Online-Only Appendix for
An Empirical Evaluation of Some Long-Horizon Macroeconomic Forecasts

Kurt G. Lunsford* Kenneth D. West†

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Contents

*Federal Reserve Bank of Cleveland, P.O. Box 6387, Cleveland, OH 44101; kurt.lunsford@clev.frb.org

†Department of Economics, University of Wisconsin, Madison, WI 53706; kdwest@wisc.edu

A Data Countries and Samples

The following bullets list the countries for each variable and their respective samples used in Sections 4 and 5 of the paper:

- **Per capita real GDP growth, CPI inflation, and population growth:** We use 17 countries: AUS, BEL, CAN, CHE, DEU, DNK, ESP, FIN, FRA, GBR, ITA, JPN, NLD, NOR, PRT, SWE, and USA. All samples are 1871-2020.

- **Labor productivity growth:** We use 18 countries: BEL, CAN, CHE, CHL, DEU, DNK, ESP, FIN, FRA, GBR, GRC, IRL, ITA, JPN, NLD, SWE, and USA have samples of 1871-2022. NOR has a sample of 1876-2022.

- **Broad money growth:** We use 12 countries. The countries and respective samples are AUS (1871-2020), CAN (1872-2020), CHE (1881-2020), DNK (1871-2020), FIN (1871-2020), GBR (1871-2020), ITA (1871-2020), JPN (1871-2020), NOR (1871-2020), PRT (1871-2020), SWE (1872-2020), USA (1871-2020).

- **Total equity returns:** We use 11 countries. The countries and respective samples are AUS (1870-2020), BEL (1870-2020), DEU (1870-2020), DNK (1873-2020), FRA (1870-2020), GBR (1871-2020), ITA (1870-2020), NOR (1881-2020), PRT (1871-2020), SWE (1871-2020), USA (1872-2020).

- **Short-term nominal interest rate:** We use 9 countries. The countries and respective samples are CHE (1870-2020), DNK (1875-2020), ESP (1870-2020), FIN (1870-2020), GBR (1870-2020), NLD (1870-2020), PRT (1880-2020), SWE (1870-2020), USA (1870-2020).

- **Long-term nominal interest rate:** We use 12 countries: AUS, CAN, DNK, FRA, GBR, ITA, JPN, NOR, PRT, SWE, and USA have samples of 1870-2020. CHE has a sample of 1880-2020.

- **Real exchange rate:** We use 16 countries. AUS, BEL, CAN, CHE, DEU, DNK, ESP, FIN, FRA, GBR, ITA, NLD, NOR, PRT, and SWE have samples of 1870-2020. JPN has a sample of 1873-2020. We compute the real exchange rate as US CPI times the nominal exchange rate (expressed in local currency over US currency) divided by home country CPI.

- **Investment to GDP ratio:** We use 7 countries: ESP, FIN, BGR, ITA, SWE, and USA have samples of 1870-2020. CAN has a sample of 1871-2020.

B Additional Tables

In this appendix, we show tables with additional results from our pseudo out-of-sample analysis. They are as follows:

- Table B.1 shows coverage rates for a different categorization of variables that are plausibly stationary and plausibly non-stationary. To categorize the variables, we use the maximum likelihood estimate of d from the MWd model estimated on the longest estimation sample (the $h = 10$ recursive sample). For example, for per capita GDP growth for the USA, this sample is 1871-2010. If the estimate of d is less than 0.5, then we say the variable is plausibly stationary. Otherwise, we say the variable is plausibly non-stationary.

- Table B.2 shows the number of countries per variable that are plausibly stationary using the method in the previous bullet.

- Table B.3 shows actual coverage rates for each of our 10 variables by forecasting model and forecast horizon. As in our main results, we show medians and IQRs. Take per capita real GDP growth as an example. We have 17 countries times 2 sampling schemes to give 34 coverage rates. We then show the median and IQR across these 34 coverage rates.

- Table B.4 shows test results for coverage rates. It shows the average coverage rates for a given model and forecast horizon, where the average is computed via the pooled regression of $d_{i,\tau}$ on a constant, in which $d_{i,\tau} = \mathbf{1}(\bar{x}_{i,\tau} \in FI^{68}_{i,\tau})$ indicates when the realization for series i averaged over $t = \tau+1, \ldots, \tau+h$ falls into the 68 percent forecast interval made in period τ. Table B.4 also shows the corresponding t-statistics, which are $\bar{d} - 0.68$ divided by standard errors that are computed following Driscoll and Kraay (1998) and using the Bartlett kernel (Newey and West, 1987) with a bandwidth of h. Fixed-b critical values are also shown. In Table 5.1 in the paper, we use †s to mark when the absolute value of the t-statistics in Table B.4 are less than the corresponding fixed-b critical values.

- Table B.5 shows probability integral transform (PIT) rates by forecasting model and forecast horizon. For this table, we collect the PITs separately for the stationary and non-stationary variables. By construction, the value of a PIT is between 0 and 1. We then report the fraction of the PITs that fall in the intervals $[0, 0.2)$, $[0.2, 0.4)$, $[0.4, 0.6)$, $[0.6, 0.8)$, and $[0.8, 1.0]$.

Because the PITs for a well-calibrated model are uniformly distributed on $[0, 1]$, the ideal value is 0.2 for each interval. The values in this table are shown in Figure 5.1 of the paper.

- Table B.6 shows what we call PIT distances by forecasting model and forecast horizon. For the 5 intervals in the previous table, let r_k be the realized fraction of the PITs that fall into the kth interval. Then, we compute the PIT distance as $(1/5)\sum_{k=1}^{5}|r_k - 0.2|$, which is the distance from the realized PIT rates to the ideal PIT rates.

  We make six comments about the distances reported in Table B.6. First, for plausibly stationary variables for $h = 10$, the MW0, AR(1), and MWd models have the smallest distances, consistent with our coverage rate results. Second, for plausibly stationary variables for $h = 25$, the AR(1) and MWd have the smallest PIT distances with little increase in distance compared to $h = 10$. Third, for plausibly stationary variables, all models show big increases in PIT distances from $h = 25$ to $h = 50$. Fourth, for plausibly non-stationary variables for $h = 10$, the random walk and MW1 models have reasonably small PIT distances (comparable to the AR(1) model for stationary variables), consistent with our coverage rate results. Fifth, for plausibly non-stationary variables, the random walk model shows little increase in PIT distance from $h = 10$ to $h = 25$ and has the same PIT distance at $h = 25$ as the AR(1) and MWd models for the stationary variables. Sixth, for plausibly non-stationary variables, all models except the iid model show big increases in PIT distances from $h = 25$ to $h = 50$.

- Table B.7 shows test results for the PITs. Let $z_{i,\tau}$ denote the PIT for series i with a forecast distribution made in period τ for a given model and forecast horizon. Then, Table B.7 shows estimates of the the "average," $\bar{m}_1 = P^{-1}\sum_{\tau=R}^{T-h}\left(n_\tau^{-1}\sum_{i\in S_\tau} z_{i,\tau}\right)$, and "variance," $\bar{m}_2 = P^{-1}\sum_{\tau=R}^{T-h}\left(n_\tau^{-1}\sum_{i\in S_\tau}(z_{i,\tau} - 0.5)^2\right)$. It also shows Wald statistics, computed from $[\bar{m}_1, \bar{m}_2]' - [1/2, 1/12]'$ and a long-run variance-covariance matrix that uses the Bartlett kernel (Newey and West, 1987) with a bandwidth of h. Fixed-b critical values are also shown. In Figure 5.1 in the paper, we use ‡s to mark when the Wald statistics in Table B.7 are less than the corresponding fixed-b critical values.

- Table B.8 summarizes the continuous ranked probability score (CRPS) results. Let $F_{\tau,h}(\cdot)$ be the cumulative distribution function for a forecast distribution made with sample $\{x_t\}_{t=1}^{\tau}$

for horizon h. Then, we compute

$$CRPS_{\tau,h} = \int_{-\infty}^{\infty} [F_{\tau,h}(y) - \mathbf{1}(y \geq \bar{x}_{\tau,h})]^2 dy$$

for $\tau = R, \ldots, T-h$ and take the average. For the iid, AR(1), and random walk models, which have normal distributions, we use the CRPS formula on page 367 of Gneiting and Raftery (2007). For the MW0 and MW1 models, which have Student-t distributions, we use the formulas on page 25 of Jordan, Krüger, and Lerch (2019). For the MWd model, we use numerical integration. As with the Winkler score table in the body of the paper, we report CRPS results in values relative to the iid model. We also report the fraction of samples for a given forecast horizon in which each model has the lowest CRPS.

- Table B.9 summarizes the root mean squared prediction error (RMSPE) results. We compute RMSPE with $\sqrt{P^{-1}\sum_{\tau=R}^{T-h}(\bar{x}_{\tau,h} - \hat{f}_{\tau,h})^2}$, in which the point forecast, $\hat{f}_{\tau,h}$, is the mean of the forecast distribution. Point forecasts for the iid and MW0 models are the same, and we report results for these models jointly. As with the Winkler score table in the body of the paper, we report RMSPE results in values relative to the iid model. We also report the fraction of samples for a given forecast horizon in which each model has the lowest RMSPE.

- Table B.10 summarizes the mean absolute prediction error (MAPE) results. We compute the MAPE with $P^{-1}\sum_{\tau=R}^{T-h}|\bar{x}_{\tau,h} - \hat{f}_{\tau,h}|$ in which the point forecast, $\hat{f}_{\tau,h}$, is the median of the forecast distribution. We use the median because the MAPE is a consistent scoring (or loss) function for the median of the forecast distribution (while mean squared errors are a consistent scoring function for the mean) (Gneiting, 2011). Use of the median rather than the mean of the forecast distributions only affects forecasts of the MWd model; mean and median are the same in all other models. Point forecasts for the iid and MW0 models are the same, and we report results for these models jointly. As with the Winkler score table in the body of the paper, we report MAPE results in values relative to the iid model. We also report the fraction of samples for a given forecast horizon in which each model has the lowest MAPE.

- Table B.11 summarizes the absolute forecast bias results. We compute absolute forecast bias as $|P^{-1}\sum_{\tau=R}^{T-h}(\bar{x}_{\tau,h} - \hat{f}_{\tau,h})|$ in which the point forecast, $\hat{f}_{\tau,h}$, is the mean of the forecast distribution. Point forecasts for the iid and MW0 models are the same, and we report results for these models jointly. As with the Winkler score table in the body of the paper, all absolute bias results are reported in values relative to the iid model. We also report the fraction of samples for a given forecast horizon in which each model has the lowest absolute bias.

Table B.1: Coverage rates of nominal 68 percent forecast intervals: medians and IQRs

	(1)	(2)	(3a)	(3b)	(4a)	(4b)
(1)			Stationary variables		Non-stationary variables	
(2)			180 samples		92 samples	
(3)	horizon	model	median coverage	IQR	median coverage	IQR
(4)	10	iid	0.50	(0.38, 0.66)	0.14	(0.11, 0.26)
(5)	10	MW0	0.63	(0.53, 0.73)	0.36	(0.27, 0.51)
(6)	10	AR(1)	0.69	(0.61, 0.78)	0.56	(0.46, 0.69)
(7)	10	MWd	0.72	(0.65, 0.80)	0.58	(0.45, 0.70)
(8)	10	RW	0.95	(0.88, 0.97)	0.72	(0.59, 0.84)
(9)	10	MW1	0.76	(0.71, 0.83)	0.63	(0.51, 0.75)
(10)	25	iid	0.42	(0.26, 0.63)	0.10	(0.06, 0.16)
(11)	25	MW0	0.57	(0.41, 0.69)	0.24	(0.16, 0.37)
(12)	25	AR(1)	0.65	(0.53, 0.77)	0.41	(0.28, 0.61)
(13)	25	MWd	0.65	(0.54, 0.76)	0.45	(0.27, 0.63)
(14)	25	RW	0.98	(0.95, 0.99)	0.75	(0.42, 0.89)
(15)	25	MW1	0.86	(0.79, 0.91)	0.65	(0.33, 0.82)
(16)	50	iid	0.28	(0.13, 0.59)	0.04	(0.00, 0.13)
(17)	50	MW0	0.49	(0.22, 0.69)	0.13	(0.00, 0.23)
(18)	50	AR(1)	0.54	(0.32, 0.79)	0.30	(0.21, 0.49)
(19)	50	MWd	0.57	(0.32, 0.79)	0.35	(0.26, 0.56)
(20)	50	RW	1.00	(0.98, 1.00)	0.79	(0.46, 0.94)
(21)	50	MW1	0.92	(0.87, 0.94)	0.69	(0.37, 0.88)

Notes:

1. Stationary variables are defined as those having a maximum likelihood value of d in the MWd model less than 0.5 over the longest available sample.

2. Non-stationary variables are defined as those having a maximum likelihood value of d in the MWd model larger than 0.5 over the longest available sample.

3. We show the number of countries per variable that have plausibly stationary series in Table B.2.

4. The number of samples for each group in row (2) is the number of variables in that group times the two sampling schemes (rolling and recursive).

5. In the list of models in column (2), we use the shorthand "RW" for the random walk model.

6. Medians and IQRs (interquartile ranges) were constructed as described in the notes to Table 5.1 in the body of the paper.

Table B.2: Frequency of stationarity by variable

	(1)	(2)	(3)
(1)	variable	no. of countries	no. of countries with $\hat{d} < 0.5$
(2)	GDP growth	17	17
(3)	productivity growth	18	18
(4)	CPI inflation	17	15
(5)	money growth	12	10
(6)	population growth	17	9
(7)	equity returns	11	11
(8)	short-term interest	9	0
(9)	long-term interest	12	0
(10)	real exchange rate	16	10
(11)	I/Y ratio	7	0

Notes:

1. This table reports the number of countries per variable that have plausibly stationary series.

2. $\hat{d}$ is the maximum likelihood estimate of d using the longest available sample of data. For example, for per capita GDP growth, we estimate $\hat{d}$ on 1871-2010 (the recursive sample for $h = 10$).

Table B.3: Coverage rates of nominal 68 percent forecast intervals: medians and IQRs

	(1)	(2)	(3a)	(3b)	(4a)	(4b)	(5a)	(5b)	(6a)	(6b)	(7a)	(7b)
(1)			GDP growth		Productivity growth		CPI inflation		Money growth		Population growth	
(2)			17 countries		18 countries		17 countries		12 countries		17 countries	
(3)			34 samples		36 samples		34 samples		24 samples		34 samples	
(4)	horizon	model	median coverage	IQR	median coverage	IQR	median coverage	IQR	median coverage	IQR	median coverage	IQR
(5)	10	iid	0.65	(0.55, 0.71)	0.56	(0.47, 0.72)	0.48	(0.36, 0.56)	0.41	(0.37, 0.45)	0.28	(0.20, 0.40)
(6)	10	MW0	0.69	(0.58, 0.80)	0.61	(0.52, 0.69)	0.68	(0.62, 0.79)	0.62	(0.56, 0.68)	0.58	(0.42, 0.64)
(7)	10	AR(1)	0.70	(0.61, 0.80)	0.63	(0.57, 0.71)	0.79	(0.73, 0.84)	0.66	(0.62, 0.74)	0.70	(0.56, 0.76)
(8)	10	MWd	0.71	(0.63, 0.78)	0.71	(0.61, 0.77)	0.81	(0.76, 0.85)	0.75	(0.68, 0.80)	0.71	(0.62, 0.76)
(9)	10	RW	0.96	(0.95, 0.98)	0.97	(0.95, 0.99)	0.95	(0.93, 0.96)	0.94	(0.90, 0.97)	0.88	(0.80, 0.92)
(10)	10	MW1	0.76	(0.74, 0.80)	0.77	(0.74, 0.83)	0.83	(0.78, 0.88)	0.80	(0.76, 0.84)	0.77	(0.70, 0.83)
(11)	25	iid	0.70	(0.49, 0.76)	0.58	(0.41, 0.69)	0.32	(0.26, 0.51)	0.30	(0.22, 0.39)	0.24	(0.17, 0.32)
(12)	25	MW0	0.69	(0.55, 0.84)	0.57	(0.43, 0.71)	0.60	(0.44, 0.72)	0.56	(0.43, 0.67)	0.50	(0.33, 0.59)
(13)	25	AR(1)	0.72	(0.58, 0.83)	0.59	(0.49, 0.71)	0.67	(0.58, 0.73)	0.65	(0.52, 0.72)	0.67	(0.48, 0.79)
(14)	25	MWd	0.65	(0.55, 0.78)	0.65	(0.55, 0.77)	0.71	(0.63, 0.83)	0.66	(0.64, 0.73)	0.69	(0.60, 0.76)
(15)	25	RW	0.99	(0.97, 1.00)	0.99	(0.98, 1.00)	0.97	(0.96, 0.99)	0.99	(0.97, 1.00)	0.94	(0.88, 0.98)
(16)	25	MW1	0.88	(0.85, 0.92)	0.86	(0.84, 0.91)	0.88	(0.80, 0.91)	0.86	(0.79, 0.92)	0.85	(0.81, 0.91)
(17)	50	iid	0.43	(0.22, 0.65)	0.57	(0.26, 0.79)	0.20	(0.08, 0.47)	0.13	(0.08, 0.27)	0.15	(0.07, 0.32)
(18)	50	MW0	0.58	(0.26, 0.75)	0.50	(0.30, 0.77)	0.49	(0.19, 0.84)	0.48	(0.28, 0.69)	0.41	(0.17, 0.58)
(19)	50	AR(1)	0.58	(0.33, 0.76)	0.58	(0.33, 0.79)	0.53	(0.37, 0.87)	0.56	(0.22, 0.87)	0.55	(0.25, 0.77)
(20)	50	MWd	0.45	(0.28, 0.67)	0.50	(0.33, 0.74)	0.70	(0.56, 0.91)	0.65	(0.51, 0.85)	0.76	(0.38, 0.87)
(21)	50	RW	1.00	(1.00, 1.00)	1.00	(0.98, 1.00)	0.98	(0.98, 1.00)	0.99	(0.98, 1.00)	0.96	(0.94, 0.98)
(22)	50	MW1	0.92	(0.89, 0.95)	0.91	(0.87, 0.97)	0.94	(0.88, 0.94)	0.94	(0.92, 0.96)	0.92	(0.89, 0.94)
(23)			Equity returns		Short-term rates		Long-term rates		Real exchange rates		I/Y ratio	
(24)			11 countries		9 countries		12 countries		16 countries		7 countries	
(25)			22 samples		18 samples		24 samples		32 samples		14 samples	
(26)	horizon	model	median coverage	IQR	median coverage	IQR	median coverage	IQR	median coverage	IQR	median coverage	IQR
(27)	10	iid	0.62	(0.57, 0.73)	0.11	(0.10, 0.13)	0.14	(0.11, 0.17)	0.20	(0.12, 0.26)	0.05	(0.04, 0.12)
(28)	10	MW0	0.68	(0.61, 0.73)	0.26	(0.23, 0.28)	0.36	(0.32, 0.40)	0.44	(0.31, 0.59)	0.30	(0.25, 0.36)
(29)	10	AR(1)	0.72	(0.63, 0.80)	0.46	(0.40, 0.50)	0.43	(0.35, 0.54)	0.65	(0.59, 0.71)	0.62	(0.55, 0.71)
(30)	10	MWd	0.68	(0.65, 0.75)	0.42	(0.39, 0.48)	0.50	(0.39, 0.57)	0.66	(0.64, 0.70)	0.59	(0.56, 0.77)
(31)	10	RW	0.90	(0.88, 0.93)	0.66	(0.62, 0.73)	0.46	(0.35, 0.66)	0.75	(0.72, 0.78)	0.74	(0.62, 0.85)
(32)	10	MW1	0.63	(0.59, 0.69)	0.50	(0.44, 0.55)	0.51	(0.41, 0.57)	0.70	(0.67, 0.74)	0.71	(0.63, 0.81)
(33)	25	iid	0.55	(0.40, 0.63)	0.09	(0.08, 0.11)	0.10	(0.07, 0.14)	0.08	(0.07, 0.11)	0.04	(0.00, 0.08)
(34)	25	MW0	0.59	(0.46, 0.66)	0.21	(0.19, 0.24)	0.25	(0.21, 0.33)	0.21	(0.15, 0.33)	0.12	(0.05, 0.23)
(35)	25	AR(1)	0.63	(0.52, 0.72)	0.30	(0.26, 0.34)	0.25	(0.16, 0.30)	0.56	(0.47, 0.65)	0.59	(0.53, 0.66)
(36)	25	MWd	0.55	(0.49, 0.64)	0.25	(0.22, 0.33)	0.27	(0.23, 0.36)	0.54	(0.46, 0.60)	0.54	(0.51, 0.65)
(37)	25	RW	0.96	(0.94, 0.99)	0.48	(0.36, 0.76)	0.24	(0.16, 0.44)	0.82	(0.77, 0.87)	0.83	(0.68, 0.89)
(38)	25	MW1	0.76	(0.74, 0.85)	0.34	(0.27, 0.42)	0.27	(0.21, 0.44)	0.77	(0.72, 0.82)	0.75	(0.65, 0.85)
(39)	50	iid	0.40	(0.09, 0.57)	0.06	(0.04, 0.12)	0.00	(0.00, 0.06)	0.06	(0.03, 0.19)	0.00	(0.00, 0.06)
(40)	50	MW0	0.44	(0.16, 0.60)	0.12	(0.08, 0.15)	0.07	(0.00, 0.17)	0.19	(0.10, 0.34)	0.00	(0.00, 0.10)
(41)	50	AR(1)	0.54	(0.25, 0.76)	0.26	(0.16, 0.30)	0.26	(0.23, 0.35)	0.44	(0.32, 0.56)	0.38	(0.27, 0.43)
(42)	50	MWd	0.36	(0.20, 0.56)	0.32	(0.23, 0.36)	0.31	(0.22, 0.37)	0.51	(0.30, 0.63)	0.28	(0.28, 0.46)
(43)	50	RW	1.00	(0.99, 1.00)	0.62	(0.43, 0.87)	0.29	(0.25, 0.63)	0.87	(0.85, 0.91)	0.55	(0.50, 0.85)
(44)	50	MW1	0.93	(0.86, 0.96)	0.44	(0.35, 0.52)	0.33	(0.28, 0.56)	0.83	(0.77, 0.85)	0.69	(0.43, 0.76)

Notes:
1. The number of samples is the number of countries times the two sampling schemes (rolling and recursive).
2. In the list of models in column (2), we use the shorthand "RW" for the random walk model.
3. Median refers to the median across the number of samples of a given variable. Consider the figure of 0.65 in row (5), column (3a). Per columns (1) and (2), the iid model is used to produce 34 sets of pseudo out-of-sample forecasts of GDP growth for $h = 10$. Actual coverage rates of forecast intervals with nominal 68 percent coverage are computed for each set of pseudo out-of-sample forecasts. 0.65 is the median value across these 34 coverage rates.
4. IQR is interquartile range. It shows the 25th and 75th percentiles of the coverage rates across the number of samples in rows (3) and (25). These percentiles are computed analogously to the median in the previous note.

Table B.4: Tests of coverage rates

	(1)	(2)	(3a)	(3b)	(3c)	(4a)	(4b)	(4c)
(1)				Stationary variables			Non-stationary variables	
(2)				184 samples			88 samples	
(3)	horizon	model	average	*t*-statistic	critical value	average	*t*-statistic	critical value
(4)	10	iid	0.52	-6.69	2.26	0.15	-26.91	2.26
(5)	10	MW0	0.64	-1.76	2.26	0.37	-9.23	2.26
(6)	10	AR1	0.69	0.50	2.26	0.55	-4.36	2.26
(7)	10	MWd	0.72	1.42	2.26	0.56	-4.07	2.26
(8)	10	RW	0.92	11.98	2.26	0.66	-0.70	2.26
(9)	10	MW1	0.76	1.79	2.26	0.61	-2.33	2.26
(10)	25	iid	0.45	-12.12	2.99	0.09	-51.91	2.99
(11)	25	MW0	0.57	-7.61	2.99	0.24	-20.07	2.99
(12)	25	AR1	0.64	-2.54	2.99	0.43	-10.88	2.99
(13)	25	MWd	0.67	-0.29	2.99	0.42	-10.91	2.99
(14)	25	RW	0.96	22.57	2.99	0.62	-3.85	2.99
(15)	25	MW1	0.85	4.00	2.99	0.56	-4.94	2.99
(16)	50	iid	0.36	-10.91	4.77	0.09	-37.60	4.77
(17)	50	MW0	0.48	-5.80	4.77	0.16	-19.42	4.77
(18)	50	AR1	0.54	-3.89	4.77	0.37	-17.84	4.77
(19)	50	MWd	0.57	-2.60	4.77	0.39	-8.44	4.77
(20)	50	RW	0.98	46.30	4.77	0.67	-0.54	4.77
(21)	50	MW1	0.91	8.84	4.77	0.60	-2.60	4.77

Notes:

1. Stationary variables are per capita GDP growth, labor productivity growth, CPI inflation, money growth, population growth, and total equity returns.

2. Non-stationary variables are long-term nominal interest rates, short-term nominal interest rates, real exchange rates, and investment to GDP ratios.

3. The number of samples for each group in row (2) is the sum of the number of variables in that group times two (for rolling and recursive samples).

4. "average" is $P^{-1}\sum_{\tau=R}^{T-h}\left(n_\tau^{-1}\sum_{i\in S_\tau} d_{i,\tau}\right)$, in which $d_{i,\tau}$ is an indicator equal to 1 if the realized data are in the forecast interval and S_τ is the set and n_τ is the number of series available at date τ.

5. The t-statistic is computed as $(average - 0.68)/\sqrt{\hat{\omega}^2/P}$, in which $\hat{\omega}^2$ is the Driscoll and Kraay (1998) long-run variance computed with the Bartlett kernel (Newey and West, 1987) with bandwidth equal to h.

6. "critical value" is the 95 percent fixed-b critical value computed from Sun (2014) for $h = 10$ and 25, and taken from Kiefer and Vogelsang (2002) for $h = 50$.

7. In the list of models in column (2), we use the shorthand "AR1" and "RW" for the AR(1) and random walk models.

Table B.5: Rates at which PITs fall into different intervals

(1)			Plausibly stationary variables					Plausibly non-stationary variables				
(2)	horizon	model	$[0.0, 0.2)$	$[0.2, 0.4)$	$[0.4, 0.6)$	$[0.6, 0.8)$	$[0.8, 1.0]$	$[0.0, 0.2)$	$[0.2, 0.4)$	$[0.4, 0.6)$	$[0.6, 0.8)$	$[0.8, 1.0]$
(3)	10	iid	0.29	0.16	0.15	0.14	0.26	0.42	0.05	0.04	0.04	0.45
(4)	10	MW0	0.22	0.20	0.20	0.17	0.21	0.32	0.13	0.08	0.10	0.38
(5)	10	AR(1)	0.18	0.22	0.23	0.17	0.20	0.26	0.20	0.16	0.13	0.25
(6)	10	MWd	0.15	0.24	0.23	0.18	0.19	0.26	0.22	0.14	0.12	0.26
(7)	10	RW	0.05	0.17	0.58	0.15	0.05	0.19	0.20	0.26	0.15	0.20
(8)	10	MW1	0.16	0.22	0.31	0.18	0.13	0.23	0.20	0.20	0.14	0.23
(9)	25	iid	0.24	0.14	0.12	0.14	0.37	0.38	0.03	0.02	0.03	0.54
(10)	25	MW0	0.19	0.16	0.16	0.18	0.30	0.33	0.07	0.05	0.07	0.48
(11)	25	AR1(1)	0.16	0.18	0.19	0.20	0.27	0.30	0.15	0.10	0.11	0.34
(12)	25	MWd	0.14	0.20	0.19	0.20	0.27	0.30	0.15	0.09	0.12	0.35
(13)	25	RW	0.03	0.15	0.66	0.14	0.03	0.20	0.17	0.23	0.16	0.24
(14)	25	MW1	0.10	0.24	0.36	0.19	0.10	0.23	0.17	0.18	0.15	0.28
(15)	50	iid	0.16	0.08	0.09	0.13	0.53	0.25	0.02	0.02	0.03	0.67
(16)	50	MW0	0.12	0.11	0.12	0.19	0.47	0.22	0.04	0.05	0.05	0.64
(17)	50	AR(1)	0.10	0.12	0.14	0.20	0.43	0.19	0.09	0.10	0.12	0.49
(18)	50	MWd	0.09	0.14	0.13	0.22	0.42	0.19	0.08	0.08	0.15	0.50
(19)	50	RW	0.02	0.11	0.73	0.13	0.01	0.08	0.17	0.24	0.19	0.32
(20)	50	MW1	0.05	0.23	0.41	0.23	0.07	0.10	0.16	0.21	0.17	0.36

Notes:

1. See Table 4.1 in the paper for categorization of variables as plausibly stationary or plausibly non-stationary.

2. In the list of models, we use the shorthand “RW” for the random walk model.

3. Values in the table show the fraction of PITs that fall into each interval. For each collection of variables, plausibly stationary and plausibly non-stationary variables, the values sum to 1 across the rows (aside from rounding).

4. For plausibly stationary variables, the number of PITs used to calculate the rates is 17,134 for $h = 10$, 14,374 for $h = 25$, and 9,774 for $h = 50$. For plausibly non-stationary variables, the number of PITs used to calculate the rates is 8,214 for $h = 10$, 6,894 for $h = 25$, and 4,694 for $h = 50$.

Table B.6: Distance from actual PIT rates to ideal PIT rates

(1) (3)	(1) horizon	(2) model	(3) Plausibly stationary variables	(4) Plausibly non-stationary variables
(4)	10	iid	0.06	0.19
(5)	10	MW0	0.01	0.12
(6)	10	AR(1)	0.02	0.04
(7)	10	MWd	0.03	0.06
(8)	10	RW	0.15	0.02
(9)	10	MW1	0.05	0.02
(10)	25	iid	0.08	0.21
(11)	25	MW0	0.04	0.16
(12)	25	AR(1)	0.03	0.10
(13)	25	MWd	0.03	0.10
(14)	25	RW	0.18	0.03
(15)	25	MW1	0.08	0.04
(16)	50	iid	0.13	0.21
(17)	50	MW0	0.11	0.18
(18)	50	AR(1)	0.09	0.12
(19)	50	MWd	0.10	0.12
(20)	50	RW	0.21	0.06
(21)	50	MW1	0.11	0.07

Notes:

1. See Table 4.1 in the paper for categorization of variables as plausibly stationary or plausibly non-stationary.

2. In the list of models in column (2), we use the shorthand "RW" for the random walk model.

3. For the 5 intervals in Table B.5, let r_k be the value reported for the kth interval in Table B.5 for a given forecast horizon and model. This table shows the distance from those values to 0.2 for each model and forecast horizon, measured with $(1/5)\sum_{k=1}^{5} |r_k - 0.2|$.

Table B.7: Tests of first two moments of the PITs

	(1)	(2)	(3a)	(3b)	(3c)	(3d)	(4a)	(4b)	(4c)	(4d)
(1)			Stationary variables				Non-stationary variables			
(2)			184 samples				88 samples			
(3)	horizon	model	average	variance	Wald statistic	critical value	average	variance	Wald statistic	critical value
(4)	10	iid	0.48	0.122	62.41	9.04	0.52	0.205	651.76	9.04
(5)	10	MW0	0.49	0.091	2.64	9.04	0.53	0.147	82.82	9.04
(6)	10	AR1	0.50	0.083	0.02	9.04	0.48	0.113	24.63	9.04
(7)	10	MWd	0.51	0.076	2.58	9.04	0.48	0.111	29.35	9.04
(8)	10	RW	0.49	0.029	115.28	9.04	0.49	0.088	0.87	9.04
(9)	10	MW1	0.48	0.068	3.79	9.04	0.49	0.100	7.57	9.04
(10)	25	iid	0.56	0.134	204.97	22.50	0.58	0.222	2650.09	22.50
(11)	25	MW0	0.56	0.106	108.37	22.50	0.58	0.180	466.67	22.50
(12)	25	AR1	0.55	0.094	19.22	22.50	0.52	0.141	254.59	22.50
(13)	25	MWd	0.56	0.086	4.06	22.50	0.52	0.138	370.67	22.50
(14)	25	RW	0.50	0.019	751.75	22.50	0.52	0.097	25.96	22.50
(15)	25	MW1	0.49	0.049	27.31	22.50	0.52	0.108	42.70	22.50
(16)	50	iid	0.68	0.155	1137.47	51.41	0.71	0.227	6351.26	51.41
(17)	50	MW0	0.67	0.127	140.16	51.41	0.71	0.200	1705.23	51.41
(18)	50	AR1	0.66	0.114	21.19	51.41	0.65	0.150	216.48	51.41
(19)	50	MWd	0.66	0.104	133.77	51.41	0.66	0.144	110.50	51.41
(20)	50	RW	0.50	0.014	2618.43	51.41	0.61	0.089	105.79	51.41
(21)	50	MW1	0.51	0.036	258.60	51.41	0.62	0.098	60.45	51.41

Notes:

1. Stationary variables are per capita GDP growth, labor productivity growth, CPI inflation, money growth, population growth, and total equity returns.

2. Non-stationary variables are long-term nominal interest rates, short-term nominal interest rates, real exchange rates, and investment to GDP ratios.

3. The number of samples for each group in row (2) is the sum of the number of variables in that group times two (for rolling and recursive samples).

4. "average" is $P^{-1}\sum_{\tau=R}^{T-h}\left(n_\tau^{-1}\sum_{i\in S_\tau} z_{i,\tau}\right)$, in which $z_{i,\tau}$ is the PIT for series i wih a forecast distribution made in period τ and S_τ is the set and n_τ is the number of series available at date τ.

5. "variance" is $P^{-1}\sum_{\tau=R}^{T-h}\left(n_\tau^{-1}\sum_{i\in S_\tau}(z_{i,\tau}-0.5)^2\right)$, in which $z_{i,\tau}$ is the PIT for series i wih a forecast distribution made in period τ and S_τ is the set and n_τ is the number of series available at date τ.

6. The Wald statistic is computed using ($average - 0.5$) and ($variance - 1/12$), in which $1/12 \approx 0.083$, with a long-run variance-covariance matrix that uses a Bartlett kernel (Newey and West, 1987) with a bandwidh of h.

8. "critical value" is the 95 percent fixed-b critical value computed from Sun (2014) for $h = 10$ and 25, and taken from Kiefer, Vogelsang, and Bunzell (2000) for $h = 50$.

9. In the list of models in column (2), we use the shorthand "AR1" and "RW" for the AR(1) and random walk models.

Table B.8: CRPSs: medians and IQRs of relative values and fraction with minimum value

	(1)	(2)	(3a)	(3b)	(3c)	(4a)	(4b)	(4c)
(1)			Stationary variable			Non-stationary variables		
(2)			184 samples			88 samples		
(3)	horizon	model	median relative CRPS	IQR	fraction with min CRPS	median relative CRPS	IQR	fraction with min CRPS
(4)	10	iid	1.00	(1.00, 1.00)	0.28	1.00	(1.00, 1.00)	0.00
(5)	10	MW0	0.99	(0.94, 1.01)	0.09	0.87	(0.86, 0.90)	0.06
(6)	10	AR1	0.99	(0.91, 1.04)	0.33	0.57	(0.49, 0.70)	0.32
(7)	10	MWd	0.99	(0.90, 1.07)	0.27	0.62	(0.54, 0.74)	0.03
(8)	10	RW	2.45	(1.37, 3.55)	0.02	0.54	(0.48, 0.70)	0.50
(9)	10	MW1	1.54	(1.17, 2.01)	0.01	0.58	(0.50, 0.74)	0.09
(10)	25	iid	1.00	(1.00, 1.00)	0.25	1.00	(1.00, 1.00)	0.00
(11)	25	MW0	0.97	(0.92, 1.01)	0.29	0.90	(0.88, 0.92)	0.22
(12)	25	AR1	1.00	(0.94, 1.08)	0.27	0.84	(0.72, 0.94)	0.23
(13)	25	MWd	1.03	(0.92, 1.09)	0.17	0.83	(0.71, 0.94)	0.27
(14)	25	RW	3.85	(2.16, 6.89)	0.01	0.89	(0.73, 1.04)	0.22
(15)	25	MW1	2.07	(1.53, 3.01)	0.02	0.92	(0.77, 1.08)	0.07
(16)	50	iid	1.00	(1.00, 1.00)	0.27	1.00	(1.00, 1.00)	0.01
(17)	50	MW0	0.98	(0.91, 1.02)	0.28	0.92	(0.88, 0.94)	0.19
(18)	50	AR1	1.02	(0.93, 1.11)	0.23	0.87	(0.76, 0.99)	0.17
(19)	50	MWd	1.04	(0.93, 1.12)	0.20	0.84	(0.75, 0.93)	0.27
(20)	50	RW	5.60	(2.95, 11.28)	0.01	0.97	(0.77, 1.42)	0.20
(21)	50	MW1	2.85	(1.77, 4.29)	0.01	0.95	(0.76, 1.40)	0.15

Notes:

1. See Table 4.1 in the paper for categorization of variables as plausibly stationary or plausibly non-stationary.
2. In the list of models in column (2), we use the shorthand "RW" for the random walk model.
3. For each model in each sample, CRPS is expressed relative to the CRPS for the iid model in that sample. Medians and IQRs (interquartile ranges) of relative CRPSs were constructed as described in the notes to Table 5.1 in the body of the paper.
4. "fraction with min CRPS" reports the fraction of the samples for a given horizon in which the corresponding model has the lowest CRPS among the six models.

Table B.9: RMSPEs: medians and IQRs of relative values and fraction with minimum value

	(1)	(2)	(3a)	(3b)	(3c)	(4a)	(4b)	(4c)
(1)			Stationary variables			Non-stationary variables		
(2)			184 samples			88 samples		
(3)	horizon	model	median relative RMSPE	IQR	fraction with min RMSPE	median relative RMSPE	IQR	fraction with min RMSPE
(4)	10	iid/MW0	1.00	(1.00, 1.00)	0.40	1.00	(1.00, 1.00)	0.08
(5)	10	AR(1)	1.00	(0.97, 1.06)	0.27	0.71	(0.59, 0.84)	0.26
(6)	10	MWd	1.02	(0.96, 1.08)	0.31	0.74	(0.64, 0.86)	0.10
(7)	10	RW	2.01	(1.37, 2.89)	0.03	0.69	(0.56, 0.88)	0.48
(8)	10	MW1	1.70	(1.32, 2.18)	0.00	0.71	(0.59, 0.97)	0.08
(9)	25	iid/MW0	1.00	(1.00, 1.00)	0.56	1.00	(1.00, 1.00)	0.31
(10)	25	AR(1)	1.01	(0.99, 1.12)	0.24	0.98	(0.91, 1.10)	0.16
(11)	25	MWd	1.04	(1.00, 1.14)	0.18	0.98	(0.86, 1.10)	0.26
(12)	25	RW	2.68	(1.88, 3.93)	0.02	1.11	(0.94, 1.31)	0.23
(13)	25	MW1	2.25	(1.70, 2.94)	0.01	1.14	(0.96, 1.35)	0.05
(14)	50	iid/MW0	1.00	(1.00, 1.00)	0.61	1.00	(1.00, 1.00)	0.59
(15)	50	AR(1)	1.02	(1.00, 1.21)	0.22	1.05	(0.98, 1.26)	0.10
(16)	50	MWd	1.07	(1.01, 1.24)	0.16	1.05	(0.94, 1.16)	0.13
(17)	50	RW	3.36	(2.29, 5.35)	0.01	1.24	(1.02, 1.83)	0.14
(18)	50	MW1	2.94	(2.07, 3.79)	0.00	1.24	(1.02, 1.81)	0.05

Notes:

1. See Table 4.1 in the paper for categorization of variables as plausibly stationary or plausibly non-stationary.

2. In the list of models in column (2), we use the shorthand "RW" for the random walk model.

3. For each model in each sample, RMSPE is expressed relative to the RMSPE for the iid/MW0 model in that sample. Medians and IQRs (interquartile ranges) of the resulting relative RMSPEs were constructed as described in the notes to Table 5.1 in the body of the paper.

4. "fraction with min RMSPE" reports the fraction of the samples for a given horizon in which the corresponding model has the lowest RMSPE among the six models.

Table B.10: MAPEs: medians and IQRs of relative values and fraction with minimum value

	(1)	(2)	(3a)	(3b)	(3c)	(4a)	(4b)	(4c)
(1)			Stationary variables			Non-stationary variables		
(2)			184 samples			88 samples		
(3)	horizon	model	median relative MAPE	IQR	fraction with min MAPE	median relative MAPE	IQR	fraction with min MAPE
(4)	10	iid/MW0	1.00	(1.00, 1.00)	0.26	1.00	(1.00, 1.00)	0.05
(5)	10	AR(1)	0.99	(0.95, 1.02)	0.38	0.64	(0.57, 0.80)	0.30
(6)	10	MWd	1.00	(0.93, 1.07)	0.29	0.73	(0.66, 0.85)	0.02
(7)	10	RW	1.43	(1.10, 2.14)	0.07	0.61	(0.54, 0.77)	0.60
(8)	10	MW1	1.44	(1.16, 1.89)	0.01	0.67	(0.60, 0.85)	0.03
(9)	25	iid/MW0	1.00	(1.00, 1.00)	0.45	1.00	(1.00, 1.00)	0.39
(10)	25	AR(1)	1.01	(0.99, 1.07)	0.27	0.97	(0.88, 1.11)	0.22
(11)	25	MWd	1.03	(0.99, 1.09)	0.24	1.02	(0.87, 1.13)	0.16
(12)	25	RW	2.07	(1.54, 2.94)	0.03	1.07	(0.84, 1.22)	0.22
(13)	25	MW1	1.90	(1.52, 2.45)	0.01	1.10	(0.93, 1.24)	0.02
(14)	50	iid/MW0	1.00	(1.00, 1.00)	0.49	1.00	(1.00, 1.00)	0.47
(15)	50	AR(1)	1.02	(0.99, 1.13)	0.28	1.03	(0.94, 1.24)	0.10
(16)	50	MWd	1.04	(1.00, 1.09)	0.20	1.02	(0.92, 1.09)	0.18
(17)	50	RW	2.47	(1.81, 3.70)	0.02	1.16	(0.96, 1.69)	0.14
(18)	50	MW1	2.42	(1.73, 3.21)	0.01	1.17	(0.98, 1.70)	0.11

Notes:

1. See Table 4.1 in the paper for categorization of variables as plausibly stationary or plausibly non-stationary.
2. In the list of models in column (2), we use the shorthand “RW” for the random walk model.
3. For each model in each sample, MAPE is expressed relative to the MAPE for the iid/MW0 model in that sample. Medians and IQRs (interquartile ranges) of the resulting relative MAPEs were constructed as described in the notes to Table 5.1 in the body of the paper.
4. “fraction with min MAPE” reports the fraction of the samples for a given horizon in which the corresponding model has the lowest MAPE among the six models.

Table B.11: Absolute biases: medians and IQRs of relative values and fraction with minimum value

	(1)	(2)	(3a)	(3b)	(3c)	(4a)	(4b)	(4c)
(1)			Stationary variables			Non-stationary variables		
(2)			184 samples			88 samples		
(3)	horizon	model	median relative bias	IQR	fraction with min bias	median relative bias	IQR	fraction with min bias
(4)	10	iid/MW0	1.00	(1.00, 1.00)	0.09	1.00	(1.00, 1.00)	0.10
(5)	10	AR(1)	0.94	(0.64, 1.12)	0.10	0.33	(0.17, 0.61)	0.15
(6)	10	MWd	0.79	(0.46, 1.27)	0.16	0.37	(0.19, 0.62)	0.18
(7)	10	RW	0.31	(0.11, 0.98)	0.35	0.21	(0.11, 0.47)	0.48
(8)	10	MW1	0.43	(0.17, 1.00)	0.30	0.23	(0.12, 0.53)	0.09
(9)	25	iid/MW0	1.00	(1.00, 1.00)	0.05	1.00	(1.00, 1.00)	0.09
(10)	25	AR(1)	0.96	(0.81, 1.03)	0.07	0.44	(0.14, 0.76)	0.18
(11)	25	MWd	0.79	(0.56, 1.06)	0.14	0.45	(0.18, 0.69)	0.16
(12)	25	RW	0.36	(0.14, 0.81)	0.32	0.35	(0.08, 0.57)	0.35
(13)	25	MW1	0.29	(0.10, 0.81)	0.43	0.35	(0.08, 0.57)	0.22
(14)	50	iid/MW0	1.00	(1.00, 1.00)	0.08	1.00	(1.00, 1.00)	0.27
(15)	50	AR(1)	0.98	(0.92, 1.02)	0.04	0.88	(0.72, 1.15)	0.17
(16)	50	MWd	0.92	(0.78, 1.01)	0.14	0.90	(0.75, 1.03)	0.11
(17)	50	RW	0.58	(0.26, 0.94)	0.30	0.88	(0.65, 1.56)	0.31
(18)	50	MW1	0.51	(0.26, 0.88)	0.43	0.88	(0.65, 1.52)	0.14

Notes:

1. See Table 4.1 in the paper for categorization of variables as plausibly stationary or plausibly non-stationary.

2. In the list of models in column (2), we use the shorthand "RW" for the random walk model.

3. For each model in each sample, absolute bias is expressed relative to the absolute bias for the iid/MW0 model in that sample. Medians and IQRs (interquartile ranges) of relative biases were constructed as described in the notes to Table 5.1 in the body of the paper.

4. "fraction with min bias" reports the fraction of the samples for a given horizon in which the corresponding model has the lowest absolute bias among the six models.

Table C.1: Coverage rates of US GDP growth with real-time and final-version data, $h = 10$

	(1)	(2)	(3)	(4)	(5)
		Rolling samples		Recursive samples	
	model	real-time	final-version	real-time	final-version
(1)	iid	0.77	0.77	1.00	1.00
(2)	MW0	0.79	0.77	1.00	1.00
(3)	AR(1)	0.79	0.79	1.00	1.00
(4)	MWd	0.81	0.81	1.00	1.00
(5)	RW	0.98	0.98	1.00	1.00
(6)	MW1	0.89	0.91	1.00	1.00

Notes:

1. Real-time columns, (2) and (4), use real-time data from the Federal Reserve Bank of Philadelphia to make forecasts.
2. Final-version columns, (3) and (5), use data from Jordà, Schularick, and Taylor (2017) to make forecasts as in the body of the paper.
3. In the list of models in column (1), we use the shorthand "RW" for the random walk model.

C Data Revisions

We used real-time data on US GNP/GDP growth to consider the possible effects of data revisions. We first summarize results and then explain how we constructed the real-time data. Throughout "JST" refers to (Jordà, Schularick, and Taylor, 2017), which is the data that we use in the paper.

The first vintage for our real-time GDP data was 1965 and the last vintage was 2011. We made $P = 47$ real-time 68 percent forecast intervals at a horizon of $h = 10$. The first forecast interval was for 1965-1974 and the last for 2011-2020. We used JST data to decide whether the realized observation fell within our 68 percent real-time forecast intervals. That is, we take the JST data as the final version of GDP growth.

Table C.1 shows the coverage rate results. It has four coverage rates for each of our models. Two are coverage rates using the real-time data to make the forecasts for both rolling and recursive samples (shown in columns (2) and (4)). The other two are coverage rates using the JST data, which is what we use in the paper, to make the forecasts for both rolling and recursive samples (shown in columns (3) and (5)).

Table C.1 shows that using real-time data has essentially no effect on the coverage rates that we compute. The only differences between real-time and final-version coverage rates are for the MW0 model (row (2)) and the MW1 model (row (6)) in columns (2) and (3), and these differences are only 2 percentage points.

Table C.2: Correlations of PITs with real-time and final-version data for US GDP growth, $h = 10$

	(1) model	(2) rolling	(3) recursive
(1)	iid	0.99	1.00
(2)	MW0	0.98	1.00
(3)	AR(1)	0.99	1.00
(4)	MWd	0.96	1.00
(5)	RW	0.97	0.97
(6)	MW1	0.98	0.99

Notes:
1. For each model, forecasts and the corresponding PITs are computed with either real-time or final-version data. The table displays the correlations between the real-time and final-version PITs.
3. In the list of models in column (1), we use the shorthand "RW" for the random walk model.

In addition to coverage rates, we also compare PITs when forecasts are made with real-time and final-version data. For each model and for the rolling and recursive estimation schemes, Table C.2 shows the correlations between the PITs when real-time and final-version data are used to make the forecasts. These correlations are either very close or round to 1, again showing that using real-time data has essentially no effect on forecast calibration.

Details on construction of our real-time data: Our source for real-time data on US GNP/GDP was the real-time database of the Federal Reserve Bank of Philadelphia `www.philadelphiafed.org/surveys-and-data/real-time-data-research/routput`, downloaded 2025-10-11. The oldest vintage in that database was 1965, with data going back to 1947. Prior to 1947, we use JST data. The real-time data are available quarterly. We assume that our annual forecasts are made using the fourth quarter vintage of GDP. For example, we assume the 1965 real-time forecast is made in the fourth quarter of 1965, at which point real-time annual data exist through 1964. Our forecasts do not use data that were available through the first three quarters of the year of forecast.

The Federal Reserve Bank of Philadelphia data are GNP/GDP while the JST data are GNP/GDP per capita. We use JST's population data to convert the Federal Reserve Bank of Philadelphia data to per capita terms.

We make three additional comments. First, we do not attempt to adjust for any possible switch between GNP and GDP. Second, apart from an exception about to be noted, we do not account for different base years in different vintages: an adjustment for a change in base year that simply scales real GDP levels by a constant would drop out when we compute GDP growth via log differences. Third, a few vintages were missing data for 1947-1958 or 1947-1959. We use data from the last vintage that had data for those years, adjusting for a possible change in deflator base year so that all the data in a given vintage would be from the same base year. This adjustment was made by multiplying by the ratio of GDP values for a post-1958 quarter that was available in both the current and the last vintage that had data for 1947-1958 or 1947-1959.

D Derivations of Forecast Distributions for the iid, AR(1), and Random Walk Models

In the paper, we use two sample schemes to estimate the parameters of the forecasting models: a recursive scheme and a rolling scheme. In this appendix, we only show parameter estimates with the recursive sample notation, with the understanding that estimates with the rolling sample notation take a parallel form.

The iid Model. The model is $x_t = \mu + u_t$, in which u_t is iid with mean zero and variance σ^2. We use the estimates $\hat{\mu}_\tau = \tau^{-1} \sum_{t=1}^{\tau} x_t$ and $\hat{\sigma}_\tau^2 = (\tau - 1)^{-1} \sum_{t=1}^{\tau} (x_t - \hat{\mu}_\tau)^2$. Then, we treat h and τ as sufficiently large so that $h^{1/2}[(x_{\tau+1} + \cdots + x_{\tau+h})/h - \mu]$ and $\tau^{1/2}(\hat{\mu}_\tau - \mu)$ are each normally distributed with $h^{1/2}[(x_{\tau+1} + \cdots + x_{\tau+h})/h - \mu] \sim N(0, \sigma^2)$ and $\tau^{1/2}(\hat{\mu}_\tau - \mu) \sim N(0, \sigma^2)$. We rearrange terms so that $(x_{\tau+1} + \cdots + x_{\tau+h})/h - \mu \sim N(0, \sigma^2/h)$ and $\hat{\mu}_\tau - \mu \sim N(0, \sigma^2/\tau)$.

With u_t being iid, $(x_{\tau+1} + \cdots + x_{\tau+h})/h - \mu$ and $\hat{\mu}_\tau - \mu$ are independent, yielding

$$[(x_{\tau+1} + \cdots + x_{\tau+h})/h - \mu] - [\hat{\mu}_\tau - \mu] \sim N(0, [(1/h) + (1/\tau)]\sigma^2).$$

Then,

$$(x_{\tau+1} + \cdots + x_{\tau+h})/h \sim N(\hat{\mu}_\tau, [(1/h) + (1/\tau)]\sigma^2),$$

and we plug in $\hat{\sigma}_\tau^2$ for σ^2 to compute the forecast distribution.

The Random Walk Model. The model is $x_t = x_{t-1} + u_t$, in which u_t is iid with mean zero and variance σ^2. We estimate σ^2 with $\hat{\sigma}_\tau^2 = (\tau-1)^{-1}\sum_{t=2}^{\tau}(x_t - x_{t-1})^2$. It is the case that

$$
\begin{aligned}
(x_{\tau+1} + \cdots + x_{\tau+h})/h - x_\tau &= [(x_{\tau+1} - x_\tau) + \cdots + (x_{\tau+h} - x_\tau)]/h \\
&= [u_{\tau+1} + (u_{\tau+1} + u_{\tau+2}) + \cdots + (u_{\tau+1} + \cdots u_{\tau+h})]/h \\
&= hu_{\tau+1}/h + (h-1)u_{\tau+2}/h + \cdots u_{\tau+h}/h \\
&= v_{\tau+1} + v_{\tau+2} + \cdots v_{\tau+h}.
\end{aligned}
$$

In the last line, we use $v_{\tau+j} = (h-j+1)u_{\tau+j}/h$ so that $v_{\tau+j}$ and $v_{\tau+i}$ are independent for $j \neq i$ with $E(v_{\tau+j}) = 0$ and $E(v_{\tau+j}^2) = [(h-j+1)/h]^2\sigma^2$. Then, we assume that $v_{\tau+j}$ for $j = 1, 2, \ldots$ satisfies Lindeberg's condition and that h is sufficiently large to yield

$$
\frac{v_{\tau+1} + v_{\tau+2} + \cdots v_{\tau+h}}{\sqrt{\sum_{j=1}^{h}[(h-j+1)/h]^2\sigma^2}} \sim N(0,1).
$$

Using $\sum_{j=1}^{h} j^2 = h(h+1)(2h+1)/6$ from Equation 16.1.10 in Hamilton (1994), we compute $\sum_{j=1}^{h}[(h-j+1)/h]^2\sigma^2 = (h+1)(2h+1)\sigma^2/(6h)$ so that $v_{\tau+1} + v_{\tau+2} + \cdots v_{\tau+h} \sim N(0, (h+1)(2h+1)\sigma^2/(6h))$. Hence, $(x_{\tau+1} + \cdots + x_{\tau+h})/h - x_\tau \sim N(0, (h+1)(2h+1)\sigma^2/(6h))$ and

$$
(x_{\tau+1} + \cdots + x_{\tau+h})/h \sim N(x_\tau, (h+1)(2h+1)\sigma^2/(6h)),
$$

and we plug in $\hat{\sigma}_\tau^2$ for σ^2 to compute the forecast distribution.

The AR(1) Model. The model is $x_t = \rho_0 + \rho_1 x_{t-1} + u_t$, in which u_t is iid with mean zero and variance σ^2. We compute $\hat{\rho}_0$ and $\hat{\rho}_1$ with ordinary least squares, suppressing notational dependence on τ for convenience. Ordinary least squares estimates imply

$$
\hat{\rho}_0 = \frac{1}{\tau-1}\left(\sum_{t=2}^{\tau} x_t - \hat{\rho}_1 \sum_{t=2}^{\tau} x_{t-1}\right). \tag{D.1}
$$

Then, using $x_t = \rho_0 + \rho_1 x_{t-1} + u_t$, we have

$$
\hat{\rho}_0 = \rho_0 + (\rho_1 - \hat{\rho}_1)\frac{1}{\tau-1}\sum_{t=2}^{\tau} x_{t-1} + \frac{1}{\tau-1}\sum_{t=2}^{\tau} u_t. \tag{D.2}
$$

Next, we bias-adjust the ordinary least squares estimates. Yamamoto and Kunitomo (1984) show that the asymptotic bias of $\hat{\rho}_0$ is $(1+3\rho_1)\rho_0/((\tau-1)(1-\rho_1))$ and that the asymptotic bias of $\hat{\rho}_1$ is $-(1+3\rho_1)/(\tau-1)$. Then, we compute

$$\begin{aligned}
\tilde{\rho}_0 &= \hat{\rho}_0 - (1+3\hat{\rho}_1)\hat{\rho}_0/((\tau-1)(1-\hat{\rho}_1)) \\
\tilde{\rho}_1 &= \hat{\rho}_1 + (1+3\hat{\rho}_1)/(\tau-1).
\end{aligned}$$

These bias adjustments imply

$$\tilde{\rho}_0 = \left(\frac{1-\tilde{\rho}_1}{1-\hat{\rho}_1}\right)\hat{\rho}_0. \tag{D.3}$$

Hence, the mean of x_t implied by the ordinary least squares estimates, given by $\hat{\rho}_0/(1-\hat{\rho}_1)$, is unchanged by the bias adjustment.

As noted in the paper, we only forecast with the AR(1) model if $\tilde{\rho}_1 < 1$. If $\tilde{\rho}_1 \geq 1$, we forecast with the random walk model. If $\tilde{\rho}_1 < 1$, we compute $\tilde{u}_t = x_t - \tilde{\rho}_0 - \tilde{\rho}_1 x_{t-1}$ and $\tilde{\sigma}_\tau^2 = (\tau-3)^{-1}\sum_{t=2}^{\tau}\tilde{u}_t^2$. Then, we compute the period-by-period forecasts recursively, using $\tilde{x}_{\tau+1} = \tilde{\rho}_0 + \tilde{\rho}_1 x_\tau$ for the one-step-ahead forecast and $\tilde{x}_{\tau+s} = \tilde{\rho}_0 + \tilde{\rho}_1\tilde{x}_{\tau+s-1}$ for the multi-step-ahead forecasts. Hence, we can write the s-step-ahead forecast error as

$$x_{\tau+s} - \tilde{x}_{\tau+s} = \rho_0\sum_{j=0}^{s-1}\rho_1^j - \tilde{\rho}_0\sum_{j=0}^{s-1}\tilde{\rho}_1^j + (\rho_1^s - \tilde{\rho}_1^s)x_\tau + \sum_{j=0}^{s-1}\rho_1^j u_{\tau+s-j}. \tag{D.4}$$

To simplify the analysis, we then assume that $\tilde{\rho}_1 = \rho_1$, yielding

$$x_{\tau+s} - \tilde{x}_{\tau+s} = -\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)(\tilde{\rho}_0 - \rho_0) + \sum_{j=0}^{s-1}\tilde{\rho}_1^j u_{\tau+s-j}. \tag{D.5}$$

The first term on the right-hand side can then be manipulated as follows

$$\begin{aligned}
-\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)(\tilde{\rho}_0-\rho_0) &= -\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)[(\tilde{\rho}_0-\hat{\rho}_0)-(\hat{\rho}_0-\rho_0)] \\
&= \left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\left[\frac{(\tilde{\rho}_1-\hat{\rho}_1)\hat{\rho}_0}{1-\hat{\rho}_1}-(\rho_1-\hat{\rho}_1)\frac{1}{\tau-1}\sum_{t=2}^{\tau}x_{t-1}-\frac{1}{\tau-1}\sum_{t=2}^{\tau}u_t\right] \\
&= \left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)(\tilde{\rho}_1-\hat{\rho}_1)\left[\frac{\hat{\rho}_0}{1-\hat{\rho}_1}-\frac{1}{\tau-1}\sum_{t=2}^{\tau}x_{t-1}\right]-\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\frac{1}{\tau-1}\sum_{t=2}^{\tau}u_t \\
&= \left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\frac{\tilde{\rho}_1-\hat{\rho}_1}{1-\hat{\rho}_1}\frac{1}{\tau-1}\sum_{t=2}^{\tau}(x_t-x_{t-1})-\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\frac{1}{\tau-1}\sum_{t=2}^{\tau}u_t \\
&= \left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\frac{1+3\tilde{\rho}_1}{(1-\tilde{\rho}_1)\tau+(3+\tilde{\rho}_1)}\frac{x_\tau-x_1}{\tau-1}-\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\frac{1}{\tau-1}\sum_{t=2}^{\tau}u_t \\
&= \left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\frac{(1+3\tilde{\rho}_1)(x_\tau-x_1)}{(1-\tilde{\rho}_1)\tau^2+2(1+\tilde{\rho}_1)\tau-(3+\tilde{\rho}_1)}-\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\frac{1}{\tau-1}\sum_{t=2}^{\tau}u_t,
\end{aligned}$$

in which the second line uses (D.2) to substitute out $\hat{\rho}_0-\rho_0$ and (D.3) to substitute out $\tilde{\rho}_0$, the third line again imposes $\tilde{\rho}_1=\rho_1$, the fourth line uses (D.1) to substitute out $\hat{\rho}_0$, the fifth line uses $\hat{\rho}_1=((\tau-1)\tilde{\rho}_1-1)/(\tau+2)$ to substitute out $\hat{\rho}_1$, and the sixth line rearranges terms. There is a τ^2 term in the denominator of the first term. Because $\tilde{\rho}<1$, we set the first term to zero, yielding

$$-\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)(\tilde{\rho}_0-\rho_0)=-\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\frac{1}{\tau-1}\sum_{t=2}^{\tau}u_t,$$

and (D.5) becomes

$$x_{\tau+s}-\tilde{x}_{\tau+s}=-\left(\sum_{j=0}^{s-1}\tilde{\rho}_1^j\right)\left(\frac{1}{\tau-1}\sum_{t=2}^{\tau}u_t\right)+\sum_{j=0}^{s-1}\tilde{\rho}_1^j u_{\tau+s-j}. \tag{D.6}$$

Hence,

$$\begin{aligned}(x_{\tau+1}+\cdots+x_{\tau+h})/h &- (\tilde{x}_{\tau+1}+\cdots+\tilde{x}_{\tau+h})/h \\ &= h^{-1}[1+(1+\tilde{\rho}_1)+\cdots+(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-1})]\left(\frac{1}{\tau-1}\sum_{t=2}^{\tau}u_t\right) \\ &\quad + h^{-1}[(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-1})u_{\tau+1}+(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-2})u_{\tau+2}+\cdots+u_{\tau+h}].\end{aligned} \tag{D.7}$$

For the first term on the right-hand side of (D.7), we use

$$h^{-1}[1+(1+\tilde{\rho}_1)+\cdots+(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-1})]\left(\frac{1}{\tau-1}\sum_{t=2}^{\tau}u_t\right)\sim N(0,V_1) \tag{D.8}$$

in which $V_1=[1+(1+\tilde{\rho}_1)+\cdots+(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-1})]^2\sigma^2/((\tau-1)h^2)$.

For the second term on the right-hand side of (D.7), we define the new variables $v_{\tau+1}=(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-1})u_{\tau+1}/h$, $v_{\tau+2}=(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-2})u_{\tau+2}/h$, and so on. Hence, the second term on the right-hand side of (D.7) becomes $v_{\tau+1}+v_{\tau+2}+\cdots+v_{\tau+h}$, in which $v_{\tau+j}$ and $v_{\tau+i}$ are independent for $j\neq i$ with $E(v_{\tau+j})=0$ and $E(v_{\tau+j}^2)=(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-j})^2\sigma^2/h^2$. Then, we assume that $v_{\tau+j}$ for $j=1,2,\dots$ satisfies Lindeberg's condition and that h is sufficiently large to yield

$$\frac{v_{\tau+1}+v_{\tau+2}+\cdots v_{\tau+h}}{\sqrt{\sum_{j=1}^{h}E(v_{\tau+j}^2)}}\sim N(0,1),$$

implying that

$$h^{-1}[(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-1})u_{\tau+1}+(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-2})u_{\tau+2}+\cdots+u_{\tau+h}]\sim N(0,V_2), \tag{D.9}$$

in which $V_2=[(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-1})^2+(1+\tilde{\rho}_1+\cdots+\tilde{\rho}_1^{h-2})^2+\cdots+1]\sigma^2/h^2$.

Note that the first and second terms on the right-hand side of (D.7) are based on the non-overlapping samples $\{u_2,\dots,u_\tau\}$ and $\{u_{\tau+1},\dots,u_{\tau+h}\}$. Because u_t is iid, these two terms are independent and we have

$$(x_{\tau+1}+\cdots+x_{\tau+h})/h-(\tilde{x}_{\tau+1}+\cdots+\tilde{x}_{\tau+h})/h\sim N(0,V), \tag{D.10}$$

in which

$$V = [1 + (1 + \tilde{\rho}_1) + \cdots + (1 + \tilde{\rho}_1 + \cdots + \tilde{\rho}_1^{h-1})]^2 \sigma^2 / (h^2(\tau - 1)) \\ + [(1 + \tilde{\rho}_1 + \cdots + \tilde{\rho}_1^{h-1})^2 + (1 + \tilde{\rho}_1 + \cdots + \tilde{\rho}_1^{h-2})^2 + \cdots + 1] \sigma^2 / h^2. \tag{D.11}$$

Then, we use $\tilde{x}_{\tau+s} = \tilde{\rho}_0 \sum_{j=0}^{s} \tilde{\rho}_1^j + \tilde{\rho}_1^s x_\tau = \tilde{\rho}_0/(1 - \tilde{\rho}_1) + \tilde{\rho}_1^s (x_\tau - \tilde{\rho}_0/(1 - \tilde{\rho}_1))$ so that we forecast $(x_{\tau+1} + \cdots + x_{\tau+h})/h$ with a Normal distribution with a mean of

$$\frac{\tilde{\rho}_0}{1 - \tilde{\rho}_1} + \frac{1}{h}(\tilde{\rho}_1 + \tilde{\rho}_1^2 + \cdots + \tilde{\rho}_1^h)\left(x_\tau - \frac{\tilde{\rho}_0}{1 - \tilde{\rho}_1}\right)$$

and a variance of V in Equation (D.11), plugging $\tilde{\sigma}_\tau^2$ in for σ^2.

E Formulas for the Müller and Watson (2016) Models

The Müller and Watson (2016) (MW) models are based on the asymptotic properties of the sample averages. Let $\hat{\beta}_{\tau,1:q} = [\hat{\beta}_{1,\tau}, \ldots, \hat{\beta}_{q,\tau}]'$ be the vector of cosine-weighted averages and $y_{\tau,h} = \bar{x}_{\tau,h} - \hat{\mu}_\tau$ be the future average of x_t centered on the in-sample average. Then, we use

$$\tau^{1-\kappa} \begin{bmatrix} \hat{\beta}_{\tau,1:q} \\ y_{\tau,h} \end{bmatrix} \sim N(0, \Sigma), \quad \Sigma = \begin{bmatrix} \Sigma_{\beta\beta} & \Sigma_{\beta y} \\ \Sigma_{y\beta} & \Sigma_{yy} \end{bmatrix}, \tag{E.1}$$

in which κ is a scaling factor that depends on the relevant model for x_t: $\kappa = 1/2$ for the MW0 model, $\kappa = 3/2$ for the MW1 model, and $\kappa = 1/2 + d$ for the MWd model. The covariance matrix, Σ, also depends on the model for x_t, and we provide details for computing Σ in Appendix F.

The MW0 Model. For this model, MW compute the covariance matrix, Σ, in (E.1) analytically. Let σ_{lrv}^2 be the long-run variance of u_t. Then, $\Sigma_{\beta\beta} = \sigma_{lrv}^2 I_q$, in which I_q is the $(q \times q)$ identity matrix, $\Sigma_{y\beta} = \Sigma_{\beta y}'$ is a $(1 \times q)$ matrix of zeros, and $\Sigma_{yy} = [(1/h) + (1/\tau)](\tau \sigma_{lrv}^2)$. MW then show that $\bar{x}_{\tau,h}$ has a generalized Student-t distribution with q degrees of freedom, a location parameter of $\hat{\mu}_\tau$ and a scale parameter of $\sqrt{[(1/h) + (1/\tau)](\tau \hat{\beta}_{\tau,1:q}' \hat{\beta}_{\tau,1:q}/q)}$.

The MW1 Model. Unlike for the MW0 model, MW do not provide an analytical form for every element of Σ when u_t is $I(1)$. Because of this, we use an approximation of Σ, providing formulas in Appendix F. Given the appropriate form of Σ, MW show that $\bar{x}_{\tau,h}$ has a generalized Student-t density with q degrees of freedom, a location parameter of $\hat{\mu}_\tau + \Sigma_{y\beta}\Sigma_{\beta\beta}^{-1}\hat{\beta}_{\tau,1:q}$ and a scale parameter of $\sqrt{(\Sigma_{yy} - \Sigma_{y\beta}\Sigma_{\beta\beta}^{-1}\Sigma_{\beta y})(\hat{\beta}'_{\tau,1:q}\Sigma_{\beta\beta}^{-1}\hat{\beta}_{\tau,1:q}/q)}$.

The MWd Model. The DGP for x_t is $x_t = \mu + u_t$, in which u_t is a mean zero and fractionally integrated or $I(d)$ process with fractional parameter $d \in (-0.5, 1)$. We treat d as unknown and use a Bayesian approach to construct the forecast density. Following MW, we set a grid of potential values of d, $\{-0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8, 1\}$, and use a prior of uniform mass on each grid point. The resulting Bayes predictive density is a weighted average of generalized Student-t densities with q degrees of freedom. We provide further details in Appendix F.

F Details for the Müller and Watson (2016) Models

F.1 Covariance Approximations

Begin with $\hat{\beta}_{\tau,1:q}$ and $y_{\tau,h}$ being jointly normally distributed as in Equation (E.1). MW's forecasting approach relies on knowing the form of Σ in Equation (E.1). For the MW0 model, MW provide analytical values for every element of Σ. However, for the MW1 and MWd models, we use numerical approximations from Section 3.2 of Müller and Watson (2020). To start, let $r = h/\tau$ be the ratio of the forecast horizon to the sample size. We use $N = 1000$ and compute the integer $H = \text{round}(rN)$. Using the notation $\psi_{j,t} = \sqrt{2}\cos(\pi j(t - 1/2)/N)$, we write the $(N \times q)$ matrix

$$\Psi = \begin{bmatrix} \psi_{1,1} & \psi_{2,1} & \cdots & \psi_{q,1} \\ \psi_{1,2} & \psi_{2,2} & \cdots & \psi_{q,2} \\ \vdots & \vdots & & \vdots \\ \psi_{1,N} & \psi_{2,N} & \cdots & \psi_{q,N} \end{bmatrix}.$$

Then, we write the $((N + H) \times (q + 1))$ matrix

$$\Xi = \begin{bmatrix} \Psi & -\mathbf{1}_{N\times 1} \\ \mathbf{0}_{H\times q} & (N/H)\mathbf{1}_{H\times 1}, \end{bmatrix}$$

in which $\mathbf{1}_{m\times n}$ denotes an $(m \times n)$ matrix of ones and $\mathbf{0}_{m\times n}$ denotes an $(m \times n)$ matrix of zeros. Next, let L be a lower-triangular $((N+H)\times(N+H))$ matrix with ones on and below the main diagonal. Then, we approximate Σ for the MW1 model with $\Sigma = \sigma^2_{lrv}(\Xi' L L' \Xi)/N^3$, in which σ^2_{lrv} denotes the long-run variance of Δu_t. The distribution of $(x_{\tau+1} + \cdots + x_{\tau+h})/h$ is generalized Student-t with q degrees of freedom and has a location parameter of $\hat{\mu}_\tau + \Sigma_{y\beta}\Sigma^{-1}_{\beta\beta}\hat{\beta}_{\tau,1:q}$ and a scale parameter of $\sqrt{[\Sigma_{yy} - \Sigma_{y\beta}\Sigma^{-1}_{\beta\beta}\Sigma_{\beta y}](\hat{\beta}'_{\tau,1:q}\Sigma^{-1}_{\beta\beta}\hat{\beta}_{\tau,1:q}/q)}$. Any value of $\sigma^2_{lrv} > 0$ cancels out of both the location and the scale parameters; hence, we set $\sigma^2_{lrv} = 1$ and compute

$$\Sigma = (\Xi' L L' \Xi)/N^3, \tag{F.1}$$

for the MW1 model.

For the MWd model, if the value of d is such that $-0.5 < d < 0.5$, define a $((N+H)\times(N+H))$ matrix Λ in which the (i,j) element is given by

$$\lambda_{i,j} = \frac{\Gamma(k+d)\Gamma(1-2d)}{\Gamma(k+1-d)\Gamma(1-d)\Gamma(d)},$$

in which $k = |i-j|$ and $\Gamma(\cdot)$ denotes the gamma function. Then, we set $\sigma^2_{lrv} = 1$ as in the MW1 model[1] and compute

$$\Sigma = (\Xi' \Lambda \Xi)/N^{1+2d}. \tag{F.2}$$

If the value of d is such that $0.5 < d < 1.5$, compute $\tilde{d} = d - 1$ and define a $((N+H)\times(N+H))$ matrix Λ in which the (i,j) element is given by

$$\lambda_{i,j} = \frac{\Gamma(k+\tilde{d})\Gamma(1-2\tilde{d})}{\Gamma(k+1-\tilde{d})\Gamma(1-\tilde{d})\Gamma(\tilde{d})},$$

in which $k = |i-j|$ and $\Gamma(\cdot)$ denotes the gamma function. Then, we set $\sigma^2_{lrv} = 1$ and compute

$$\Sigma = (\Xi' L \Lambda L' \Xi)/N^{1+2d}. \tag{F.3}$$

[1]For the MWd model, σ^2_{lrv} denotes the long-run variance of $(1-B)^d u_t$ with B being the backshift or lag operator.

F.2 The Distribution of the MWd Model

For the MWd model, we treat $d \in (-0.5, 1.5)$ as unknown and use the Bayesian approach in MW. We allow d to take values in a discrete grid, $\mathcal{G} = \{d_1, d_2, \ldots, d_N\}$, and use prior weights, $\{\omega_1, \omega_2, \ldots, \omega_N\}$ subject to $\omega_n \in (0,1)$ for $n = 1, \ldots, N$ and $\sum_{n=1}^{N} \omega_N = 1$. As in MW, we choose $\mathcal{G} = \{-0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8, 1.0\}$ and our weights are $\omega_n = 1/8$ for $n = 1, \ldots, 8$.

MW redistrict the model so that the forecast densities are invariant to location and scale. This means using $\hat{\beta}^s_{\tau,1:q} = \hat{\beta}_{\tau,1:q}/\sqrt{\hat{\beta}'_{\tau,1:q}\hat{\beta}_{\tau,1:q}}$ to construct the forecast densities instead of just using $\hat{\beta}_{\tau,1:q}$. It also means that the model is set up to initially predict $y^s_{\tau,h} = y_{\tau,h}/\sqrt{\hat{\beta}'_{\tau,1:q}\hat{\beta}_{\tau,1:q}}$ before then making predictions about $y_{\tau,h}$. That is, the Bayes predictive density is constructed to predict $y^s_{\tau,h}$ conditional on $\hat{\beta}^s_{\tau,1:q}$:

$$f^{bayes}(y^s_{\tau,h}|\hat{\beta}^s_{\tau,1:q}) = \frac{\sum_{n=1}^{N} f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h})\omega_n}{\sum_{n=1}^{N} f_{d_n}(\hat{\beta}^s_{\tau,1:q})\omega_n}, \tag{F.4}$$

in which $f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h})$ is the joint density of $\hat{\beta}^s_{\tau,1:q}$ and $y^s_{\tau,h}$ with a covariance matrix associated with fractional integration parameter d_n, $\Sigma(d_n)$, and $f_{d_n}(\hat{\beta}^s_{\tau,1:q})$ is the marginal density of $\hat{\beta}^s_{\tau,1:q}$ implied by $f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h})$. To ease notation going forward, we write $\Sigma_n = \Sigma(d_n)$ for indexing the covariance matrices for the different values of d in $\mathcal{G}$. Then, the joint density of $\hat{\beta}^s_{\tau,1:q}$ and $y^s_{\tau,h}$ is

$$f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h}) = \frac{1}{2}\pi^{-(q+1)/2}|\Sigma_n|^{-1/2}\Gamma((q+1)/2)\left(\begin{bmatrix}\hat{\beta}^{s\prime}_{\tau,1:q} & y^s_{\tau,h}\end{bmatrix}\Sigma_n^{-1}\begin{bmatrix}\hat{\beta}^s_{\tau,1:q} \\ y^s_{\tau,h}\end{bmatrix}\right)^{-(q+1)/2}, \tag{F.5}$$

in which Γ denotes the gamma function. We write the submatrices of Σ_n as $\Sigma_{n,\beta\beta}$, $\Sigma_{n,y\beta} = \Sigma'_{n,\beta y}$, and $\Sigma_{n,yy}$. Then, the implied marginal density of $\hat{\beta}^s_{\tau,1:q}$ is

$$f_{d_n}(\hat{\beta}^s_{\tau,1:q}) = \frac{1}{2}\pi^{-q/2}|\Sigma_{n,\beta\beta}|^{-1/2}\Gamma(q/2)\left(\hat{\beta}^{s\prime}_{\tau,1:q}\Sigma^{-1}_{n,\beta\beta}\hat{\beta}^s_{\tau,1:q}\right)^{-q/2}. \tag{F.6}$$

We can then compute the maximum likelihood value of d by checking which value of d in $\mathcal{G}$ maximizes $f_{d_n}(\hat{\beta}^s_{\tau,1:q})$ in (F.6). We re-write Equation (F.4) as

$$f^{bayes}(y^s_{\tau,h}|\hat{\beta}^s_{\tau,1:q}) = \sum_{n=1}^{N} \frac{f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h})}{f_{d_n}(\hat{\beta}^s_{\tau,1:q})} \frac{f_{d_n}(\hat{\beta}^s_{\tau,1:q})\omega_n}{\sum_{k=1}^{N} f_{d_k}(\hat{\beta}^s_{\tau,1:q})\omega_k}.$$

We then use $|\Sigma_n| = |\Sigma_{n,\beta\beta}||\Sigma_{n,yy} - \Sigma_{n,y\beta}\Sigma_{n,\beta\beta}^{-1}\Sigma_{n,\beta y}| = |\Sigma_{n,\beta\beta}|(\Sigma_{n,yy} - \Sigma_{n,y\beta}\Sigma_{n,\beta\beta}^{-1}\Sigma_{n,\beta y})$, where the second equation follows because $\Sigma_{n,yy} - \Sigma_{n,y\beta}\Sigma_{n,\beta\beta}^{-1}\Sigma_{n,\beta y}$ is scalar, and

$$\begin{bmatrix} \Sigma_{n,\beta\beta} & \Sigma_{n,\beta y} \\ \Sigma_{n,y\beta} & \Sigma_{n,yy} \end{bmatrix}^{-1} = \begin{bmatrix} \Sigma_{n,\beta\beta}^{-1} + \Sigma_{n,\beta\beta}^{-1}\Sigma_{n,\beta y}\Sigma_{n,y\beta}\Sigma_{n,\beta\beta}^{-1}\nu_n^{-1} & \Sigma_{n,\beta\beta}^{-1}\Sigma_{n,\beta y}\nu_n^{-1} \\ \nu_n^{-1}\Sigma_{n,y\beta}\Sigma_{n,\beta\beta}^{-1} & \nu_n^{-1} \end{bmatrix},$$

in which $\nu_n = \Sigma_{n,yy} - \Sigma_{n,y\beta}\Sigma_{n,\beta\beta}^{-1}\Sigma_{n,\beta y}$. Defining $m_n(\hat{\beta}^s_{\tau,1:q}) = \Sigma_{n,y\beta}\Sigma_{n,\beta\beta}^{-1}\hat{\beta}^s_{\tau,1:q}$ and $s_n^2(\hat{\beta}^s_{\tau,1:q}) = (\Sigma_{n,yy} - \Sigma_{n,y\beta}\Sigma_{n,\beta\beta}^{-1}\Sigma_{n,\beta y})(\hat{\beta}^{s\prime}_{\tau,1:q}\Sigma_{n,\beta\beta}^{-1}\hat{\beta}^s_{\tau,1:q})/q$, we have

$$\frac{f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h})}{f_{d_n}(\hat{\beta}^s_{\tau,1:q})} = \frac{1}{\sqrt{s_n^2(\hat{\beta}^s_{\tau,1:q})}} \frac{1}{\sqrt{\pi q}} \frac{\Gamma((q+1)/2)}{\Gamma(q/2)} \left(1 + \frac{1}{q}\frac{(y^s_{\tau,h} - m_n(\hat{\beta}^s_{\tau,1:q}))^2}{s_n^2(\hat{\beta}^s_{\tau,1:q})}\right)^{-(q+1)/2}.$$

Hence, $f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h})/f_{d_n}(\hat{\beta}^s_{\tau,1:q})$ is a generalized Student-t density with q degrees of freedom, a location parameter of $m_n(\hat{\beta}^s_{\tau,1:q})$ and a scale parameter of $\sqrt{s_n^2(\hat{\beta}^s_{\tau,1:q})}$. This result then implies that $f^{bayes}(y^s_{\tau,h}|\hat{\beta}^s_{\tau,1:q})$ is a weighted average of generalized Student-t^q densities with weights given by $f_{d_n}(\hat{\beta}^s_{\tau,1:q})\omega_n/(\sum_{k=1}^N f_{d_k}(\hat{\beta}^s_{\tau,1:q})\omega_k)$. Using $\hat{\beta}^s_{\tau,1:q} = \hat{\beta}_{\tau,1:q}/\sqrt{\hat{\beta}'_{\tau,1:q}\hat{\beta}_{\tau,1:q}}$ and $y^s_{\tau,h} = y_{\tau,h}/\sqrt{\hat{\beta}'_{\tau,1:q}\hat{\beta}_{\tau,1:q}}$, we can push the above result further and write

$$\frac{f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h})}{f_{d_n}(\hat{\beta}^s_{\tau,1:q})} = \frac{\sqrt{\hat{\beta}'_{\tau,1:q}\hat{\beta}_{\tau,1:q}}}{\sqrt{s_n^2(\hat{\beta}_{\tau,1:q})}} \frac{1}{\sqrt{\pi q}} \frac{\Gamma((q+1)/2)}{\Gamma(q/2)} \left(1 + \frac{1}{q}\frac{(y_{\tau,h} - m_n(\hat{\beta}_{\tau,1:q}))^2}{s_n^2(\hat{\beta}_{\tau,1:q})}\right)^{-(q+1)/2},$$

so that $f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h})/f_{d_n}(\hat{\beta}^s_{\tau,1:q})$ can be written in terms of $y_{\tau,h}$ and $\hat{\beta}_{\tau,1:q}$. Let $t(y_{\tau,h}, m, s^2, q)$ be the generalized Student-t density with location m, scale s, and degrees of freedom q. Then,

$$\begin{aligned} \frac{f_{d_n}(\hat{\beta}^s_{\tau,1:q}, y^s_{\tau,h})}{f_{d_n}(\hat{\beta}^s_{\tau,1:q})} &= \sqrt{\hat{\beta}'_{\tau,1:q}\hat{\beta}_{\tau,1:q}}\; t(y_{\tau,h}, m_n(\hat{\beta}_{\tau,1:q}), s_n^2(\hat{\beta}_{\tau,1:q}), q) \\ &= \sqrt{\hat{\beta}'_{\tau,1:q}\hat{\beta}_{\tau,1:q}}\; t((x_{\tau+1} + \cdots + x_{\tau+h})/h, \hat{\mu}_\tau + m_n(\hat{\beta}_{\tau,1:q}), s_n^2(\hat{\beta}_{\tau,1:q}), q), \end{aligned}$$

and we define $f^{bayes}((x_{\tau+1}+\cdots+x_{\tau+h})/h|\hat{\mu}_\tau,\hat{\beta}_{\tau,1:q})$ as

$$
\begin{aligned}
&f^{bayes}((x_{\tau+1}+\cdots+x_{\tau+h})/h|\hat{\mu}_\tau,\hat{\beta}_{\tau,1:q})\\
&\quad=\frac{1}{\sqrt{\hat{\beta}'_{\tau,1:q}\hat{\beta}_{\tau,1:q}}}f^{bayes}(y^s_{\tau,h}|\hat{\beta}^s_{\tau,1:q})\\
&\quad=\sum_{n=1}^{N}t((x_{\tau+1}+\cdots+x_{\tau+h})/h,\hat{\mu}_\tau+m_n(\hat{\beta}_{\tau,1:q}),s_n^2(\hat{\beta}_{\tau,1:q}),q)\frac{f_{d_n}(\hat{\beta}_{\tau,1:q})\omega_n}{\sum_{k=1}^{N}f_{d_k}(\hat{\beta}_{\tau,1:q})\omega_k}.
\end{aligned}
\tag{F.7}
$$

Hence, the density of $(x_{\tau+1}+\cdots+x_{\tau+h})/h$ conditional on $\hat{\mu}_\tau$ and $\hat{\beta}_{\tau,1:q}$ is a weighted average of generalized Student-t^q densities in which the weights are functions of the prior weights and the likelihoods of the values of d, determined by the marginal density in Equation (F.6).

To compute the point forecast, which is the expectation of $(x_{\tau+1}+\cdots+x_{\tau+h})/h$ over $f((x_{\tau+1}+\cdots+x_{\tau+h})/h|\hat{\mu}_\tau,\hat{\beta}_{\tau,1:q})^{bayes}$, we first note that the expectation of $t((x_{\tau+1}+\cdots+x_{\tau+h})/h,\hat{\mu}_\tau+m(\hat{\beta}_{\tau,1:q},d_n),s^2(\hat{\beta}_{\tau,1:q},d_n),q)$ is $\hat{\mu}_\tau+m(\hat{\beta}_{\tau,1:q},d_n)$. Then, we have

$$
\hat{f}^{MWd}_{\tau,h}=\sum_{n=1}^{N}(\hat{\mu}_\tau+\Sigma_{n,y\beta}\Sigma^{-1}_{n,\beta\beta}\hat{\beta}_{\tau,1:q})\frac{f_{d_n}(\hat{\beta}_{\tau,1:q})\omega_n}{\sum_{k=1}^{N}f_{d_k}(\hat{\beta}_{\tau,1:q})\omega_k}. \tag{F.8}
$$

We compute medians and equal-tailed forecast intervals using the cumulative distribution function (CDF) that corresponds with Equation (F.7). The CDF is

$$
\begin{aligned}
&F^{bayes}((x_{\tau+1}+\cdots+x_{\tau+h})/h|\hat{\mu}_\tau,\hat{\beta}_{\tau,1:q})\\
&\quad=\sum_{n=1}^{N}T\left(\frac{(x_{\tau+1}+\cdots+x_{\tau+h})/h-\hat{\mu}_\tau-m_n(\hat{\beta}_{\tau,1:q})}{\sqrt{s_n^2(\hat{\beta}_{\tau,1:q})}},q\right)\frac{f_{d_n}(\hat{\beta}_{\tau,1:q})\omega_n}{\sum_{k=1}^{N}f_{d_k}(\hat{\beta}_{\tau,1:q})\omega_k},
\end{aligned}
\tag{F.9}
$$

in which $T(\cdot,q)$ is the CDF for a standard Student-t distribution with q degrees of freedom. Taking $\hat{\mu}_\tau$ and $\hat{\beta}_{\tau,1:q}$ as given, we use the method of bisection to solve for the values of $(x_{\tau+1}+\cdots+x_{\tau+h})/h$ that yield $F^{bayes}=0.16$, $F^{bayes}=0.5$, and $F^{bayes}=0.84$.

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