\theta + \lambda \Big(\sum_r c_r^{\theta+1} - \sum_r c_r^\theta z_r \Big).$$

The FOC for c_r is

$$1 + \lambda \big[(\theta+1) c_r - \theta z_r \big] = 0,$$

so $(\theta+1)c_r - \theta z_r$ is constant in r. Denote this constant by C:

$$(\theta+1) c_r^0 - \theta z_r = C \quad \Rightarrow \quad c_r^0 = \frac{\theta}{\theta+1} z_r + \frac{C}{\theta+1}.$$

Multiply by L_r^0 and sum over r:

$$\sum_r L_r^0 c_r^0 = \frac{\theta}{\theta+1} \sum_r L_r^0 z_r + \frac{C}{\theta+1} \sum_r L_r^0.$$

Using $\sum_r L_r^0 = 1$ and (54),

$$\sum_r L_r^0 c_r^0 = \frac{\theta}{\theta+1} \sum_r L_r^0 c_r^0 + \frac{C}{\theta+1},$$

so

$$C = \sum_r L_r^0 c_r^0.$$

Substitute back to obtain equation (33) in the text:

$$c_r^0 = \frac{\theta}{\theta+1} z_r + \frac{1}{\theta+1} \sum_s L_s^0 c_s^0. \tag{55}$$

Proposition 12. *If $z_1 \neq z_2$, then the "optimal" allocation implied by* (55) *is Pareto-inefficient and $A > 1$ if the status quo is the "optimal" allocation in* (33).

In words, starting at the "optimal" allocation (33), there exist a set of lump-sum transfers such that every agent can be made better off.

Proof. Applying Theorem 1 and the fact that in this simple one good economy $w_r = z_r$ and $p_r = 1$, we obtain the expression for A in equation (15) of Example 2:

$$A = \frac{\sum_r \int z_r \mathbf{1}[l_h^{\text{comp}} = r] dh}{\sum_r \int \bar{c}_{hr} \mathbf{1}[l_h^{\text{comp}} = r] dh}, \tag{56}$$

where

$$l_h^{\text{comp}} \in \arg\max_{l \in R} \sum_r \left[\frac{z_r}{A} - \bar{c}_{hr} \right] \mathbf{1}\left[l = r \right]. \tag{57}$$

By Lemma 1, we can also write

$$A = \max_{\boldsymbol{l}} \frac{\sum_r \int z_r \mathbf{1}[l_h = r] dh}{\sum_r \int \bar{c}_{hr} \mathbf{1}[l_h = r] dh}, \tag{58}$$

where the location of each agent is directly chosen. This problem can be solved by Dinkelbach's method. Define the function $F(x)$ to be

$$F(x) = \sum_h \max_{l_h = 1,2} \left\{ z_{l_h} - x \frac{\varepsilon_{hl_h^0} c_{l_h^0}^0}{\varepsilon_{hl_h}} \right\}.$$

By Dinkelbach (1967), the solution to (58) must have the property that $F(A) = 0$. We show that $A > 1$ by verifying that $F(1) > 0$. This proves that A cannot be equal to one and since $F(x)$ is decreasing in x, the true solution must feature $A > 1$. To see this, consider $x = 1$, and define

$$\phi_h(r) = z_r - \frac{\varepsilon_{hl_h^0} c_{l_h^0}^0}{\varepsilon_{hr}},$$

so

$$F(1) = \sum_h \max_{l_h=1,2} \phi_h(l_h).$$

Under the initial choice l_h^0,

$$\phi_h(l_h^0) = z_{l_h^0} - c_{l_h^0}^0,$$

and from the resource constraint (54)

$$\sum_h \phi_h(l_h^0) = 0.$$

Hence $F(1) \geq 0$. To show $A > 1$ it suffices to prove $F(1) > 0$ since $F(x)$ is strictly decreasing in x and must satisfy $F(A) = 0$.

Without loss, assume $z_1 > z_2$. Consider any agent with initial location $l_h^0 = 2$:

$$l_h^0 = 2 \quad \Longleftrightarrow \quad c_2^0 \varepsilon_{h2} \geq c_1^0 \varepsilon_{h1} \quad \Longleftrightarrow \quad \frac{\varepsilon_{h2}}{\varepsilon_{h1}} \geq \frac{c_1^0}{c_2^0}.$$

For such h we have

$$\phi_h(2) = z_2 - c_2^0, \qquad \phi_h(1) = z_1 - c_2^0 \frac{\varepsilon_{h2}}{\varepsilon_{h1}}.$$

The condition

$$\phi_h(1) > \phi_h(2)$$

is equivalent to

$$z_1 - c_2^0 \frac{\varepsilon_{h2}}{\varepsilon_{h1}} > z_2 - c_2^0 \quad \Longleftrightarrow \quad \frac{\varepsilon_{h2}}{\varepsilon_{h1}} < 1 + \frac{z_1 - z_2}{c_2^0}.$$

Consider the set of h with

$$l_h^0 = 2 \quad \text{and} \quad \phi_h(1) > \phi_h(2),$$

or from the conditions above

$$\frac{\varepsilon_{h2}}{\varepsilon_{h1}} \in \left[\frac{c_1^0}{c_2^0},\ 1 + \frac{z_1 - z_2}{c_2^0} \right).$$

This interval is nonempty because $(\varepsilon_{h1}, \varepsilon_{h2})$ has a continuous positive density and

$$\left(1 + \frac{z_1 - z_2}{c_2^0} \right) - \frac{c_1^0}{c_2^0} = \frac{c_1^0 - c_2^0}{c_2^0\, \theta} > 0,$$

using $c_1^0 - c_2^0 = \frac{\theta}{\theta+1}(z_1 - z_2)$ from (55) and $z_1 > z_2$. For those h,

$$\max_r \phi_h(r) = \phi_h(1) > \phi_h(2) = \phi_h(l_h^0),$$

and for all other h we have $\max_r \phi_h(r) \geq \phi_h(l_h^0)$. Therefore

$$F(1) = \sum_h \max_{l_h} \phi_h(l_h) > \sum_h \phi_h(l_h^0) = 0.$$

Since $F(1) > 0$, and $F(x)$ is decreasing, the root of $F(x)$ must be greater than 1. The root of $F(x)$ is A, by Dinkelbach (1967) and Lemma 1. The case $z_2 > z_1$ is symmetric.

□