# Supplementary Material for "Trading and Liquidity with Limited Cognition"

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December 13, 2010

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This supplementary appendix establishes results to complement and extend the main analysis of Biais, Hombert, and Weill (2010) (henceforth BHW). Each sections is self-contained and can be read separately.

# I Comparison with Lagos and Rocheteau (2009)

One key addition of our new paper (Biais, Hombert, and Weill, 2010, henceforth BHW) over the earlier work of Biais and Weill (2009) is to relax the restrictive assumption that traders can only hold one or zero units of the asset.

While we allow unrestricted asset holdings, we restrict attention to a particular functional form of the utility flow function (see Section 2.1, page 6 in BHW). The reader may be concerned that our particular functional form is bringing new undesirable restrictions through the back door. The goal of this section is to demonstrate that, as far as we can tell, this concern is unwarranted.

To make this argument, we show that our preference specification is consistent with the main implications of allowing unrestricted asset holdings that have been documented in the literature. Our benchmark is the analysis of Lagos and Rocheteau (2009, hereafter LR). LR derive new results about the distribution of asset holdings and measures of liquidity when investors are allowed unrestricted asset holdings. Some of their results are proved under general twice continuously differentiable and strictly concave preferences, and others are shown under a particular iso-elastic preference specification. In this note we show that their findings, the ones derived for differentiable preferences, and the ones derived for particular iso-elastic preferences, also hold with BHW's preference specification.

## I.1 Setup

We consider the steady state setup of LR with the preference specification of BHW. First, investors switch indefinitely between a high valuation type and a low valuation type. As in BHW, high–valuation investors' utility flow is $v(h, q) = q$ for $q \leq 1$ and $v(h, q) = 1$ for $q \geq 1$. Low–valuation investors' utility flow is $v(\ell, q) = q - \delta q^{1+\sigma}/(1 + \sigma)$ for $q \leq 1$ and $v(\ell, q) = 1 - \delta/(1 + \sigma)$ for $q \geq 1$. High–valuation (low–valuation) investors switch to low–valuation (high–valuation) at rate $\gamma_\ell$ ($\gamma_h$). Second, as in LR, the market is a dealer market where investors can only submit market orders. Specifically, investors meet dealers according to a Poisson process with arrival rate $\rho$. When an investor and a dealer meet, they bargain over the size of the market order, and over a trading fee. The outcome of the bargaining process is given by the generalized Nash-bargaining solution, where the dealer's bargaining power is $\eta$. In all what follows, we let $\kappa \equiv \rho(1 - \eta)$. Table 1 provides the correspondence between LR and BHW's notations.

| | Lagos and Rocheteau | Biais, Hombert and Weill |
|---|---|---|
| Asset supply | $A \geq 0$ | $s \in [0,1)$ |
| Asset holdings | $a$ | $q$ |
| Set of preference types | $i \in \mathbb{X}$ | $\theta \in \{h, \ell\}$ |
| No. of preference types | $I$ | 2 |
| Preference switching rate $i \to j$ | $\delta\pi_j$ | $\gamma_j$ |
| Discount rate | $r$ | $r$ |
| Meeting rate with dealers | $\alpha$ | $\rho$ |
| Dealers bargaining power | $\eta$ | $\eta$ |

**Table 1:** Correspondence between notations

## I.2 Steady state equilibrium

Let us start by deriving the steady state equilibrium. We cannot apply directly the results of LR because they require twice continuously differentiable and strictly concave utility flows. Our utility flows function, by contrast, are not twice continuously differentiable because of the kink in $q = 1$, and they are only weakly concave since they are constant for $q \geq 1$. However, it is straightforward to characterize the equilibrium following the same steps as in LR.

In a steady state equilibrium, $q_\theta$ is the asset holding chosen by a type–$\theta$ when she meets a dealer; the asset price is $p$; lastly, $\phi_\theta(q)$ is the equilibrium fee paid to the dealer by a type–$\theta$ investor holding $q$ unit of the asset before meeting the dealer.

With BHW's preferences, equilibrium allocations come in only two flavors: either $q_h$ is strictly lower than 1 ("interior" equilibrium allocation) or equal to 1 ("corner" equilibrium allocation). The following Lemma, proved in Section IX.1.1, characterizes $q_h$ and $q_\ell$ in each case.

**Lemma I.1** (Steady state allocation)**.** *There exists a unique steady–state equilibrium. If* $s < (\gamma_h + \varepsilon\gamma_\ell)/(\gamma_h + \gamma_\ell)$*, then*

$$q_\ell = \frac{\gamma_h + \gamma_\ell}{\gamma_h/\varepsilon + \gamma_\ell} s, \qquad \text{and} \qquad q_h = \frac{q_\ell}{\varepsilon}, \tag{I.1}$$

*where* $\varepsilon \equiv (\gamma_\ell/(r + \kappa + \gamma_\ell))^{1/\sigma}$*. Otherwise if* $(\gamma_h + \varepsilon\gamma_\ell)/(\gamma_h + \gamma_\ell) \leq s < 1$

$$q_\ell = \frac{\gamma_h + \gamma_\ell}{\gamma_\ell} s - \frac{\gamma_h}{\gamma_\ell}, \qquad \text{and} \qquad q_h = 1 \tag{I.2}$$

As intuition suggests, a corner equilibrium allocation, $q_h = 1$, arise when the asset supply is large enough.[1]

[1] It is straightforward (but somewhat uninteresting) to extend the analysis to $s \geq 1$: in this case all investors hold more than one unit of the asset with zero marginal utility, and hence $p = 0$.

## I.3 Counterpart of Proposition 2-4 in LR

To derive their results for trading and liquidity LR assume the Inada condition $v_i'(0) = +\infty$. It is merely a simplifying assumption to keep equilibrium asset holdings strictly positive. With our preference specification, this condition is not satisfied since $v_q(h, 0) = v_q(\ell, 0) = 1$, but this causes no complication since, by Lemma I.1, our equilibrium asset holdings are also strictly positive.

First, LR establish (Proposition 2) that the dispersion of asset holdings increases with the trading frictions. The following Proposition, proved in Section IX.1.2, reproduce their results with our preference specification:[2]

**Proposition I.1** (Dispersion of asset holdings)**.** *Holding either $\rho$ or $\eta$ fixed:*

*(i)* $q_h \to s$ *and* $q_\ell \to s$ *as* $r + \kappa \to 0$.

*(ii) An increase in $r + \kappa$ causes the distribution of asset holdings to become more dispersed.*

Second, LR show (Proposition 3) that trade volume increases when trading frictions vanish. The following Proposition, proved in Section IX.1.3, reproduce their results with our preference specification:[3]

**Proposition I.2** (Trade volume)**.**

*(i) Trade volume goes to zero as $r + \kappa \to 0$.*

*(ii) Trade volume increases with $\kappa$.*

*(iii) For $\kappa' > \kappa$ the distribution of trade sizes associated with $\kappa'$ first-order stochastically dominates the one associated with $\kappa$.*

Lastly, LR show (Lemma 4) that fees – both total and per unit of asset traded – increase with the size of the trade, and (Proposition 4) that trading frictions have a nonmonotonic effect on fees. The following Proposition, proved in Section IX.1.4, reproduce their results with our preference specification:

**Proposition I.3** (Transaction costs)**.**

*(i) For $i \in \{h, \ell\}$ and $q \neq q_i$, $\partial/\partial q[\phi_i(q)]$ and $\partial/\partial q[\phi_i(q)/|q_i - q|]$ have the same sign as $q - q_i$.*

*(ii) There exists $\overline{r}$ such that for $r < \overline{r}$ and $i \neq j$, $\phi_i(q_j)$ is nonmonotonic in $\kappa$ and is largest for some $\kappa \in (0, +\infty)$.*

[2] LR prove point *(ii)* with iso-elastic preferences and many types. Proposition I.1 shows that it also holds with two types under our preference specification.

[3] LR prove *(ii)* for iso-elastic preferences only, and point *(iii)* for logarithmic preferences. Both hold with our preference specification.

*(iii) There exists $\overline{r}$ such that for $r < \overline{r}$, the expected fee earned by a dealer conditional on meeting an investor is nonmonotonic in $\kappa$ and is largest for some $\kappa \in (0, +\infty)$.*

# II Equilibrium with only market orders

In this section we consider the setup of our paper (Biais, Hombert, and Weill, 2010, henceforth BHW) with one modification: we shut down algorithms and limit order books. Precisely, as in Section 4, page 25 in BHW, we assume that traders can only submit orders when their information process jump. Differently from BHW, we assume that traders can only submit market orders. In this context, we show that the price recovers faster to its fundamental value than in the equilibria of BHW. However, at the same time, social welfare is lower. We also provide a discussion of traders' incentives to submit limit orders, and link our result to earlier findings from the literature.

In all what follows we call first equilibrium of BHW (Proposition 1, page 18 in BHW) an Algorithmic Trading Equilibrium, or "ATE", because it is implemented using algorithms. Similarly, we call the second equilibrium shown in BHW (Proposition 9, page 30 in BHW) a Limit Order Equilibrium, or "LOE", because it is implemented using limit orders only. Lastly, the equilibrium we are about to solve for, where traders only use market orders, is called a Market Order Equilibrium, or "MOE."

## II.1 Solving for an equilibrium

We first solve for a MOE. The reader may want to skip the step-by-step analysis and go directly to Proposition II.1, which describes the MOE.

Because we maintain the imperfect cognition friction, the entire preliminary analysis of Section 3 in BHW goes through, under the maintained assumption that the price is bounded, continuous, and piecewise continuously differentiable. The key difference with BHW is that, upon an information event, a trader can only submit market orders to buy and sell or, equivalently, that a trader's asset holding has to stay constant in between information events, $q_{t,u} = q_{t,t}$ for all $u \geq t$. Plugging this restriction into the inter-temporal payoff, equation (6) page 14 in BHW, we obtain:

$$V(q) = \mathbb{E}_0 \left[ \int_0^\infty e^{-rt} \int_t^\infty e^{-(r+\rho)(u-t)} \left\{ \mathbb{E}_t \left[ v(\theta_u, q_{t,t}) \right] - \xi_u q_{t,t} \right\} du \, \rho dt \right]. \tag{II.1}$$

As is the case in the LOE of BHW, it is clear from the above expression that an asset holding plan is optimal if and only if it maximizes the expected utility of a trader from one information event to the next. That is, upon an information event at time $t$, the trader picks a constant asset holding plan, $q_{t,t}$, in order to maximize:

$$(r+\rho)\int_t^\infty e^{-(r+\rho)(u-t)}\left\{\mathbb{E}_t\left[v(\theta_u, q_{t,t})\right] - \xi_u q_{t,t}\right\} du = \overline{v}(\theta, q_{t,t}) - \bar{\xi}_t q_{t,t}, \tag{II.2}$$

where

$$\bar{\xi}_t \equiv (r+\rho)\int_t^\infty e^{-(r+\rho)(u-t)}\xi_u \, du \tag{II.3}$$

is the average holding cost incurred by the trader until her next information event and where direct calculations[4] show that

$$\begin{aligned}
\overline{v}(h,q) &= v(h,q) \\
\overline{v}(\ell,q) &= \frac{r+\rho}{r+\rho+\gamma}v(\ell,q) + \frac{\gamma}{r+\rho+\gamma}v(h,q).
\end{aligned}$$

With market order, the market clearing condition becomes:

$$\rho\mu_{ht}\mathbb{E}\big[q_{t,t}\,|\,\theta_t = h\big] + \rho(1-\mu_{ht})\mathbb{E}\big[q_{t,t}\,|\,\theta_t = \ell\big] \quad = \quad \rho s, \tag{II.4}$$

which is obtained by differentiating equation (8) in BHW. The intuition for (II.4) is straightforward. At each point in time, there is a flow $\rho\mu_{ht}$ of high-valuation investors who experience an information event, with a gross asset demand equal to $\mathbb{E}\left[q_{t,t}\,|\,\theta_t = h\right]$, which leads to the first term on the left–hand side of (II.4). The second term is, symmetrically, the gross demand of low–valuation investors. To calculate the gross supply, we note that, since the investors experiencing an information event at time $t$ are drawn at random, their average asset holding is equal to $s$, the economy–wide per capita asset holding. This results in the (flow) gross supply $\rho s$, on the right–hand side of (II.4).

After canceling $\rho$ from both sides of (II.4), one finds a market–clearing condition which is formally the same as the market-clearing condition in the perfect cognition case.[5] Taken together with the objective (II.2), this remark implies that the equilibrium equations for a MOE are *the same as in the perfect cognition case* but after replacing $\xi_t$ by $\bar{\xi}_t$ and $v(\theta,q)$ by $\overline{v}(\theta,q)$. Then, all the analysis of Section 2.3 in BHW goes trough. In particular, investors who

[4]These calculations are special cases of the ones conducted at the beginning of Section IX.1.1, page 72, after letting $\gamma_h = \gamma$ and $\gamma_\ell = 0$.

[5]Its meaning is different, of course: with imperfect cognition and market orders, the market is not clearing among all investors, but only among the flow of investors experiencing an information event.

experience an information event with a low utility hold:

$$q_{\ell,t} = \begin{cases} (s-\mu_{ht})/(1-\mu_{ht}) & \text{if } t \leq T_s \\ 0 & \text{if } t > T_s, \end{cases} \tag{II.5}$$

while investors who experience an information event with a high utility hold:

$$q_{h,t} = \begin{cases} 1 & \text{if } t \leq T_s \\ s/q_{ht}, \text{ on average,} & \text{if } t > T_s, \end{cases} \tag{II.6}$$

Also the average holding cost is:

$$\bar{\xi}_t = \overline{v}_q(\ell, q_{\ell,t}) = \begin{cases} 1 - \delta \frac{r+\rho}{r+\rho+\gamma} \left(\frac{s-\mu_{ht}}{1-\mu_{ht}}\right)^{\sigma} & \text{if } t < T_s \\ 1 & \text{if } t \geq T_s. \end{cases}$$

To recover $p_t$ from $\bar{\xi}_t$, we take the derivative of (II.3). These calculations lead to:

**Proposition II.1** (Market order equilibrium.)**.** *There exists a MOE. The equilibrium allocation is described by equations (II.5) and (II.6), and is unique up to the distribution of asset holdings among high–valuation investors after $T_f$. The price is continuous and satisfies the ODE*

$$t < T_s: \quad \xi_t = rp_t - \dot{p}_t = 1 - \delta \frac{r+\rho}{r+\rho+\gamma} \left(\frac{s-\mu_{ht}}{1-\mu_{ht}}\right)^{\sigma} \left[1 + \frac{\sigma\gamma(1-s)}{r+\rho} \frac{1}{s-\mu_{ht}}\right],$$
$$t > T_s: \quad \xi_t = rp_t = 1.$$

Note that, if $\sigma \in (0,1)$, $\xi_t \to -\infty$ when $t \to T_s^-$. Nevertheless, the integral $p_t = \int_t^\infty e^{-r(u-t)} \xi_u \, du$ remains well defined, because in a left neighborhood of $T_s$, $\xi_t = O\left([T_s - t]^{-(1-\sigma)}\right)$.

## II.2 Properties of the MOE

**Price.** One sees that, in the MOE, the price recovers to its long run value of $1/r$ at time $T_s < T_f$, earlier than in either the ATE or the LOE. In some sense, the price appear more "resilient" in the MOE than in the ATE or the LOE. Also, when the price reaches $1/r$, it grows very quickly: $\dot{p}_t/p_t \geq r$ at $t = T_s$. This property is illustrated in Figure 1 which plots $\dot{p}_t/p_t$ in the case where $\sigma = 0.3$. Since $\sigma < 1$, the holding cost $\xi_t$ goes to $-\infty$ when $t$ approaches $T_s$ from the left by Proposition II.1, thus the growth rate of the price becomes infinite. In particular, the price grows at a much higher pace in the MOE than in the ATE or the LOE just before $T_s$.

This larger growth rate underlies traders' incentives to submit limit orders. Indeed, if let us

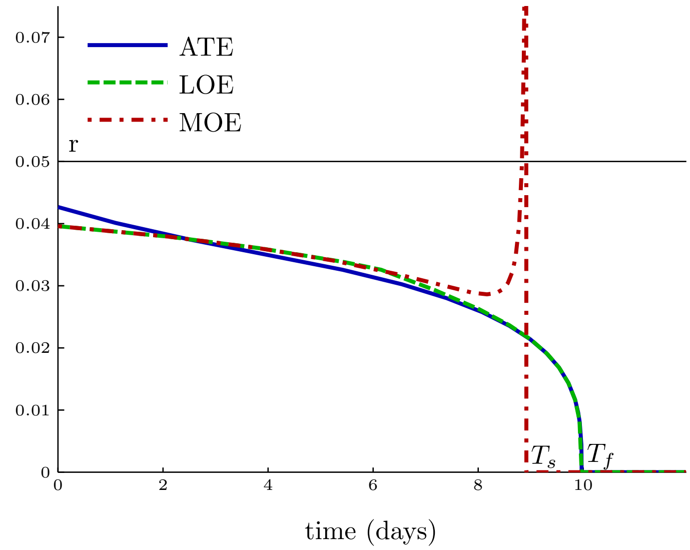


**Figure 1:** Growth rate of the price, when $\sigma = 0.3$

allow a single low–valuation investor in the MOE to submit limit orders. Upon an information event around time $T_s$, the investor anticipates that the price will grow very quickly. To reap the associated capital gains, he will find it optimal to buy lots of assets with a market order, and re-sell them with a limit–order to sell at $p_{T_s}$, executed at time $T_s$.

If all investors are allowed to submit such limit–orders and engage in the above described buy-low sell-high trading strategy, two general equilibrium effects arise. First, there is an increase in demand before $T_s$ and, second, there is an increase in supply after $T_s$. The first effect tends to increase the price before $T_s$, while the second effect tends to decrease it. This second effect explains why, in the ATE or the LOE, the price takes more than $T_s$ periods to recover. Taken together, the two effects reduce the growth rate of the price around $T_s$.

Another property which is worth mentioning is that social welfare in the ATE or the LOE is strictly higher than in the MOE.[6] At the same time, price recovery is slower.

**Trading volume.** The trading volume in the MOE, in the ATE, and in LOE, are plotted in Figure 2. We observe that, for $u < T_\phi$, the trading volume in the MOE and in the LOE are exactly equal. This is because no limit orders are executed before $T_\phi$ in the LOE, so the allocation and the trading volume do not depend upon whether limit orders are available or not. For $u \in (T_\phi, T_f)$, the trading volume is lower in the MOE than in the LOE, because limit sell orders are executed in the LOE but not in the MOE.

After $T_f$, the volume falls rapidly to zero in the ATE and in the LOE, while it remains well above zero for a while in the MOE. This is because in the former cases, low–valuation traders

[6]This is because the MOE allocation is feasible for the planner in the social welfare maximization problems of Proposition 2, page 19 in BHW, and for the one of Lemma VII.20, page 43 in this Addendum.

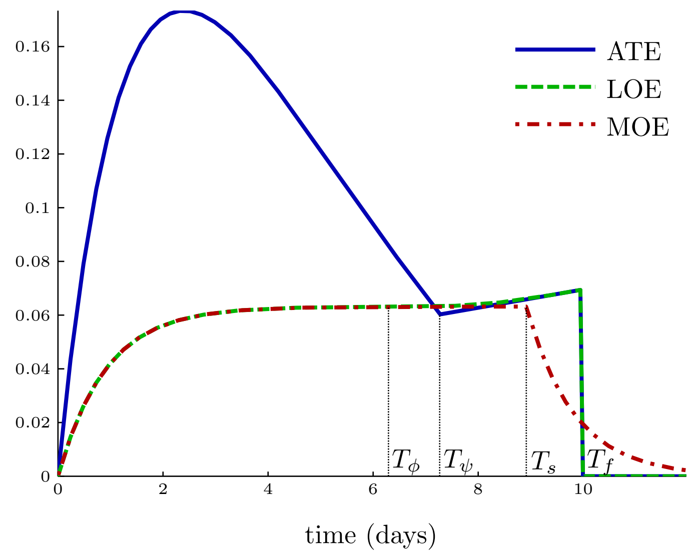


**Figure 2:** Trading volume, when $\sigma = 0.3$

who already have had an information event have already sold their asset with limit sell orders. By contrast, in the MOE, they have to wait another information event to sell, which explains why trading does not stop after $T_f$. Although the price converges faster in the MOE, trading lasts longer and the reallocation of asset takes more time.

# III Indivisible assets

In Biais and Weill (2009, henceforth BW), we solved for an equilibrium with limit orders only, under the restrictive assumption that traders can hold either zero or one unit of the asset. In this section we compare the predictions of this indivisible asset model with the predictions of our new divisible asset model (Biais, Hombert, and Weill, 2010, henceforth BHW).

In all what follows we call the first equilibrium of BHW (Proposition 1, page 18 in BHW) an Algorithmic Trading Equilibrium, or "ATE", because it is implemented using algorithms. Similarly, we call the second equilibrium shown in BHW (Proposition 9, page 30 in BHW) a Limit Order Equilibrium, or "LOE", because it is implemented using limit orders only. We show that the ATE and LOE equilibria of BW and BHW can differ in important ways. First, the asset holding restriction of BW implies that traders cannot split their orders. In BHW, by contrast, there is order splitting in equilibrium: traders submit entire sequence of orders so as to slowly and continuously unwind their holdings. Second, we find that in the ATE of BW, asset holding plans are always hump shaped, while in some cases of BHW they can be strictly decreasing. In other cases, however, the asset holding plans and price path of BW and BHW are very similar: in particular, we show that when $\sigma \to 0$, the divisible asset equilibrium of BHW converges to the indivisible asset equilibrium of BW.

## III.1 BW vs BHW's preferences

In BW, traders can hold either zero or one unit of the asset. If they hold zero unit, their utility is normalized to zero. If they hold one unit, their utility is equal to 1 when in the high state, $\theta_t = h$, and equal to $1 - \delta$ when in the low state, $\theta_t = \ell$.

In BHW, by contrast, investors can hold any positive quantity of the asset. When in the high state, $\theta_t = h$, their flow utility is:

$$v(h, q) = q, \text{ for all } q \leq 1, \text{ and } v(h, q) = 1, \text{ for all } q > 1.$$

When in the low state, $\theta_t = \ell$, the flow utility is:

$$v(\ell, q) = q - \delta \frac{q^{1+\sigma}}{1+\sigma}, \text{ for all } q \leq 1, \text{ and } v(\ell, q) = 1 - \delta/(1+\sigma), \text{ for all } q > 1,$$

where $\delta \in (0, 1)$ and $\sigma > 0$.

To understand the convergence results of this section, it useful to note that, when $\sigma \to 0$ in BHW, the flow utilities becomes $v(h, q) = \min\{q, 1\}$ and $v(\ell, q) = (1 - \delta) \min\{q, 1\}$. This limiting "Leontief" specification is evidently closely related to the "indivisible asset" specification of BW. On the one hand, because of zero marginal utility for $q > 1$, in equilibrium traders find it optimal to keep their holdings in $[0, 1]$.[7] On the other hand, because of linear utility over $q \in [0, 1]$, in equilibrium traders find it optimal to hold either zero or one unit of the asset. With this in mind, then, it is not surprising that the equilibria derived in BHW converge to their BW's counterparts as $\sigma \to 0$.

## III.2 The ATE

We start by solving for the ATE when assets are indivisible, an equilibrium concept which was not considered in BW.

**Proposition III.1** (ATE with BW's preferences)**.** *For each $u \in [0, T_f]$, let $\psi_u^*$ be the unique solution of:*

$$\int_0^{\psi_u^*} \rho e^{-\rho(u-t)} \left(1 - \mu_{ht}\right) dt = \int_0^u e^{-\rho(u-t)} \left(s - \mu_{ht}\right) dt.$$

*Let $p_u^*$ be the continuous price path solving the ODE:*

$$\begin{aligned} u < T_f: \quad & r p_u^* - \dot{p}_u^* = 1 - \delta \frac{1 - \mu_{hu}}{1 - \mu_{h\psi_u^*}} \\ u \geq T_f: \quad & r p_u^* = 1. \end{aligned}$$

*Lastly consider the time–$t$ asset holding plan when $\theta_t = \ell$:*

$$\begin{aligned} q_{t,u}^* &= \mathbb{I}_{\{t \leq \psi_u^*\}} && \textit{if} \quad t \in (0, T_f], \ \textit{for all} \quad u \in [t, T_f) \\ &= 0 && \textit{if} \quad t \in (0, \infty), \ \textit{for all} \quad u \in [T_f \vee t, \infty), \end{aligned}$$

*and, when $\theta_t = h$:*

$$\begin{aligned} q_{t,u}^* &= 1 && \textit{if} \quad t \in (0, T_f], \ \textit{for all} \quad u \in [t, \infty) \\ &= 1 \ \textit{with proba} \ \frac{s}{\mu_{ht}}, && \textit{if} \quad t \in [T_f, \infty), \ \textit{for all} \quad u \in [t, \infty). \end{aligned}$$

*Then, the price $p_t^*$ and asset holding plan $q_{t,u}^*$ is an ATE with BW's preferences.*

In the proof of the proposition, in Section IX.2.1, page 77 of this supplementary appendix, we establish two additional results. First, as in BHW, the price is strictly increasing until $T_f$,

[7] This is not true in general, however: zero marginal utility flow above a certain threshold *does not imply* that equilibrium asset holding are always less than the threshold. See Weill (2007) for an example.

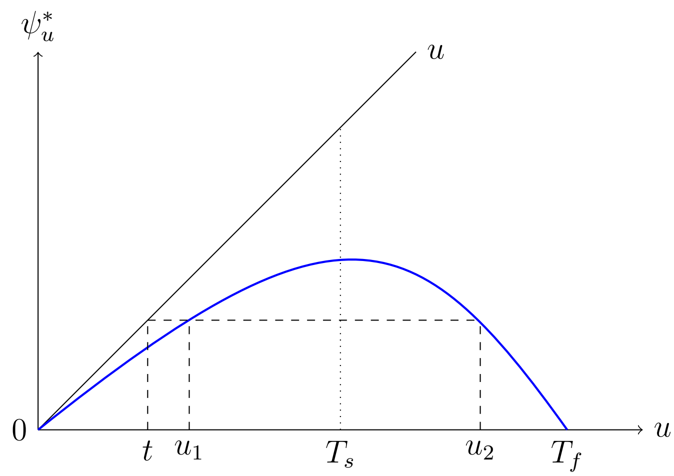


**Figure 3:** The function $\psi_u^*$.

and constant thereafter. Second, as illustrated in Figure 3, the function $\psi_u^*$ is less than $u$, hump-shaped, and achieves its maximum at $u = T_s$.

Before turning to the comparison with BHW, let us describe the holding plan of a low–valuation trader. The only interesting case is when the information event occurs at $t \leq T_f$ – the other cases are essentially the same as in BHW. There are two sub-cases:

- If $t \leq \psi_{T_f}^*$: from Figure 3 one sees that, when $t < \psi_{T_s}^*$, there exists two times $u_1 < u_2$ such that $\psi_{u_1}^* = \psi_{u_2}^* = t$. For all $u \in (u_1, u_2)$, we have $t \leq \psi_u^*$, while for $u \in [t, u_1]$ and $u \in [u_2, \infty)$, we have $t > \psi_u^*$. This means that the trader holds zero assets from time $t$ to time $u_1$, one unit from time $u_1$ to time $u_2$, and zero units after time $u_2$. Because the price is strictly increasing, this asset holding plan is implemented as follows: sell all your assets with a market order at the information event time $t$, buy one unit with a market order automatically triggered at time $u_1$, and sell one unit with a limit sell order at price $p_{u_2}$, executed at time $u_2$.
- If $t \in (\psi_{T_f}^*, T_f]$: from Figure 3, one sees that $\psi_u^* < t$ for all $u \in [t, T_f)$, and so $q_{t,u} = 0$.

**Comparison with BHW.** Perhaps the main substantive difference is that, in BW, traders never split their order. This is because BW exogenously restrict asset holdings to be either zero or one, implying that all trades must have the same block size of one. In BHW we relax this restrictive assumption and obtain equilibrium trading strategies featuring order splitting: traders find it optimal to submit entire sequence of limit and triggers order, so as to slowly and continuously unwind their asset holdings.[8]

[8]Technically, in BW asset holdings are discontinuous, while in BHW they are continuous. Continuity in

Another difference concerns the asset holding plans of low–valuation traders. In the ATE of BW, for $t < \psi^*_{T_s}$, the asset holdings take the form of a hump–shaped step function: first zero, then one, and then zero, as illustrated in Figure 4. When $s \leq \sigma/(1+\sigma)$, this step function is qualitatively different from the asset holding plans of BHW: indeed, with these parameters, BHW's asset holdings plans are strictly decreasing, instead of hump–shaped in BW. When $s > \sigma/(1+\sigma)$, however, BW's asset holdings can be viewed an extreme version of the smooth hump–shaped asset holding of BHW. In particular, Figure 4 illustrate that in BHW, for some parameters, low–valuation traders' asset holdings increase continuously until reaching $q_{t,u} = 1$, then stays equal to one for some time, and then continuously decrease their asset holdings until reaching $q_{t,u} = 0$ at time $u = T_f$. The similarity with BW's step function is clear from the figure. Also, in both BHW and BW, there is undercutting: low–valuation traders submit limit sell orders at lower and lower prices.

As argued before, similarities between BW and BHW should be expected for small $\sigma$. The next proposition makes that point formally and shows that for $\sigma$ close enough to zero, the smoothly increasing and decreasing portion of the BHW's asset holding plan become arbitrarily close to vertical lines, and BHW's equilibrium asset holdings converges to that of BW:

**Proposition III.2** (Convergence of the ATE.)**.** *Consider the ATE in BHW and let $\sigma \to 0$. Then the average asset holding plans, $\mathbb{E}\left[q_{t,u} \,|\, \theta_t\right]$, and the price path, $p_t$, converge pointwise, almost everywhere, towards the average asset holding plans $\mathbb{E}\left[q^*_{t,u} \,|\, \theta_t\right]$ and price path $p^*_t$ of Proposition III.1.*

We have convergence "in average holding plan" because, at $t \geq T_f$, high–valuation traders are indifferent between any holding plan $q_{t,u} \in [0,1]$, and so only the average asset holding plan is determinate.

Lastly, let us note that an obvious difference between BW and BHW concerns the asset holding plan of high–valuation traders after time $T_f$: because they can only hold zero or one unit, market clearing require that they randomize between a market order for one unit, and no market order, with probability $s/\mu_{ht}$. But this difference is inessential: in BHW, such randomization is also an optimal strategy, given that high–valuation have linear utility and, after $T_f$, are indifferent between holding any quantity $q \in [0,1]$.

## III.3 The LOE

Turning to the LOE, we start by recalling the main result of BW:

BHW arises because marginal utility decreases strictly and continuously in $q \in [0,1]$: this implies that, in response to continuous changes in the holding cost $\xi_u$, traders change their asset holdings, $q_{t,u}$, continuously.

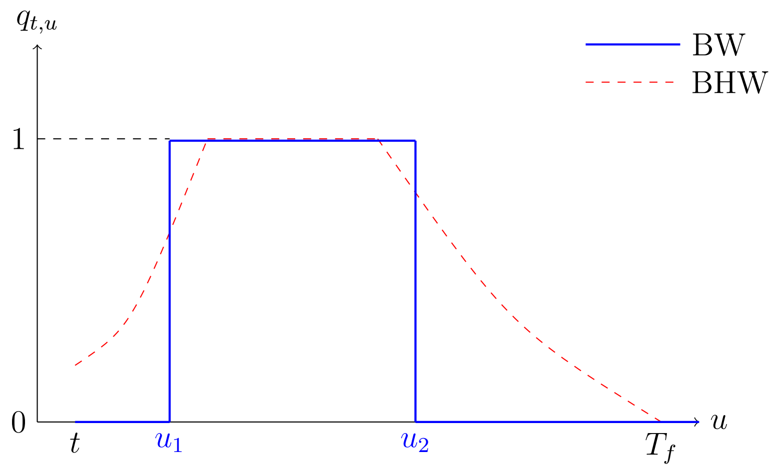


**Figure 4:** ATE asset holdings for low–valuation traders who experience an information event at time $t$. The red dashed curve represent a typical equilibrium asset holding from BHW, and the blue thick curve represent the typical equilibrium asset holding in BW.

**Proposition III.3** (LOE in BW.). *For each $t \in [0, T_f]$, let $\phi_t^*$ be the unique solution of*

$$\int_t^{\phi_t^*} \rho e^{-\rho(\phi_t^* - u)} \left(s - \mu_{hu}\right) du = 0.$$

*Let $p_u^*$ be the continuous price path solving the ODE:*

$$\begin{aligned}
t < T_s: \quad & rp_t^* - \dot{p}_t^* = 1 - \delta + \delta \frac{d}{dt}\left[\frac{1}{1-\mu_{ht}}\right] \int_t^{\phi_t^*} e^{-(r+\rho)(u-t)}(1-\mu_{hu})\, du \\
t \in (T_s, T_f): \quad & rp_t^* - \dot{p}_t^* = 1 - \delta \frac{1-\mu_{ht}}{1-\mu_{h(\phi^*)_t^{-1}}} \\
t \geq T_f: \quad & rp_u^* = 1.
\end{aligned}$$

*Lastly consider the time–$t$ asset holding plan when $\theta_t = \ell$:*

$$\begin{aligned}
q_{t,u}^* &= \mathbb{I}_{\{u \leq \phi_t^*\}} \text{ with proba } \frac{s-\mu_{ht}}{1-\mu_{ht}} && \text{if} \quad t \in (0, T_s), \text{ for all} \quad u \in [t, \infty) \\
&= 0 && \text{if} \quad t \in [T_s, \infty), \text{ for all} \quad u \in [t, \infty),
\end{aligned}$$

*and, for $\theta_t = h$:*

$$\begin{aligned}
q_{t,u}^* &= 1 && \text{if} \quad t \in (0, T_f], \text{ for all} \quad u \in [t, \infty) \\
&= 1 \text{ with proba } \frac{s}{\mu_{ht}}, && \text{if} \quad t \in [T_f, \infty), \text{ for all} \quad u \in [t, \infty).
\end{aligned}$$

*Then, the price $p_t^*$ and asset holding plan $q_{t,u}^*$ is an LOE with BW's preferences.*

The comparison between BW and BHW goes along the same lines as in the previous section. First, in BW, traders never split their orders. In BHW, by contrast, there is order splitting: traders submit entire sequences of limit sell orders so as to be able to unwind their asset holding slowly and continuously. Second, in both BW and BHW (for small enough $\sigma$), there is undercutting: limit sell orders are being submitted at lower and lower prices.

Lastly, in BW, we have randomization: to clear the market with indivisible assets, it is sometimes necessary to have identical investors behave differently. But, as argued earlier, this difference is somewhat inessential.[9] We conclude this section with the convergence result:

**Proposition III.4** (Convergence of the LOE.). *Consider the LOE in BHW and let $\sigma \to 0$. Then the average asset holding plans, $\mathbb{E}\left[q_{t,u} \,|\, \theta_t\right]$, and the price path, $p_t$, converge pointwise, almost everywhere, to the average asset holding plans $\mathbb{E}\left[q_{t,u}^* \,|\, \theta_t\right]$ and price path $p_t^*$ of Proposition III.3.*

We have convergence "in average holding plan" for two reasons. First, as before, after time $T_f$ high–valuation traders are indifferent between any holding plan $q_{t,u} \in [0,1]$, so only the average asset holding is determinate. Second, before time $T_s$, low–valuation traders choose the same holding plan in BHW, while they randomize between different holding plans in BW.

[9] Randomization occurs, as in the ATE, for high–valuation traders who experience an information event at time $t \in (T_f, \infty)$. But, differently from the ATE, it also occurs for low–valuation traders who experience an information event at time $t \in (0, T_f)$. As shown in BW, a fraction of low–valuation trader hold on to their asset and submit a limit sell order, while the complementary fraction sells. In BHW, by contrast, since low–valuation traders utility flow is strictly concave, randomization is strictly suboptimal. In particular, instead of randomizing between 0 and 1, all low–valuation traders choose the same asset holding plan. But this difference is somewhat inessential: if one were to replace the indivisible asset preference of BW by the essentially equivalent "Leontieff" specification, $v(h,q) = \min\{q,1\}$ and $v(\ell,q) = (1-\delta)\min\{q,1\}$, then we could construct an equilibrium based on *the same* price path as in Proposition III.3 and *without randomization*.

# IV Equilibrium with recurrent liquidity shocks

In this section we propose an extension Biais, Hombert, and Weill's (2010, henceforth BHW) model with recurrent aggregate liquidity shocks. We solve for a limited cognition equilibrium (as in Proposition 1, page 18 in BHW), and we provide closed–form expressions for all equilibrium objects. A numerical example illustrates that the results of the basic model are robust to the introduction of recurrent aggregate liquidity shocks. Our example also suggests that recurrent liquidity shocks have quantitatively important effects on the long–run level of the asset price.

BHW makes the simplifying assumption that high–valuation traders derive *linear* utility flows for the asset forever. With recurrent shocks, this assumption is relaxed, as high–valuation traders "effective" utility flow becomes non linear. Indeed, they anticipate the arrival of periodic liquidity shock, causing them to derive strictly concave utility for the asset. We find however that our results remain qualitatively similar with linear and non–linear high–valuation utility flow. This suggests that BHW's conclusion are robust to the introduction of non–linear utility flows.

## IV.1 The setup

We consider the model of Biais, Hombert, and Weill (2010, henceforth BHW) with one modification: instead of assuming that the liquidity shock occurs only once and is unanticipated, we assume liquidity shocks occur recurrently at random times, and are rationally anticipated by traders.

Our model of recurrent aggregate liquidity shocks is similar to that of Duffie, Gârleanu, and Pedersen (2007). We assume that aggregate liquidity shocks hit the economy at Poisson arrival times with intensity $\kappa > 0$. As in our basic model, when a shock hits, investors switch to the low–valuation state and recover later at independent exponential times with intensity $\gamma$. Differently from the basic model, however, traders rationally anticipate a new liquidity shock may hit at any time.

We consider the market setup of Section 3, page 11 in BHW. That is, the only constraint on traders' asset holding plans is the limited cognition friction. We assume, however, that when an aggregate liquidity shock occurs, a trader cancels all of her unfilled orders, and keep her asset holding constant until her next information event. This assumption simplifies the analysis, and it also captures the intuitive notion that, when a large aggregate liquidity event occurs, institutions may "withdraw" from the market in order to analyze the new shock until they reach a trading decision. This is in line with evidence from the "flash crash" that hit the US equity markets on May $6^{\text{th}}$, 2010: the Securities and Exchanges Commission (SEC, 2010) reports that automated trading systems paused in reaction to the sudden price decline in order to allow traders and risk managers to fully assess the risks before trading was resumed.

In all what follows, the time index, either "$t$" or "$u$", denotes the time elapsed since the last aggregate shock. We focus on stationary equilibria in which:

- The price only depends on the time $t$ elapsed since the last aggregate shock.
- Time–$t$ low–valuation traders choose the same asset holding plan, $q_{\ell,t,u}$.
- Time–$t$ high–valuation traders choose the same asset holding plan, $q_{h,t,u}$.

## IV.2 Market clearing

Consider the economy at time $u$, i.e., $u$ periods after the last liquidity shock. The population of traders can be partitioned in two sub-population:

- First, there is a measure $1-e^{-\rho u}$ of traders who have not yet received an information event. By assumption these traders have kept their asset holding constant since the last liquidity shock. Thus, they constitute a representative sample of the asset holding distribution one instant before the last liquidity shock. In particular, since the market clears one instant before the last liquidity shock, their average asset holding must be equal to $s$.
- Second, there is a density $\rho e^{-\rho(u-t)}$ of investors who had their last information event $t$ periods after the last liquidity shock. Among these traders, a fraction $1-\mu_{ht}$ have a low valuation and hold $q_{\ell,t,u}$ at time $u$, and the complementary fraction $\mu_{ht}$ has a high valuation and hold $q_{h,t,u}$ at time $u$.

Taken together, the above remarks imply that the market clearing condition is:

$$\left(1-e^{-\rho u}\right)s+\int_0^u \rho e^{-\rho(u-t)}\left\{(1-\mu_{ht})q_{\ell,t,u}+\mu_{ht}q_{h,t,u}\right\}dt=s,$$

which becomes, after rearranging:

$$\int_0^u \rho e^{-\rho(u-t)}\left\{(1-\mu_{ht})q_{\ell,t,u}+\mu_{ht}q_{h,t,u}\right\}dt=\int_0^u \rho e^{-\rho(u-t)}s\,dt, \tag{IV.1}$$

the exact same market–clearing condition as in BHW.

## IV.3 The trader's problem

When an information event occurs at time $t$, a trader picks her asset holding plan until her next information event. The plan is unrestricted, as long as no further liquidity shock occurs.

If a liquidity shock occurs, the trader cancels all her unfilled orders and keeps her asset holding constant until her next information event. With this in mind, we show in Section IX.3.1 that a trader's expected utility from time $t$ until her next information event is:

$$\int_t^\infty e^{-(r+\rho+\kappa)(u-t)} \left\{ \mathbb{E}_t \left[ v(\theta_u, q_{t,u}) \right] + \kappa W(q_{t,u}) - \xi_u q_{t,u} \right\} du, \tag{IV.2}$$

$$\text{where } \xi_u = rp_u - \dot{p}_u - \kappa(p_0 - p_u) \tag{IV.3}$$

is an adjusted holding cost at time $u$, and where $W(q)$ is the continuation value (net of holding costs) of a trader who holds $q$ units of assets from the beginning of a liquidity shock, until her next information event. As was the case in BHW, an optimal asset holding plan maximize the objective (IV.2) pointwise. That is, $q_{t,u}$ maximizes:

$$\mathbb{E}\left[ v(\theta_u, q_{t,u}) \,|\, \theta_t \right] + \kappa W(q_{t,u}) - \xi_u q_{t,u}.$$

Note that this problem is very similar to that of BHW, with adjustments reflecting the trader's rational expectations about future liquidity shocks. Namely, when a liquidity shock occurs at time $u$ with intensity $\kappa$, the trader's continuation utility is $W(q_{t,u})$, and the drop in asset price results in the capital loss $p_u - p_0$.

We conclude this section with an explicit expression for the marginal continuation value, $W_q(q)$, derived in Section IX.3.1, page 81:

**Lemma IV.1** (An expression for $W_q(q)$)**.** *The derivative of $W(q)$ with respect to $q$ writes:*

$$W_q(q) = -C + \frac{1}{r+\rho} - \frac{\delta(r+\rho+\kappa)}{r+\rho+\kappa+\gamma} q^\sigma \;\; \textit{when } q \leq 1; \;\; \textit{and } W_q(q) = -C \textit{ when } q > 1, \tag{IV.4}$$

*where* $C \equiv \frac{r+\rho+\kappa}{r+\rho} \int_0^\infty e^{-(r+\rho+\kappa)u} \xi_u \, du$.

The discontinuity of the marginal continuation value, $W_q(q)$, arises because traders' utility flow functions have a kink at $q = 1$. This is just as in BHW. Note however that, in contrast with BHW, the continuation value $W(q)$ injects some curvature in the problem of a high–valuation investor.

## IV.4 Solving for equilibrium

We already noted two striking similarities with BHW: the market–clearing condition is the same, and the trader's problem takes a similar form. This suggests that the equilibrium with anticipated recurrent shock is likely to resemble the equilibrium with a one–time unanticipated shock. To make this point more formally, in this section we provide closed–form formulas for equilibrium objects.

First, let us note that, if $\theta_t = h$, the trader's objective does not depend on $t$, which allows us to write $q_{h,t,u} = q_{h,u}$. Also, as in BHW:

**Lemma IV.2** (Bounded holdings)**.** *In equilibrium, at all times, $q_{h,u}$ and $q_{\ell,t,u}$ lie in $[0,1]$.*

Otherwise, if some trader found it optimal to hold a quantity strictly greater than 1, then given our preference specification all investors would find it optimal to hold at least 1, which would contradicting market clearing. Therefore

$$q_{h,u} = \min\{Q_{h,u}\,,\,1\} \text{ and } q_{\ell,u,t} = \min\{Q_{\ell,t,u}\,,\,1\},$$

where $Q_{h,u}$ and $Q_{\ell,t,u}$ solve the first–order condition of an "unconstrained" trader's problem:[10]

$$\begin{aligned}
0 &= 1 && +\kappa\left(-C + \frac{1}{r+\rho} - \frac{\delta(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)}Q_{h,u}^{\sigma}\right) - \xi_u \\
0 &= 1 - \delta e^{-\gamma(u-t)}Q_{\ell,t,u}^{\sigma} && +\kappa\left(-C + \frac{1}{r+\rho} - \frac{\delta(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)}Q_{\ell,t,u}^{\sigma}\right) - \xi_u
\end{aligned}$$

Subtracting one equation from the other, one immediately sees that:

**Lemma IV.3.** *In equilibrium, at all times, $Q_{\ell,t,u} = \alpha_{t,u}Q_{h,u}$, where:*

$$\alpha_{t,u} \equiv \left(1 + \frac{(r+\rho)(r+\rho+\kappa+\gamma)}{\kappa(r+\rho+\kappa)}e^{-\gamma(u-t)}\right)^{-1/\sigma}. \tag{IV.5}$$

The next step is to substitute $q_{h,u} = \min\{Q_{h,u}\,,\,1\}$ and $q_{\ell,t,u} = \min\{\alpha_{t,u}Q_{h,u}\,,\,1\}$ into the market–clearing condition (IV.1). This leads to a simple one–equation–in-one–unknown problem for $Q_{h,u}$:

$$\int_0^u \rho e^{-\rho(u-t)}\left\{(1-\mu_{ht})\min\{\alpha_{t,u}Q_{h,u}\,,\,1\} + \mu_{ht}\min\{Q_{h,u}\,,\,1\} - s\right\}du = 0. \tag{IV.6}$$

This equation is easily shown to have a unique solution – all the details are in Section IX.3.2, page 83. Next, using the first–order condition for $Q_{h,u}$, we obtain that the price solves the ODE:

$$rp_u - \dot{p}_u - \kappa(p_0 - p_u) = \xi_u = \frac{r+\rho+\kappa}{r+\rho} - \frac{\delta\kappa(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)}Q_{h,u}^{\sigma} - \kappa C. \tag{IV.7}$$

[10] Precisely, this "unconstrained problem" ignores the kink at $q = 1$ and artificially assumes that, for $q \geq 1$, the utility flow and the continuation value have the same functional form as for $q \in [0,1]$.

We are not done yet, however: indeed, $p_0$ appears on the left–hand side of the equation, and the constant $C$ is a function of the the entire path of $\xi_t$, which is itself a function of $C$. In Section IX.3.3, page 84, we show that these fixed–point problems can be solved analytically, leading to:

**Lemma IV.4.** *The price process satisfies the ODE, for all $t$:*

$$\begin{aligned}(r+\kappa)p_u - \dot{p}_u =& 1 + \frac{\kappa}{r} - \frac{\delta\kappa(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)} Q^{\sigma}_{h,u} \\ & - \frac{\delta\kappa^2(r+\kappa)(r+\rho+\kappa)}{r(r+\rho)(r+\rho+\kappa+\gamma)} \int_0^{\infty} \left(e^{-(r+\kappa)z} - e^{-(r+\rho+\kappa)z}\right) Q^{\sigma}_{h,z}\, dz.\end{aligned}$$

## IV.5 Comparison with BHW

We can solve numerically for the equilibrium $Q_{h,u}$ by following the steps outlined in Section IX.3.2. To solve numerically the ODE for the price, we go in two steps: first, we compute $p_{T_{\max}}$ for some $T_{\max} > 0$ large enough. Second, we solve the ODE for $t \in [0, T_{\max}]$ using a Runge-Kutta algorithm. In both steps we need to integrate the function of $Q_{h,u}$ in the neighborhood of infinity. For this we solve numerically for $Q_{h,u}$ for all $u \in [0, T_{\max}]$, and we use a first-order approximation for $u > T_{\max}$. The details are explained in Section IX.3.4, page 84.

### IV.5.1 When aggregate shocks occur on average every 4 months

We plot in Figure 5 the equilibrium strategies and the price path for aggregate shocks occurring at a quarterly frequency on average: $\kappa = 4$. We let $\sigma = 0.3$ and otherwise choose the same parameter values as in Table 1. Our computations illustrate that, although equilibrium objects are analytically more complicated than in BHW, they are qualitatively very similar. The computations also indicate that the effects of recurrent liquidity shocks on the long-run price level are quantitatively significant.

**High–valuation traders.** As in BHW, high–valuation traders hold one unit before $T_f$, and have a decreasing average holding after $T_f$. There is one difference with BHW: because of the curvature induced by the continuation value $W(q)$, high–valuation traders are no longer indifferent between any asset holding after $T_f$.

**Low–valuation traders.** The holdings of low–valuation traders are hump–shaped, just as in BHW.

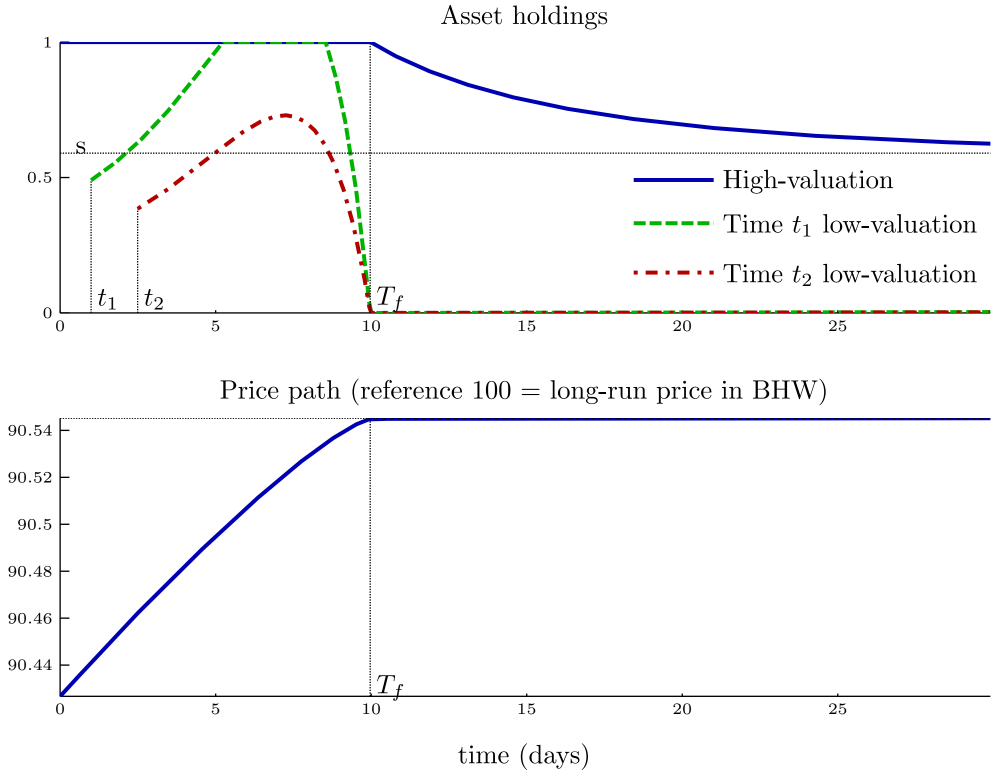


**Figure 5:** Asset holdings (top panel) and price path (bottom panel) with recurrent shocks when $\kappa = 4$

**Price path.** There are some notable differences for the price path. First, the expectation of future liquidity shock results in a permanent negative level effect – the long–run price level decreases by 9.5%. Another difference arises because of the curvature due to the continuation value $W(q)$ in the utility flow: after time $T_f$, high–valuation traders' holdings decrease, their marginal utility flow increases, and hence the price path continues to increase. This last feature of the price path is, however, not discernible at the scale of the figure. This is because shocks are not very frequent, so the prospect of future liquidity shocks injects very little curvature into high–valuation traders' utility flow.

### IV.5.2 When aggregate shocks occur on average every 10 days

Figure 6 plots the same objects when aggregate shocks occur every 10 days on average: $\kappa = 25$, keeping all other parameter values the same as before. Since the intensity at which traders switch from low– to high–valuation is also $\gamma = 25$, a trader has a 50% chance to recover from a liquidity shock before the next liquidity shock hits, and a 50% chance that a new aggregate shock occurs before he has recovered from the previous one. The equilibrium objects are quite similar to the previous case, with two noticeable difference. First, recurrent shocks have a quantitatively large impact on the long-run price level: it is now about 42% lower than in BHW.

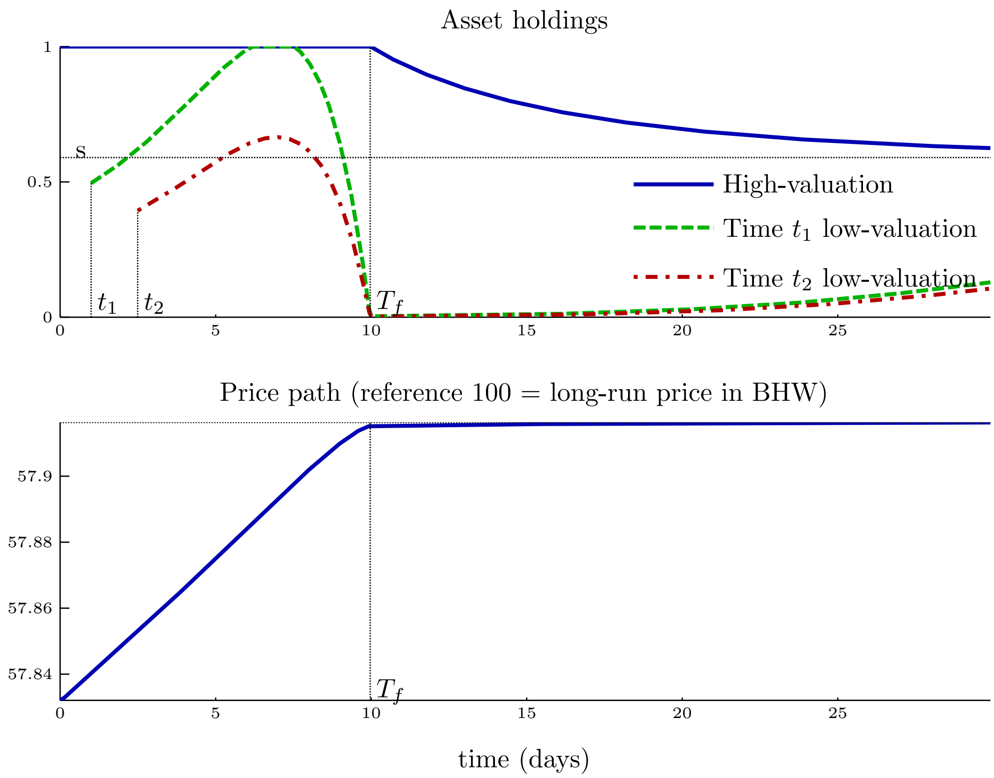


**Figure 6:** Asset holdings (top) and price path (bottom) with recurrent shocks when $\kappa = 25$

Second, low–valuation traders have strictly positive and increasing asset holdings after time $T_f$. Intuitively, the long-run price level is lower than before and so low–valuation investors find it profitable to hold some asset. To put it differently, all traders anticipate to receive aggregate shocks frequently and have their valuation "re–set" to the low state. This reduces the difference between the expected utility flows of high– and low–valuation traders. As a result equilibrium holdings of high–valuation traders decrease and the holdings of low–valuation traders increase.

# V Positive liquidity shocks

In this section we show that our setup features a natural symmetry: an equilibrium with *positive* liquidity shock can be deduced from the equilibrium with *negative* liquidity shock after a simple change of variable. In particular, the price with positive liquidity shock is just the symmetric of the price with negative liquidity shock, with respect to the long run value of $1/r$.

In all what follows, we use the tilde " $\tilde{}$ " notation to distinguish variables in the positive liquidity shock model from their negative liquidity shock counterparts. To simplify the exposition we consider a setup where asset holdings must belong to the interval $[0, 1]$. Clearly, this is without loss of generality in Biais, Hombert, and Weill (2010, henceforth BHW) as traders always find it optimal to keep their asset holdings less than one.

## V.1 The positive liquidity shock model

The setup is exactly the same as BHW's, except for the fact that the liquidity shock is positive instead of negative: at time zero, investors make a transition to a high-marginal valuation state, and keep a high–valuation for independent random exponential times with intensity $\gamma$. When in the high state, an investor's flow utility for holding $q \in [0, 1]$ shares of the asset is:

$$\tilde{v}(\tilde{h}, \tilde{q}) = \tilde{q} + \delta \frac{1 - (1 - \tilde{q})^{1+\sigma}}{1 + \sigma} \tag{V.1}$$

When in the low state, it is $\tilde{v}(\tilde{\ell}, \tilde{q}) = \tilde{q}$. Relative to the low state, the high state has both higher utility and higher marginal utility. Also, note that the high and the low state play opposite role as in BHW.

As in BHW, after defining the holding cost $\tilde{\xi} \equiv r\tilde{p}_u - \dot{\tilde{p}}_u$, we obtain the trader's intertemporal valuation net of the cost of buying and selling assets:

$$\mathbb{E}\left[\int_0^\infty e^{-rt} \int_t^\infty \mathbb{E}_t \left[\tilde{v}\left(\tilde{\theta}_u, \tilde{q}_{t,u}\right) - \tilde{\xi}_u q_{t,u}\right] du \, \rho \, dt\right]. \tag{V.2}$$

The market-clearing condition is exactly as in BHW after replacing $h$ by $\tilde{\ell}$ and $\ell$ by $\tilde{h}$:

$$\int_0^u \rho e^{-\rho(u-t)} \left\{(1 - \mu_{\tilde{\ell}t})\mathbb{E}\left[\tilde{q}_{t,u} \,|\, \tilde{\theta}_t = \tilde{h}\right] + \mu_{\tilde{\ell}t}\mathbb{E}\left[\tilde{q}_{t,u} \,|\, \tilde{\theta}_t = \tilde{\ell}\right] - \tilde{s}\right\} du = 0, \tag{V.3}$$

where $\tilde{s} \equiv 1 - s$. Equilibria with limited cognition are defined in the same way as in BHW.

| Positive liquidity shock | | Negative liquidity shock |
|---|---|---|
| $\tilde{\theta}_u = \tilde{h}$ | $\rightarrow$ | $\theta_u \equiv \ell$ |
| $\tilde{\theta}_u = \tilde{\ell}$ | $\rightarrow$ | $\theta_u \equiv h$ |
| $\tilde{\xi}_u$ | $\rightarrow$ | $\xi_u \equiv 1 + (1 - \tilde{\xi}_u)$ |
| $\tilde{s}$ | $\rightarrow$ | $s \equiv 1 - \tilde{s}$ |
| $\tilde{q}_{t,u}$ | $\rightarrow$ | $q_{t,u} \equiv 1 - \tilde{q}_{t,u}$ |

**Table 2:** The change of variables.

## V.2 The change of variables

To solve for an equilibrium, we make the change of variables summarized in Table 2. The details are intuitive: we interchange the role of the high and the low state, we replace $1 - \tilde{q}$ in the utility flow functions by $q$, and we take the symmetric of the holding cost $\tilde{\xi}_u$ around the long run value of 1, i.e. $\xi_u = 1 + (1 - \tilde{\xi}_u)$.

First, let us make the change of variables in the utility flow net of cost, $\tilde{v}(\tilde{\theta}, \tilde{q}_{t,u}) - \tilde{\xi}_u \tilde{q}_{t,u}$, in terms of our newly defined variables.

$$\begin{aligned}
\tilde{v}(\tilde{\theta}_u, \tilde{q}_{t,u}) - \tilde{\xi}_u \tilde{q}_{t,u} &= \tilde{q}_{t,u} + \mathbb{I}_{\{\tilde{\theta}_u = \tilde{h}\}} \delta \frac{1 - (1 - \tilde{q}_{t,u})^{1+\sigma}}{1+\sigma} - \tilde{\xi}_u \tilde{q}_{t,u} \\
&= (1 - q_{t,u}) + \mathbb{I}_{\{\theta_u = \ell\}} \delta \frac{1 - q_{t,u}^{1+\sigma}}{1+\sigma} - \left[1 + (1 - \xi_u)\right](1 - q_{t,u}) \\
&= q - \mathbb{I}_{\{\theta_u = \ell\}} \delta \frac{q^{1+\sigma}}{1+\sigma} - \xi_u q_{t,u} + \underbrace{\left(\frac{\delta}{1+\sigma} \mathbb{I}_{\{\theta_u = \ell\}} - 2 + \xi_u\right)}_{\equiv k(\theta_u)} \\
&= v(\theta_u, q_{t,u}) - \xi_u q_{t,u} + k(\theta_u).
\end{aligned}$$

But $k(\theta_u)$ is a constant function of the valuation state, over which the trader has no control. Clearly, this means that, after making the change of variable, the investors' objective is, up to a constant, the same as in BHW. Furthermore, the constraints on asset holding plans are also the same as in BHW. When the only constraint on asset holding plans is the limited cognition friction (as in Proposition 1, page 18 in BHW) this is obvious. When traders can only submit limit orders at the time of information events (as in Proposition 9, page 30 in BHW), this is also true: indeed, the change of variable simultaneously switches the monotonicity of both the price and asset holding plans. Suppose, for instance, that $\tilde{p}_u$ is strictly decreasing for some set of times. In that case only limit–buy orders can be executed and so the asset holding plan, $\tilde{q}_{t,u}$, has to be increasing. But then the transformed price, $p_u = 2/r - \tilde{p}_u$ is strictly increasing, only limit-sell orders can be executed so that the transformed holding plan, $q_{t,u} = 1 - \tilde{q}_{t,u}$, has to be decreasing.

Second, let us make the change of variable in the market–clearing condition (V.3):

$$\begin{aligned}
&\int_0^u \rho e^{-\rho(u-t)}\left\{(1-\mu_{\tilde{\ell}t})\mathbb{E}\left[\tilde{q}_{t,u}\,|\,\tilde{\theta}_t=\tilde{h}\right]+\mu_{\tilde{\ell}t}\mathbb{E}\left[\tilde{q}_{t,u}\,|\,\tilde{\theta}_t=\tilde{\ell}\right]-\tilde{s}\right\}du &&=0\\
\iff &\int_0^u \rho e^{-\rho(u-t)}\left\{(1-\mu_{ht})\mathbb{E}\left[1-q_{t,u}\,|\,\theta_t=\ell\right]+\mu_{ht}\mathbb{E}\left[1-q_{t,u}\,|\,\theta_t=h\right]-(1-s)\right\}du &&=0\\
\iff &\int_0^u \rho e^{-\rho(u-t)}\left\{(1-\mu_{ht})\mathbb{E}\left[q_{t,u}\,|\,\theta_t=\ell\right]+\mu_{ht}\mathbb{E}\left[q_{t,u}\,|\,\theta_t=h\right]-s\right\}du &&=0
\end{aligned}$$

which is the same as the market–clearing condition of BHW.

Taken together, we find that, after making the change of variables, the trader's problem and the market–clearing conditions are the same as in BHW. This allows us to conclude that:

**Proposition V.1** (Positive liquidity shocks)**.** *Given the equilibria of Proposition 1 and 9 in BHW's negative liquidity shock model, one obtains corresponding equilibria in the positive liquidity shock model after making the change of variables of Table 2.*

# VI Continuity of the price path

In this section we establish that, with limited cognition, when traders can submit algorithms and/or limit orders, the price path must to be continuous. We consider price paths which can be non-monotonic, have kinks, and jumps. But to simplify the analysis we rule out some pathological cases. Namely, we impose two regularity conditions. First, at any point, the price is either left– or right–continuous: i.e., if $p_{u^+} \neq p_u$, then $p_{u^-}$ exists and is equal to $p_u$ and vice versa if $p_{u^-} \neq p_u$. The second regularity condition is that, in any finite time, the price has finite (but possibly arbitrarily large) number of monotonicity changes, jumps, and kinks. Formally:

**Condition 1** (Well-behaved price path)**.** *At any time $u$, the price $p_u$ is either left– or right-continuous. And, for any finite time $t$, there exists $0 = t_0 < t_1 < t_2 < \ldots < t_K = t$ such that: in every interval $(t_k, t_{k+1})$, $\dot{p}_u$ exists, is continuous, has finite limit to the right of $t_k$ and to the left of $t_{k+1}$, and does not change sign, i.e., either $\dot{p}_u = 0$, $\dot{p}_u < 0$, or $\dot{p}_u > 0$.*

After imposing these regularity conditions, we are left with a broad class of admissible price paths. To the best of our knowledge it includes all the price paths arising in the various models studied in the finance–search literature. In particular, it includes the equilibrium price path of Weill (2007) which in some cases features one kink or one jump.

We let $t_0 < t_1 < t_2 \ldots < t_k < \ldots$ be the boundary points of the *maximal* intervals where the above properties hold. That is, for all $t_k$, the price has either a kink, a discontinuity, or its derivative is zero. We call these maximal intervals *spots*. We let an *increasing spot* be an interval where the price is strictly increasing. Similarly, we let a *decreasing spot* be an interval where the price is strictly decreasing. And, lastly, we let a *flat spot* be an interval where the price is constant.

In all what follows we call the first equilibrium concept of BHW (Proposition 1, page 17 in BHW) an Algorithmic Trading Equilibrium, or "ATE", because it corresponds to the case where traders can implement their asset holding plans using algorithms. Similarly, we call the second equilibrium concept of BHW (Proposition 8, page 27) a Limit Order Equilibrium, or "LOE", because it correspond to the case where traders can only submit limit and market orders when their information event process jumps.

## VI.1 Continuity of the price in a ATE

Suppose by contradiction that the price path jumps upwards at some $u > 0$. Consider for instance that $p_u < p_{u^+}$ (the case $p_{u^-} < p_u$ is identical after replacing $u$ by $u^-$ and $u^+$ by $u$ everywhere in the following arguments). We show that investors' asset demand is unbounded at time $u$ which contradicts market clearing given that investors' can't short-sell. Formally, we show that given any $K > 0$, for almost all $\omega \in \Omega$, if $\tau_u \in (0, u)$ then $q_{\tau_u,u} \geq K$. Indeed, for any given $K$ consider

$$C = \left\{ \omega \in \Omega \, : \, \tau_u > 0, \quad \text{and} \quad q_{\tau_u,u} < K \right\},$$

and the following deviation. At your first information event before $u$, submit a trigger order to buy $K$ additional unit of the asset at time $u$, executed at price $p_u$, and a trigger order to sell these assets just after time $u$, executed at price $p_{u^+}$. Then, asset holdings at time $u$ are $q_{\tau_u,u} + K \geq K$. Since the investor enjoys some positive utility from holding these extra $K$ units, the net change in expected utility is greater than the profit from buying at price $p_u$ and re-selling at price $p_{u^+}$. Thus the expected utility of the deviation is more than

$$\mathbb{E}\left[\mathbb{I}_C e^{-ru}(p_{u^+} - p_u)K\right] = P(C)e^{-ru}(p_{u^+} - p_u)K \geq 0.$$

But the expected utility of the deviation is negative when the holding plan is optimal, implying that $P(C) = 0$. Since the other investors have $\tau_u = 0$ and thus hold $s$ unit of the asset, this contradicts market clearing.

Similarly, if the price jumps downwards at some $u > 0$, we can follow the same lines of reasoning to show that for any $\varepsilon > 0$, for almost all $\omega \in \Omega$, if $\tau_u \in (0, u]$ then $q_{\tau_u,u} < \varepsilon$, implying that $q_{\tau_u,u} = 0$. Again, this contradicts market clearing.

## VI.2 Continuity of the price in an LOE

In the case of a LOE, the proof has a similar logic but turns out to be much longer. The reason is that traders have access to a smaller menu of orders than in the ATE, so it is not as easy, and sometimes not possible, to "arbitrage the jump." To see why, suppose that $p_{u^+} > p_u$, but that the price is increasing before time $u$. In the ATE, investors could "arbitrage the jump" by submitting a *trigger* order to buy at time $u$, executed at price $p_u$, and a trigger order to sell at time $u^+$, executed at price $p_{u^+}$. In a pure limit order market, while may still be possible to "sell high" with a limit sell order at price $p_{u^+}$, it is no longer possible to "buy low" with a trigger buy order just before $u$, as triggers are not available.

To rule out an upward jump, in Section VI.2.1, we restrict attention to the subpopulation of investors who have an information event shortly before $u$, and who can arbitrage the jump by submitting a *market order* to buy and a limit–sell order at price $p_{u^+}$. Namely, we show that if $p_{u^+} > p_u$, then the asset demand of this subpopulation would be unbounded, which is sufficient to contradict market clearing.

The case of downward jumps creates additional complications: indeed, because of the short-selling constraint we can't rely on making the symmetric argument that the *supply* would be unbounded. To rule out a downward jump, in Section VI.2.3 we make a different argument. We show that while traders who have an information event can't supply unbounded amount of the asset, they find it optimal to supply all of their holdings. On the other side of the market, no other trader want to buy: because of the downward jump, all these potential buyers find it optimal to buy at the lower post-jump price.

### VI.2.1 The price cannot jump up

Suppose by contradiction that $p_u < p_{u^+}$. Then, there exist some $t$ and $\eta > 0$ such that

$$e^{-rz} p_z < e^{-ru}(p_{u^+} - \eta) \quad \text{for all } z \in (t, u]. \tag{VI.1}$$

We now show that investors' asset demand is unbounded at time $u$. Given any $K > 0$, for almost all $\omega \in \Omega$, if $\tau_u \in (t, u]$ then $q_{\tau_u, u} \geq K$. Indeed, for any given $K$ consider

$$C = \left\{ \omega \in \Omega \, : \, \tau_u \in (t, u], \quad \text{and} \quad q_{\tau_u, u} < K \right\},$$

and the following deviation. Buy $K$ additional unit of the asset when the information process jumps at date $z$ for the first time during $(t, u)$, and re-sell these assets at time $u^+$ with a limit order at price $p_{u^+} - \eta/2$ which, by our choice of $t$ and $\eta$, is executed at time $u^+$. Then, asset holdings at time $u$ are $q_{\tau_u, u} + K \geq K$. Since the investor enjoys some positive utility from holding these extra $K$ units, the net change in expected utility is greater than the capital gain $(e^{-ru}(p_{u^+} - \eta/2) - e^{-rz} p_z)K > e^{-ru}(\eta/2)K$. Thus the expected utility of the deviation is more than $P(C)e^{-ru}(\eta/2)K$. Optimality of the holding plan then implies that $P(C) = 0$. Because there is a positive measure of investors whose information proces has jumped during $(t, u]$, and because of the short-selling constraint, this contradicts market clearing.

### VI.2.2 The price cannot grow at a rate greater than $r$

Before proving that the price cannot jump downwards, we establish a useful result:

**Lemma VI.1.** *In all spots, $rp_u - \dot{p}_u \geq 0$.*

Suppose by contradiction that $\dot{p}_u > rp_u$ over some interval $[u_1, u_2]$. Note that the price is strictly increasing over $[u_1, u_2]$. Then let us fix some $u \in (u_1, u_2)$. We can do the same reasoning as in the proof that the price cannot jump up, in Section VI.2.1 above: investors always want to demand an additional unit of the asset if they have an information event during $[u_1, u)$, and sell it back at time $u$. Indeed, given any $K > 0$, then for almost all $\omega$ such that the information process jumps at least once during $[u_1, u]$, then $q_{\tau_{u_2},u} > K$. Otherwise, an investor could profit from buying $K$ additional units at his first information event time during $[u_1, u]$ and selling back at $u_2$. As above, this contradicts market clearing.

### VI.2.3 The price cannot jump down

Consider by contradiction some $u > 0$ such that $p_u > p_{u^+}$ (as before the case $p_{u^-} > p_u$ is identical after replacing $u$ by $u^-$ and $u^+$ by $u$). Then, we pick $t < u$ close to $u$, and $\eta > 0$ and $K > 0$ small enough so that: $z \mapsto p_z$ is continuous and either strictly increasing, strictly decreasing, or constant on the interval $[t, u]$; and, for all $z \in [t, u]$, $p_z > p_{u^+} + \eta$; and

$$-\int_t^u e^{-(r+\rho)z}\,dz + e^{-(r+\rho)u}(p_u - p_{u^+} - \eta) > K. \tag{VI.2}$$

Keeping in mind that marginal utility is bounded above by one, the intuition of this inequality is the following: at any possible margin and at any time $z \in [t, u]$, it is optimal to decrease the asset holding by one unit until the next jump of the information process, and buy back at time $u^+$ with a limit sell order at price $p_{u^+} + \eta$ or at the next jump of the information process, whichever comes first. In Section IX.4.1, page 86, we prove the following two results:

**Lemma VI.2.** *For almost all $\omega \in \Omega$:*

1. *If $\tau_u \in [t, u]$, then $q_{\tau_u,u} = 0$.*
2. *If $\tau_u \in (0, t)$, then $q_{\tau_u,u} \leq q_{\tau_u,t}$*

Point 1 says that, if the trader has an information event at a date sufficiently close to $u$, then she wants to take advantage of the price jump by reducing his asset holding as much as possible, and buying everything back after the jump with a limit order. Point 2 says that, by the same token, if the trader does not have an information event at a date sufficiently close to $u$, she will prefer to delay all his purchases until after the jump.

Lemma VI.2 implies that the market cannot clear. Indeed,

$$\begin{aligned}\mathbb{E}\left[q_{\tau_u,u}\right] &< \mathbb{E}\left[q_{\tau_t,t}\mathbb{I}_{\{\tau_u<t\}}\right] = \mathbb{E}\left[q_{\tau_t,t}\mathbb{I}_{\{\tau_t'-\tau_t>u-\tau_t\}}\right] = \mathbb{E}\left[q_{\tau_t,t}e^{-\rho(u-\tau_t)}\right] \\ &\leq \mathbb{E}\left[q_{\tau_t,t}e^{-\rho(u-t)}\right] = se^{-\rho(u-t)} < s,\end{aligned}$$

where $\tau_t'$ denote the next information event time after $t$. In the first line, the first inequality comes from the fact that, by Point 1 and 2 of Lemma VI.2, $q_{\tau_u,u} = 0$ if $\tau_u > t$, $q_{\tau_u,u} \leq q_{\tau_u,t}$ if $\tau_u < t$ and, of course, $\tau_u = \tau_t$ if $\tau_u \leq t$. The following equality on the first line follows from the fact that $\tau_u \leq t \Leftrightarrow \tau_t' > u$, and the next equality uses the fact that the random inter-arrival time $\tau_t' - \tau_t$ is independent from $\mathcal{F}_{\tau_t}$ while $q_{\tau_t,t}$ belongs to $\mathcal{F}_{\tau_t}$.

# VII Uniqueness of a Markov Limit Order Equilibrium

In this section we assume, as in Proposition 9, page 30 in Biais, Hombert, and Weill (2010) (henceforth BHW), that traders can only submit market and limit orders when their information jump process jumps. But, in contrast with BHW, we do not make any *a priori* monotonicity restriction on the shape of the price path. In this context, we show that there exists a unique Markov equilibrium, i.e., an equilibrium where traders' holding plan, $q_{t,u}$, only depend on time (the aggregate state), and on the trader idiosyncratic state, $\theta_t$, at the information event time. In all what follows we assume that traders face a pure limit order book operating according to the price priority, time priority, and volume maximization rules explained in the first paragraph of Section 4, page 25 in BHW.

## VII.1 Preliminary comments and overview

We start with some general comments on the proof and an overview of the results. In all what follows, we let a Limit Order Equilibrium, or "LOE", be the equilibrium concept of Proposition 9 page 30 in BHW, where traders can only submit market and limit orders when their information jump process jumps.

### VII.1.1 What makes proving uniqueness difficult

As noted in BHW, in a pure limit order market, the shape of the price path imposes constraints on the holding plan of traders. For instance, when the price path is strictly increasing (decreasing), then the price priority rule implies that only limit sell (buy) orders can be executed, and so holding plan have to be decreasing (increasing). When the price path is flat, then the type of orders that can be executed is determined by time priority and volume maximization. Namely, if there are limit sell (buy) orders in the book, then only limit sell (buy) orders can be executed, in a first–in–first–out basis.

With some *a priori* monotonicity restrictions on the price, it can be relatively easy to prove uniqueness. For instance, suppose that we restrict attention to prices that are strictly increasing. Then, only limit sell orders can be executed, and asset holding plans have to be decreasing. In that context, one can show that the equilibrium allocation must solve a social planning problem, subject to the constraints that asset holding plans are decreasing, and by concavity arguments that this social planning problem has a unique solution. This, of course, would only provide a partial result: it would establish uniqueness of a LOE with a strictly increasing price path.

In this section, instead of imposing *a priori* monotonicity restrictions, we establish monotonicity properties directly using elementary optimality and market clearing arguments. Once sufficiently many properties are established, we can prove uniqueness based on a social–planning argument similar to the one outlined above.

#### VII.1.2 Markov versus non-Markov LOE

Why is it sometimes easier to establish results for Markov LOE? Our proofs often require to exhibit profitable deviations from a candidate equilibrium holding plan. As we explain below, it is easier to construct one–stage deviations starting from a Markovian candidate equilibrium holding plan than from a non–Markovian one.

A typical deviation consists in changing the orders submitted at some information event, and reverting to the candidate equilibrium holding plan at some later information event. To make things concrete, suppose for instance that the deviation requires that, at the information event $\tau_k$, a trader *does not submit* some limit order to sell at the ask price $a$. In order to revert to the original holding plan at the next information event, $\tau_{k+1}$, it may be necessary to resubmit these limit sell orders. However, it can be the case that the "new" limit order to sell at price $a$, submitted at $\tau_{k+1}$, has *a different time priority* than an "old" limit sell orders at price $a$ submitted at time $\tau_k$: it will be executed at a later time because it has been submitted later. Clearly, in this example, it is not possible to revert to the original holding plan at $\tau_{k+1}$.[11]

Note however that, if the candidate equilibrium holding plan is Markov, the problem described in the previous paragraph never arises: that is, it is always possible to engineer "one stage" deviations, which are started at some information event $\tau_k$ and reverted at the next information event $\tau_{k+1}$. Indeed, in a Markov holding plan, at any information event time $\tau_k$, the trader's order only depend on his current type, not on his particular history up to time $\tau_k$. Therefore, with a Markov holding plan, we can always assume that a trader cancels all her previously submitted orders and submits new ones, as if it was her first information event time ever. In particular, the holding plan at time $\tau_{k+1}$ does not rely on the time priority of previously submitted limit orders, so the problem identified in the previous paragraph does not arise.

#### VII.1.3 Overview of the proof

We start by establishing basic results on the price path that must hold in any LOE: we show that the price is always less than $1/r$, that it is weakly increasing for $t \geq T_s$, and that it reaches its long run value of $1/r$ in finite time, $T_f$. Then, we move to a result that we were only able to

[11]We encountered such situation in the continuity proof of Section VI and had to engineer more complex deviation, we were reverted after a multiple, and sometimes random, number of subsequent information event.

prove in the case of a Markov LOE: the price is weakly increasing before $T_s$ as well. Together with features of the dynamics of preferences, this last result allows us to prove that, in a Markov LOE, investor asset holding plans are decreasing. Depending on the equilibrium price path, however, they may be subject to additional constraints: namely, traders' holding plan cannot decrease in an arbitrary fashion when the price has a flat spot.

Then, we temporarily abstract from these additional constraints and study a "relaxed" problem, where traders can choose *any* decreasing holding plan. This is a relaxed problem because limit orders may impose additional constraints on the holding plan during flat spot (for instance, that it can only decrease at specific times). We show however that, due to the dynamics of preferences, these additional constraints are not binding: even if traders were allowed to choose from *any* kind of decreasing holding plan, they would *choose* to keep their holding plan flat when the price has a flat spot. In other words, traders do not need limit orders during flat spot. This shows that a Markov LOE is a "relaxed" equilibrium, i..e, an equilibrium for a "relaxed" economy where traders can choose *any* decreasing holding plan. Then, based on the social planning argument outline above, we show that such a relaxed equilibrium is unique. This establishes the uniqueness of a Markov LOE.

## VII.2 Intermediate results

First, we establish in Section IX.5.1, page 93, that:

**Lemma VII.1.** *In all spots, $\dot{p}_t \geq 0$ or $1 - rp_t + \dot{p}_t \geq 0$.*

The intuition is the following. If there is an interval where $\dot{p}_t < 0$ and $1 - rp_t + \dot{p}_t < 0$, then since marginal utility is bounded by 1, investors prefer to postpone any purchase until the end of the interval, which is possible since the price decreases. This contradicts market clearing. A corollary of Lemma VII.1 is:

**Corollary VII.1.** *If $p_t > 1/r$ for some $t$, then $\dot{p}_u \geq 0$ in all subsequent spots.*

Indeed, consider by contradiction the earlier time interval $(u_1, u_2)$ after $t$ such that $\dot{p}_u < 0$. Since $p_u$ is increasing before that interval, we have $p_u > 1/r$ in the right neighborhood of $u_1$. This implies that $\dot{p}_u < 0$ and $1 - rp_u + \dot{p}_u < 0$, which is a contradiction by Lemma VII.1.

We then prove in Section IX.5.2, page 95, that:

**Lemma VII.2.** *For all $t$, $p_t \leq 1/r$.*

The idea of the proof is the following. Suppose there is some $t$ such that $p_t > 1/r$. First note that, by Corollary VII.1, the price is increasing for all $u \geq t$. But since the price is bounded, it follows that it converges to some finite limit $p_\infty > 1/r$. At the same time, the capital gains from buying and selling becomes very small, so the benefit from speculative buy-low-sell-high strategies vanish, and investors demand is mostly driven by the value of holding the asset. Since the PV of utility flows from holding are always less than $1/r$, we find that, eventually, investors' demand has to be equal to zero, which contradicts market clearing.

We continue with a corollary of Lemma VII.2:

**Corollary VII.2.** *In all spots,* $1 - rp_t + \dot{p}_t \geq 0$.

Indeed, Lemma VII.2 implies that if $1 - rp_t + \dot{p}_t < 0$, then $\dot{p}_t < 0$. But, at the same time, Lemma VII.1 implies that, if $1 - rp_t + \dot{p}_t < 0$, then $\dot{p}_t \geq 0$, which is a contradiction. Another corollary we prove in Section IX.5.3, page 99, is:

**Corollary VII.3.** *Consider some* $[u_1, u_3]$ *where the price is either strictly increasing, strictly decreasing, or flat, and such that* $1 - rp_z + \dot{p}_z > 0$ *for all* $z \in [u_1, u_3]$*. Then, for all* $u_2 \in (u_1, u_3)$, $\tau_{u_2} \in [u_1, u_2)$ *and* $\theta_{\tau_{u_2}} = h$ *imply that* $q_{\tau_{u_2}, u_2} \geq 1$ *almost surely.*

In other words, all high-valuation investors with an information event during $[u_1, u_3]$ find it optimal to hold at least one unit during that time interval. This is intuitive: they derive positive net utility, $1 - rp_z + \dot{p}_z$, from holding the asset, and strictly positive utility for $z \in [u_1, u_3]$.

## VII.3 The price is weakly increasing for $t \geq T_s$

We show in Section IX.5.4, page 100, that:

**Lemma VII.3.** *In all spots after* $T_s$, $\dot{p}_t \geq 0$.

To show this result, we consider the following two cases.

**Case 1.** Suppose that there is a decreasing spot followed by either a flat or an increasing spot. Consider $u_1 < u_2 < u_3$ such that $[u_1, u_2]$ is at the end of the decreasing spot, and $[u_2, u_4]$ is at the beginning of the subsequent flat-or-increasing spot. Choose $u_1$ such that $p_{u_1} < 1/r$, which is feasible because the price is strictly decreasing to the left of $u_2$. And, if the subsequent spot is increasing, choose $u_1$ and $u_4$ such that $p_{u_1} < p_{u_4}$, and let $u_3$ be the solution of $p_{u_3} = p_{u_1}$. Note that all investors who had an information event before $u_1$ have increasing asset holdings over $[u_1, u_3]$. Indeed, their asset holdings can only increase over $[u_1, u_2]$. Moreover, by price priority, their asset holding cannot decrease over $[u_2, u_3]$ because any limit order to sell at at price $p_z \in [p_{u_2}, p_{u_3}]$ must have been executed before $u_1$. Now, because $p_z < 1/r$ and $\dot{p}_z \geq 0$, $1 - rp_z + \dot{p}_z > 0$ for $z \in [u_2, u_3]$. Corollary VII.3 then implies that all high-valuation investors with an information event at time $z \in [u_2, u_3]$ hold more than one unit at time $u_3$. But this is also true for high-valuation with an information event during $[u_1, u_2]$, because they can submit limit order to buy just before $u_2$ (we confirm this in Section IX.5.4). Therefore, all high-valuation investors who had an information event during $[u_1, u_2]$ hold one unit at time $u_3$. Since $\mu_{hz} > s$ for $z \in [u_1, u_3]$, the only way this can happen is if limit sell order submitted before $u_1$ are executed. But this is impossible since the price is strictly decreasing for $z \in [u_1, u_2]$, and remains below $p_{u_1}$ for $z \in [u_2, u_3]$ (we confirm the corresponding violation of market clearing in Section IX.5.4).

**Case 2.** The other case to consider is when the price decreases forever after $T_s$. Then, we use the following Lemma, proved in Section IX.5.4, page 100:

**Lemma VII.4.** *Suppose that the price is continuously differentiable in the neighborhood of some $t > T_s$. Then, either $\dot{p}_t \geq 0$, or $\dot{p}_t < 0$ and $1 - rp_t + \dot{p}_t = 0$.*

The intuition is the following. Recall that $1 - rp_z + \dot{p}_t \geq 0$. Suppose there is some $t > T_s$ such that $\dot{p}_t < 0$ and $1 - rp_t + \dot{p}_t \neq 0$. Then since by Corollary VII.2, $1 - \dot{p}_t + \dot{p}_t \geq 0$, we must have that $1 - rp_t + \dot{p}_t > 0$. Because the price is continuously differentiable in a neighborhood of $t$, there exists some interval $[u_1, u_3]$ around $t$ such that these two strict inequalities are a satisfied: for all $z \in [u_1, u_3]$, $\dot{p}_z$ and $1 - rp_z + \dot{p}_z > 0$. But by Corollary VII.3 we know that high-valuation investors with an information event during $[u_1, t)$ hold more than one unit at time $t$. But this contradicts market clearing since $\mu_{hz} > s$ over $[u_1, u_3]$ and, because the price is strictly decreasing during $[u_1, t]$, no limit sell orders can be executed.

Now if the price decrease forever after $T_s$, the above Lemma shows that $1 - rp_t + \dot{p}_t = 0$ for all $t > T_s$. Moreover, since $p_t \leq 1/r$ and $\dot{p}_t < 0$, we have that $p_z < 1/r$ to the left of $t$. Integrating this ODE implies that the price goes to minus infinity, which is a contradiction.

## VII.4 The price reaches $1/r$ in finite time

We define

$$T_f \equiv \inf\{t \geq T_s \,:\, p_t = 1/r\},$$

with the convention that $T_f = \infty$ if the set is empty. In Section IX.5.5, page 102, we prove that:

**Lemma VII.5.** *Then, high-valuation traders who have an information event during $(T_s, T_f)$ hold at least one unit at all times until $T_f^-$, except perhaps at the boundary points of maximal spots.*

Indeed since by Lemma VII.3, $\dot{p}_t \geq 0$ for $t \in (T_s, T_f)$, it follows that $1 - rp_t + \dot{p}_t > 0$ for all $t \in (T_s, T_f)$, and thus, by Corollary VII.3, that high-valuation investors with an information event after $T_s$ hold more than one unit in the interior of all maximal spots during $(T_s, T_f)$. From this result it follows that:

**Lemma VII.6.** *There is some $T_f \geq T_s$ such that $p_t = 1/r$ for all $t \geq T_f$.*

First, by combining Lemma VII.2 and Lemma VII.3, it is clear that if the price reaches $1/r$ at some time after $T_s$, it stays equal to $1/r$ forever after. Now suppose that the price never reaches $1/r$. Then $T_f = \infty$ which implies, by Lemma VII.5, that all high-valuation investors with an information event after $T_s$ hold one unit. But the asymptotic measure of high-valuation investors is one, and the asset supply is strictly less than one, which contradicts market clearing.

## VII.5 The price grows at a rate strictly less than $r$

We prove in Section IX.5.7, page 103, that:

**Lemma VII.7.** *Suppose that the price is continuously differentiable in the neighborhood of some time $t$ and that $\dot{p}_t > 0$. Then, $rp_t - \dot{p}_t \geq 1 - \delta$.*

Otherwise, $rp_t - \dot{p}_t < 1 - \delta$, i.e., the holding cost is strictly less than the minimum flow utility from the asset. Then the two inequalities $\dot{p}_z > 0$ and $rp_z - \dot{p}_z < 1 - \delta$ hold in a neighborhood $[u_1, u_3]$ of $t$. Every investor with an information event during $[u_1, t]$ wants to hold at least one unit of the asset, and perhaps re–sell during $(t, u_3)$ with a limit sell order. By the same token, no investor whose information process last jumped prior to time $u_1$ wants to sell during $[u_1, t]$. This contradicts market clearing at time $t$.

## VII.6 The price is increasing for $t \leq T_s$: a partial result

Suppose there are decreasing spots before $T_s$ and consider the latest one. Because this is the latest one, and because the price is increasing after $T_s$, it follows that this decreasing spot is followed by either a flat or a increasing spot. We prove in Section IX.5.8, page 105 that:

**Lemma VII.8.** *In an equilibrium, the last strictly decreasing spot cannot be followed by an increasing spot.*

The intuition is the following. A low-valuation investor with an information event during the decreasing spot anticipates that, during the subsequent increasing spot, his expected valuation will rise but he will not be able to buy. This gives him incentive to place a large limit buy order at the end of the decreasing spot. On the aggregate, this results in a positive measure of limit buy orders to be executed exactly at the end of the decreasing spot. But this cannot be the basis of an equilibrium because no limit sell order can be executed and so the measure of asset supplied at that precise time is zero.

## VII.7 Properties of Markov LOE

Next, we derive properties specific to *Markov* LOE.

### VII.7.1 The price is increasing for $t \leq T_s$

We already know from Lemma VII.8 that the last decreasing spot cannot be followed by an increasing spot. We now show that, in a Markov LOE, it cannot be followed by a flat spot either:

**Lemma VII.9.** *In a Markov LOE, a strictly decreasing spot cannot be followed by a flat spot.*

The proof, shown in Section IX.5.8 page 105, follows a similar logic, but for now we need to restrict attention to Markov equilibrium. Clearly, a Corollary of Lemma VII.8 and VII.9 is:

**Corollary VII.4.** *In a Markov LOE, the price is weakly increasing.*

### VII.7.2 Trading strategies in a Markov LOE

In a Markov equilibrium, traders' holding plan are "Markovian": they only depend on the information event time and on their valuation type at the information event time. Therefore, holding plans are fully described by functions $q_{\ell,t,u}$ and $q_{h,t,u}$ prescribing the time $u$ asset holdings of an investor who last contacted the market at time $t$ with a low ("$\ell$") or high ("$h$") valuation.

For any Markovian holding plan, the value of the investor's objective can be simplified further, since with a Markov holding plan $\mathbb{E}_t\left[v(\theta_u, q_{\theta,t,u})\right]$ only depends on time and on the investor's type at time $t$. Therefore, after conditioning with respect to the type at time $t$, we obtain

$$\int_0^\infty e^{-rt} \sum_\theta \Pr(\theta_t = \theta) \int_t^\infty e^{-(r+\rho)(u-t)} \left\{ \mathbb{E}_\theta\left[v(\theta_u, q_{\theta,t,u})\right] - q_{\theta,t,u}\left(rp_u - \dot{p}_u\right) \right\} du\, dt,$$

where, in the above, $\mathbb{E}_\theta\left[\,\cdot\,\right]$ is a shorthand for the expectation conditional on $\theta(t)$ being equal to $\theta$. It then immediately follows that:

**Lemma VII.10** (Necessary condition for Optimality)**.** *If some Markovian asset holding plan* $\{q^*_{\ell,t,u}, q^*_{h,t,u}\}$ *is optimal, then it maximizes*

$$\int_t^\infty e^{-(r+\rho)(u-t)} \left\{ \mathbb{E}_\theta\left[v(\theta_u, q_{\theta,t,u})\right] - q_{\theta,t,u}\left(rp_u - \dot{p}_u\right) \right\} du \qquad \text{(VII.1)}$$

*for* $\theta \in \{\ell, h\}$ *and for almost all* $t \geq 0$*, subject to the constraint of being implementable with limit and market orders submitted at time* $t$*.*

Suppose indeed that $\{q^*_{\ell,t,u}, q^*_{h,t,u}\}$ does not maximize (VII.2) for some positive measure set of time $\mathcal{T}$ and some $\theta \in \{\ell, h\}$. Then, for all $t \in \mathcal{T}$ and $\theta_t = \theta$, switch to a holding plan that achieves a higher value in the objective (VII.2), and keep your holding plan for $t \notin \mathcal{T}$ otherwise. It is important to note that the Markov restriction ensures that it is feasible to keep the holding plan for some $t \notin \mathcal{T}$, even if it has been modified for some $t' < t$. Indeed, a trader who has an information event at time $t$ behaves "as if" it was her first information ever. In particular her asset holding can be implemented without using any order she may have submitted at earlier information events. Note also that the resulting holding plan is Markovian and clearly achieves a higher value.

Finally, one should keep in mind that Lemma VII.10 only provides a necessary condition. To prove optimality, one also needs to compare the holding plan $q^*_{t,u}$ to other holding plans which are not Markov.

### VII.7.3 High–valuation holdings in a Markov LOE

We start with the following Lemma, proved in Section IX.5.9, page 111:

**Lemma VII.11.** *In a Markov LOE, almost surely, high-valuation investors who have an information event before (after) $T_f$ hold one unit (less than one unit) of the asset.*

And, obviously, this implies that:

**Corollary VII.5.** *In a Markov LOE, almost surely, high-valuation investors who have an information event before $T_f$ demand one unit of the asset.*

### VII.7.4 Low–valuation holdings in a Markov LOE

First, we have:

**Lemma VII.12.** *In any Markov LOE, low-valuation investors hold zero unit after $T_f$.*

This result is proved in Section IX.5.10, page 111. We already know that, for a low–valuation trader who has an information event at some time $t$ during an increasing spot, $q_{t,u}$ must be a decreasing function of time. To show that the same is true if the information event occurs during a flat spot, in Section IX.5.11, page 112, we show the following Lemma:

**Lemma VII.13.** *Suppose there exists a Markov LOE such that the price path has a flat spot. Then, at almost all times $t$ during the flat spot, low-valuation traders do not submit any limit order to buy.*

The reason is that, if limit buy orders were submitted and executed after their submission times, then strategies would be non-Markovian.

To see why, note that a low-valuation trader's expected utility flow is increasing and, during a flat spot, the price is constant. Thus, a low-valuation trader aspires to asset holdings that are smoothly increasing during the flat spot. But a perfectly smooth increasing holding is not feasible. Indeed, the first information event time during the flat spot, a low-valuation trader receives the opportunity to submit a limit buy order at (at most) one execution time, so her asset holding can only be a step function with (at most) one step. The second information event time during the flat spot, this low-valuation trader receives the opportunity to submit a limit-buy order at some "new" execution time. Because of concave utility she wants to smooth her holding, and so she has incentives to use her previously submitted limit buy order: this allows her asset holdings to to be "smoother" with two steps instead of one. Obviously, such a trading strategy is *not* Markov: the trader is not behaving as if she was contacting the market for the first time. Therefore, in a Markov LOE, at almost all times during flat spots, low–valuation traders do not submit limit buy orders. But, by Corollary VII.5, this is also true for high–valuation traders. Therefore:

**Corollary VII.6.** *In a Markov LOE, at any time during a flat spot, there is a measure zero of limit buy order outstanding at the the current market price. Consequently, a limit order to buy at the current market price is executed immediately.*

The corollary shows that, for a low–valuation trader who has an information event at some time $t$ during a flat spot, $u \mapsto q_{t,u}$ is decreasing. We already know that this is also true for information event outside of flat spots. Therefore:

**Corollary VII.7.** *In a Markov LOE, low–valuation traders choose decreasing holding plans.*

## VII.8 A relaxed Equilibrium

The above results show that, in a Markov equilibrium, high– and low–valuation traders choose decreasing holding plans. But note that, depending on the price path, there may be additional constraints: for instance, if an information event occurs during an increasing spot, asset holding have to be constant during subsequent flat spot.

In this section, we temporarily abstract from these additional constraints: we study a "relaxed equilibrium" arising when traders can choose *any* decreasing asset holdings plan. We show that such a relaxed equilibrium exists, is unique, and that any Markov LOE is a relaxed equilibrium. Clearly, this shows that a Markov LOE is unique as well.

### VII.8.1 The relaxed problem

Our first result is:

**Lemma VII.14.** *For each time, a holding plan $q_{t,u}$ solves the relaxed problem if and only if it maximizes*

$$\int_t^\infty e^{-(r+\rho)(u-t)} \left\{ \mathbb{E}_t \left[ v(\theta_u, q_{t,u}) \right] - q_{t,u} \left( r p_u - \dot{p}_u \right) \right\} du \tag{VII.2}$$

*for almost all $(t, \omega) \in \mathbb{R}_+ \times \Omega$.*

The "if" part is obvious. To prove the "only if" part, we proceed by contrapositive. Suppose that $\{q_{t,u}\}$ does not maximize (VII.2) for some positive measure set of $\mathbb{R} \times \Omega$. Then, for all times and events in that set, switch to a plan that achieves a higher value in the objective (VII.2), and keep your holding plan the same otherwise. This is feasible because, in the relaxed problem, earlier choices of holding plans do not constraint subsequent ones. Clearly, because of the expression (6) for the investor's objective (page 14 in BHW) this new plan achieves a strictly higher utility.

**Lemma VII.15.** *Assume that the price satisfies Condition 1 as well as all the properties derived so far. Then, in the relaxed problem, for any optimal asset holding plan and almost all $(t, \omega) \in \mathbb{R}_+ \times \Omega$:*

- $q_{t,u} \in [0, 1]$*;*
- *If $t \in [0, T_f]$, $\theta_t = h$, then $q_{t,u} = 1$ for all $u \in [t, T_f]$.*
- *If $t \in [0, \infty)$, $\theta_t = \ell$, then $q_{t,u} = 0$ for all $u \in [T_f, \infty]$.*
- *If $t \in [0, T_f)$, $\theta_t = \ell$, then $\{q_{t,u} : u \in [t, T_f)\}$ solves the problem:*

$$(R) : \quad \sup_{q_{t,u}} \int_t^{T_f} e^{-(r+\rho)(u-t)} \left\{ \mathbb{E}_t \left[ v(\theta_u, q_{t,u}) \right] - q_{t,u} \left( r p_u - \dot{p}_u \right) \right\} du, \tag{VII.3}$$

*subject to the constraint that $q_{t,u} \in [0, 1]$ and is decreasing over $[t, T_f)$.*

Thus, we are left with the problem of maximizing (VII.3) with respect to some $[0, 1]$-valued decreasing function. To study the existence of a maximizer, we relax the problem further: we allow investor to choose a holding plan $q_{t,u} \in L^2([t, T_f])$ which lie almost everywhere in $[0, 1]$ and is almost everywhere decreasing. Formally, there exists a set $S \subseteq [t, T_f]$ of full measure such that $q_{t,u} \in [0, 1]$ and is decreasing over $S$, i.e., for all $(u, u') \in S^2$, $u \leq u'$ implies that $q_{t,u} \geq q_t(u')$. Note that we alter the constraint set in two ways, first, we constraint holdings to be bounded by 1, but we know from VII.15 that this constraint is not binding. Second, instead of optimizing within the set of $[0,1]$-valued decreasing functions, which is included in $L^2([t,T_f])$ (see, e.g., Theorem 10.11 in Apostol, 1974), we optimize within the larger set of $L^2([t,T_f])$ functions which are decreasing and $[0,1]$-valued almost everywhere instead of everywhere. This ensures that the constraint set is closed under the $L^2$ norm. Given that the objective is concave, continuous for the $L^2$ norm, and that the constraint set is clearly convex and bounded, we obtain that:

**Lemma VII.16.** *The supremum of (VII.3) is achieved by some $q_{t,u}$ such that $S = [t, T_f]$.*

The details are in Section IX.5.13, page 116, but the result follows basically from an application of Proposition 1.2, Chapter II in Eckland and Téman (1987). In principle, the maximizer found in Lemma VII.16 is only decreasing almost everywhere. However, it is easy to show that given any maximizer of (VII.15), one can construct another maximizer which is decreasing everywhere. Next, in Section IX.5.14, page 116, we show:

**Lemma VII.17.** *Any maximizer of Lemma VII.16 is constant during flat spots.*

This means that low–valuation traders finds it optimal to hold constant asset holding during flat spot, even when allowed to choose any decreasing asset holding plan. This immediately implies that:

**Lemma VII.18.** *Suppose there exists a Markov LOE. Then, for almost all information event times $t \in (0, T_f)$, the holding plan of a low–valuation trader solves the relaxed problem. Conversely, any solution of the relaxed problem is an optimal holding plan for a low–valuation trader in a Markov LOE.*

Indeed, we already know from Lemma VII.13 that, in a Markov equilibrium, low-valuation traders with an information event time before $T_f$ during a flat spot only submit limit sell orders. Outside of flat spots, the price is strictly increasing so, evidently, limit buy orders are never submitted because they would be either immediately executed, or never executed. Therefore, a low-valuation trader's asset holding plan has to be decreasing. It can be an arbitrary decreasing function during increasing spots, but it has to stay flat during all or part of flat spots – depending on when limit orders can be executed during flat spot. But we know from Lemma VII.17 that if we allow the trader to solve the relaxed problem, i.e. to choose from *any* decreasing function, she would find it optimal to keep her holdings constant during flat spot anyway. Thus, the solution of the traders' problem must be a solution of the relaxed problem, and vice versa.

### VII.8.2 The relaxed equilibrium

Continuing with the above, we can define a *relaxed* equilibrium in the obvious way: it is a piecewise continuously differentiable price path $p_t$ and a feasible asset holding plan $q_{t,u}$ which is decreasing in $u$ for each $t > 0$, such that given the price the asset holding plan solves the relaxed problem. We then have the following two properties:

**Lemma VII.19.** *Any Markov LOE is a relaxed equilibrium. In particular, the LOE of Proposition 9, page 30 in BHW, is a relaxed equilibrium.*

Consider a Markov equilibrium. Then the strategies of all types of traders solve the relaxed problem. For high-valuation traders this follows from Lemma VII.11 and VII.15. For low-valuation traders who have an information event time after $T_f$, this follows from Lemma VII.12 and VII.15. And, finally, for low-valuation traders with an information event before $T_f$, this follows from Lemma VII.18. Lastly, the asset holding plan of a Markov equilibrium is, obviously, feasible.

Next, consider the relaxed planning problem consisting in choosing decreasing asset holding plans, $q_{t,u} \in [0,1]$, in order to maximize:

$$W(q) = \mathbb{E}_0 \left[ \int_0^\infty e^{-ru} v(\theta_u, q_{\tau_u,u})\, du \right], \tag{VII.4}$$

subject to the feasibility constraints (8) for all $u$, page 15 in BHW. Then, we have:

**Lemma VII.20.** *A relaxed equilibrium solves the relaxed planning problem.*

The proof is omitted as it follows the exact same argument as in the proof of Proposition 2. Next, we establish (essential) uniqueness of a planning solution, and hence of a Markov equilibrium:

**Lemma VII.21.** *Consider the BHW-LOE asset holding plan $q$ and any other solution $q'$ of the planning problem. Then*

- *For almost all $(t,u,\omega) \in \mathbb{R}^2_+ \times \Omega$ such that $0 < t \leq u$, if $\theta_t = \ell$, then $q'_{t,u} = q_{t,u}$.*
- *For almost all $(t,u,\omega) \in \mathbb{R}^2_+ \times \Omega$ such that $0 < t < u < T_f$, if $\theta_t = h$, then $q'_{t,u} = 1$.*

The first point follows from the strict concavity of low-valuation traders' objective. The second point follows from the fact that, once the allocation of low-valuation traders is set, then given that $q_{t,u} \leq 1$ feasibility implies that all high-valuation traders hold one unit.

# VIII Proofs omitted in the appendix of the paper

## VIII.1 Proof of Lemma A.1

We first note that, by the law of iterated expectations:

$$\mathbb{E}\left[v(\theta_u, q_{t,u}) - \xi_u q_{t,u} \,|\, \tau_u = t\right] = \mathbb{E}\left[\mathbb{E}\left[v(\theta_u, q_{t,u}) - \xi_u q_{t,u} \,|\, \mathcal{F}_{t^-}, \tau_u = t\right] \,\middle|\, \tau_u = t\right] \tag{VIII.1}$$

where, as usual, $\mathcal{F}_{t^-}$ is the sigma algebra generated by all the $\mathcal{F}_z$, $z < t$, representing the trader information "one instant prior to $t$." Now recall that:

$$v(\theta_u, q_{t,u}) = \min\{q_{t,u}, 1\} - \mathbb{I}_{\{\theta_u = \ell\}}\delta\frac{\min\{q_{t,u}, 1\}^{1+\sigma}}{1+\sigma}.$$

Therefore, the inner expectation on the right-hand side of (VIII.1) writes as:

$$\begin{aligned}
&\mathbb{E}\left[\min\{q_{t,u}, 1\} - \mathbb{I}_{\{\theta_u = \ell\}}\delta\frac{\min\{q_{t,u}, 1\}^{1+\sigma}}{1+\sigma} - \xi_u q_{t,u} \,\middle|\, \mathcal{F}_{t^-}, \tau_u\right] \\
=&\min\{q_{t,u}, 1\} - \mathbb{E}\left[\mathbb{I}_{\{\theta_u = \ell\}} \,|\, \mathcal{F}_{t^-}, \tau_u\right]\frac{\min\{q_{t,u}, 1\}^{1+\sigma}}{1+\sigma} - \xi_u q_{t,u} \\
=&\min\{q_{t,u}, 1\} - \mathbb{E}\left[\mathbb{I}_{\{\theta_u = \ell\}} \,|\, \mathcal{F}_{t^-}\right]\frac{\min\{q_{t,u}, 1\}^{1+\sigma}}{1+\sigma} - \xi_u q_{t,u} \\
=&\mathbb{E}\left[\min\{q_{t,u}, 1\} - \mathbb{I}_{\{\theta_u = \ell\}}\frac{\min\{q_{t,u}, 1\}^{1+\sigma}}{1+\sigma} - \xi_u q_{t,u} \,\middle|\, \mathcal{F}_{t^-}\right]
\end{aligned} \tag{VIII.2}$$

where the first equality follows because $q_{t,u}$ is $\mathcal{F}_t$-predictable, and thus measurable with respect to $\mathcal{F}_{t^-}$ (see Exercise E10, Chapter I, in Brémaud, 1981). The second equality, on the other hand, follows because the type process is independent from the information event process: this allows to freely add or remove any information generated by the information event process from the conditioning information.

Now the random variable of equation (VIII.2) is $\mathcal{F}_{t^-}$-measurable. Since $\{\tau_u = t\} = \{N_t - N_{t^-} = 1 \text{ and } N_u - N_t = 0\}$ and because the information event process has independent increment and is independent from the type process, it follows that $\{\tau_u = t\}$ is independent $\mathcal{F}_{t^-}$. Thus, the expectation of (VIII.2) conditional on $\{\tau_u = t\}$, is equal to its unconditional expectation, which proves the claim.

## VIII.2 Proof of Lemma A.2

The left-hand side of (17) is continuous, strictly increasing for $Q_u < (1 - \mu_{hu})^{-1/\sigma}$ and constant for $Q_u \geq (1 - \mu_{hu})^{-1/\sigma}$. It is zero when $Q_u = 0$, and, when $Q_u = (1 - \mu_{hu})^{-1/\sigma}$, it is equal to:

$$\int_0^u (1 - \mu_{ht})\,\rho e^{-\rho(u-t)}\,dt > S_u = \int_0^u (s - \mu_{ht})\,\rho e^{-\rho(u-t)}\,dt,$$

since $s < 1$. Therefore, equation (17) has a unique solution, $Q_u$, and the solution satisfies $0 \leq Q_u < (1 - \mu_{hu})^{-1/\sigma}$. To prove that $Q_u$ is continuously differentiable we apply the Implicit Function Theorem (see, e.g., Theorem 13.7 Apostol, 1974). We note that (17) writes $K(u, Q_u) = 0$, where

$$K(u, Q) \equiv \int_0^u e^{\rho t}(1-\mu_{ht}) \min\{(1-\mu_{ht})^{1/\sigma} Q, 1\}\, dt - \int_0^u e^{\rho t}\,(s-\mu_{ht})\; dt. \tag{VIII.3}$$

Since we know that $Q_u < (1-\mu_{hu})^{-1/\sigma}$, we restrict attention to the domain $\{(u, Q) \in \mathbb{R}^2_+ \,:\, u > 0 \text{ and } Q < (1-\mu_{hu})^{-1/\sigma}\}$. In this domain, $\Psi(Q) < u$, and so equation (VIII.3) can be written, using the definition of $\Psi(Q)$:

$$\begin{aligned} K(u, Q) &= \int_0^{\Psi(Q)} e^{\rho t}(1-\mu_{ht})\, dt + \int_{\Psi(Q)}^u e^{\rho t}\,(1-\mu_{ht})^{1+1/\sigma}\, Q\, dt - \int_0^u e^{\rho t}\,(s-\mu_{ht})\; dt \\ &= \int_0^u e^{\rho t}(1-s)\, dt - \int_{\Psi(Q)}^u e^{\rho t}\,(1-\mu_{ht}) \left[1-(1-\mu_{ht})^{1/\sigma}\, Q\right] dt, \end{aligned} \tag{VIII.4}$$

To apply the Implicit Function Theorem, we need to show that $K(u, Q)$ is continuously differentiable. To see this, first note that the partial derivative of $K(u, Q)$ with respect to $u$ is, using (VIII.3):

$$\frac{\partial K}{\partial u} = e^{\rho u}(1-\mu_{hu}) \min\{(1-\mu_{hu})^{1/\sigma} Q, 1\} - e^{\rho u}(s-\mu_{hu}).$$

and is clearly continuous. To calculate the partial derivative with respect to $Q$, we consider two cases. When $Q \in [0, 1]$, then $\Psi(Q) = 0$, and so, using (VIII.4):

$$\frac{\partial K}{\partial Q} = \int_0^u e^{\rho t}\,(1-\mu_{ht})^{1+1/\sigma}\; dt = \int_{\Psi(Q)}^u e^{\rho t}\,(1-\mu_{ht})^{1+1/\sigma}\; dt,$$

When, on the other hand, $Q \in [1, (1-\mu_{hu})^{-1/\sigma})$, on the other hand, $\Psi(Q)$ solves $(1-\mu_{h\Psi(Q)})^{-1/\sigma} = Q$ and hence is continuously differentiable. Bearing this in mind when differentiating (VIII.4), we obtain again that

$$\frac{\partial K}{\partial Q} = \int_{\Psi(Q)}^u e^{\rho t}\,(1-\mu_{ht})^{1+1/\sigma}\; dt.$$

Since $\Psi(Q)$ is continuous, the above calculations show that $\partial K/\partial Q$ is continuous for all $(u, Q)$ in its domain. Next, note that because $(1-\mu_{hu})^{1/\sigma} Q_u < 1$, we have $\Psi(Q_u) < u$ and therefore $\partial K/\partial Q > 0$ at $(u, Q_u)$. Taken together, these observations allow to apply the Implicit Function Theorem and state that

$$Q_u' = -\frac{\partial K/\partial u}{\partial K/\partial Q} = \frac{e^{\rho u}(s-\mu_{hu}) - e^{\rho u}(1-\mu_{hu})^{1+1/\sigma} Q_u}{\int_{\psi_u}^u e^{\rho t}\,(1-\mu_{ht})^{1+1/\sigma}},$$

where we used that $\psi_u \equiv \Psi(Q_u)$ and $Q_u(1-\mu_{hu})^{1/\sigma} < 1$.

## VIII.3 Proof of Lemma A.4

The continuity of $\overline{Q}_u$ is obvious. That $\overline{Q}_{0+} = s$ follows from an application of l'Hôpital rule, and $\overline{Q}_{T_f} = 0$ follows by definition of $T_f$. Next, after taking derivatives with respect to $u$ we find that $\text{sign}\left[\overline{Q}'_u\right] = \text{sign}\left[F_u\right]$, where:

$$F_u \equiv (s - \mu_{hu}) \int_0^u e^{\rho t} \left(1 - \mu_{ht}\right)^{1+1/\sigma} dt - \left(1 - \mu_{hu}\right)^{1+1/\sigma} \int_0^u e^{\rho t} \left(s - \mu_{ht}\right) dt, \tag{VIII.5}$$

is continuously differentiable. Taking derivatives once more, we find that $\text{sign}\left[F'_u\right] = \text{sign}\left[G_u\right]$ where:

$$G_u \equiv \left(1 + \frac{1}{\sigma}\right) \left(1 - \mu_{hu}\right)^{1/\sigma} \int_0^u e^{\rho t} \left(s - \mu_{ht}\right) dt - \int_0^u e^{\rho t} \left(1 - \mu_{ht}\right)^{1+1/\sigma} dt, \tag{VIII.6}$$

is continuously differentiable. Now suppose that $\overline{Q}'_u = 0$. Then $F_u = 0$ and, after rearranging (VIII.5):

$$\left(1 - \mu_{hu}\right)^{1/\sigma} \int_0^u e^{\rho t} \left(s - \mu_{ht}\right) dt = \frac{s - \mu_{hu}}{1 - \mu_{hu}} \int_0^u e^{\rho t} \left(1 - \mu_{ht}\right)^{1+1/\sigma} dt.$$

Plugging this back into $G_u$ we find that:

**R VIII.1.** *Suppose that $F_u = 0$ for some $u > 0$. Then sign $\left[F'_u\right] =$ sign $\left[s\left(1 + \frac{1}{\sigma}\right) - 1 - \frac{\mu_{hu}}{\sigma}\right]$.*

Now note that $G_0 = 0$ and $G'_0 = s(1 + 1/\sigma) - 1$. Thus,

**R VIII.2.** *If $s \leq \sigma/(1 + \sigma)$, then $F_u < 0$ for all $u > 0$.*

To see this, first note that, from repeated application of the Mean Value Theorem (see, e.g., Theorem 5.11 in Apostol, 1974), it follows that $F_u < 0$ for small $u$. Indeed, since $F_0 = 0$, $F_u = uF'_v$, for some $v \in (0, u)$. But $\text{sign}\left[F'_v\right] = \text{sign}\left[G_v\right]$. Now, since $G_0 = 0$, $G_v = vG'_w$ for some $w \in (0, v)$. But $G'_0 < 0$ so $G'_w$ is negative as long as $u$ is small enough. But if $F_u$ is negative for small $u$, it has to stay negative for all $u$. Otherwise, it would need to cross the $x$-axis from below at some $u > 0$, which is impossible given Result RVIII.1 and the assumption that $s \leq \sigma/(1 + \sigma)$.

**R VIII.3.** *If $s > \sigma/(1 + \sigma)$, then $F_u > 0$ for small $u$, and $F_u < 0$ for $u \in [T_s, T_f]$.*

The first part follows from applying the same reasoning as in the above paragraph, since when $s > \sigma/(1 + \sigma)$ we have $G'_0 > 0$. The second part follows from noting that, when $u \in [T_s, T_f]$, the first term of $F_u$ is negative, and strictly negative when $u \in (T_s, T_f]$, while the second term is negative, and strictly negative when $u \in [T_s, T_f)$. So $F_u$ changes sign in the interval $(0, T_s)$. We now show that:

**R VIII.4.** *If $s > \sigma/(1 + \sigma)$, $F_u$ changes sign only once in the interval $(0, T_s)$.*

Consider some $u_0$ such that $F_{u_0} = 0$. We can rewrite this equation as:

$$0 = -\int_0^{u_0} g(\mu_{ht}, \mu_{hu_0}) e^{\rho t}\, dt, \quad \text{where} \quad g(x, y) \equiv (s - x)(1 - y)^{1+1/\sigma} - (1 - x)^{1+1/\sigma}(s - y).$$

The function $x \mapsto g(x, y)$ is strictly concave, and it is such that $g(y, y) = 0$. Note that, for the above equation to hold, the function $x \mapsto g(x, \mu_{hu_0})$ has to change sign in the interval $(0, \mu_{hu_0})$. In particular, it must be the case that $\partial g/\partial x(\mu_{hu_0}, \mu_{hu_0}) < 0$. Otherwise, suppose that $\partial g/\partial x(\mu_{hu_0}, \mu_{hu_0}) \geq 0$. Then, by strict concavity, $g(x, \mu_{hu_0})$ lies strictly below its tangent at $x = \mu_{hu_0}$. But since $g(x, \mu_{hu_0}) = 0$ and is increasing when $x = \mu_{hu_0}$, the tangent is negative for $x \leq \mu_{hu_0}$, and so $g(x, \mu_{hu_0}) < 0$ for all $x \in (0, \mu_{hu_0})$, a contradiction. After calculating the partial derivative, we find:

$$\frac{\partial g}{\partial x}(\mu_{hu_0}, \mu_{hu_0}) < 0 \Leftrightarrow s\left(1 + \frac{1}{\sigma}\right) - 1 - \frac{\mu_{hu}}{\sigma} < 0.$$

Together with Result RVIII.1 this shows that if $F_{u_0} = 0$ for some $u_0 \in (0, T_s)$, then $F'_{u_0} < 0$, implying Result RVIII.4. We conclude that, over $(0, T_f]$, $F_u$ is first strictly positive and then strictly negative, which shows that $\overline{Q}_u$ is hump-shaped.

## VIII.4 Proof of Lemma A.5

Given that $\Delta_u = (1 - \mu_{hu})^{1/\sigma} Q_u$, we have

$$\Delta'_u = -\frac{1}{\sigma}\frac{\mu'_{hu}}{1 - \mu_{hu}}(1 - \mu_{hu})^{1/\sigma} Q_u + (1 - \mu_{hu})^{1/\sigma} Q'_u.$$

Using the formula (A.3) for $Q'_u$, in Lemma A.2, we obtain:

$$\begin{aligned} \text{sign}\left[\Delta'_u\right] &= \text{sign}\left[-\frac{1}{\sigma}\frac{\mu'_{hu}}{1 - \mu_{hu}} Q_u + Q'_u\right] \\ &= \text{sign}\left[-\frac{e^{-\rho u}}{\sigma}\frac{\mu'_{hu}}{1 - \mu_{hu}} Q_u \int_{\psi_u}^u e^{\rho t}(1 - \mu_{ht})^{1+1/\sigma}\, dt + s - \mu_{hu} - (1 - \mu_{hu})^{1+1/\sigma} Q_u\right]. \end{aligned} \tag{VIII.7}$$

We first show:

**RVIII.5.** $\Delta'_u < 0$ *for* $u$ *close to zero.*

To show this result, first note that when $u$ is close to zero, $\overline{Q}_u \simeq s < 1$. Therefore $\psi_u = 0$ and, by Lemma A.3, $Q_u = \overline{Q}_u$. Plugging in $\psi_u = 0$ and the the expression (A.4) for $\overline{Q}_u$ in (VIII.7), one obtains:

$$\begin{aligned} &\text{sign}\left[\Delta'_u\right] \\ &= \text{sign}\Big\{ -\frac{e^{-\rho u}}{\sigma}\frac{\mu'_{hu}}{1 - \mu_{hu}} \int_0^u e^{\rho t}(s - \mu_{ht})\, dt \int_0^u e^{\rho u}(1 - \mu_{ht})^{1+1/\sigma}\, dt \\ &\qquad + (s - \mu_{hu}) \int_0^u e^{\rho t}(1 - \mu_{ht})^{1+1/\sigma}\, dt - (1 - \mu_{hu})^{1+1/\sigma} \int_0^u e^{\rho t}(s - \mu_{ht})\, dt \Big\}. \end{aligned} \tag{VIII.8}$$

Now let $\gamma \equiv \mu'_{h0}$. Now, for the various functions appearing in the above formula, we calculate the first- and second derivatives at $u = 0$, and we obtain the following Taylor expansions:

$$\begin{aligned}
&\frac{e^{-\rho u}}{\sigma}\frac{\mu'_{hu}}{1-\mu_{hu}} = \frac{\gamma}{\sigma}\left(1+o(1)\right)\\
&\int_0^u e^{\rho t}(s-\mu_{ht})\,dt = u\left(s+[\rho s-\gamma]\,\frac{u}{2}\right) = u\left(s+o(1)\right)\\
&\int_0^u e^{\rho t}(1-\mu_{ht})^{1+1/\sigma}\,dt = u\left(1+\left[\rho-\gamma\left(1+\frac{1}{\sigma}\right)\right]\frac{u}{2}\right)+o(u) = u\left(1+o(1)\right)\\
&s-\mu_{hu} = s-\gamma u+o(u)\\
&(1-\mu_{ht})^{1+1/\sigma} = 1-\gamma\left(1+\frac{1}{\sigma}\right)+o(u).
\end{aligned}$$

Plugging these into (VIII.8) we obtain:

$$\begin{aligned}
\text{sign}\left[\Delta'_u\right] = \text{sign}\Bigg[ &-\frac{\gamma}{\sigma}u^2\left(1+o(1)\right)\left(s+o(1)\right)\\
&+u\left(s-\gamma u+o(u)\right)\left(1+\left[\rho-\gamma\left(1+\frac{1}{\sigma}\right)\right]\frac{u}{2}\right)\\
&-u\left(1-\gamma\left(1+\frac{1}{\sigma}\right)+o(u)\right)\left(s+[\rho s-\gamma]\,\frac{u}{2}\right)\Bigg].
\end{aligned}$$

After developing and rearranging, we obtain

$$\text{sign}\left[\Delta'_u\right] = -\frac{\gamma u^2}{2}\times\text{sign}\left[(1-s)+\frac{s}{\sigma}\right] < 0$$

establishing Result RVIII.5. Next, we show:

**R VIII.6.** *Suppose $\Delta'_{u_0} = 0$ for some $u_0 \in (0, T_f]$. Then, $\Delta_u$ is strictly decreasing at $u_0$.*

For this we first manipulate (VIII.7) as follows:

$$\begin{aligned}
\text{sign}\left[\Delta'_u\right] &= \text{sign}\left[-\frac{e^{-\rho u}}{\sigma}\frac{\mu'_{hu}}{1-\mu_{hu}}\frac{\Delta_u}{(1-\mu_{hu})^{1/\sigma}}\int_{\psi_u}^u e^{\rho t}(1-\mu_{ht})^{1+1/\sigma}\,dt + s-\mu_{hu}-(1-\mu_{hu})\Delta_u\right]\\
&= \text{sign}\left[-\frac{1}{\sigma}\frac{\mu'_{hu}}{1-\mu_{hu}}\Delta_u\int_{\psi_u}^u e^{-\rho(u-t)}\left(\frac{1-\mu_{ht}}{1-\mu_{hu}}\right)^{1+1/\sigma}dt+\frac{s-\mu_{hu}}{1-\mu_{hu}}-\Delta_u\right]\\
&= \text{sign}\left[-\frac{\gamma}{\sigma}\Delta_u\int_{\psi_u} e^{\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right](u-t)}\,dt+1-(1-s)e^{\gamma u}-\Delta_u\right]\\
&= \text{sign}\left[-\frac{\gamma}{\sigma}\Delta_u G(u-\psi_u)+1-(1-s)e^{-\gamma u}-\Delta_u\right], \qquad \text{where} \qquad G(y)\equiv\int_0^y e^{\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]x}\,dx,
\end{aligned}$$

and where we obtain the first equality after substituting in the expression for $Q_u$; the second equality after dividing by $1-\mu_{hu}$ and bringing $e^{-\rho u}$ inside the first integral; the third equality by using the functional form $\mu_{ht} = 1-e^{-\gamma t}$; and the fourth equality by changing variable ($x = u-t$) in the integral.

Now suppose $\Delta'_u = 0$ at some $u_0$. From the above we have:

$$H_{u_0} \equiv -\frac{\gamma}{\sigma}\Delta_{u_0} G(u_0 - \psi_{u_0}) + 1 - (1-s)e^{-\gamma u_0} - \Delta_{u_0} = 0.$$

If $Q_{u_0} < 1$ then $\psi_{u_0} = 0$ and $\psi'_{u_0} = 0$. Together with the fact that $\Delta'_{u_0} = 0$, this implies that

$$H'_{u_0} = -\frac{\gamma}{\sigma}\Delta_{u_0} G'(u_0) - (1-s)\gamma e^{-\gamma u_0} < 0,$$

since $G(u)$ is, clearly, a strictly increasing function. If $Q_u = 1$, then $\psi_{u_0} = 0$ and the left-derivative $\psi'_{u_0^-} = 0$, so the same calculation implies that $H'_{u_0^-} < 0$. If $Q_{u_0} > 1$ we first note that, around $u_0$,

$$Q_u = (1 - \mu_{h\psi_u})^{-1/\sigma} \Rightarrow \Delta_u = \left(\frac{1 - \mu_{h\psi_u}}{1 - \mu_{hu}}\right)^{1/\sigma} = e^{-\gamma \frac{\psi_u - u}{\sigma}}.$$

So if $\Delta'_{u_0} = 0$, we must have that $\psi'_u = 1$. Plugging this back into $H'_u$ we obtain that $H'_u = (1-s)e^{-\gamma u_0} < 0$. Lastly, if $Q_{u_0} = 1$, then the same calculation leads to $\psi_{u_0^+} = 1$ and so $H_{u_0^+} < 0$. In all cases, we find that $H_u$ has strictly negative left- and right-derivatives when $H_{u_0} = 0$. Thus, whenever it is equal to zero, $\Delta'_u$ is strictly decreasing. With Result RVIII.6 in mind, we then obtain:

**R VIII.7.** $\Delta'_u$ *cannot change sign over* $(0, T_f]$.

Suppose it did and let $u_0$ be the first time in $(0, T_f]$ where $\Delta'_u$ changes sign. Because $\Delta'_u$ is continuous, we have $\Delta'_{u_0} = 0$. But recall that $\Delta'_u < 0$ for $u \simeq 0$, implying that at $u = u_0$, $\Delta'_u$ crosses the $x$-axis from below and is therefore increasing, contradicting Result RVIII.6.

## VIII.5 Proof of Lemma A.7

For $u \in (T_1, T_2)$, we have $Q_u \neq \overline{Q}_u$ and therefore and therefore $\Psi(Q_u) = \psi_u > 0$. By definition of $\psi_u$, we also have

$$Q_u = (1 - \mu_{h\psi_u})^{-1/\sigma}. \tag{VIII.9}$$

Replacing into equation (A.3) for $Q'_u$ of Lemma A.2 , one obtains that:

$$\text{sign}\left[Q'_u\right] = \text{sign}\left[X_u\right] \text{ where } X_u \equiv s - \mu_{hu} - (1 - \mu_{hu})\left(\frac{1 - \mu_{hu}}{1 - \mu_{h\psi_u}}\right)^{1/\sigma}.$$

As noted above, $Q_u$ and thus $X_u$ changes sign at least once over $(T_1, T_2)$. Now, for any $u_0$ such that $X_{u_0} = 0$, we have $Q'_{u_0} = 0$ and, given (VIII.9), $\psi'_{u_0} = 0$. Taking the derivative of $X_u$ at such $u_0$, and using $X_{u_0} = 0$, leads:

$$\text{sign}\left[X'_{u_0}\right] = \text{sign}\left[-1 + \left(1+\frac{1}{\sigma}\right)\left(\frac{1-\mu_{hu_0}}{1-\mu_{h\psi_{u_0}}}\right)^{1/\sigma}\right] = \text{sign}\left[Y_{u_0}\right],$$
$$\text{where } Y_u \equiv -1 + \left(1+\frac{1}{\sigma}\right)\frac{s-\mu_{hu}}{1-\mu_{hu}},$$

where the second equality follows by using $X_{u_0} = 0$. Now take $u_0$ to be the *first* time $X_u$ changes sign during $(T_1, T_2)$. Since $X_{u_0} = 0$, $X_u$ strictly positive to the left of $u_0$, and $X_u$ strictly negative to the right of $u_0$, we must have that $X'_{u_0} \leq 0$. Suppose, then, that $X_u$ changes sign once more during $(T_1, T_2)$ at some time $u_1$. The same reasoning as before implies that, at $u_1$, $X'_{u_1} \geq 0$. But this is impossible $Y_u$ is strictly decreasing.

## VIII.6 Proof of Lemma A.8

**Proof of the limit of $T_f(\rho)$, in equation (A.7).** The defining equation for $T_f(\rho)$ is

$$H(\rho, T_f(\rho)) = 0 \quad \text{ where } H(\rho, u) \equiv \int_0^u e^{\rho t}\left(s - \mu_{ht}\right)\, dt = 0.$$

Since $T_f > T_s$, we have

$$\frac{\partial H}{\partial u}(\rho, T_f(\rho)) = e^{\rho T_f(\rho)}\left(s - \mu_{hT_f(\rho)}\right) < 0.$$

Turning to the partial derivative with respect to $\rho$ we note that since $\mu_{ht} - s$ changes sign at $T_s$:

$$\begin{aligned}\frac{\partial H}{\partial \rho}(\rho, T_f(\rho)) &= \int_0^{T_f(\rho)} t \times e^{\rho t}\left(\mu_{ht} - s\right)\, dt \\ &< \int_0^{T_s} T_s e^{\rho t}\left(s - \mu_{ht}\right)\, dt + \int_0^{T_s} T_s e^{\rho t}\left(s - \mu_{ht}\right)\, dt = T_s H(\rho, T_f) = 0.\end{aligned}$$

Taken together, $\partial H/\partial u < 0$ and $\partial H/\partial \rho < 0$ imply that $T_f(\rho)$ is strictly decreasing in $\rho$. In particular, it has a limit, $T_f(\infty)$, as $\rho$ goes to infinity. To determine the limit, we integrate by part $H(\rho, T_f)$:

$$0 = H(\rho, T_f(\rho)) = s - \mu_{hT_f(\rho)} - se^{-\rho T_f} + \int_0^\infty \mathbb{I}_{\{t\in[0,T_f(\rho)]\}}\mu'_{ht} e^{-\rho(T_f - t)}\, dt.$$

Because $T_f(\rho)$ is bounded below by $T_s$, the second term goes to zero as $\rho \to \infty$. The integrand of the third term is bounded and goes to zero for all $t$ except perhaps at $t = T_f(\infty)$. Thus, but dominated convergence, the third term goes to zero as $\rho \to \infty$. We conclude that $\mu_{hT_f(\infty)} = s$ and hence that $T_f(\infty) = T_s$.

**Proof of the first–order expansion, in equation (A.8).** Let

$$f(t,\rho) \equiv (1-\mu_{ht}) \min\left\{(1-\mu_{ht})^{1/\sigma} Q_u(\rho), 1\right\} + \mu_{ht} - s. \tag{VIII.10}$$

By its definition, $Q_u(\rho)$ solves: $\int_0^u \rho e^{-\rho(u-t)} f(t,\rho)\,dt = 0$. Note that, for each $\rho$, $f(t,\rho)$ is continuously differentiable with respect to $t$ except at $t = \psi_u(\rho)$ such that $(1-\mu_{h\psi(\rho)})^{1/\sigma} Q_u(\rho) = 1$. Thus, we can integrate the above by part and obtain:

$$0 = \int_0^u \rho e^{-\rho(u-t)} f(t,\rho)\,dt = f(u,\rho) - e^{-\rho u} f(0,\rho) - \int_0^u e^{-\rho(u-t)} f_t(t,\rho)\,dt, \tag{VIII.11}$$

where $f_t(t,\rho)$ denotes the partial derivative of $f(t,\rho)$ with respect to $t$. Now consider a sequence of $\rho$ going to infinity and the associated sequence of $Q_u(\rho)$. Because $Q_u(\rho)$ is bounded above by $(1-\mu_{hu})^{-1/\sigma}$, this sequence has at least one accumulation point $Q_u(\infty)$. Taking the limit in (VIII.11) along a subsequence converging to this accumulation point, we obtain that $Q_u(\infty)$ solves the equation

$$(1-\mu_{hu}) \min\{(1-\mu_{hu})^{1/\sigma} Q_u(\infty), 1\} + \mu_{hu} - s = 0.$$

whose unique solution is $Q_u(\infty) = (s-\mu_{hu})/(1-\mu_{hu})^{1+1/\sigma}$. Thus $Q_u(\rho)$ has a unique accumulation point, and therefore converges towards it. To obtain the asymptotic expansion, we proceed with an additional integration by part in equation (VIII.11):

$$\begin{aligned} 0 =& f(u,\rho) - f(0,\rho)e^{-\rho u} - \frac{1}{\rho} f_t(u,\rho) + \frac{1}{\rho} f_t(0,\rho) e^{-\rho u} + \frac{1}{\rho}\int_0^u f_{tt}(t,\rho) e^{-\rho(u-t)}\,dt \\ &+ \frac{1}{\rho} e^{-\rho(u-\psi_u(\rho))} \left[f_t(\psi_u(\rho)^+,\rho) - f_t(\psi_u(\rho)^-,\rho)\right]. \end{aligned}$$

where the term on the second line arises because $f_t$ is discontinuous at $\psi_u(\rho)$. Given that $Q_u(\rho)$ converges and is therefore bounded, the third, fourth and fifth terms on the first line are $o(1/\rho)$. For the second line we note that, since $Q_u(\rho)$ converges to $Q_u(\infty)$, $\psi_u(\rho)$ converges to $\psi_u(\infty)$ such that $(1-\mu_{h\psi_u(\infty)})^{1/\sigma} Q_u(\infty) = 1$. In particular, one easily verifies that $\psi_u(\infty) < u$. Therefore $e^{-\rho(u-\psi_u(\rho))}$ goes to zero as $\rho \to \infty$, so the term on the second line is also $o(1/\rho)$. Taken together, this gives:

$$0 = f(u,\rho) - \frac{1}{\rho} f_t(u,\rho) + o\left(\frac{1}{\rho}\right). \tag{VIII.12}$$

Equation (A.8) obtains after substituting in the expressions for $f(u,\rho)$ and $f_t(u,\rho)$, using that $\mu'_{ht} = \gamma(1-\mu_{ht})$.

**Proof of the convergence of the argmax, in equation (A.9).** First one easily verify that $Q_u(\infty)$ is hump–shaped (strictly decreasing) if and only if $Q_u(\rho)$ is hump–shaped (strictly decreasing). So if $s(1+1/\sigma) \leq 1$, then both $Q_u(\rho)$ and $Q_u(\infty)$ are strictly increasing, achieve their maximum at $u = 0$, and the result follows. Otherwise, if $s(1+1/\sigma) > 1$, consider any sequence of $\rho$ going to infinity and the associated sequence of $T_\psi(\rho)$. Since $T_\psi(\rho) < T_f(\rho) < T_f(0)$, the sequence of $T_{\psi(\rho)}$ is bounded and, therefore, it has at least one accumulation point, $T_{\psi(\infty)}$. At each point along the sequence, $T_\psi(\rho)$ maximizes $Q_u(\rho)$. Using equation (A.3) to write the corresponding first–order condition, $Q'_{T_\psi(\rho)} = 0$, we obtain after rearranging that

$$Q_{T_\psi(\rho)}(\rho) = \frac{s - \mu_{hT_\psi(\rho)}}{1 - \mu_{hT_\psi(\rho)}} = Q_{T_\psi(\rho)}(\infty) \geq Q_{T_\psi^*}(\rho).$$

where $T_\psi^*$ denotes the unique maximizer of $Q_u(\infty)$. Letting $\rho$ go to infinity on both sides of the equation, we find

$$Q_{T_\psi(\infty)}(\infty) \geq Q_{T_\psi^*}(\infty).$$

But since $T_\psi^*$ is the unique maximizer of $Q_u(\infty)$, $T_\psi(\infty) = T_\psi^*$. Therefore, $T_\psi(\rho)$ has a unique accumulation point, and converges towards it.

## VIII.7 Proof of Lemma A.9

**Proof of convergence for low valuation, $u \leq T_s$, in equation (A.10)**. We first introduce the following notation: for $t < T_f$ and $u \in [t, T_f)$, $q_{\ell,t,u}(\rho) = \min\{(1-\mu_{ht})^{1/\sigma} Q_u(\rho), 1\}$ is the time–$u$ asset holding of a time–$t$ low–valuation trader. Now pick $u_\varepsilon < u$ such that, for all $t \in [u_\varepsilon, u]$,

$$\frac{s - \mu_{hu}}{1 - \mu_{hu}} \leq \left(\frac{1 - \mu_{ht}}{1 - \mu_{hu}}\right)^{1/\sigma} \frac{s - \mu_{hu}}{1 - \mu_{hu}} < \min\left\{1, \frac{s - \mu_{hu}}{1 - \mu_{hu}} + \frac{\varepsilon}{2}\right\}. \tag{VIII.13}$$

Then, note that, for $t \in [u_\varepsilon, u]$, by (A.8), as $\rho \to \infty$:

$$q_{\ell,t,u}(\rho) \to \min\left\{\left(\frac{1 - \mu_{ht}}{1 - \mu_{hu}}\right)^{1/\sigma} \frac{s - \mu_{hu}}{1 - \mu_{hu}}, 1\right\} = \left(\frac{1 - \mu_{ht}}{1 - \mu_{hu}}\right)^{1/\sigma} \frac{s - \mu_{hu}}{1 - \mu_{hu}} \tag{VIII.14}$$

since, by (VIII.13), the left–hand side of the "min" is less than one. Moreover, since $q_{\ell,t,u}(\rho)$ is decreasing in $t$, (VIII.14) implies that $q_{\ell,t,u}(\rho) \leq q_{\ell,u_\varepsilon,u}(\rho) < 1$ for $\rho$ large enough. Put differently, for $\rho$ large enough, $q_{\ell,t,u}(\rho) = (1 - \mu_{ht})^{1/\sigma} Q_u(\rho)$ for all $t \in [u_\varepsilon, u]$. Clearly, this implies that the convergence of $q_{\ell,t,u}(\rho)$ is uniform in $t \in [u_\varepsilon, u]$. Together with (VIII.13), this implies that:

$$\left|q_{\ell,t,u}(\rho) - \frac{s - \mu_{hu}}{1 - \mu_{hu}}\right| \leq \left|q_{\ell,t,u}(\rho) - \left(\frac{1 - \mu_{ht}}{1 - \mu_{hu}}\right)^{1/\sigma} \frac{s - \mu_{hu}}{1 - \mu_{hu}}\right| + \left|\left(\frac{1 - \mu_{ht}}{1 - \mu_{hu}}\right)^{1/\sigma} \frac{s - \mu_{hu}}{1 - \mu_{hu}} - \frac{s - \mu_{hu}}{1 - \mu_{hu}}\right| \leq \varepsilon,$$

for $\rho$ large enough, for all $t \in [u_\varepsilon, u]$. Keeping in mind that all traders who have a low–valuation at time $u$ must have had a low–valuation at their last information event, and going back to equation (A.10), this implies that, for $\rho$ large enough:

$$\text{Proba}\left(\left| q_{\tau_u,u} - \frac{s - \mu_{hu}}{1 - \mu_{hu}} \right| > \varepsilon \,\middle|\, \theta_u = \ell\right) \leq \text{Proba}\left(\tau_u < u_\varepsilon \,\middle|\, \theta_u = \ell\right) = e^{-\rho(u - u_\varepsilon)} \to 0$$

as $\rho \to \infty$.

**Proof of convergence for high valuation, $u \leq T_s$, in equation (A.11)**. With limited cognition, if $\theta_u = h$ and $\theta_{\tau_u} = h$, then $q_{\tau_u,u} = 1$. Thus, a necessary condition for $\theta_u = h$ and $|q_{\tau_u,u} - 1| > \varepsilon$ is that $\theta_{\tau_u} = \ell$. This implies that:

$$\text{Proba}\left(\left|q_{\tau_u,u} - 1\right| > \varepsilon \,\middle|\, \theta_u = h\right) \leq \text{Proba}\left(\theta_{\tau_u} = \ell \,\middle|\, \theta_u = h\right) = e^{-\rho u} + \int_0^u \rho e^{-\rho(u-t)} \left(1 - \frac{\mu_{ht}}{\mu_{hu}}\right) dt$$
$$= \int_0^u \frac{\mu'_{ht}}{\mu_{hu}} e^{-\rho(u-t)} \, dt \to 0.$$

where, in the first equality, $1 - \mu_{ht}/\mu_{hu}$ is the probability of a low–valuation at time $t$ conditional on a low–valuation at time $u$, and where the second equality follows after integrating by parts. Convergence of the integral to zero follows by dominated convergence, since the integrand is bounded and converges to zero for all $t < u$.

**Convergence of the distribution of asset holdings for $u > T_s$.** With unlimited cognition, all traders with a low–valuation at time $u > T_s$ hold zero asset. With limited cognition, low–valuation traders hold zero asset if $\tau_u \geq T_f(\rho)$ and $\theta_{\tau_u} = \ell$. Moreover, for $\rho$ large enough and $\eta$ small enough, $T_s < T_f(\rho) < T_s + \eta < u$. Thus:

$$\text{Proba}\big(q_{\tau_u,u} > \varepsilon \,\big|\, \theta_u = \ell\big) \leq \text{Proba}\big(\tau_u < T_f(\rho) \,\big|\, \theta_u = \ell\big) = e^{-\rho(u - T_f(\rho))} \leq e^{-\rho(u - T_s - \eta)} \to 0$$

as $\rho \to \infty$. Lastly, let us turn to traders with a high–valuation at time $u > T_s$. With unlimited cognition, the distribution of asset holdings is indeterminate with a mean of $s/\mu_{hu}$. With limited cognition, take $\rho$ large enough so that $T_f(\rho) < u$. The distribution of asset holdings is also indeterminate with mean

$$\frac{\int_0^u \rho e^{-\rho(u-t)} s \, dt}{\int_0^u \rho e^{-\rho(u-t)} \mu_{ht} \, dt}.$$

Integrating the numerator and denominator by part shows that, as $\rho \to \infty$, this mean asset holding converges to $s/\mu_{ht}$, its unlimited cognition counterpart.

## VIII.8 Proofs Lemma A.10, A.11 and A.12

### VIII.8.1 Proof of Lemma A.10

In the perfect cognition case:

$$I(s) = \int_0^{+\infty} \mathbb{I}_{\{u < T_s\}} e^{-ru} \left(1 - (1-s)e^{\gamma u}\right)^{\sigma} \, du.$$

Since $T_s = -\gamma \log(1-s)$ goes to $+\infty$ when $s$ goes to 1, then the integrand of $I(s)$ converges pointwise towards $e^{-ru}$. Moreover, the integrand is bounded by $e^{-ru}$. Therefore, by an application of the

Dominated Convergence Theorem, $I(s)$ goes to $\int_0^{+\infty} e^{-ru}\, du = 1/r$ when $s \to 1$.

In the market with limited cognition, for $u > 0$, we note that $Q_u(s)$ is an increasing function of $s$ and is bounded above by $(1 - \mu_{hu})^{-1/\sigma}$. Letting $s \to 1$ in the market clearing condition (17) then shows that $Q_u \to (1 - \mu_{hu})^{-1/\sigma} > 1$. Using that $T_f > T_s$ goes to $+\infty$ when $s \to 1$, we obtain that the integrand of $J(s)$ goes to $e^{-ru}$. Moreover, the integrand is bounded by $e^{-ru}$. Therefore, by dominated convergence, $J(s)$ goes to $1/r$.

### VIII.8.2 Proof of Lemma A.11

In the market with perfect cognition, we can compute:

$$I'(s) = \int_0^{T_s} e^{-ru} \sigma e^{\gamma u} \left(1 - (1-s)e^{\gamma u}\right)^{\sigma - 1} du + \frac{\partial T_s}{\partial s} \left(1 - (1-s)e^{\gamma T_s}\right)^{\sigma}. \tag{VIII.15}$$

The second term is equal to 0 since $e^{\gamma T_s} = (1 - \mu_{hT_s})^{-1} = (1-s)^{-1}$. We then compute an approximation of the first term when $s$ goes to 1.

Consider first the case when $r > \gamma$. Equation (VIII.15) rewrites:

$$I'(s) = \sigma \int_0^{+\infty} \mathbb{I}_{\{u < T_s\}} e^{-(r-\gamma)u} \left(1 - (1-s)e^{\gamma u}\right)^{\sigma-1} du.$$

Since $T_s$ goes to infinity when $s$ goes to 1, the integrand goes to, and is bounded by, $e^{-(r-\gamma)u}$. Therefore, by dominated convergence, $I'(s)$ goes to $\sigma/(r - \gamma)$.

When $r = \gamma$, equation (VIII.15) becomes:

$$I'(s) = \sigma \int_0^{T_s} \left(1 - (1-s)e^{\gamma u}\right)^{\sigma-1} du = \sigma \int_0^{T_s} \left(1 - (1-s)e^{\gamma(T_s - z)}\right)^{\sigma-1} dz = \sigma \int_0^{T_s} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz.$$

where we make the change of variable $z \equiv T_s - u$ to obtain the second equality, and we use that $1 - s = e^{-\gamma T_s}$ to obtain the third equality. The integrand goes to 1 when $T_s$ goes to infinity. Thus, the Cesàro mean $I'(s)/T_s$ converges to $\sigma$, i.e.:

$$I'(s) \sim \sigma T_s = -\sigma\gamma \log(1-s).$$

Consider now that $r < \gamma$. We make the change of variable $z \equiv T_s - u$ in equation (VIII.15):

$$\begin{aligned} I'(s) &= \sigma \int_0^{T_s} e^{(\gamma-r)(T_s-z)} \left(1 - (1-s)e^{\gamma(T_s-z)}\right)^{\sigma-1} dz \;= \sigma e^{(\gamma-r)T_s} \int_0^{T_s} e^{-(\gamma-r)z} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz \\ &= \sigma e^{(\gamma-r)T_s} \int_0^{+\infty} \mathbb{I}_{\{z<T_s\}} e^{-(\gamma-r)z} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz, \end{aligned}$$

where we use that $1 - s = e^{-\gamma T_s}$ to obtain the second equality in the first line. The integrand in the second line goes to, and is bounded by, $e^{-(\gamma-r)z}(1 - e^{-\gamma z})^{\sigma-1}$, which in integrable. Therefore, by dominated convergence, the integral goes to $\int_0^{+\infty} e^{-(\gamma-r)z}\left(1-e^{-\gamma z}\right)^{\sigma-1} dz$ when $s$ goes to 1. Finally, using that $e^{-\gamma T_s} = 1-s$, we obtain:

$$I'(s) \sim \sigma \left(\frac{1}{1-s}\right)^{1-r/\gamma} \int_0^{+\infty} e^{-(\gamma-r)z}\left(1-e^{-\gamma z}\right)^{\sigma-1} dz.$$

### VIII.8.3 Proof of Lemma A.12

Throughout all the proof and the intermediate results therein, we work under the maintained assumption

$$\gamma + \gamma/\sigma - \rho > 0 \Longleftrightarrow \gamma + \sigma(\gamma-\rho) > 0, \tag{VIII.16}$$

which is without loss of generality since we want to compare prices when $\sigma$ is close to zero. We start by differentiating $J(s)$:

$$J'(s) = \frac{\partial T_f}{\partial s} e^{-rT_f} e^{-\gamma T_f} Q^{\sigma}_{T_f^-} + \int_0^{T_f} e^{-ru} e^{-\gamma u} \frac{\partial Q^{\sigma}_u}{\partial s}\, du > \int_{T_1}^{T_2} e^{-ru} e^{-\gamma u} \frac{\partial Q^{\sigma}_u}{\partial s}\, du,$$

where the inequality follows from the following facts: the first term is zero since $Q_{T_f^-} = 0$; the integrand in the second term is positive since $Q_u$ is increasing in $s$ by equation (17); and $0 < T_1 < T_2 < T_f$ are defined as in the proof of Proposition 3, as follows. We consider that $s$ is close to 1 so that $Q_u > 1$ for some $u$. Then, $T_1 < T_2$ are defined as the two solutions of $Q_{T_1} = Q_{T_2} = 1$. Note that $T_1$ and $T_2$ are also the two solutions of $\overline{Q}_{T_1} = \overline{Q}_{T_2}$. Because both $Q_u$ and $\overline{Q}_u$ are hump shaped, we know that $Q_u$ and $\overline{Q}_u$ are strictly greater than one for $u \in (T_1, T_2)$, and less than one otherwise. For $u \in (T_1, T_2)$, we can define $\psi_u > 0$ as in Section IX.2.2: $Q_u = (1-\mu_{h\psi_u})^{-1/\sigma}$. By construction, $\psi_u \in (0, u)$, and, as shown in Section VIII.8.4:

$$\frac{\partial \psi_u}{\partial s} = \frac{\gamma + \sigma(\gamma-\rho)}{\gamma\rho} \frac{\left(1-e^{-\rho u}\right) e^{\gamma u}}{e^{-(\rho-\gamma)(u-\psi_u)} - e^{-(\gamma/\sigma)(u-\psi_u)}}. \tag{VIII.17}$$

Plugging $Q^{\sigma}_u = (1-\mu_{h\psi_u}) = e^{\gamma\psi_u}$ in the expression of $J'(s)$, we obtain:

$$J'(s) > \frac{\gamma + \sigma(\gamma-\rho)}{\rho} \int_{T_1}^{T_2} e^{-ru} \frac{\left(1-e^{-\rho u}\right) e^{\gamma\psi_u}}{e^{-(\rho-\gamma)(u-\psi_u)} - e^{-(\gamma/\sigma)(u-\psi_u)}}\, du. \tag{VIII.18}$$

**When $r > \gamma$.** For this case fix some $\overline{u} > 0$ and pick $s$ close enough to one so that that $Q_{\overline{u}} > 1$. Such $s$ exists since, as argued earlier in Section VIII.8.1, for all $u > 0$, $Q_u \to (1-\mu_{hu})^{-1/\sigma}$ as $s \to 1$. Since the integrand in (VIII.18) is strictly positive, we have:

$$\begin{aligned} J'(s) &> \frac{\gamma + \sigma(\gamma-\rho)}{\rho} \int_0^{\overline{u}} \mathbb{I}_{\{u>T_1\}} e^{-ru} \frac{\left(1-e^{-\rho u}\right) e^{\gamma\psi_u}}{e^{-(\rho-\gamma)(u-\psi_u)} - e^{-(\gamma/\sigma)(u-\psi_u)}}\, du \\ &> \frac{\gamma + \sigma(\gamma-\rho)}{\rho} \frac{1}{e^{|\rho-\gamma|(\overline{u}-\psi_{\overline{u}})} - e^{-(\gamma/\sigma)(\overline{u}-\psi_{\overline{u}})}} \int_0^{\overline{u}} \mathbb{I}_{\{u>T_1\}} e^{-ru}\left(1-e^{-\rho u}\right) e^{\gamma\psi_u}\, du. \end{aligned}$$

where the second line follows from the fact, proven is Section VIII.8.4, that $u-\psi_u$ is strictly increasing in $u$ when $\psi_u > 0$. In Section VIII.8.4 we also prove that $T_1 \to 0$ and that, for all $u > 0$, $\psi_u \to u$ when $s$ goes to 1. Therefore, in the above equation, the integral remains bounded away from zero, and the whole expression goes to infinity.

**When $r \leq \gamma$.** In this case we make the change of variable $z \equiv T_s - u$ in equation (VIII.18) and we use that $e^{-\gamma T_s} = (1-s)$:

$$\begin{aligned}
J'(s) &> \frac{\gamma+\sigma(\gamma-\rho)}{\rho} \int_{T_s-T_2}^{T_s-T_1} (1-s)^{\frac{r}{\gamma}} e^{rz} \frac{\left(1-e^{-\rho(T_s-z)}\right) e^{\gamma\psi_{T_s-z}}}{e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}}\, dz \\
&> \frac{\gamma+\sigma(\gamma-\rho)}{\rho} \int_0^{+\infty} \mathbb{I}_{\{\max\{T_s-T_2,0\}<z<T_s-T_1\}} (1-s)^{\frac{r}{\gamma}} e^{rz} \frac{\left(1-e^{-\rho(T_s-z)}\right) e^{\gamma\psi_{T_s-z}}}{e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}}\, dz.
\end{aligned}$$

where the second line follows from the addition of the max operator in the indicator variable and the fact that the integrand is strictly positive. We show in Section VIII.8.4 that, if $\psi_{T_s-z} > 0$, then:

$$e^{\gamma\psi_{T_s-z}} > \begin{cases} \left(\frac{\gamma+\sigma(\gamma-\rho)}{\rho}\right)^{\frac{\gamma}{\rho-\gamma}} (1-s)^{-1} e^{-\gamma z} & \text{if } \rho \neq \gamma, \\ e^{-(1+\sigma)}(1-s)^{-1} e^{-\gamma z} & \text{if } \rho = \gamma, \end{cases} \tag{VIII.19}$$

and:

$$\left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}\right)^{-1} > \frac{\gamma}{\gamma+\sigma(\gamma-\rho)} \frac{\rho}{\max\{2\rho-\gamma,\gamma\}}. \tag{VIII.20}$$

When $\gamma \neq \rho$, we obtain:

$$\begin{aligned}
J'(s) > \left(\frac{\gamma+\sigma(\gamma-\rho)}{\rho}\right)^{\frac{\gamma}{\rho-\gamma}} & \frac{\gamma}{\max\{2\rho-\gamma,\gamma\}} (1-s)^{-1+\frac{r}{\gamma}} \\
& \times \int_0^{+\infty} \mathbb{I}_{\{\max\{T_s-T_2,0\}<z<T_s-T_1\}} e^{-(\gamma-r)z} \left(1-e^{-\rho(T_s-z)}\right) dz.
\end{aligned} \tag{VIII.21}$$

In Section VIII.8.4 we show that $T_s - T_2 < 0$ when $s$ is close to 1 and that $T_1$ goes to 0 when $s$ goes to 1. Since $T_s$ goes to infinity, these facts imply that the integrand goes to, and is bounded above by, $e^{-(\gamma-r)z}$ when $s \to 1$. Therefore, by dominated convergence, the integral goes to $1/(\gamma-r)$. A similar computation obtains when $\gamma = \rho$.

Consider now the case $\gamma = r$. When $\gamma \neq \rho$, equation (VIII.21) rewrites:

$$\begin{aligned}
J'(s) &> \left(\frac{\gamma+\sigma(\gamma-\rho)}{\rho}\right)^{\frac{\gamma}{\rho-\gamma}} \frac{\gamma}{\max\{2\rho-\gamma,\gamma\}} \int_{\max\{T_s-T_2,0\}}^{T_s-T_1} \left(1-e^{-\rho(T_s-z)}\right) dz \\
&= \left(\frac{\gamma+\sigma(\gamma-\rho)}{\rho}\right)^{\frac{\gamma}{\rho-\gamma}} \frac{\gamma}{\max\{2\rho-\gamma,\gamma\}} \left(T_s - T_1 - \max\{T_s-T_2,0\} - \frac{e^{-\rho T_1} - e^{-\rho\min\{T_2,T_s\}}}{\rho}\right).
\end{aligned}$$

Since $T_s - T_2 < 0$ and $T_1 \to 0$ when $s$ goes to 1, the last term in large parenthesis is equivalent to $T_s = \log((1-s)^{-1})/\gamma$ when $s$ goes to 1. A similar computation obtains when $\gamma = \rho$.

### VIII.8.4 Intermediate results for the proofs of Lemma A.10, A.11 and A.12

**Derivative of the $\psi_u$ function when $\psi_u > 0$.** When $\psi_u > 0$, time–$t$ low–valuation traders hold $q_{t,u} = 1$ if $t < \psi_u$, and $q_{t,u} = (1-\mu_{ht})^{1/\sigma}(1-\mu_{h\psi_u})^{-1/\sigma}$ if $t > \psi_u$. The market clearing condition (17) rewrites:

$$\int_0^{\psi_u} e^{\rho t}(1-\mu_{ht})\,dt + \int_{\psi_u}^{u} e^{\rho t}(1-\mu_{ht})^{1+1/\sigma}(1-\mu_{h\psi_u})^{-1/\sigma}\,dt = \int_0^u e^{\rho t}(s-\mu_{ht})\,dt. \tag{VIII.22}$$

We differentiate this equation with respect to $s$:

$$\frac{\partial \psi_u}{\partial s}\frac{\gamma}{\sigma}\int_{\psi_u}^{u} e^{\rho t}(1-\mu_{ht})^{1+1/\sigma}(1-\mu_{h\psi_u})^{-1/\sigma}\,dt = \int_0^u e^{\rho t}\,dt.$$

After computing the integrals and rearranging the terms we obtain equation (VIII.17).

**Limits of $T_1$ and $T_2$ when $s \to 1$.** For any $u > 0$, when $s$ is close enough to 1 we have $Q_u > 1$ and thus $T_1 < u < T_2$. Therefore $T_1 \to 0$ and $T_2 \to \infty$, when $s \to 1$. To obtain that $T_2 > T_s$ when $s$ is close to 1, it suffices to show that $\overline{Q}_{T_s} > 1$ for $s$ close to 1. After computing the integrals in equation (17) and using that $e^{-\gamma T_s} = 1-s$, we obtain, when $\gamma \neq \rho$:

$$\overline{Q}_{T_s} = \left(\frac{1}{\gamma-\rho} + \frac{\gamma(1-s)^{1-\frac{\rho}{\gamma}}}{\rho(\rho-\gamma)} + \frac{1-s}{\rho}\right)\frac{\gamma+\gamma/\sigma-\rho}{1-(1-s)^{\frac{\gamma+\gamma/\sigma-\rho}{\gamma}}}.$$

When $\gamma < \rho$, $\overline{Q}_{T_s}$ goes to infinity when $s$ goes to 1. When $\gamma > \rho$, $\overline{Q}_{T_s}$ goes to $(\gamma+\gamma/\sigma-\rho)/(\gamma-\rho) > 1$. When $\gamma = \rho$, a similar computation shows that $\overline{Q}_{T_s} \sim (\gamma+\gamma/\sigma-\rho)T_s$, which goes to infinity.

**Proof that $u - \psi_u$ is strictly increasing in $u$ when $\psi_u > 0$.** When $\rho \neq \gamma$, after computing the integrals in (VIII.22) and rearranging, we obtain:

$$\begin{aligned}&\left(\frac{1}{\rho-\gamma} + \frac{1}{\gamma+\gamma/\sigma-\rho}\right)\left(1-e^{-(\rho-\gamma)(u-\psi_u)}\right) - \frac{1}{\gamma+\gamma/\sigma-\rho}\left(1-e^{-(\gamma/\sigma)(u-\psi_u)}\right)\\ =&(1-s)\frac{e^{\gamma u}-e^{-(\rho-\gamma)u}}{\rho}.\end{aligned} \tag{VIII.23}$$

Taking the derivative of the left-hand side with respect to $u-\psi_u$ we easily obtain that the left-hand side of that equation is strictly increasing in $u-\psi_u$. Since the right-hand side is strictly increasing in $u$, then $u-\psi_u$ is a strictly increasing function of $u$.

When $\rho = \gamma$, the same computation leads to:

$$\int_0^{u-\psi_u}\left(e^{-\gamma/\sigma z}-1\right)dz + \int_0^u e^{-\rho t}(1-s)\,dt = 0$$

which is strictly decreasing in $u - \psi_u$ and strictly increasing in $u$, implying that $u - \psi_u$ is a strictly increasing function of $u$.

**Proof that $\psi_u \to u$ when $s \to 1$.** As noted earlier in Section VIII.8.1, for any $u$, $Q_u \to (1-\mu_{hu})^{-1/\sigma}$ as $s \to 1$. Together with the defining equation of $\psi_u$, $Q_u = (1-\mu_{h\psi_u})^{1/\sigma}$, this implies that $\psi_u \to u$ as $s \to$.

**Proof of equation (VIII.19).** When $\gamma \neq \rho$, we make the change of variable $z \equiv T_s - u$ in the market clearing condition (VIII.23):

$$\begin{aligned}
&\left(\frac{1}{\rho-\gamma} + \frac{1}{\gamma+\gamma/\sigma-\rho}\right) e^{(\rho-\gamma)\psi_{T_s-z}} - \frac{(1-s)^{\frac{\gamma+\gamma/\sigma-\rho}{\gamma}} e^{(\gamma+\gamma/\sigma-\rho)z}}{\gamma+\gamma/\sigma-\rho} e^{(\gamma/\sigma)\psi_{T_s-z}} \\
&= (1-s)^{-\frac{\rho-\gamma}{\gamma}} \left(\frac{e^{-(\rho-\gamma)z}}{\rho-\gamma} - \frac{e^{-\rho z}}{\rho}\right) + \frac{1-s}{\rho},
\end{aligned} \tag{VIII.24}$$

where we have used that $e^{-\gamma T_s} = (1-s)$. This implies that:

$$\begin{aligned}
&\left(\frac{1}{\rho-\gamma} + \frac{1}{\gamma+\gamma/\sigma-\rho}\right) e^{(\rho-\gamma)\psi_{T_s-z}} > (1-s)^{-\frac{\rho-\gamma}{\gamma}} \left(\frac{e^{-(\rho-\gamma)z}}{\rho-\gamma} - \frac{e^{-\rho z}}{\rho}\right) \\
\implies \quad &\frac{\gamma}{(\rho-\gamma)[\gamma+\sigma(\gamma-\rho)]} e^{(\rho-\gamma)\psi_{T_s-z}} > (1-s)^{-\frac{\rho-\gamma}{\gamma}} \frac{e^{-(\rho-\gamma)z}}{\rho-\gamma}\frac{\gamma}{\rho},
\end{aligned}$$

where, to move from the first to the second line, we have collected terms on the left–hand side and used $e^{-\rho z} < e^{-(\rho-\gamma)z}$ on the right-hand side. Equation (VIII.19) then follows after applying to both sides the increasing transformation:

$$x \mapsto \left(\frac{(\rho-\gamma)[\gamma+\sigma(\gamma-\rho)]}{\gamma} x\right)^{\frac{\gamma}{\rho-\gamma}}$$

Finally, when $\gamma = \rho$, after computing the integrals in the market clearing condition (VIII.22), making the change of variable $z \equiv u - T_s$, and using that $e^{-\gamma T_s} = (1-s)$, we obtain:

$$T_s - z - \psi_{T_s-z} - \frac{1 - e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}}{\gamma/\sigma} = \frac{e^{-\gamma z}}{\gamma} - \frac{1-s}{\gamma}.$$

This implies:

$$\psi_{T_s-z} > T_s - z - \frac{\sigma}{\gamma} - \frac{e^{-\gamma z}}{\gamma} > T_s - z - \frac{\sigma+1}{\gamma},$$

from which equation (VIII.19) follows by multiplying by $\gamma$, taking the exponent of the expression, and using that $e^{-\gamma T_s} = (1-s)$.

**Proof of equation (VIII.20).** When $\gamma \neq \rho$, we make the change of variable $z \equiv T_s - u$ in equation

(VIII.23):

$$\left(\frac{1}{\rho-\gamma}+\frac{1}{\gamma+\gamma/\sigma-\rho}\right)e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})}-\frac{1}{\gamma+\gamma/\sigma-\rho}e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}$$
$$=\frac{1}{\rho-\gamma}-\frac{e^{-\gamma z}}{\rho}+\frac{e^{-\gamma z-\rho(T_s-z)}}{\rho}. \tag{VIII.25}$$

When $\rho>\gamma$, we add $-1/(\rho-\gamma)\times e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}$, which is negative, to the left–hand side of (VIII.25) and obtain:

$$\begin{aligned}
&\gamma\frac{e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})}-e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}}{(\rho-\gamma)[\gamma+\sigma(\gamma-\rho)]}<\frac{1}{\rho-\gamma}-\frac{e^{-\gamma z}}{\rho}+\frac{e^{-\gamma z-\rho(T_s-z)}}{\rho}\\
\Longrightarrow\quad&\left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})}-e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}\right)>\frac{(\rho-\gamma)[\gamma+\sigma(\gamma-\rho)]}{\gamma}\left(\frac{1}{\rho-\gamma}+\frac{1}{\rho}\right)\\
\Longrightarrow\quad&\left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})}-e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}\right)>\frac{(2\rho-\gamma)[\gamma+\sigma(\gamma-\rho)]}{\rho\gamma},
\end{aligned} \tag{VIII.26}$$

where we move from the first to the second line by multiplying both sides by $(\rho-\gamma)[\gamma+\sigma(\gamma-\rho)]/\gamma$, which is a positive number since $\rho>\gamma$. Equation (VIII.20) then follows.

When $\rho<\gamma$, we also add $-1/(\rho-\gamma)\times e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}$ to the left–hand side of (VIII.25). But, since $\rho<\gamma$ this term is negative so we obtain the opposite inequality. This inequality is reversed when we multiply both sides of the equation by $(\rho-\gamma)[\gamma+\sigma(\gamma-\rho)]/\gamma$, which is a negative number since $\rho<\gamma$. Thus, we end up with the same inequality, (VIII.26), and equation (VIII.20) follows.

Finally, when $\gamma=\rho$, equation (VIII.20) follows since $1-e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}<1$.

## VIII.9 Proof of Lemma A.13

One sees easily that, since the left–hand side of (A.16) is strictly increasing and $s>\sigma/(1+\sigma)$, equation (A.16) has a unique solution. Moreover, since $s<1$, the left–hand side of (A.16) is greater than the right–hand side when evaluated at $\mu_{hT_s}$, implying that $T_\phi<T_s$. Next, define

$$H(t,\phi)\equiv\int_t^\phi e^{\rho u}g(\mu_{ht},\mu_{hu})\,du,$$
$$\text{where}\quad g(x,y)\equiv(1-y)^{1+1/\sigma}(s-x)-(1-x)^{1+1/\sigma}(s-y).$$

Let $t<T_\phi$ and $x=\mu_{ht}$. Since $x=\mu_{ht}<\mu_{hT_\phi}<\mu_{hT_s}=s$, the function $y\mapsto g(x,y)$ is strictly convex. Moreover, $g(x,x)=0$, and

$$\frac{\partial g}{\partial y}(x,x)=(1-x)^{1/\sigma}\left[\frac{x}{\sigma}-\left(1+\frac{1}{\sigma}\right)s+1\right].$$

This partial derivative is strictly negative since $x<\mu_{hT_\phi}$. Therefore $g(x,y)$ is strictly negative for $y$ just above $x$. Since $g(x,1)=(1-x)^{1+1/\sigma}(1-s)>0$, this implies that $y\mapsto g(x,y)$ has a root in

$(x, 1)$. Because of strict convexity it is the only root; we denote it by $\mu_{ht'}$ with $t' > t$. It follows that $\phi \mapsto H(t, \phi)$ is strictly decreasing over $[t, t']$ and strictly increasing over $[t', +\infty)$. Now we note that $H(t, t) = 0$. Moreover, $H(t, \phi)$ goes to $+\infty$ when $\phi \to +\infty$: indeed, $\mu_{hu}$ converges to 1 when $u \to +\infty$ and $g(x, 1) > 0$. Taken together, this means that $\phi \mapsto H(t, \phi)$ has a unique root $\phi_t > t' > t$.

We now establish that $t \mapsto \phi_t$ is a strictly decreasing function. First we note that $\partial H/\partial\phi(t, \phi_t) = e^{\rho\phi_t} g(\mu_{ht}, \mu_{h\phi_t}) > 0$ since $\phi_t > t'$. Then,

$$\frac{\partial H}{\partial t}(t, \phi_t) = -e^{\rho t} g(\mu_{ht}, \mu_{ht}) + \mu'_{ht} \int_t^{\phi_t} e^{\rho u} \frac{\partial g}{\partial x}(\mu_{ht}, \mu_{hu})\, du.$$

The first term is equal to zero because $g(x, x) = 0$ for all $x \in [0, 1]$. To evaluate the sign of the second term, we note that

$$\int_t^{\phi_t} e^{\rho u} \frac{\partial g}{\partial x}(\mu_{ht}, \mu_{hu})\, du = \int_t^{\phi_t} e^{\rho u} \left[ -(1 - \mu_{hu})^{1+1/\sigma} + \left(1 + \frac{1}{\sigma}\right)(1 - \mu_{ht})^{1/\sigma}(s - \mu_{hu}) \right] du.$$

But, since $H(t, \phi_t) = 0$, we have

$$\int_t^{\phi_t} e^{\rho u}(1 - \mu_{hu})^{1+1/\sigma}\, du = \frac{(1 - \mu_{ht})^{1+1/\sigma}}{s - \mu_{ht}} \int_t^{\phi_t} e^{\rho u}(s - \mu_{hu})\, du.$$

Plugging this into the equation just before, we obtain that $\partial H/\partial t(t, \phi_t)$ has the same sign as

$$-(1 - \mu_{ht}) + \left(1 + \frac{1}{\sigma}\right)(s - \mu_{ht}) = -\frac{\mu_{ht}}{\sigma} + \left(1 + \frac{1}{\sigma}\right) s - 1,$$

which is strictly positive since $t < T_\phi$. An application of the Implicit Function Theorem shows, then, that $\phi_t$ is strictly decreasing and continuously differentiable.

It remains to show that $\lim_{t \to T_\phi} \phi_t = T_\phi$ and that $\phi_0 < T_f$. First, note that since $\phi_t$ is strictly decreasing for $t \in [0, T_\phi)$, it has a well defined limit as $t \to T_\phi$. Moreover, it must be that $\phi_t \geq T_\phi$. Indeed, if $\phi_{t_1} < T_\phi$ for some $t_1$, then for all $t_2 \in (\phi_{t_1}, T_\phi)$ we have $\phi_{t_2} > t_2 > \phi_{t_1}$, which is impossible since $\phi_t$ is strictly decreasing. In particular, we must have that $\lim_{t \to T_\phi} \phi_t \geq T_\phi$. Now, towards a contradiction, assume that $\lim_{t \to T_\phi} \phi_t > T_\phi$. Note that, given $\partial g/\partial y(\mu_{hT_\phi}, \mu_{hu}) > 0$ for all $u > T_\phi$ and $g(\mu_{hT_\phi}, \mu_{hT_\phi}) = 0$, we have $g(\mu_{hT_\phi}, \mu_{hu}) > 0$ for all $u > T_\phi$. Therefore

$$0 < \int_{T_\phi}^{\lim_{t \to T_\phi} \phi_t} e^{\rho u} g(\mu_{hT_\phi}, \mu_{hu})\, du = H(T_\phi, \lim_{t \to T_\phi} \phi_t) = \lim_{t \to T_\phi} H(t, \phi_t) = 0,$$

by continuity of $H$, which is a contradiction. Therefore, we conclude that $\lim_{t \to T_\phi} \phi_t = T_\phi$.

In order to show that $\phi_0 < T_f$ we only need to show that $H(0, T_f) > 0$ because $H(0, \phi) \leq 0$ for all $\phi \leq \phi_0$. But we have

$$H(0, T_f) = \int_0^{T_f} e^{\rho u}(1 - \mu_{hu})^{1+1/\sigma} s\, du - \int_0^{T_f} e^{\rho u}(s - \mu_{hu})\, du > 0$$

since the first integral is strictly positive and the second integral is equal to zero by definition of $T_f$.

## VIII.10 Proof of Lemma A.14

Direct calculations show that

$$\frac{d}{dt}\left[\frac{s-\mu_{ht}}{(1-\mu_{ht})^{1+1/\sigma}}\right] = \frac{\mu'_{ht}}{(1-\mu_{ht})^{2+1/\sigma}}\left[s\left(1+\frac{1}{\sigma}\right)-1-\frac{\mu_{ht}}{\sigma}\right] \tag{VIII.27}$$

The first multiplicative term on the right–hand side is always strictly positive because $\mu'_{ht} > 0$. When $t = T_\phi$, the second multiplicative term is zero by definition of $T_\phi$. When $t < T_\phi$, it is strictly positive given that $\mu_{ht}$ is strictly increasing.

## VIII.11 Proof of Lemma A.15

Let us now turn to the proof of Lemma A.15. For $t < T_\phi$, $rp_t - \dot{p}_t$ is given by equation (A.18). Since $\delta < 1$, $(s-\mu_{ht})/(1-\mu_{ht}) < 1$, and using Lemma A.14, we obtain that $rp_t - \dot{p}_t > 0$. In order to show that it is strictly below 1, we need some additional computations. First, note that equation (VIII.27) implies that:

$$\frac{d}{dt}\left[\left(\frac{s-\mu_{ht}}{(1-\mu_{ht})^{1+1/\sigma}}\right)^{\sigma}\right] = \sigma\left(\frac{s-\mu_{ht}}{(1-\mu_{ht})^{1+1/\sigma}}\right)^{\sigma-1}\frac{\mu'_{ht}}{(1-\mu_{ht})^{2+1/\sigma}}\left[s\left(1+\frac{1}{\sigma}\right)-1-\frac{\mu_{ht}}{\sigma}\right].$$

Next, we plug $\mu'_{ht} = \gamma(1-\mu_{ht})$ in the above, and then we plug the resulting expression in equation (A.18). After some algebraic manipulations, we obtain:

$$\begin{aligned} & rp_t - \dot{p}_t \\ =& 1-\delta\left(\frac{s-\mu_{ht}}{1-\mu_{ht}}\right)^{\sigma}\left\{1-\left[1+\sigma-\sigma\frac{1-\mu_{ht}}{s-\mu_{ht}}\right]\frac{\gamma}{1-\mu_{ht}}\int_t^{\phi_t} e^{-(r+\rho)(u-t)}(1-\mu_{hu})\,du\right\}. \end{aligned} \tag{VIII.28}$$

One easily check that the term in brackets, $1+\sigma-\sigma(1-\mu_{ht})/(s-\mu_{ht})$, is strictly smaller than 1 because $s < 1$, and greater than zero because $t \le T_\phi$. On the other hand, after multiplying the integral term by $\gamma/(1-\mu_{ht}) = \gamma e^{\gamma t}$ we find:

$$\frac{\gamma}{r+\rho+\gamma}\left(1-e^{-(r+\rho+\gamma)(\phi_t - t)}\right) < 1. \tag{VIII.29}$$

Taken together, these inequalities imply that $rp_t - \dot{p}_t \in (0,1)$.

For $t \in (T_\phi, \phi_0)$, $rp_t - \dot{p}_t$ is given by equation (A.19). We have $(1-\mu_{h\phi_t^{-1}})/(1-\mu_{ht}) \in (0,1)$ since $\phi_t^{-1} < T_\phi < t$, and $(s-\mu_{h\phi_t^{-1}})/(1-\mu_{h\phi_t^{-1}}) \in (0,1)$ since $\phi_t^{-1} < T_\phi < T_s$. Therefore $rp_t - \dot{p}_t \in (0,1)$.

For $t \in (\phi_0, T_\phi)$, $rp_t - \dot{p}_t$ is the same in the ATE, thus it also lies in $(0,1)$.

## VIII.12 Proof of Lemma A.16

Plugging equation (VIII.28) into the definition of $Q_u$, we obtain that, for $u \in (0, T_\phi)$,

$$Q_u = \frac{s - \mu_{hu}}{(1 - \mu_{hu})^{1+1/\sigma}} \left[ 1 - \frac{\gamma}{r + \rho + \gamma} \left( 1 - e^{-(r+\rho+\gamma)(\phi_u - u)} \right) \left( 1 + \sigma - \sigma \frac{1 - \mu_{hu}}{s - \mu_{hu}} \right) \right]^{1/\sigma}.$$

From Lemma A.14 we know that $(s - \mu_{hu})/((1 - \mu_{hu})^{1+1/\sigma}$ is strictly increasing in $u$ over $(0, T_\phi)$. In the term in brackets, the first term in parentheses is strictly positive and strictly decreasing in $u$ since $\phi_u$ is strictly decreasing. The second term in parentheses is strictly positive because $u < T_\phi$ and it is strictly decreasing in $u$ since $\mu_{hu} < s < 1$ when $u < T_\phi < T_s$.

For $t \in (T_\phi, \phi_0)$,

$$Q_u = \frac{s - \mu_{h\phi_u^{-1}}}{(1 - \mu_{h\phi_u^{-1}})^{1+1/\sigma}}.$$

This is a strictly decreasing function of $u$ because $u \mapsto \phi_u^{-1}$ is strictly decreasing and belongs to $(0, T_\phi)$, and $x \mapsto (s - x)/(1 - x)^{1+1/\sigma}$ is strictly increasing over $(0, \mu_{hT_\phi})$ by Lemma A.14.

For $t \in (\phi_0, T_f)$, $Q_u = \overline{Q}_u$. Since $u \mapsto \overline{Q}_u$ is hump-shaped by Lemma A.4 all we need to show is $\overline{Q}'_{\phi_0} < 0$. To that end, note first that $H(0, \phi_0) = 0$ writes as

$$\int_0^{\phi_0} e^{\rho t} (1 - \mu_{ht})^{1+1/\sigma} s \, dt = \int_0^{\phi_0} e^{\rho t} (s - \mu_{ht}) \, dt.$$

From the proof of Lemma A.4, in Section VIII.3, equation (VIII.5) we know that $\overline{Q}'_{\phi_0}$ has the same sign as

$$\int_0^{\phi_0} e^{\rho t} \left[ (1 - \mu_{ht})^{1+1/\sigma} (s - \mu_{h\phi_0}) - (1 - \mu_{h\phi_0})^{1+1/\sigma} (s - \mu_{ht}) \right] dt.$$

Replacing the first equation into the second, we find that $\overline{Q}'_{\phi_0}$ as the same sign as

$$s - \mu_{h\phi_0} - s(1 - \mu_{h\phi_0})^{1+1/\sigma} = -g(0, \mu_{h\phi_0}),$$

where the function $g$ is defined in the proof of Lemma A.13. But we already know from this proof that $g(0, \mu_{h\phi_0}) > 0$ has to hold for $H(0, \phi_0) = 0$, which from the above imply that $\overline{Q}'_{\phi_0} < 0$.

It remains to show that $u \mapsto Q_u$ is continuous at $u = T_\phi$ and $u = \phi_0$. Starting from equations (A.18) and (A.19), continuity at $u = T_\phi$ follows from $\phi_{T_\phi}^{-1} = T_\phi$ and Lemma A.14. Turning to continuity at $u = \phi_0$, (A.19) evaluated at $t = \phi_0$ yields $Q_{\phi_0^-} = s$. On the other hand, plugging $H(0, \phi_0) = 0$ into the definition (A.4) of $\overline{Q}_u$ we obtain that $Q_{\phi_0^+} = \overline{Q}_{\phi_0} = s$.

## VIII.13 Proof of Lemma A.17

By construction the price is continuously differentiable in all the open intervals $(0, T_\phi)$, $(T_\phi, \phi_0)$, $(\phi_0, T_f)$, and $(T_f, \infty)$. Let us show that it is also continuously differentiable at the boundary points of these intervals. First note that, by definition of $Q_t$, in equation (A.22), it follows that

$$rp_u = 1 - \delta(1 - \mu_{hu})Q_u^\sigma + \dot{p}_u. \tag{VIII.30}$$

Since by Lemma A.16, $Q_u$ is continuous over $(0, T_f)$, and since the price is continuous by construction, it follows that $\dot{p}_u$ is continuous over $(0, T_f)$ as well. Turning to $t = T_f$, we have $Q_{T_f} = \overline{Q}_{T_f} = 0$ by definition of $T_f$ and of $\overline{Q}_u$ in equation (A.4). Plugging $Q_{T_f} = 0$ in (VIII.30), it follows that $rp_{T_f} = 1 + \dot{p}_{T_f^-}$. Since $p_{T_f} = 1/r$ it follows that $\dot{p}_{T_f^-} = 0$. Since $p_t$ is constant for $t > T_f$, this shows that $\dot{p}_t$ is continuous at $T_f$.

Next, we show that the price is strictly increasing over $(0, T_f)$. We start with the time interval $(T_\phi, T_f)$. Since by Lemma A.16, $Q_u$ is strictly decreasing over $(T_\phi, T_f)$, it follows that $\Delta_u = (1 - \mu_{hu})^{1/\sigma} Q_u$ is strictly decreasing over $(T_\phi, T_f)$. Using the same argument as in the proof Proposition 1 in Section A.3, it follows that the price is strictly increasing over $(T_\phi, T_f)$.

The proof is more difficult for the initial time interval, $[0, T_\phi]$. We start by defining, for $t \in [0, T_\phi]$:

$$\delta_t \equiv \delta \left( \frac{s - \mu_{ht}}{1 - \mu_{ht}} \right)^\sigma.$$

Clearly, since $\mu_{ht}$ is strictly increasing, we have $\delta_t' < 0$. Also, using $\mu_{ht}' = \gamma(1 - \mu_{ht})$ one easily sees after some algebra that:

$$\gamma \delta_t + \delta_t' = \gamma \sigma \delta \frac{(s - \mu_{ht})^{\sigma - 1}}{(1 - \mu_{ht})^\sigma} \left[ s \left( 1 + \frac{1}{\sigma} \right) - 1 - \frac{\mu_{ht}}{\sigma} \right] \geq 0 \tag{VIII.31}$$

for $t \in [0, T_\phi]$, by definition of $T_\phi$. With the definition of $\delta_t$, and keeping in mind that $1 - \mu_{ht} = e^{-\gamma t}$, ODE (A.18) writes:

$$\begin{aligned} rp_t - \dot{p}_t &= 1 - \delta_t + \frac{d}{dt}\left[\delta_t e^{\gamma t}\right] e^{-\gamma t} \int_t^{\phi_t} e^{-(r+\rho+\gamma)(u-t)} \, du \\ &= 1 - \delta_t + \frac{\delta_t + \gamma \delta_t'}{r + \rho + \gamma} \left( 1 - e^{-(r+\rho+\gamma)(\phi_t - t)} \right) \end{aligned} \tag{VIII.32}$$

And, ODE (A.19) writes:

$$rp_t - \dot{p}_t = 1 - e^{-\gamma(t - \phi_t^{-1})} \delta_{\phi_t^{-1}}. \tag{VIII.33}$$

Next, we differentiate equations (VIII.32) and (VIII.33) to find ODEs for $d_t \equiv \dot{p}_t$:

$$t \in (0, T_\phi): \quad rd_t - \dot{d}_t = -\delta'_t + \left[\gamma\delta_t + \delta'_t\right] e^{-(r+\rho+\gamma)(\phi_t - t)} \left(\phi'_t - 1\right) + \frac{\gamma\delta'_t + \delta''_t}{r+\rho+\gamma}\left(1 - e^{-(r+\rho+\gamma)(\phi_t - t)}\right),$$

$$t \in (T_\phi, \phi_0): \quad rd_t - \dot{d}_t = -\delta'_{\phi_t^{-1}} e^{-\gamma(t-\phi_t^{-1})} + \left[\gamma\delta_{\phi_t^{-1}} + \delta'_{\phi_t^{-1}}\right]\left(1 - \frac{1}{\phi'_{\phi_t^{-1}}}\right) e^{-\gamma(t-\phi_t^{-1})}.$$

We already know that the price is continuously differentiable and hence that $d_t = \dot{p}_t$ is continuous over $[0, T_f]$. This allows to write:

$$d_t = \int_t^{\phi_t} e^{-r(u-t)}\left(rd_u - \dot{d}_u\right) du + e^{-r(\phi_t - t)} d_{\phi_t}.$$

Since $\phi_t \geq T_\phi$, we already know that $d_{\phi_t} \geq 0$. So, in order to show that $d_t \geq 0$, it suffices to show that the integral is positive. To that end, equipped with the above analytical expressions of $rd_t - \dot{d}_t$, the integral can be written as sum of five terms:

$$\text{Term (I)}: -\int_t^{T_\phi} e^{-r(u-t)}\delta'_u \, du$$

$$\text{Term (II)}: -\int_{T_\phi}^{\phi_t} e^{-r(u-t)}\delta'_{\phi_u^{-1}} e^{-\gamma(u-\phi_u^{-1})} \, du$$

$$\text{Term (III)}: \int_t^{T_\phi} e^{-r(u-t)} \frac{\gamma\delta'_u + \delta''_u}{r+\rho+\gamma}\left(1 - e^{-(r+\rho+\gamma)(\phi_u - u)}\right) du$$

$$\text{Term (IV)}: \int_t^{T_\phi} e^{-r(u-t)}\left[\gamma\delta_u + \delta'_u\right]\left(\phi'_u - 1\right) e^{-(r+\rho+\gamma)(\phi_u - u)} \, du$$

$$\text{Term (V)}: \int_{T_\phi}^{\phi_t} e^{-r(u-t)}\left[\gamma\delta_{\phi_u^{-1}} + \delta'_{\phi_u^{-1}}\right]\left(1 - \frac{1}{\phi'_{\phi_u^{-1}}}\right) e^{-\gamma(u-\phi_u^{-1})} \, du.$$

We make the change of variable $u = \phi_z$ in Term (V) and obtain that

$$\text{Term (V)} = \int_t^{T_\phi} e^{-r(z-t)}\left[\gamma\delta_z + \delta'_z\right]\left(1 - \phi'_z\right) e^{-(r+\gamma)(\phi_z - z)} \, dz.$$

Since, by equation (VIII.31), $\gamma\delta_z + \delta'_z \geq 0$ for $z \leq T_\phi$, and since by Lemma A.13, $\phi'_z \leq 0$, this implies that the sum of terms (IV) and (V) is positive.

Next, since

$$\frac{1 - e^{-(r+\rho+\gamma)(\phi_u - u)}}{r+\rho+\gamma} = \int_u^{\phi_u} e^{-(r+\rho+\gamma)(z-u)} \, dz$$

we can rewrite Term (III) as

$$\int_t^{T_\phi} [\gamma\delta'_u + \delta''_u] e^{(\rho+\gamma)u} \int_u^{\phi_u} e^{-(\rho+\gamma)z - r(z-t)} \, dz \, du.$$

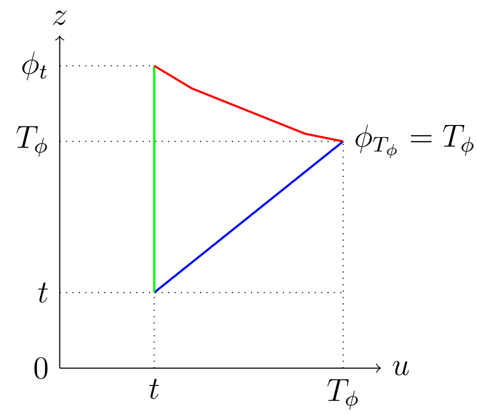


**Figure 7:** The green vertical line is the segment $u = t$, from $u = 0$ to $u = \phi_t$. The blue upward slopping line is the segment $z = u$, from $u = t$ to $u = T_\phi$. The red downward slopping curve is the function $z = \phi_u$, from $u = t$ to $u = T_\phi$. Thus, the domain of integration is the area enclosed in the "triangle-shaped" area between the green line, the blue line, and the red curve.

Using the graphical representation of the domain of integration shown in Figure 7, we switch round the two integrals and obtain

$$\begin{aligned}\text{Term (III)} = &\int_t^{T_\phi} \left[\int_t^z [\gamma\delta'_u + \delta''_u] e^{(\rho+\gamma)u}\, du\right] e^{-(\rho+\gamma)z - r(z-t)}\, dz \\ &+ \int_{T_\phi}^{\phi_t} \left[\int_t^{\phi_z^{-1}} [\gamma\delta'_u + \delta''_u] e^{(\rho+\gamma)u}\, du\right] e^{-(\rho+\gamma)z - r(z-t)}\, dz.\end{aligned}$$

Since $\delta'_t < 0$, it follows that

$$[\gamma\delta'_u + \delta''_u] e^{(\rho+\gamma)u} > [(\rho+\gamma)\delta'_u + \delta''_u] e^{(\rho+\gamma)u} = \frac{d}{du}\left[\delta'_u e^{(\rho+\gamma)u}\right].$$

Plugging this inequality into the last expression we found for Term (III) and explicitly integrating with respect to $u$, we find

$$\begin{aligned}\text{Term (III)} > &\int_t^{T_\phi} \left[\delta'_z e^{(\rho+\gamma)z} - \delta'_t e^{(\rho+\gamma)t}\right] e^{-(\rho+\gamma)z - r(z-t)}\, dz \\ &+ \int_{T_\phi}^{\phi_t} \left[\delta'_{\phi_z^{-1}} e^{(\rho+\gamma)\phi_z^{-1}} - \delta'_t e^{(\rho+\gamma)t}\right] e^{-(\rho+\gamma)z - r(z-t)}\, dz \\ > &\int_t^{T_\phi} \delta'_z e^{-r(z-t)}\, dz + \int_{T_\phi}^{\phi_t} \delta'_{\phi_z^{-1}} e^{-(\rho+\gamma)(z-\phi_z^{-1}) - r(z-t)}\, dz - \int_t^{\phi_t} \delta'_t e^{-(r+\rho+\gamma)(z-t)}\, dz.\end{aligned}$$

The last term is positive since $\delta'_t < 0$. The first term cancels out with Term (I). Adding up the second term in the above equation with Term (II), we obtain

$$\int_{T_\phi}^{\phi_t} e^{-r(z-t)}\delta'_{\phi_z^{-1}}\left[e^{-(\rho+\gamma)(z-\phi_z^{-1})} - e^{-\gamma(z-\phi_z^{-1})}\right] dz,$$

which is positive given that $\delta'_u < 0$. Taken together, this all show that the sum of Terms (I) through (V) is positive, and hence that $d_t = \dot{p}_t > 0$ for all $t \in [0, T_\phi]$.

## VIII.14 Proof that condition $(i)$ and $(ii)$ hold

From the definition of $M(u,q)$ in equation (A.27), and keeping in mind that $\xi_u = 1 - \delta(1-\mu_{hu})Q_u^\sigma$ by definition of $Q_u$, we have

$$\frac{\partial M}{\partial q}(u,q) = 1 - \xi_u - \delta\frac{1-\mu_{h_u}}{1-\mu_{ht}}q^\sigma = \delta\frac{1-\mu_{hu}}{1-\mu_{ht}}\left[(1-\mu_{ht})Q_u^\sigma - q^\sigma\right]. \tag{VIII.34}$$

By the definition of $q_{t,u}$ in equation (A.22) and (A.24) this implies that

$$\frac{\partial M}{\partial q}(u, q_{t,u}) = 0 \quad \text{for all } t \in (0, T_\phi],\ u \in [\phi_t, T_f);\ \text{and for all } t \in [T_\phi, T_f),\ u \in [t, T_f]. \tag{VIII.35}$$

This clearly implies that condition $(i)$ holds for $t \geq T_\phi$. For $t \in (0, T_\phi]$, we define

$$N_t = \int_t^{\phi_t} e^{-(r+\rho)(u-t)}\frac{\partial M}{\partial q}(u, q_{t,u})\, du.$$

First, we note that $\phi_{T_\phi} = T_\phi$ implies $N_{T_\phi} = 0$. Then, we differentiate $N_t$ with respect to $t$

$$\begin{aligned}
N_t' &= (r+\rho)N_t - \frac{\partial M}{\partial q}(t, q_{t,t}) + \int_t^{\phi_t} e^{-(r+\rho)(u-t)}\frac{\partial}{\partial t}\left[\frac{\partial M}{\partial q}(u, q_{t,t})\right] du \\
&= (r+\rho)N_t - \left(1 - \xi_t - \delta q_{t,t}^\sigma\right) - \delta\int_t^{\phi_t} e^{-(r+\rho)(u-t)}(1-\mu_{hu})\frac{\partial}{\partial t}\left[\frac{q_{t,t}^\sigma}{1-\mu_{ht}}\right] du \\
&= (r+\rho)N_t,
\end{aligned}$$

where: for the first equality we used that $\partial M/\partial q(u, q_{t,\phi_t}) = 0$, and in the integral we have substituted $q_{t,u} = q_{t,t}$ since $u \in [t, \phi_t]$; we obtain the second equality by evaluating $\partial M/\partial q(u,q)$, in equation (VIII.34), at $(t, q_{t,t})$; we obtain the third equality from equation (A.18) and $q_{t,t} = (s - \mu_{ht})/(1-\mu_{ht})$. Therefore we have a differential equation for $N_t$. Given the boundary condition $N_{T_\phi} = 0$, we have

$$N_t = 0 \quad \text{for all } t \in [0, T_\phi]. \tag{VIII.36}$$

With this in mind, we turn to condition $(i)$ for $t \in [0, T_\phi]$ and note that:

$$\begin{aligned}\int_t^{T_f} e^{-(r+\rho)(u-t)} \frac{\partial M}{\partial q}(u, q_{t,u}) q_{t,u}\, du =& q_{t,t} \int_t^{\phi_t} e^{-(r+\rho)(u-t)} \frac{\partial M}{\partial q}(u, q_{t,u})\, du + \int_{\phi_t}^{T_f} \frac{\partial M}{\partial q}(u, q_{t,u}) q_{t,u}\, du \\ =& q_{t,t} N_t + \int_{\phi_t}^{T_f} \frac{\partial M}{\partial q}(u, q_{t,u}) q_{t,u}\, du = 0,\end{aligned}$$

where: we obtain the first equality by breaking the interval of integration into $[t, \phi_t]$ and $[\phi_t, T_f]$, and noting that $q_{t,u} = q_{t,t}$ over $[t, \phi_t]$; we obtain the second equality by recognizing that the first integral is equal to $N_t$; the last equality follows from (VIII.35) and (VIII.36). This establishes condition $(i)$ holds.

Let us now turn to condition $(ii)$. Note first that this condition holds for $t \in [T_\phi, T_f]$ since in that case $\partial M/\partial q(u, q_{t,u}) = 0$. To show that it also holds for $t \in (0, T_\phi)$, we show:

**RVIII.8.** *There exists $u_1 \in (t, \phi_t)$ such that $u \mapsto \partial M/\partial q(u, q_{t,u}) < 0$ for $u \in (t, u_1)$ and $\partial M/\partial q(u, q_{t,u}) > 0$ for $u \in (u_1, \phi_t)$.*

Indeed, by equation (VIII.34):

$$\text{sign}\left[\frac{\partial M}{\partial q}(u, q_{t,u})\right] = \text{sign}\left[F_u\right] \qquad \text{where } F_u \equiv (1 - \mu_{ht}) Q_u^\sigma - q_{t,u}^\sigma.$$

By Lemma A.16 we know that $Q_u$ is strictly increasing before $T_\phi$ and strictly decreasing after $T_\phi$. Also, by construction of the candidate LOE, $q_{t,u}$ is constant over $[t, \phi_t]$. Thus $F_u$ is strictly increasing over $[t, T_\phi)$, and strictly decreasing over $(T_\phi, \phi_t]$. Second, when $u = t$, we have:

$$F_t = (1 - \mu_{ht}) Q_t^\sigma - q_{t,t}^\sigma = \frac{1 - \xi_t}{\delta} - q_{t,t}^\sigma = \frac{1}{\delta}\left[1 - \delta\left(\frac{s - \mu_{ht}}{1 - \mu_{ht}}\right)^\sigma - \xi_t\right],$$

where the first equality follows by definition of $Q_u$, in equation (A.22), and the second equality follows by definition of $q_{t,t}$ for $t \in (0, T_\phi)$, in equation (A.21). Now, by equation (A.18) and Lemma A.14, this last expression is strictly negative for $t < T_\phi$, i.e., $F_t < 0$. Third, the asset holding plan is continuous at $u = \phi_t$ so $F_{\phi_t} = (1 - \mu_{ht}) Q_{\phi_t}^\sigma - q_{t,t}^\sigma = (1 - \mu_{ht}) Q_{\phi_t}^\sigma - q_{t,\phi_t}^\sigma$. But this last expression is equal to zero by equation (A.22). Taken together, the above shows that $F_u$ is increasing over $[t, T_\phi]$, decreasing over $[T_\phi, \phi_t]$, negative at $u = t$, and zero at $u = \phi_t$. This shows that there exists $u_1 \in (t, \phi_t)$ such that $F_u < 0$ for $u \in (t, u_1)$ and $F_u > 0$ for $u \in (u_1, \phi_t)$. Because $u \mapsto \partial M/\partial q(u, q_{t,u})$ has the same sign as $F_u$, result RVIII.8 follows.

Next, for any decreasing function $\tilde{q}_{t,u}$, we have

$$\begin{aligned}&\int_t^{T_f} e^{-(r+\rho)(u-t)} \frac{\partial M}{\partial q}(u, q_{t,u}) \tilde{q}_{t,u}\, du \\ \leq& \int_t^{u_1} e^{-(r+\rho)(u-t)} \frac{\partial M}{\partial q}(u, q_{t,u}) \tilde{q}_{t,u_1}\, du + \int_{u_1}^{\phi_t} e^{-(r+\rho)(u-t)} \frac{\partial M}{\partial q}(u, q_{t,u}) \tilde{q}_{t,u_1}\, du = N_t \tilde{q}_{t,u_1},\end{aligned}$$

but $N_t = 0$ because of (VIII.36), and therefore condition $(ii)$ holds.

## VIII.15 Proof of Lemma A.19

**For $t \in (0, \phi_u^{-1})$.** In the ATE, time–$t$ low–valuation traders hold:

$$q_{t,u}^{ATE} = \min\{(1-\mu_{ht})^{1/\sigma} Q_u^{ATE}, 1\}. \tag{VIII.37}$$

In the LOE, on the other hand:

$$q_{t,u}^{LOE} = \frac{s-\mu_{ht}}{1-\mu_{ht}},$$

which is strictly less than 1. Thus, if $q_{t,u}^{ATE} = 1$, we have that $q_{t,u}^{LOE} < q_{t,u}^{ATE}$. Now, if $q_{t,u}^{ATE} < 1$, we write

$$\begin{aligned} q_{t,u}^{LOE} &= (1-\mu_{ht})^{1/\sigma} \frac{s-\mu_{ht}}{(1-\mu_{ht})^{1+1/\sigma}} \\ &< (1-\mu_{ht})^{1/\sigma} \frac{s-\mu_{h\phi_u^{-1}}}{(1-\mu_{h\phi_u^{-1}})^{1+1/\sigma}} \\ &= (1-\mu_{ht})^{1/\sigma} Q_u^{LOE} \end{aligned}$$

where the first line follows by multiplying and dividing by $(1-\mu_{ht})^{1/\sigma}$, the second line from Lemma A.14, and the third line by combining equations (A.19) and (A.22). Therefore, if $Q_u^{ATE} \geq Q_u^{LOE}$ implies that $q_{t,u}^{LOE} < (1-\mu_{ht})^{1/\sigma} Q_u^{ATE} = q_{t,u}^{ATE}$ given equation (VIII.37) and our assumption that $q_{t,u}^{ATE} < 1$.

**For $t \in (\phi_u^{-1}, u)$.** In the ATE, time–$t$ low–valuation traders holdings are still determined by equation (VIII.37). In the LOE, their holdings are given by

$$q_{t,u}^{LOE} = (1-\mu_{ht})^{1/\sigma} Q_u^{LOE}.$$

Since $Q_u^{ATE} \geq Q_u^{LOE}$ and $q_{t,u}^{LOE} \leq 1$, it follows that $q_{t,u}^{ATE} \geq q_{t,u}^{LOE}$.

## VIII.16 Proof of Lemma A.20

For $\rho > 0$, $T_f$ is defined as the unique $u > 0$ such that:

$$K(u,\rho) \equiv \int_0^u e^{\rho t}(s-\mu_{ht})\, dt = 0.$$

This equation has a unique strictly positive solution when $\rho \geq 0$. When $\rho > 0$ it is equal to $T_f$. When $\rho = 0$ we denote it by $\hat{T}_f$. Moreover, $K(\cdot\,,\cdot)$ is continuously differentiable in $u$ and $\rho$ with

$\partial K/\partial u(\hat{T}_f, 0) \neq 0$. Thus, by the Implicit Function Theorem, the unique strictly positive solution of $K(u,\rho) = 0$ is a continuous function of $\rho$ in a neighborhood of $\rho = 0$. In particular, $T_f \to \hat{T}_f$ as $\rho \to 0$.

We now prove an analogous result for $Q_u^{ATE}$. The only subtlety is that the support of $Q_u^{ATE}$, which is $(0, T_f)$, implicitly depends on $\rho$. We first note that, since $\partial K/\partial u(T_f, \rho) < 0$ and $\partial K/\partial \rho(T_f, \rho) < 0$, then $T_f$ is decreasing in $\rho$, i.e., $T_f$ increases to $\hat{T}_f$ when $\rho$ decreases to 0. Therefore, for all $u < \hat{T}_f$, $Q_u^{ATE}$ is well defined for $\rho$ close enough to zero. Using the same argument as for $T_f$, in this neighborhood of $\rho = 0$, $Q_u^{ATE}$ goes to a well-defined limit $\hat{Q}_u^{ATE}$ satisfying:

$$\int_0^u (1-\mu_{ht}) \min\{(1-\mu_{ht})^{1/\sigma} \hat{Q}_u^{ATE}, 1\}\, dt = \int_0^u (s - \mu_{ht})\, dt.$$

The price $p_u^{ATE}$ is obtained by integrating the ODE (18) over $t \in [u, T_f]$:

$$p_u^{ATE} = e^{-r(T_f - u)} \frac{1}{r} + \int_u^{T_f} e^{-r(t-u)} \left[1 - \delta(1-\mu_{ht})(Q_t^{ATE})^\sigma\right] dt.$$

By continuity, $p_u^{ATE}$ goes to:

$$\hat{p}_u^{ATE} = e^{-r(\hat{T}_f - u)} \frac{1}{r} + \int_u^{\hat{T}_f} e^{-r(t-u)} \left[1 - \delta(1-\mu_{ht})(\hat{Q}_t^{ATE})^\sigma\right] dt.$$

Using similar arguments, we obtain analogous results in the LOE. Time $T_\phi$ does not depend on $\rho$, thus $\hat{T}_\phi = T_\phi$. For all $t \in [0, T_\phi)$, $\phi_t$ goes to the unique $\hat{\phi}_t \in (T_\phi, \hat{T}_f)$ such that:

$$\int_t^{\hat{\phi}_t} \left[(1-\mu_{hu})^{1+1/\sigma}(s-\mu_{ht}) - (1-\mu_{ht})^{1+1/\sigma}(s-\mu_{hu})\right] du = 0.$$

The price path also goes to a well-defined limit $\hat{p}_u^{LOE}$ satisfying the same ODE as when $\rho > 0$ after letting $\rho = 0$.

## VIII.17 Proof of Lemma A.21

In the ATE, after integrating the ODE for the price over $u \in [0, \phi_0]$ and taking the limit $\rho \to 0$, we obtain:

$$\hat{p}_0^{ATE} = e^{-r\hat{\phi}_0} \hat{p}_{\hat{\phi}_0}^{ATE} + \int_0^{\hat{\phi}_0} e^{-ru} \left[1 - \delta(1-\mu_{hu})(\hat{Q}_u^{ATE})^\sigma\right] du.$$

Inspecting the proof of Proposition 3, one notes that it does not use the fact that $\rho > 0$. Therefore it still holds when $\rho = 0$ and $u \mapsto \hat{Q}_u^{ATE}$ is hump-shaped. We also have $\hat{Q}_0^{ATE} = s$, and:

$$\hat{Q}_{\hat{\phi}_0}^{ATE} \geq \frac{\int_0^{\hat{\phi}_0} (s - \mu_{ht})\, dt}{\int_0^{\hat{\phi}_0} (1 - \mu_{ht})^{1+1/\sigma}\, dt} = s,$$

where the inequality follows from the market clearing condition (17) and the last equality comes from the definition of $\hat{\phi}_0$, equation (A.17). Taken together, these facts imply that:

$$\hat{p}_0^{ATE} < e^{-r\hat{\phi}_0} \hat{p}_{\hat{\phi}_0}^{ATE} + \int_0^{\hat{\phi}_0} e^{-ru} \left[1 - \delta(1 - \mu_{hu}) s^\sigma\right] du.$$

For the LOE, we integrate the ODEs (A.18) and (A.19) over $u \in [0, \phi_0]$ and we take the limit as $\rho \to 0$. It follows that:

$$\begin{aligned}
\hat{p}_0^{LOE} =& e^{-r\hat{\phi}_0} \hat{p}_{\hat{\phi}_0}^{LOE} + \int_0^{\hat{T}_\phi} e^{-ru} \left[1 - \delta_u + \frac{\partial}{\partial u} \left[\frac{\delta_u}{1 - \mu_{hu}}\right] \int_u^{\hat{\phi}_u} e^{-r(z-u)} (1 - \mu_{hz})\, dz\right] du \\
&+ \int_{\hat{T}_\phi}^{\hat{\phi}_0} e^{-ru} \left[1 - \delta_{\hat{\phi}_u^{-1}} \frac{1 - \mu_{hu}}{1 - \mu_{h\hat{\phi}_u^{-1}}}\right] du,
\end{aligned}$$

where $\delta_u \equiv \delta((s - \mu_{hu})/(1 - \mu_{hu}))^\sigma$. In the first line, we can compute the double integral by switching the order of integration as in the proof of Lemma A.17 (see Figure 7, page 65 in this supplementary appendix):

$$\begin{aligned}
&\int_0^{\hat{T}_\phi} \int_u^{\hat{\phi}_u} e^{-ru} \frac{\partial}{\partial u} \left[\frac{\delta_u}{1 - \mu_{hu}}\right] e^{-r(z-u)} (1 - \mu_{hz})\, dz\, du \\
=& \int_0^{\hat{T}_\phi} \int_0^z e^{-rz} \frac{\partial}{\partial u} \left[\frac{\delta_u}{1 - \mu_{hu}}\right] (1 - \mu_{hz})\, du\, dz + \int_{\hat{T}_\phi}^{\hat{\phi}_0} \int_0^{\hat{\phi}_z^{-1}} e^{-rz} \frac{\partial}{\partial u} \left[\frac{\delta_u}{1 - \mu_{hu}}\right] (1 - \mu_{hz})\, du\, dz \\
=& \int_0^{\hat{T}_\phi} e^{-rz} (1 - \mu_{hz}) \left[\frac{\delta_z}{1 - \mu_{hz}} - \delta_0\right] dz + \int_{\hat{T}_\phi}^{\hat{\phi}_0} e^{-rz} (1 - \mu_{hz}) \left[\frac{\delta_{\hat{\phi}_z^{-1}}}{1 - \mu_{h\hat{\phi}_z^{-1}}} - \delta_0\right] dz.
\end{aligned}$$

Plugging this back into the expression of $\hat{p}_0^{LOE}$ and substituting $\delta_0 = \delta s^\sigma$, we obtain:

$$\hat{p}_0^{LOE} = e^{-r\hat{\phi}_0} \hat{p}_{\hat{\phi}_0}^{LOE} + \int_0^{\hat{\phi}_0} e^{-ru} \left(1 - \delta(1 - \mu_{hu}) s^\sigma\right) du.$$

Since $\hat{p}_{\hat{\phi}_0}^{ATE} = \hat{p}_{\hat{\phi}_0}$, we obtain that $\hat{p}_0^{ATE} < \hat{p}_0^{LOE}$.

## VIII.18 Constructing the limit order book

Consider, at some time $u$ and for some ask price $a \in (p_{T_\psi}, p_{T_f})$, the volume of limit sell orders with ask price in $[a, a + da]$. These order have been submitted by traders who had an information events before $u$, in order to reduce their holding during the interval $[z, z+dz]$, where $p_z = a$ and, by the chain rule, $\dot{p}_z\, dz = da$. Namely, a trader who had her last information event at $t \in (0, u]$ plans to execute

$$\left| \frac{\partial}{\partial z} \min\{(1-\mu_{ht})^{1/\sigma} Q_z, 1\} \right| dz$$

during $[z, z+dz]$. Note that the partial derivative is well-defined except possibly for one value $t$ such that $(1-\mu_{ht})^{1/\sigma} Q_z = 1$. Integrating over all times $t \in (0, u]$, and using that $dz = da/\dot{p}_z$, we obtain that the volume of sell orders with ask in $[a, a+da]$ is:

$$S_{u,a}\, da = \int_0^u \rho e^{-\rho(u-t)} (1-\mu_{ht}) \left| \frac{\partial}{\partial z} \min\{(1-\mu_{ht})^{1/\sigma} Q_z, 1\} \right| \frac{da}{\dot{p}_z}\, dt,$$

The volume of limit sell orders outstanding in the book as of time $u < T_f$ is:

$$S_u = \int_{\max\{p_u, p_{T_\psi}\}}^{p_{T_f}} S_{u,a}\, da. \tag{VIII.38}$$

$S_u$ is represented in Figure 6 (lower panel) as well as in Figure 8 (lower panel).

# IX Proofs omitted in the supplementary appendix

## IX.1 Proof for Section I

### IX.1.1 Proof of Lemma I.1

We define the expected discounted utility as in LR

$$\overline{v}(\theta, q) \equiv (r+\kappa) \int_0^{+\infty} e^{-(r+\kappa)s} \mathbb{E}[v(\theta_{t+s}, q) \,|\, \theta_t = \theta] \, ds,$$

for $\theta \in \{h, \ell\}$ and $q \geq 0$. Using that

$$\text{Prob}[\theta_{t+z} = h \,|\, \theta_t = h] = e^{-(\gamma_h+\gamma_\ell)z} + \frac{\gamma_h}{\gamma_h+\gamma_\ell}\left(1 - e^{-(\gamma_h+\gamma_\ell)z}\right) \tag{IX.1}$$

$$\text{Prob}[\theta_{t+z} = \ell \,|\, \theta_t = h] = \frac{\gamma_\ell}{\gamma_h+\gamma_\ell}\left(1 - e^{-(\gamma_h+\gamma_\ell)z}\right) \tag{IX.2}$$

we can compute the integral in $\overline{v}(h, q)$ to obtain

$$\overline{v}(h, q) = \frac{r+\kappa}{r+\kappa+\gamma_h+\gamma_\ell} v(h, q) + \frac{1}{r+\kappa+\gamma_h+\gamma_\ell}\Big(\gamma_h v(h, q) + \gamma_\ell v(\ell, q)\Big). \tag{IX.3}$$

And symmetrically:

$$\overline{v}(\ell, q) = \frac{r+\kappa}{r+\kappa+\gamma_h+\gamma_\ell} v(\ell, q) + \frac{1}{r+\kappa+\gamma_h+\gamma_\ell}\Big(\gamma_h v(h, q) + \gamma_\ell v(\ell, q)\Big). \tag{IX.4}$$

Plugging in the analytical expression for the utility flows, we find that $\overline{v}(h, q)$ and $\overline{v}(\ell, q)$ are strictly concave over $q \in [0, 1]$, with derivatives

$$\begin{aligned}
\overline{v}_q(h, q) &= 1 - \frac{\gamma_\ell}{r+\kappa+\gamma_h+\gamma_\ell} q^\sigma \\
\overline{v}_q(\ell, q) &= 1 - \frac{r+\kappa+\gamma_\ell}{r+\kappa+\gamma_h+\gamma_\ell} q^\sigma,
\end{aligned}$$

and constant over $q \in [1, +\infty)$.

Similarly to Lagos and Rocheteau (2009), given some price $p > 0$,[12] traders' optimal asset holdings are given by the first–order condition

$$q_\theta = \begin{cases} 0 & \text{if } \; \overline{v}_q(\theta, 0) \leq rp \\ \left(\overline{v}_q\right)^{-1}(\theta, rp) & \text{if } \; \overline{v}_q(\theta, 1) < rp < \overline{v}_q, (\theta, 0) \\ 1 & \text{if } \; \overline{v}_q(\theta, 1) \geq rp, \end{cases}$$

[12] One easily rules out $p = 0$: in that case all traders would want to hold more than one unit, which would violate market clearing given that the aggregate supply is strictly less than 1.

for $\theta \in \{h, \ell\}$. They also satisfy the market clearing condition

$$\frac{\gamma_h}{\gamma_h + \gamma_\ell} q_h + \frac{\gamma_\ell}{\gamma_h + \gamma_\ell} q_\ell = s.$$

First, both $q_h > 0$ and $q_\ell > 0$. Otherwise, if either $q_\ell = 0$ or $q_h = 0$, the fact that $\overline{v}_q(h, 0) = \overline{v}_q(\ell, 0)$ would imply that $q_\ell = q_h = 0$. Otherwise, the first–order conditions would imply that $q_h = 0$ *and* $q_\ell = 0$, which would contradict market clearing. Second, we have that $q_\ell < 1$. Otherwise, if $q_\ell = 1$, then the fact that $\overline{v}_q(h, 1) > \overline{v}_q(\ell, 1)$ would imply that $q_h = 1$ as well, which also contradicts market clearing. Taken together, these two remarks imply equilibrium allocations come in only two flavors: we can have an "interior" equilibrium allocation with $q_h \in (0, 1)$ and $q_\ell \in (0, 1)$; or a "corner" equilibrium allocation where $q_h = 1$ and $q_\ell \in (0, 1)$.

If the equilibrium allocation is interior, then the high– and low–valuation marginal utilities are equalized, implying that $q_\ell = \varepsilon q_h$, where

$$\varepsilon \equiv \left( \frac{\gamma_\ell}{r + \kappa + \gamma_\ell} \right)^{1/\sigma} \in (0, 1).$$

Substituting into the market clearing condition leads to equation (I.1). Using the formula for $q_h$, one finds that $q_h < 1$, if and only if

$$s < \frac{\gamma_h + \varepsilon \gamma_\ell}{\gamma_h + \gamma_\ell}. \tag{IX.5}$$

If the equilibrium allocation is at a corner, then $q_h = 1$ and the market clearing condition then implies the left part of equation (I.2) for the expression of $q_\ell$. For this allocation to be the basis of an equilibrium, we must have $v_q(h, 1) \geq v_q(\ell, 1)$, which after some simple algebra is easily shown to be equivalent to

$$s \geq \frac{\gamma_h + \varepsilon \gamma_\ell}{\gamma_h + \gamma_\ell}, \tag{IX.6}$$

which is the opposite of (IX.5), implying equilibrium uniqueness.

### IX.1.2 Proof of Proposition I.1

For Point $(i)$, note that $\varepsilon \to 1$ when $r + \kappa \to 0$. But when $\varepsilon$ is close enough to 1, condition (IX.5) implies that the equilibrium is interior since $s < 1$, therefore $q_\ell$ and $q_h$ are given by equations (I.1). In turns $\varepsilon \to 1$ then implies $q_\ell \to s$ and $q_h \to s$.

For point $(ii)$ note first that $\varepsilon$ decreases with $r + \kappa$ and that the equilibrium is interior if $r + \kappa$ is small enough. Therefore there exists $\overline{c}$ (possibly $+\infty$) such that the equilibrium is interior if $r + \kappa < \overline{c}$ and corner if $\kappa \geq \overline{c}$. When $\kappa < \overline{c}$, $q_\ell < s$ is strictly decreasing in $r + \kappa$, while $q_h > s$ and is strictly increasing in $r + \kappa$. Therefore the dispersion of asset holdings $q_h - q_\ell$ increases strictly with $r + \kappa$. When $r + \kappa \geq \overline{c}$, $q_h$ and $q_\ell$ do not depend on $r + \kappa$. Overall, this shows that the dispersion of asset holdings is weakly increasing in $r+\kappa$.

### IX.1.3 Proof of Proposition I.2

Trade volume is equal to the measure of high-valuation investors meeting a dealer, times the probability that a high–valuation investor was in the low state the last time he met a dealer, times the trade size $q_h - q_\ell$. The flow of high–valuation investors meeting a dealer, and who had a low–valuation at their last contact is:

$$\rho \int_0^{+\infty} \rho e^{-\rho t} \frac{\gamma_\ell}{\gamma_h+\gamma_\ell} \frac{\gamma_h}{\gamma_h+\gamma_\ell} \left(1 - e^{-(\gamma_h+\gamma_\ell)t}\right) \, dt;$$

where: $\rho$ is the flow of investors currently meeting dealers; $e^{-\rho t}\, dt$ is the probability density that an investor had her last contact time $t$ periods ago; $\gamma_\ell/(\gamma_h+\gamma_\ell)$ is that the investor's type was low $t$ periods ago; and $\gamma_h/(\gamma_h+\gamma_\ell)(1-e^{-(\gamma_h+\gamma_\ell)t})$ is the probability of being in the high type now, conditional on being in the low–type $t$ periods ago. Computing the integral we obtain that trade volume is equal to

$$\mathcal{V} = \rho \frac{\gamma_h}{\gamma_h+\gamma_\ell} \frac{\gamma_\ell}{\rho+\gamma_h+\gamma_\ell}(q_h - q_\ell).$$

Now, to prove point $(i)$, let $r+\kappa \to 0$. As shown in Proposition I.1, both $q_\ell \to s$ and $q_h \to s$, and so $q_h - q_\ell \to 0$. Since either $\eta$ or $\rho$ is fixed for the proposition, $\rho$ must remain bounded as $r+\kappa \to 0$. Taken together these remark imply that $\mathcal{V}$ vanished to 0.

For point $(ii)$, note that since $\rho(\gamma_h\gamma_\ell)/[(\gamma_h+\gamma_\ell)(\rho+\gamma_h+\gamma_\ell)]$ is strictly increasing in $\kappa$ for a given $\eta$, and constant for given $\rho$, and since $q_h - q_\ell$ is (weakly) increasing in $r+\kappa$ according to Proposition I.1, it follows that trade volume increases strictly with $\kappa$.

Lastly, for point $(iii)$, with two types the distribution of trade sizes is degenerate: all trades have the same size, $q_h - q_\ell$, which increases with $\kappa$.

### IX.1.4 Proof of Proposition I.3

**Proof of point** $(i)$**.** Note that, as in LR the fees are equal to:

$$\phi(\theta, q) = \frac{\eta}{r+\kappa} \left[\overline{v}(\theta, q_\theta) - \overline{v}(\theta, q) - rp(q_\theta - q)\right], \tag{IX.7}$$

a fraction $\eta$ of the trading surplus, for $\theta \in \{h, \ell\}$ and $q \geq 0$. Therefore

$$\frac{\partial}{\partial q}[\phi(\theta, q)] = \frac{\eta}{r+\kappa} \left[-\overline{v}_q(\theta, q) + rp\right]. \tag{IX.8}$$

Consider first the case $\theta = \ell$. Since $q_\ell \in (0,1)$, it satisfies the investors' first–order condition $\overline{v}_q(\theta, q_\ell) = rp$. Given that $q \mapsto \overline{v}'_\ell(q)$ is strictly positive and strictly decreasing over $q \in [0,1)$ and $\overline{v}_q(\ell, q) = 0$ for $q > 1$, $\partial/\partial q[\phi(\ell, q)]$ has the same sign as $q - q_\ell$.

Consider now the case $\theta = h$. The argument is the same as for $\theta = \ell$ if the equilibrium allocation is interior. If the equilibrium allocation is at a corner, $q_h = 1$ and $rp \in (0, \overline{v}_q(h,1)]$. But $q \mapsto \overline{v}_q(h,q)$ is strictly positive, strictly decreasing over $q \in [0,1)$, and such that $\overline{v}_q(h,q) = 0$ for $q > 1$. Direct verification shows, then, that $\partial/\partial q[\phi(h,q)]$ has the same sign as $q-1$.

Second, the derivative of fees per unit of asset trade with respect to the trade size is equal to

$$\begin{aligned}
\frac{\partial}{\partial q}\left[\frac{\phi(\theta,q)}{|q_\theta - q|}\right] &= \operatorname{sign}[q-q_\theta]\left\{\frac{\partial/\partial q[\phi(\theta,q)]}{q-q_\theta} - \frac{\phi(\theta,q)}{(q-q_\theta)^2}\right\} \\
&= \frac{\operatorname{sign}[q-q_\theta]}{(q-q_\theta)^2}\frac{\eta}{r+\kappa}\left\{-\overline{v}_q(\theta,q)(q-q_\theta) - \overline{v}(\theta,q_\theta) + \overline{v}(\theta,q)\right\},
\end{aligned}$$

where we have used equations (IX.7) and (IX.8) to move from the first line to the second line. To conclude, we note that, on the second line, the term is curly brackets is always positive since $q \mapsto \overline{v}(\theta,q)$ is concave.

**Proof of point** $(ii)$**.** We proceed as in LR to show that $\phi(\theta, q_{\theta'})$, $\theta \neq \theta'$ in $\{h,\ell\}$, is non monotonic in $\kappa$ for $r$ small enough. Specifically, we denote by $\phi_{\theta',\theta}(\kappa,r)$ the equilibrium fee $\phi(\theta,q_{\theta'})$ as a function of the parameters $\kappa$ and $r$, and we show that for any $\kappa > 0$ there exists $\overline{r}$ such that for all $r < \overline{r}$ and all $\theta \neq \theta'$ in $\{h,\ell\}$:

(a) $\phi_{\theta',\theta}(\kappa,r) > \overline{\phi}_{\theta',\theta}(\kappa)$ for some strictly positive function $\overline{\phi}(\kappa) > 0$;

(b) $\phi_{\theta',\theta}(\kappa',r) \to 0$ as $\kappa' \to +\infty$;

(c) $\phi_{\theta',\theta}(0,r) \to 0$ as $r \to 0$.

Clearly (a)-(b)-(c) imply point $(ii)$: by (a) and (b) we have that $\phi(\kappa,r) > \lim_{\kappa\to\infty}\phi_{\theta',\theta}(\kappa,r) = 0$; and by (a) and (c), we have that for $r$ small enough, $\phi_{\theta',\theta}(0,r) < \overline{\phi}(\kappa) < \phi(\kappa,r)$.

To prove these three points we introduce the notations $\varepsilon(\kappa,r)$, $q_\theta(\kappa,r)$, $p(\kappa,r)$, and $\overline{v}(\theta,q;\kappa,r)$, to stress that all these functions depend on $\kappa$ and $r$. Note that these functions are all well defined, continuous when $r=0$ or $\kappa=0$.

**Proof of point** $(ii)$**–(a).** Consider $\kappa > 0$ and some arbitrary $\bar{r}$. Substituting the first–order condition $rp(\kappa,r) = \overline{v}_q(\ell, q_\ell(\kappa,r);\kappa,r)$, when $\theta = \ell$, and $rp(\kappa,r) \leq \overline{v}_q(h,q_h(\kappa,r);\kappa,r)$, when $\theta = h$, into equation (IX.7) we can write

$$\begin{aligned}
\phi_{\theta',\theta}(\kappa,r) &\geq \frac{\eta}{r+\kappa}\int_{q_{\theta'}(\kappa,r)}^{q_\theta(\kappa,r)}\left[\overline{v}_q(\theta,q;\kappa,r) - \overline{v}_q(\theta,q_\theta(\kappa,r);\kappa,r)\right] \\
&\geq \frac{\eta}{r+\kappa}\inf\{|\overline{v}_{qq}(\theta,q;\kappa,r)| : q \in [q_\ell(\kappa,r);q_h(\kappa,r)]\}\frac{(q_h(\kappa,r)-q_\ell(\kappa,r))^2}{2} \\
&\geq \frac{\eta}{r+\kappa}\inf\{|\overline{v}_{qq}(\theta,q;\kappa,r)| : q \in [q_\ell(\kappa,r);q_h(\kappa,r)]\}\frac{(q_h(\kappa,0)-q_\ell(\kappa,0))^2}{2} \\
&\geq \frac{\eta}{\overline{r}+\kappa}\inf\{|\overline{v}_{qq}(h,q;\kappa,\overline{r})| : q \in [q_\ell(\kappa,0);q_h(\kappa,0)]\}\frac{(q_h(\kappa,0)-q_\ell(\kappa,0))^2}{2} \equiv \overline{\phi}(\kappa),
\end{aligned}$$

where $v_{qq}(\theta, q)$ denote the second derivative of $\overline{v}(\theta, q)$ with respect to $q$. The third line obtains because, as shown in point $(ii)$ of Proposition I.1, $q_h(\kappa, r) - q_\ell(\kappa, r)$ increases with $r + \kappa$. The last line obtains because direct calculations show that $|\overline{v}_{qq}(\theta; \kappa, r)| \geq |\overline{v}_{qq}(h; \kappa, \overline{r})|$, and because we showed in the Proof of Proposition I.1 that $q_h(\kappa, r)$ ($q_\ell(\kappa, r)$) is increasing (decreasing) in $r + \kappa$.

Note that $\varepsilon(\kappa, 0) > 0$ and therefore $q_\ell(\kappa, 0) > 0$, implying that

$$\inf\{|\overline{v}_{qq}(h, q; \kappa, \overline{r})| : q \in [q_\ell(\kappa, 0); q_h(\kappa, 0)]\} > 0.$$

Besides, inspection of the formulas given in Lemma I.1 show that $q_h(\kappa, 0) - q_\ell(\kappa, 0) > 0$. Therefore $\overline{\phi}(\kappa) > 0$.

**Proof of point** $(ii)$**–(b).** Let $\kappa \to \infty$. By inspection of equations (I.1)-(I.2), $q_h(\kappa, r)$, $q_\ell(\kappa, r)$ and $p(\kappa, r)$ all have finite limits and $q \mapsto \overline{v}(\theta, q; \kappa, r)$ converges uniformly towards a bounded function. Therefore the term in brackets in equation (IX.7) goes to a finite limit and $\phi_{\theta', \theta}(\kappa, r)$ goes to 0.

**Proof of point** $(ii)$**–(c).** Consider $\kappa > 0$. When $r$ goes to 0, $\varepsilon(0, r)$ goes to 1, the equilibrium indexed by $(0, r)$ is interior by condition (IX.5), $q_\ell(0, r)$ and $q_h(0, r)$ are given by equation (I.1) and they both converge towards $s$. Furthermore, taking derivatives with respect to $r$ at $r = 0$ we obtain

$$\frac{\partial \varepsilon}{\partial r}(0, 0) = -\frac{1}{\sigma \gamma_\ell}, \qquad \frac{\partial q_\ell}{\partial r}(0, 0) = -\frac{s}{\sigma} \frac{1}{\gamma_h + \gamma_\ell} \frac{\gamma_h}{\gamma_\ell}, \qquad \frac{\partial q_h}{\partial r}(0, 0) = \frac{s}{\sigma} \frac{1}{\gamma_h + \gamma_\ell}.$$

And therefore,

$$q_h(0, r) - q_\ell(0, r) = \frac{1}{\sigma \gamma_\ell} r + o(r). \tag{IX.9}$$

Note also that, after plugging $v(\theta, q) - v(\theta, q') = v_q(\theta, q)(q - q') + o(q - q')$ into the formulas for $\overline{v}(\theta, q)$, it appears that

$$\overline{v}(\theta, q; 0, r) - \overline{v}(\theta, q'; 0, r) = \overline{v}_q(\theta, q; 0, r)(q - q') + K(r) o(q - q'). \tag{IX.10}$$

where $K(r)$ is a bounded function. Now evaluating the fee when $\kappa = 0$, we obtain:

$$\begin{aligned}
\phi_{\theta', \theta}(r, 0) =& \frac{\eta}{r} \Big\{ \overline{v}(\theta, q_\theta(0, r); 0, r) - \overline{v}(\theta', q_{\theta'}(0, r); 0, r) - rp(0, r) \left[ q_\theta(0, r) - q_{\theta'}(0, r) \right] \Big\} \\
=& \frac{\eta}{r} \Big\{ \left[ \overline{v}_q(\theta, q_\theta(0, r); 0, r) - rp(0, r) \right] \left[ q_\theta(0, r) - q_{\theta'}(0, r) \right] + K(r) o\big( q_\theta(0, r) - q_{\theta'}(0, r) \big) \Big\} \\
=& \frac{\eta}{r} K(r) o\big(r\big) = o(1),
\end{aligned}$$

which goes to zero as $r \to 0$. In the above, we move from the first to the second line by plugging the Taylor approximation (IX.10). We move from the second to the third line by plugging in the Taylor approximation (IX.9) into the second "little-$o$" term, and by noting that the first term is equal to zero since the equilibrium is interior for $r$ close to zero and therefore $\overline{v}_q(\theta, q_\theta(0, r); 0, r) = rp(0, r)$.

Therefore, the fee $\phi_{\theta',\theta}(0,r)$ converges to zero as $r$ goes to zero.

**Proof of point** $(iii)$**.** The expected fee earned by a dealer conditional on meeting an investor is equal to the probability that the investor has a high type and had a low type the last time he met a dealer (we computed that probability in the proof of Proposition I.2) times the fee $\phi_h(q_\ell)$, plus the same thing inverting 'high' and 'low'

$$\Phi = \frac{\gamma_h}{\gamma_h+\gamma_\ell}\frac{\gamma_\ell}{\rho+\gamma_h+\gamma_\ell}\phi_h(q_\ell) + \frac{\gamma_\ell}{\gamma_h+\gamma_\ell}\frac{\gamma_h}{\rho+\gamma_h+\gamma_\ell}\phi_\ell(q_h).$$

The proof then follows the same steps as the proof of point $(ii)$.

## IX.2 Proof for Section III

### IX.2.1 Proof of Proposition III.1

We verify optimality, market clearing, and then prove some elementary properties of equilibrium objects.

**High–valuation optimality.** First, we have that $rp_u^* - \dot{p}_u^* \in (0,1)$ for all $u \in (0,T_f)$, so the optimality of high–valuation traders holding plan follows from the same proof as in Section A.8.3 in BHW.

**Low–valuation optimality.** As in BHW, consider a low–valuation trader who experiences an information event at time $t$. The trader's holding plan is optimal if it maximizes:

$$\begin{aligned}\mathbb{E}\left[v(\theta_u,q_{t,u})\,|\,\theta_t=\ell\right] - \xi_u^* q_{t,u} &= q_{t,u} - \delta\frac{1-\mu_{hu}}{1-\mu_{ht}}q_{t,u} - \left(1-\delta\frac{1-\mu_{hu}}{1-\mu_{h\psi_u^*}}\right)q_{t,u} \\ &= \delta(1-\mu_{hu})q_{t,u}\left(\frac{1}{1-\mu_{h\psi_u^*}} - \frac{1}{1-\mu_{ht}}\right) \\ &= \delta(1-\mu_{hu})q_{t,u}\frac{\mu_{h\psi_u^*}-\mu_{ht}}{(1-\mu_{hu})(1-\mu_{h\psi_u^*})}.\end{aligned}$$

Clearly, this expression is maximized by $q_{t,u}^* = 0$ if $t > \psi_u^*$, and by $q_{t,u}^* = 1$ if $t \le \psi_u^*$.

**Market clearing.** Consider some time $u \in (0,T_f]$. Then all high–valuation traders who had an information event at $t \le u$ hold one unit, while all low–valuation traders who had an information event at some time $t \le \psi_u^*$ hold one unit. Plugging these asset holdings into the market–clearing condition (8) of BHW, we obtain:

$$\int_0^{\psi_u^*} \rho e^{-\rho(u-t)}(1-\mu_{ht})\,dt + \int_0^u \rho e^{-\rho(u-t)}\left[\mu_{ht}-s\right]\,dt = 0,$$

which clearly holds by definition of $\psi_u^*$.

**The function $\psi_u^*$ is hump-shaped.** After canceling out $\rho e^{-\rho u}$ from both sides of the equation defining $\psi_u^*$, we find that $\psi_u^*$ solves:

$$\int_0^{\psi} e^{\rho t}\left(1-\mu_{ht}\right) dt = \int_0^{u} e^{\rho t}\left(s-\mu_{ht}\right) dt.$$

Note that because $u \leq T_f$, the right–hand side is strictly positive. Moreover, the left–hand side is strictly increasing, zero at $\psi = 0$, and clearly greater than the right–hand side at $\psi = u$. Therefore, the above equation has a unique solution, $\psi = \psi_u^*$, and this solution is strictly less than $u$. Since the equation is continuously differentiable with a non-zero derivative at $\psi = \psi_u^*$, we can apply the Implicit Function Theorem and obtain that $\psi_u^*$ is continuously differentiable, with: $\psi_u^{*\prime} e^{\rho\psi_u^*}\left(1-\mu_{h\psi_u^*}\right) = e^{\rho u}\left(s-\mu_{hu}\right)$. In particular, $\psi_u^{*\prime} > 0$ if $u \in [0, T_s)$ and $\psi_u^{*\prime} < 0$ if $u \in (T_s, T_f]$: i.e., $\psi_u^*$ is hump–shaped with a maximum at $u = T_s$. Moreover, by definition of $T_f$, we have $\psi_{T_f}^* = 0$.

Next, plugging the functional form $\mu_{ht} = 1 - e^{-\gamma t}$ into the equation defining $\psi_u^*$, we obtain that $\psi_u^*$ solves:

$$\frac{1}{\rho-\gamma}\left[1-e^{(\rho-\gamma)(\psi-u)}\right] = (1-s)e^{\gamma u}\frac{1-e^{-\rho u}}{\rho}.$$

The left-hand size is a strictly decreasing function of $\psi - u$, but the right-hand side is a strictly increasing function of $u$. It follows, then, that $d/du\left[\psi_u^* - u\right] = \dot{\psi}_u^* - 1 < 0$.

**The price is strictly increasing.** The ODE for the price is:

$$rp_u^* = 1 - \delta\frac{1-\mu_{hu}}{1-\mu_{h\psi_u^*}} + \dot{p}_u^*.$$

Note that, at $u = T_f$, $rp_u^* = 1$ and $\psi_{T_f}^* = 0$, so $\dot{p}_{T_f^-}^* = \delta(1-\mu_{hu})/(1-\mu_{h\psi_u^*}) > 0$. For $u < T_f$, we let $d_u^* \equiv \dot{p}_u^*$. Differentiating the ODE for $p_u^*$, and plugging the functional form $\mu_{hu} = 1 - e^{-\gamma u}$, we obtain:

$$rd_u^* = \delta\gamma\left(1-\dot{\psi}_u^*\right)e^{-\gamma(u-\psi_u^*)} + \dot{d}_u^*.$$

And so:

$$d_t^* = \delta\gamma\int_t^{T_f}\left(1-\dot{\psi}_u^*\right)e^{-\gamma(u-\psi_u^*)}e^{-r(u-t)}\,du + e^{-r(T_f-t)}d_{T_f^-}^*.$$

But we showed in the previous paragraph that that $\dot{\psi}_u^* < 1$, and just established that $\dot{p}_{T_f^-}^* = d_{T_f^-}^* > 0$. Thus, it follows that $d_t^* > 0$ for all $t \in [0, T_f)$.

### IX.2.2 Proof of Proposition III.2

First we have the following preliminary result:

**RIX.1.** *For all $u > 0$, $\int_0^u e^{\rho t}(1-\mu_{ht})^{1+1/\sigma}\,dt \to 0$ as $\sigma \to 0$.*

Indeed, the function $e^{\rho t}(1-\mu_{ht})^{1+/\sigma}$ is bounded above by $e^{\rho t}$, and it converges pointwise to zero for $t>0$. The result then follows by an application of the dominated convergence Theorem. From the above result, it follows immediately that:

**RIX.2.** *For all $u \in (0, T_f)$, $\overline{Q}_u \to \infty$, as $\sigma \to 0$, where the function $\overline{Q}_u$ is defined in equation (A.4), page 36 in BHW.*

Next let us recall two useful notations of Section , page 79 in BHW: $\Psi(Q) \equiv \inf\{\psi \geq 0 : (1-\mu_{h\psi})^{1/\sigma} Q \leq 1\}$ and $\psi_u \equiv \Psi(Q_u)$. Note that $\psi_u < u$ because otherwise $(1-\mu_{hu})^{1/\sigma} Q_u \geq 1$ and the market clearing condition (17), page 18 in BHW, would be violated. We now show that:

**RIX.3.** *For all $u \in (0, T_f)$, $\psi_u > 0$ as long as $\sigma$ is close enough to zero.*

Indeed, for $\sigma$ close enough to zero, we have that $\overline{Q}_u > 1$, which by Lemma A.3, page 36 in BHW, implies that $Q_u > 1$. By definition of $\Psi(Q)$, this implies that $\Psi(Q_u) = \psi_u > 0$. Note that when $\psi_u > 0$, then by definition of $\Psi(Q)$ we have that $(1-\mu_{h\psi_u})^{1/\sigma} Q_u = 1 \Leftrightarrow Q_u = (1-\mu_{h\psi_u})^{-1/\sigma}$ and, by equation (16), page 18 in BHW:

$$q_{t,u} = \min\left\{\left(\frac{1-\mu_{ht}}{1-\mu_{h\psi_u}}\right)^{1/\sigma}, 1\right\}, \tag{IX.11}$$

for a low–valuation trader who experienced an information event at time $t$. Next we show that:

**RIX.4.** *For all $u \in (0, T_f)$, $\psi_u \to \psi_u^*$ as $\sigma \to 0$, where $\psi_u^*$ is the function defined in Proposition III.1.*

To see this, take $\sigma$ small enough so that $\psi_u > 0$. Then, note that by definition of $\psi_u$ the market–clearing condition (17), page 18 in BHW, can be rewritten:

$$\int_{\psi_u}^{u} \rho e^{-\rho(u-t)} \left(\frac{1-\mu_{ht}}{1-\mu_{h\psi_u}}\right)^{1/\sigma} dt + \int_0^{\psi_u} \rho e^{-\rho(u-t)} (1-\mu_{ht})\, dt = \int_0^u \rho e^{-\rho(u-t)} (s-\mu_{ht})\, dt. \tag{IX.12}$$

Now for any sequence of $\sigma$ converging to zero, the associated sequence of $\psi_u$ belongs to the compact $[0, u]$ so it has at least one converging subsequence. Denote this subsequence by $\psi_u^n$, its limit by $\psi_u^\infty$, and the associated subsequence of $\sigma$ by $\sigma^n$. Looking at the first–integral on the left–hand side of (IX.12), one sees that the function:

$$\mathbb{I}_{\{t\in[\psi_u^n, t]\}} e^{-\rho(u-t)} \left(\frac{1-\mu_{ht}}{1-\mu_{h\psi_u^n}}\right)^{1/\sigma^n} (1-\mu_{ht})$$

is bounded above by $e^{-\rho(u-t)}$ and converges to zero everywhere except perhaps at $t = \psi_u^\infty$. Thus an application of the dominated convergence Theorem implies that the first term on the left–hand side of (IX.12) goes to zero as $n \to \infty$. Going to the limit $n \to \infty$ in the other terms of (IX.12), one finds that $\psi^\infty$ solves:

$$\int_0^{\psi_u^\infty} \rho e^{-\rho(u-t)}(1-\mu_{ht})\,dt = \int_0^u \rho e^{-\rho(u-t)}(s-\mu_{ht})\,dt$$

and is thus equal to $\psi_u^*$. Thus, all convergent subsequences of $\psi_u$ the same limit, $\psi_u^*$, as $\sigma \to 0$. This implies that $\psi_u \to \psi_u^*$. The next convergence result concerns asset holding plans:

**RIX.5.** *For all $u \in (0,T_f)$ and for all $t \in (0,u)$, the holding of a time–$t$ low–valuation trader converges to $q_{t,u} = \mathbb{I}_{\{t \leq \psi_u^*\}}$ as $\sigma \to 0$.*

This result follows directly from equation (IX.11) given that $\psi_u \to \psi_u^*$. Note that the rest of the holding plans are identical in BW and BHW: the asset holding plan of low–valuation traders at $u \geq T_f$, the asset holding plan of high–valuation $u \leq T_f$, and the average asset holding plan of high–valuation traders at $u > T_f$. Turning the price path in BHW, we have, when $\sigma$ is small enough and $u \leq T_f$:

$$rp_u - \dot{p}_u = 1 - \delta(1-\mu_{hu})Q_u^\sigma = 1 - \delta\frac{1-\mu_{hu}}{1-\mu_{h\psi_u}}$$

which, since $\psi_u$ converges to $\psi_u^*$, clearly converges to $rp_u^* - \dot{p}_u^*$ as $\sigma \to 0$. For $u > T_f$, we have that $rp_u = rp_u^* = 1$. Integrating up and applying the dominated convergence Theorem leads to:

**RIX.6.** *For all $u$, $p_u \to p_u^*$ as $\sigma \to 0$.*

### IX.2.3 Proof of Proposition III.4

From equation (A.16) page 41 in BHW it follows that:

**RIX.7.** *As $\sigma \to 0$, $T_\phi \to T_s$.*

Let us turn, then to the defining equation of $\phi_t$, (A.17) page 41 in BHW. Dividing through by $(1-\mu_{ht})^{1+1/\sigma}$ and applying the same reasoning as in the proof of R.IX.4, we obtain that:

**RIX.8.** *For all $t \in [0,T_s)$, $\phi_t \to \phi_t^*$ as $\sigma \to 0$, where $\phi_t^*$ is the function defined in Proposition III.3. Similarly, for all $u \in (T_s,T_f]$, $\phi_u^{-1} \to \phi_u^{*-1}$ as $\sigma \to 0$.*

Consider, then, the limiting holding plan of a low–valuation trader. For $t \in (0,T_s)$ and $u < \phi_t^*$: by R.IX.8 , we have that $u < \phi_t$ for $\sigma$ small enough, and hence that $q_{t,u} = (s-\mu_{ht})/(1-\mu_{ht})$. For $t \in (0,T_s)$ and $u > \phi_t^*$: by R.IX.8 we have that $u > \phi_t > T_\phi$ for $\sigma$ small enough. Hence, $\xi_u$ is given by equation (A.19) page 42 in BHW, and

$$q_{t,u} = (1-\mu_{ht})^{1/\sigma}Q_u = (1-\mu_{ht})^{1/\sigma}\left(\frac{1-\xi_u}{\delta(1-\mu_{hu})}\right)^{1/\sigma} = \left(\frac{1-\mu_{ht}}{1-\mu_{h\phi_u^{-1}}}\right)^{1/\sigma}\frac{s-\mu_{h\phi_u^{-1}}}{1-\mu_{h\phi_u^{-1}}}.$$

But, since $u > \phi_t^*$ and since $\phi_t^*$ is strictly decreasing, we have that $\phi_u^{*-1} < t$. Hence, as $\sigma \to 0$, the ratio $(1-\mu_{ht})/(1-\mu_{h\phi_u^{-1}})$ converges to a limit that is strictly less than 1, implying that $q_{t,u}$ converges to zero. Lastly, consider some $t \in (T_s, T_f)$ and $u \in (t, T_f)$. Then for $\sigma$ small enough, we have that $t \in (T_\phi, \phi_0)$ and so $q_{t,u}$ is given by the same equation as above, and we also obtain that $q_{t,u}$ converges to zero. Lastly, we note that the rest of the holding plans are identical in BW and BHW: the asset holding plan of low–valuation traders at $u \geq T_f$, the asset holding plan of high–valuation $u \leq T_f$, and the average asset holding plan of high–valuation traders at $u > T_f$.

The last thing to verify is that the price path converges. For $t > T_f$, we have that $rp_t = rp_t^* = 1$. For $t \in (T_s, T_f)$, we can go to the limit in equation (A.19), page 42 in BHW, and we find that $rp_t - \dot{p}_t \to rp_t^* - \dot{p}_t^*$. For $t \in (0, T_s)$, note that:

$$\frac{d}{dt}\left[\left(\frac{s-\mu_{ht}}{(1-\mu_{ht})^{1+1/\sigma}}\right)^{\sigma}\right] = \mu_{ht}'\left[-\sigma\frac{(s-\mu_{ht})^{\sigma-1}}{(1-\mu_{ht})^{1+\sigma}} + (1+\sigma)\frac{(s-\mu_{ht})^{\sigma}}{(1-\mu_{ht})^{2+\sigma}}\right]$$
$$\to \frac{\mu_{ht}'}{(1-\mu_{ht})^2} = \frac{d}{dt}\left[\frac{1}{1-\mu_{ht}}\right]$$

as $\sigma \to 0$. Going to the limit in equation (A.18), page 42 in BHW, we find that $rp_t - \dot{p}_t \to rp_t^* - \dot{p}_t^*$. Integrating up and applying the dominated convergence Theorem, we find that $p_t \to p_t^*$ for all $t$.

## IX.3 Proofs and Calculations for Section IV

### IX.3.1 Calculations of equation (IV.2) and Proof of Lemma IV.1

**Proof of Lemma IV.1.** The function $W(q)$ is the continuation value of a trader who holds $q$ units of the asset from the beginning of a liquidity shock until her next information event. Let $\tau_\kappa$ denote the random time of the next liquidity shock, and $\tau_\rho$ the random time of the next information event. By our maintained distributional assumption, $\tau_\kappa$ and $\tau_\rho$ are independent random exponential times with respective intensities $\kappa$ and $\rho$. As BHW, we calculate values net of the cost of buying and selling the asset. With this accounting convention in mind, we write:

$$\begin{aligned} W(q) = -p_0 q + \mathbb{E}\bigg[\int_0^{\tau_\rho\wedge\tau_\kappa} & e^{-ru} v(\theta_u, q)\,du + \mathbb{I}_{\{\tau_\rho<\tau_\kappa\}} e^{-r\tau_\rho} p_{\tau_\rho} q \\ & + \mathbb{I}_{\{\tau_\kappa<\tau_\rho\}} e^{-r\tau_\kappa}\Big\{p_0 q + W(q)\Big\} \,\Big|\, \theta_0 = \ell\bigg]. \end{aligned} \tag{IX.13}$$

The first term, $-p_0 q$, is the cost purchasing the asset at $u = 0$.[13] The second term is the present value of utility flows until either the next information event, $\tau_\rho$, or the next liquidity shock, $\tau_\kappa$, which ever comes first. The third term is the re-selling value of the asset at the next information event, at price $p_{\tau_\rho}$, if the information event occurs before the next liquidity shock. The fourth terms is the re-selling value of the asset and the continuation value at the next liquidity shock, if the next liquidity shock

[13] This term appears because of our accounting convention of calculating values net of holding cost.

occurs before the next information event.[14] Next, we observe that:

$$p_{\tau_\rho\wedge\tau_\kappa}e^{-r\tau_\rho\wedge\tau_\kappa} = \mathbb{I}_{\{\tau_\rho<\tau_\kappa\}}e^{-r\tau_\rho}p_{\tau_\rho} + \mathbb{I}_{\{\tau_\kappa<\tau_\rho\}}e^{-r\tau_\kappa}p_{\tau_\kappa}. \tag{IX.14}$$

Substituting the resulting expression for $\mathbb{I}_{\{\tau_\rho<\tau_\kappa\}}e^{-r\tau_\kappa}p_{\tau_\kappa}$ into (IX.13), we find:

$$\begin{aligned}
W(q) &= \mathbb{E}\left[\int_0^{\tau_\rho\wedge\tau_\kappa} e^{-ru}v(\theta_u,q)\,du - p_0 q + e^{-r\tau_\rho\wedge\tau_\kappa}p_{\tau_\rho\wedge\tau_\kappa}q + \mathbb{I}_{\{\tau_\kappa<\tau_\rho\}}e^{-r\tau_\kappa}\left\{(p_0-p_{\tau_\kappa})q + W(q)\right\}\,\middle|\,\theta_0=\ell\right]\\
&= \mathbb{E}\left[\int_0^{\tau_\rho\wedge\tau_\kappa} e^{-ru}\left\{v(\theta_u,q)-(rp_u-\dot p_u)q\right\}du + \mathbb{I}_{\{\tau_\kappa<\tau_\rho\}}e^{-r\tau_\kappa}\left\{(p_0-p_{\tau_\kappa})q + W(q)\right\}\,\middle|\,\theta_0=\ell\right]\\
&= \int_0^\infty e^{-(r+\rho+\kappa)u}\left\{\mathbb{E}\left[v(\theta_u,q)\,|\,\theta_0=\ell\right] - \xi_u q + \kappa W(q)\right\}du
\end{aligned}$$

where the second line follows from noting that $-p_0 + e^{-rt}p_t = \int_0^t e^{-ru}\left(\dot p_u - rp_u\right)du$, and the third line follows after integrating against the exponential probability densities of the independent arrival times $\tau_\rho$ and $\tau_\kappa$, and using the definition of $\xi_u$. Note that the last term can be integrated directly: $\int_0^\infty e^{-(r+\rho+\kappa)u}\kappa W(q)\,du = \kappa/(r+\rho+\kappa)W(q)$. Moving this term to the left–hand side and rearranging, we find, after differentiating with respect to $q$:

$$W_q(q) = \frac{r+\rho+\kappa}{r+\rho}\int_0^\infty e^{-(r+\rho+\kappa)u}\left\{\mathbb{E}\big[v_q(\theta_u,q_u)\,|\,\theta_0=\ell\big] - \xi_u\right\}du.$$

To obtain the formula of the lemma, note that the first term can be integrated explicitly given that $\mathbb{E}\big[v(\theta_u,q_u)\,|\,\theta_0=\ell\big] = q - \delta e^{-\gamma u}q^\sigma$ if $q\le 1$ and zero otherwise.

**Calculation of equation (IV.2).** The calculations are similar to the one for $W(q)$. First the inter-information-event utility writes, net of cost:

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_t^{\tau_\kappa\wedge\tau_\rho} e^{-r(u-t)}v(\theta_u,q_{t,u}) - p_t q_{t,t} - \int_t^{\tau_\kappa\wedge\tau_\rho} p_u dq_{t,u} + \mathbb{I}_{\{\tau_\rho<\tau_\kappa\}}e^{-r(\tau_\rho-t)}p_{\tau_\rho}q_{t,\tau_\rho}\\
&\quad + \mathbb{I}_{\{\tau_\kappa<\tau_\rho\}}e^{-r(\tau_\kappa-t)}\left\{p_0 q_{t,\tau_\kappa} + W(q_{t,\tau_\kappa})\right\}\,\Bigg|\,\theta_t\Bigg],
\end{aligned}$$

After substituting in equation (IX.14), the above can be re-written:

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_t^{\tau_\kappa\wedge\tau_\rho} e^{-r(u-t)}v(\theta_u,q_{t,u})\,du - p_{t,t}q_{t,t} - \int_t^{\tau_\kappa\wedge\tau_\rho} p_u dq_{t,u}e^{-r(u-t)} + e^{-r(\tau_\kappa\wedge\tau_\rho-t)}p_{\tau_\kappa\wedge\tau_\rho}\\
&+e^{-r(\tau_\kappa-t)}\mathbb{I}_{\{\tau_\kappa<\tau_\rho\}}\left\{W(q_{t,\tau_\kappa}) + p_0 q_{t,\tau_\kappa} - p_{\tau_\kappa}q_{t,\tau_\kappa}\right\}\,\Bigg|\,\theta_t\Bigg].
\end{aligned}$$

[14] One may wonder why we account for the re-selling-value of the asset at time $\tau_\kappa$, $p_0 q$, even though no asset is physically sold when the liquidity shock hits. The reason is that we need to cancel out the $-p_0 q$ term which is, by our accounting convention, included in the continuation value $W(q)$.

after integrating by part the second integral on the first line, we obtain:

$$\mathbb{E}\left[\int_t^{\tau_\kappa\wedge\tau_\rho} e^{-r(u-t)}\left\{v(\theta_u,q_{t,u})-(rp_u-\dot{p}_u)q_{t,u}\right\}du+e^{-r(\tau_\kappa-t)}\mathbb{I}_{\{\tau_\kappa<\tau_\rho\}}\left\{W(q_{t,\tau_\kappa})+p_0q_{t,\tau_\kappa}-p_{\tau_\kappa}q_{t,\tau_\kappa}\right\}\,\middle|\,\theta_t\right].$$

Formula (IV.2) follows after integrating against the exponential densities of the independent random times $\tau_\rho$ and $\tau_\kappa$.

### IX.3.2 Calculation of $Q_{h,u}$

**Solving for $Q_{h,u}$.** First we note that, if $Q_{h,u} > \alpha_{u,u}^{-1}$, then both $Q_{h,u} \geq 1$ and $\alpha_{t,u}Q_{h,u} \geq 1$, and so equation (IV.6) cannot hold. Therefore, $Q_{h,u} \in [0, \alpha_{u,u}^{-1}]$. But, over this interval, the left–hand side of (IV.6) is strictly increasing, negative for $Q_u = 0$, and strictly positive for $Q_u = \alpha_{u,u}^{-1}$. We conclude that equation (IV.6) has a unique solution. Keeping in mind that from Lemma IV.2, equilibrium asset holdings lie in $[0,1]$, we we can solve for $Q_{h,u}$ in three steps.

**Step 1: suppose $q_{h,u} < 1$.** Then, since $\alpha_{t,u} < 1$, $q_{\ell,t,u} < 1$ as well. Hence, the market–clearing condition (IV.6) implies:

$$Q_{h,u} = \frac{\int_0^u \rho e^{-\rho(u-t)} s\, dt}{\int_0^u \rho e^{-\rho(u-t)}\left\{\mu_{ht} + (1-\mu_{ht})\alpha_{t,u}\right\} dt}. \tag{IX.15}$$

Conversely, if the above $Q_{h,u}$ is less than 1, we have found the equilibrium $Q_{h,u}$. Otherwise, we move to the next step.

**Step 2: suppose $q_{hu} = 1$ and $q_{\ell,0,u} < 1$.** Then, given that $\alpha_{t,u} \leq \alpha_{0,u}$, we have $q_{\ell,t,u} < 1$ for all $t$. Hence, the market–clearing condition (IV.6) implies:

$$Q_{h,u} = \frac{\int_0^u \rho e^{-\rho(u-t)}(s-\mu_{ht})\, dt}{\int_0^u \rho e^{-\rho(u-t)}(1-\mu_{ht})\alpha_{t,u}\, dt}.$$

Conversely, if the above $Q_{h,u}$ is less than 1, we have found the equilibrium $Q_{h,u}$. Otherwise, we move to the last step.

**Step 3.** Having excluded the configuration of Step 1 and Step 2, we know that, in equilibrium, $q_{h,u} = 1$ and $q_{\ell,0,u} = 1$ . Note that $q_{\ell,u,u} < 1$ otherwise the market would not clear. Thus, there is some $\psi_u < u$ such that $q_{\ell,t,u} = 1$ for all $t > \psi_u$ and $q_{\ell,t,u} = 1$ for all $t \leq \psi_u$. For $t = \psi_u$, $\alpha_{\psi_u,u}Q_{h,u} = 1$. The market–clearing condition delivers $Q_{h,u}$ and $\psi_u$:

$$Q_{h,u} = \alpha_{\psi_u,u}^{-1} = \frac{\int_0^u \rho e^{-\rho(u-t)}(s-\mu_{ht})\, dt - \int_0^{\psi_u} \rho e^{-\rho(u-t)}(1-\mu_{ht})\, dt}{\int_{\psi_u}^u \rho e^{-\rho(u-t)}(1-\mu_{ht})\alpha_{t,u}\, dt}.$$

### IX.3.3 Proof of Lemma IV.4

We start by calculating $C$. We multiply both sides of equation (IV.7) by $e^{-(r+\rho+\kappa)u}$ and integrate from 0 to $\infty$. Using the definition of the constant $C$, in Lemma IV.1, this leads to:

$$\frac{r+\rho}{r+\rho+\kappa}C = \frac{1}{r+\rho} - \frac{\delta\kappa(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)}\int_0^\infty e^{-(r+\rho+\kappa)u}\, Q^\sigma_{h,u}\, du - \frac{\kappa}{r+\rho+\kappa}C$$

Solving for $C$ leads to:

$$C = \frac{1}{r+\rho} - \frac{\delta\kappa(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)}\int_0^\infty e^{-(r+\rho+\kappa)u}\, Q^\sigma_{h,u}\, du.$$

Plugging this back into equation (IV.7), we obtain:

$$\xi_u = 1 - \frac{\delta\kappa(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)}\left[Q^\sigma_{h,u} - \int_0^\infty \kappa e^{-(r+\rho+\kappa)z} Q^\sigma_{h,z}\, dz\right].$$

Now integrating the ODE $(r+\kappa)p_u = \xi_u + \kappa p_0 + \dot{p}_u$, we obtain:

$$\begin{aligned}
& p_0 = \int_0^\infty e^{-(r+\kappa)u}\xi_u\, du + \frac{\kappa}{r+\kappa}p_0 \\
\Leftrightarrow \quad & rp_0 = \int_0^\infty e^{-(r+\kappa)u}(r+\kappa)\xi_u\, du \\
\Leftrightarrow \quad & rp_0 = 1 - \frac{\delta\kappa(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)}\int_0^\infty \left[(r+\kappa)e^{-(r+\kappa)u} - \kappa e^{-(r+\rho+\kappa)u}\right] Q^\sigma_{h,u}\, du
\end{aligned}$$

where, in the first equality, $\lim_{u\to+\infty} e^{-(r+\kappa)u}p_u = 0$ follows from the no–bubble assumption, and the last equality follows from plugging in the above expression for $\xi_u$. The Lemma follows after plugging both the expression for $\xi_u$ and $p_0$ into (IV.7).

### IX.3.4 Approximation of $Q_{h,u}$ for large $u$

In this section we compute a first-order approximation for $u > T_{\max}$. Using equation (IX.15), is straightforward to verify that $Q_{h,u} < 1$ for $u > T_f$. We then consider $T_{\max} > T_f$ so that $Q_{h,u}$ is given by equation (IX.15) and for now consider $\rho > \gamma$. The denominator of (IX.15) is, using the formula (IV.5) for $\alpha_{t,u}$:

$$\begin{aligned}
& \int_0^u \rho e^{-\rho(u-t)}\left[\mu_{ht} + (1-\mu_{ht})\left(1 + \frac{r+\rho+\gamma}{\kappa}\frac{1-\mu_{hu}}{1-\mu_{ht}}\right)^{-1/\sigma}\right] dt \\
= & \int_0^u \rho e^{-\rho t}\left[1 - e^{-\gamma(u-t)} + e^{-\gamma(u-t)}\left(1 + \frac{r+\rho+\gamma}{\kappa}e^{-\gamma t}\right)^{-1/\sigma}\right] dt \\
= & 1 - e^{-\rho u} - e^{-\gamma u}\int_0^u \rho e^{-\rho t}e^{\gamma t}\left[1 - \left(1 + \frac{r+\rho+\gamma}{\kappa}e^{-\gamma t}\right)^{-1/\sigma}\right] dt = 1 - ke^{-\gamma u} + o(e^{-\gamma u})
\end{aligned}$$

where the first equality comes from the change of variable $t \to u - t$. When $u$ goes to infinity the integral in the third line is convergent since the term in brackets is equivalent to $(r+\rho+\gamma)/(\sigma\kappa)e^{-\gamma t}$ when $t \to +\infty$. The last equality follows after letting $k$ be the limit of the integral when $u$ goes to infinity, and noting that, since $\rho > \gamma$, $e^{-\rho u} = o(e^{-\gamma u})$ Next, we note that is $s(1-e^{-\rho u}) = s + o(e^{-\gamma u})$. Taken together, these two approximations allow us to write:

$$Q_{h,u} = s\left(1 + ke^{-\gamma u}\right) + o(e^{-\gamma u}).$$

We then make the approximations:

$$\begin{aligned}
Q_{h,u} &\approx s + e^{-\gamma(u-T_{\max})}(Q_{h,T_{\max}} - s) \\
Q_{h,u}^{\sigma} &\approx s^{\sigma} + e^{-\gamma(u-T_{\max})}\sigma s^{\sigma-1}(Q_{h,T_{\max}} - s)
\end{aligned}$$

for $u > T_{\max}$. Therefore we can approximate the integral in the ODE by

$$\begin{aligned}
&\int_0^{T_{\max}} \left(e^{-(r+\kappa)z} - e^{-(r+\rho+\kappa)z}\right) Q_{h,z}^{\sigma}\, dz \\
&+ s^{\sigma}\left(\frac{e^{-(r+\kappa)T_{\max}}}{r+\kappa} - \frac{e^{-(r+\rho+\kappa)T_{\max}}}{r+\rho+\kappa}\right) + \sigma s^{\sigma-1}(Q_{h,T_{\max}} - s)\left(\frac{e^{-(r+\kappa)T_{\max}}}{r+\kappa+\gamma} - \frac{e^{-(r+\rho+\kappa)T_{\max}}}{r+\rho+\kappa+\gamma}\right).
\end{aligned}$$

It remains to compute $p_{T_{\max}}$. We rewrite the ODE as

$$(r+\kappa)p_u - \dot{p}_u = \text{constant} - \frac{\delta\kappa(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)} Q_{h,u}^{\sigma},$$

where the constant can be computed using the approximation of the integral shown above. Then, after integrating and plugging in the approximation for $Q_{h,u}^{\sigma}$, we obtain:

$$\begin{aligned}
p_{T_{\max}} &= \int_{T_{\max}}^{+\infty} e^{-(r+\kappa)(u-T_{\max})}\left(\text{constant} - \frac{\delta\kappa(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)} Q_{h,u}^{\sigma}\right) du \\
&\approx \frac{\text{constant}}{r+\kappa} - \frac{\delta\kappa(r+\rho+\kappa)}{(r+\rho)(r+\rho+\kappa+\gamma)}\left(\frac{s^{\sigma}}{r+\kappa} + \frac{\sigma s^{\sigma-1}(Q_{h,T_{\max}} - s)}{r+\kappa+\gamma}\right).
\end{aligned}$$

## IX.4 Proof for Section VI

In all what follows $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ denotes the sequence of information events.

### IX.4.1 Proof of Lemma VI.2

We proceed in three steps. We start with a preliminary calculation, then we show the two points of the lemma in turn.

**A preliminary calculation.** We start with a preliminary result:

**R IX.9.** *Consider a change of the asset holding plan from $q$ to $\hat{q}$ such that, for all $\omega \in \Omega$ and all information event times $\tau_n$: $q_{\tau_n,z} - \hat{q}_{\tau_n,z} \geq 0$, $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = 0$ for $z \notin [t,u]$, and $q_{\tau_n,z} - \hat{q}_{\tau_n,z}$ is increasing in $z \in [t,u]$. Suppose that, the change in holding plan involves a limit order to buy the quantity $q_{\tau_n,u} - \hat{q}_{\tau_n,u} \geq 0$ with limit price $p_{u^+} + \eta$. Then, the change in holding plan induces a change in expected utility $\mathbb{E}[\Delta V] \geq \mathbb{E}[X]$, where*

$$X = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n < u\}} (q_{\tau_n,u} - \hat{q}_{\tau_n,u}) K, \tag{IX.16}$$

*where $K$ is the constant of equation (VI.2), page 31.*

To prove this result we first note that, for any realization of $\omega \in \Omega$, the change of asset holding plan first induces a change in the investor's utility flow for assets

$$\sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n+1}} e^{-rz} \left[ v\big(\hat{q}_{\tau_n,z}\big) - v\big(q_{\tau_n,z}\big) \right] dz \leq 0,$$

where we omit the dependence of $v(\,\cdot\,)$ on $\theta$ to simplify notations. The gains come up because the investor sells high during $(t,u]$ and buys low at $u^+$. We add together the trading gains from every time interval $[\tau_n, \tau_{n+1}]$, with our usual accounting convention that the trader buys her initial holding at time $\tau_n$, and re-sell her final holding at time $\tau_{n+1}$. We use the integration by part argument of Section A.1, page 35 in BHW, accounting for the fact that both $\Delta_{\tau_n,z}$ and $p_z$ have a discontinuity at time $u$. We find that the trading gains can be written:

$$\sum_{n=1}^{\infty} \Bigg[ \int_{\tau_n}^{\tau_{n+1}} e^{-rz} \left( rp_z - \dot{p}_z \right) \left( q_{\tau_n,z} - \hat{q}_{\tau_n,z} \right) dz \\ + \mathbb{I}_{\{\tau_n \leq u\}} \mathbb{I}_{\{\tau_{n+1} > u\}} e^{-ru} \left( q_{\tau_n,u} - \hat{q}_{\tau_n,u} \right) \left( p_u - p_{u^+} - \eta \right) \Bigg]. \tag{IX.17}$$

The first term is our usual expression for inter-temporal holding costs. The second term, on the other hand, accounts for the fact that the last purchase of $q_{\tau_n,u} - \hat{q}_{\tau_n,u}$ does not occur at price $p_u$, as implicit assumed in the first integral, but at price $p_{u^+} + \eta$. Note that many terms are equal to zero as $q_{\tau_n,z} = \hat{q}_{\tau_n,z}$ whenever $z \notin [t,u]$. The details are in Section IX.4.2, page 91.

Taken together, we find that the change in utility can be written:

$$\Delta V = \sum_{n=0}^{\infty} \Bigg[ \int_{\tau_n}^{\tau_{n+1}} e^{-rz} \big[ v(\hat{q}_{\tau_n,z}) - v(q_{\tau_n,z}) + (q_{\tau_n,z} - \hat{q}_{\tau_n,z})(rp_z - \dot{p}_z) \big] \, dz \\ + \mathbb{I}_{\{\tau_n \leq u\}} \mathbb{I}_{\{\tau_{n+1} > u\}} e^{-ru} (q_{\tau_n,u} - \hat{q}_{\tau_n,u}) \left( p_u - p_{u^+} - \eta \right) \Bigg].$$

By our maintained assumption, $q_{\tau_n,z} - \hat{q}_{\tau_n,z} \geq 0$. Moreover, by Lemma VI.1, we have that $rp_z - \dot{p}_z \geq 0$. Lastly, marginal utility is bounded above by one. This allows us to bound $\Delta V$ below by:

$$\Delta V \geq \sum_{n=0}^{\infty} \Bigg[ - \int_{\tau_n}^{\tau_{n+1}} e^{-rz} \left( q_{\tau_n,z} - \hat{q}_{\tau_n,z} \right) \, dz \\ + \mathbb{I}_{\{\tau_n \leq u\}} \mathbb{I}_{\{\tau_{n+1} > u\}} e^{-ru} (q_{\tau_n,u} - \hat{q}_{\tau_n,u}) \left( p_u - p_{u^+} - \eta \right) \Bigg].$$

Now, if $z \geq u$, then $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = 0$, while if $z < u$, $q_{\tau_n,z} - \hat{q}_{\tau_n,z}$ is increasing and is thus less than $q_{\tau_n,u} - \hat{q}_{\tau_n,u}$. Therefore:

$$\Delta V \geq \sum_{n=0}^{\infty} \mathbb{I}_{\{\tau_n \leq u\}} (q_{\tau_n,u} - \hat{q}_{\tau_n,u}) \left[ - \int_{\tau_n \vee t}^{u} e^{-rz} \mathbb{I}_{\{\tau_{n+1} \geq z\}} \, dz + \mathbb{I}_{\{\tau_{n+1} > u\}} e^{-ru} \left( p_u - p_{u^+} - \eta \right) \right].$$

Next apply the law of iterated expecations: for each term $n$ in the above expression, take expectations conditional on $\mathcal{F}_{\tau_n}$, and then unconditional expectations. This implies that $\mathbb{E}\left[\Delta V\right] \geq \mathbb{E}\left[Y\right]$, where $Y = \sum_{n=1}^{\infty} Y_n$, and $Y_n$ is the expectation of term $n$ conditional on $\mathcal{F}_{\tau_n}$. Note that $q_{\tau_n}$ and $\hat{q}_{\tau_n}$ are both $\mathcal{F}_{\tau_n}$ measurable and that, conditional on $\mathcal{F}_{\tau_n}$, $\tau_{n+1} - \tau_n$ is an exponential random variable with parameter $\rho$, so $\mathbb{E}\left[\mathbb{I}_{\{\tau_{n+1} \geq z\}}\right] = e^{-\rho(z - \tau_n)}$ for $z \geq \tau_n$. We thus obtain:

$$\begin{aligned} Y &= \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n < u\}} (q_{\tau_n,u} - \hat{q}_{\tau_n,u}) e^{\rho \tau_n} \left[ - \int_{\tau_n \vee t}^{u} e^{-(r+\rho)z} \, dz + e^{-(r+\rho)u} \left( p_u - p_{u^+} - \eta \right) \right] \\ &\geq \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n < u\}} (q_{\tau_n,u} - \hat{q}_{\tau_n,u}) \left[ - \int_{t}^{u} e^{-(r+\rho)z} \, dz + e^{-(r+\rho)u} \left( p_u - p_{u^+} - \eta \right) \right] \equiv X \end{aligned}$$

where the last inequality follows from integrating over the larger interval $[t,u]$, using equation (VI.2), and noting that $e^{\rho \tau_n} \geq 1$

**Proof of point 1 in Lemma VI.2.** There are three cases to consider.

*When the price is strictly decreasing over* $[t,u]$. Consider, for $k \geq 1$:

$$C_k = \left\{ \omega \in \Omega \, : \, \tau_k \in [t,u] \text{ and } q_{\tau_k,u} > 0 \right\} \tag{IX.18}$$

the set of events such that the information process jumps for the $k$-th time during $[t, u]$ and the investor holds a strictly positive amount of asset at time $u$. Consider the following deviation: whenever there is an information event at some $\tau_n \in [t, u]$, sell all your holdings, keep all the limit orders executed after time $u$, and submit a limit order to buy back whatever you were planning to hold after $u$. By construction, this deviation keeps asset holdings the same for all $z \notin [t, u]$. Moreover, because the price is strictly decreasing, the only limit orders that can be executed over $[t, u]$ are limit *buy* orders, implying that for all $\tau_n \in [t, u]$, $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = q_{\tau_n,z}$ has to be increasing in $z \in [\tau_n, u]$. Therefore, Result RIX.9 applies and the expected change in utility is greater than $\mathbb{E}[X]$, for $X$ of (IX.16). By construction, we have $X \geq 0$ so $\mathbb{E}[X] \geq 0$. Now if the original asset holding plan is optimal, then $0 \geq \mathbb{E}[\Delta V] \geq \mathbb{E}[X]$. Thus $\mathbb{E}[X] = 0$. Consequently, since $X \geq 0$, $X = 0$ almost surely. But, by definition of $C_k$, $X > 0$ for all $\omega \in C_k$: it thus follows that $P(C_k) = 0$. To conclude, note that if $\tau_u \geq t$, then there is some $n \geq 1$ such that $\tau_n \in [t, u]$. Therefore, the event $C = \{\tau_u \geq t \text{ and } q_{\tau_u,u} > 0\}$ is included in the union of the sets $C_k$, and $P(C) = 0$ as well.

*When the price is strictly increasing over* $[t, u]$. Consider the same set $C_n$ as in (IX.18) and the following deviation: whenever there is an information event at some $\tau_n \in [t, u]$, submit a limit order to sell all your holdings with a limit price just below $p_u$, and simultaneously a limit order to buy these holdings back at price $p_{u^+} + \eta$. For all $\tau_n \in [t, u]$, $q_{\tau_n,z} - \hat{q}_{\tau_n,z}$ equal to zero until $u$ and positive at $u$, therefore Result RIX.9 applies, and the result follows from the same argument as before.

*When the price is flat over* $[t, u]$. We have several case to consider.

1. Suppose the information process jumps for $k$-th time during $[t, u]$, and that $q_{\tau_k,u} > 0$, i.e., under the original holding plan asset holdings are strictly positive at time $u$. Suppose also that when the information process jump there is a positive measure of limit sell orders in the book at price $p_u$. Formally, consider the set of events, for $k \geq 1$:

$$C_{Ak} = \left\{\omega \in \Omega \,:\, \tau_k \in [t, u], \text{ and } q_{\tau_k,u} > 0, \text{ and at } \tau_k \;\exists \text{ limit sell order at } p_u\right\},$$

   Because of volume maximization, all limit buy orders at $p_u$ are executed immediately. So the investor's original asset holding plan $q_{\tau_k,z}$ does not involve any asset purchase during $(\tau_k, u]$, i.e., $q_{\tau_k,z}$ is decreasing during $[\tau_k, u]$. Consider, then, the following deviation:

   (a) If a limit order to sell at price $p_u$ is executed at some $Z_A \leq u$. Then, by time priority, all limit sell orders at price $p_u$ submitted under the original holding plan must have been executed by time $Z_A$. Recall that the original holding plan does not involve any purchase during $(\tau_k, u]$. So, $q_{\tau_k,z}$ is decreasing for $z \in [\tau_k, Z_A]$, and constant for $z \in (Z_A, u]$. Consider then, the following deviation: keep all orders and submit an additional limit order to sell all holdings at time $Z_A$, and a limit order to buy back $q_{\tau_k,u}$ at time $u$. If the information process jumps again before $Z_A$, cancel this order, and by doing so revert to $q_{\tau_{k+1},z}$. If the information process jumps again after $Z_A$, then buy back the assets sold at time $Z_A$, and submit the same orders as under the original plan to revert to the original holding plan $q_{\tau_{k+1},z}$. Taken together, this ensures that $q_{\tau_n,z} - \hat{q}_{\tau_n,z}$ is positive and increasing for $n = k$ and zero for $n \neq k$. Therefore Result RIX.9 applies.

(b) If a limit order to sell at price $p_u$ is executed after $u$, then keep all orders but immediately sell $q_{\tau_k,u}$, and place a limit order to buy $q_{\tau_k,u}$ back at time $u$. If the information process jumps again before $u$, buy $q_{\tau_k,u}$ back and cancel the limit buy order, and by doing so revert to $q_{\tau_{k+1},z}$. This deviation leads to $q_{\tau_n,z} - \hat{q}_{\tau_n,z}$ which is constant and strictly positive for $n = k$, and zero for $n \neq k$. Again, Result RIX.9 applies.

In both cases, Result IX.9 implies that $X \geq 0$ and $X > 0$ for all $\omega \in C_{Ak}$. The same reasoning as before then implies that $P(C_{Ak}) = 0$ for all $k$.

2. Suppose the information process jumps for $k$-th time during $[t, u]$, that $q_{\tau_k,u} > 0$, and that at time $\tau_k$ there is a positive measure of limit buy orders in the book at price $p_u$. Formally, consider the set of events, for $k \geq 1$:

$$C_{Bk} = \left\{ \omega \in \Omega \, : \, \tau_k \in [t, u], \text{ and } q_{\tau_k,u} > 0, \text{ and at } \tau_k \; \exists \text{ limit buy order at } p_u \right\}$$

Because of volume maximization, all limit sell orders at $p_u$ are executed immediately. So the investor's original asset holding plan $q_{\tau_k,z}$ does not involve any asset sale during $(\tau_k, u]$, i.e., $q_{\tau_k,z}$ is increasing. Consider, then, the following deviation: sell everything immediately and cancel all limit buy orders that would have been executed during $[t, u]$. At the same time, submit a limit buy order to buy everything back at time $u$. Continue to hold zero as long as the information process jumps at times when there are limit buy orders outstanding in the book. The first time the information process jumps and there are not limit buy orders left in the book, then all the limit buy orders the investor may have had under the original plan are executed. This allows to revert to the original plan. Note that, as long as the information process jumps at times $\tau_n \in [\tau_k, u]$ such that there are limit buy orders in the book, then the original plan only has limit buy order, and $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = q_{\tau_n,z}$ is positive and increasing for $z \in [\tau_n, u]$, and zero for $z > u$. After the first information event $\tau_n \in [\tau_k, u]$ such that there are no limit buy orders left in the book, then $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = 0$. In all cases, Result RIX.9 applies and the same reasoning as before implies that $P(C_{Bk}) = 0$.

3. Suppose the information process jumps for $k$-th time, that $q_{\tau_k,u} > 0$, and that at time $\tau_k$ there are no limit order outstanding at $p_u$. Formally, consider the set of events, for $k \geq 1$:

$$C_{0k} = \left\{ \omega \in \Omega \, : \, \tau_k \in [t, u], \text{ and } q_{\tau_k,u} > 0, \text{ and at } \tau_k \; \nexists \text{ limit order at } p_u \right\},$$

Note that, in this case, all limit orders submitted at time $\tau_k$ are executed immediately, and so the original plan $q_{\tau_k,z}$ is constant. Consider, then, the following deviation: sell everything immediately at $\tau_k$ and revert to the original plan if the information process jumps again before time $u$. Reverting to the original plan is feasible because, under both the deviation and the original plan, the investor has no limit order outstanding to buy or sell at any time in $(\tau_k, u]$. Note that the deviation leads to $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = q_{\tau_n,z}$, which is constant and positive for $n = k$ and zero for $n \neq k$. Result RIX.9 applies and the same argument as before implies that $P(C_{0k}) = 0$.

This implies point 1. of Lemma VI.2 because, clearly, any $\omega$ such that $\tau_u \geq t$ and $q_{\tau_u,u} > 0$ belongs to either $C_{Ak}$, $C_{Bk}$, or $C_{0k}$ for some $k$.

**Proof of Point 2 of Lemma VI.2.** We first show:

**RIX.10.** *All the limit orders submitted during* $(0, t)$ *to be executed during* $[t, u]$ *are of the same kind, i.e., either all are limit buy orders, or all are limit sell orders.*

The result is obvious if the price is strictly decreasing or increasing during $[t, u]$, as only one kind of order can be executed during $[t, u]$. Now if the price is flat during $[t, u]$, suppose there are two times $z \leq z' < t$ such that, at time $z$ investors submit a limit order to buy at price $p_u$, executed during $[t, u]$ and at time $z'$ they submit a limit order to sell at price $p_u$, executed during $[t, u]$. Since there cannot be orders of two kinds in the book at the same price, at time $z'$ all limit buy orders at price $p_u$ must have been executed. But this contradicts the fact that a limit buy order submitted at time $z$ is executed during $[t, u]$.

In light of the Result just above, there are only two cases to consider:

*If, during* $(0, t)$*, investors submit limit sell orders to be executed during* $[t, u]$*.* Then the result follows immediately because the asset holdings of investors whose information process last jumped during $[0, t)$ must decrease during $[t, u]$.

*If, during* $(0, t)$*, investors submit limit buy orders to be executed during* $[t, u]$*.* Then the asset holdings of investors whose information process last jumped during $(0, t)$ must increase during $[t, u]$. Consider then, the set, for $k \geq 1$:

$$C_k = \Big\{ \omega \in \Omega \, : \, \tau_k \in (0, t) \text{ and } q_{\tau_k,u} > q_{\tau_k,t} \Big\},$$

and the following deviation. Whenever, under the original plan, you submit a limit order to buy executed at some time $z \in [t, u)$, submit instead a limit order to buy the same quantity at price $p_{u^+} + \eta$, which is executed at time $u^+$. Note that this is feasible, since $p_{u^+} < p_u$. Whenever, under the original holding plan, you cancel a limit order to buy executed at time $z \in [t, u)$ then, in the deviation, you cancel the same quantity of the corresponding order executed at time $u$. This deviation leads to $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = q_{\tau_n,z} - q_{\tau_n,t}$, which is positive and increasing because, as noted at the beginning, $q_{\tau_n,z}$ is increasing during $[t, u]$. So Result RIX.9 applies, and the usual argument implies that $P(C_k) = 0$. But the event $C = \big\{ \omega \in \Omega \, : \, \tau_u \in (0, t] \text{ and } q_{\tau_u,u} > q_{\tau_u,t} \big\}$ belongs to the union of the $C_k$'s, implying that $P(C) = 0$ as well.

### IX.4.2 Calculations of trading gains

We use our usual accounting convention: we assume that a trader purchases her initial asset holding $q_{\tau_n,\tau_n}$ at time $\tau_n$, and sells her final asset holding $q_{\tau_n,\tau_{n+1}}$ at time $\tau_{n+1}$. Also, we let $q_{\tau_n,z} = 0$ for $z < \tau_n$: as will become clear shortly, this simplifies a bit the calculations, as the initial purchase of $q_{\tau_n,\tau_n}$ is accounted for by the initial jump from $q_{\tau_n,\tau_n^-} = 0$ to $q_{\tau_n,\tau_n}$.

To calculate the trading gains, all we need to do is subtract the intertemporal payments associated with $\hat{q}_{\tau_n,z}$ from the intertemporal payment associated with $q_{\tau_n,z}$. Clearly this is equivalent to calculating the payments associated with the "net" holding $\Delta_{\tau_n,z} \equiv q_{\tau_n,z} - \hat{q}_{\tau_n,z}$. We have, then, several cases to consider.

**If $\tau_n > u$, or if $\tau_{n+1} < t$.** Then the gains are zero, which is consistent with equation (IX.17) given that, in these cases, $\Delta_{\tau_n,z} = 0$ over $[\tau_n, \tau_{n+1}]$.

**If $\tau_n \leq u$ and $\tau_{n+1} > u$.** Then the gains can be written:

$$\int_{[\tau_n \vee t, u)} e^{-rz} p_z d\Delta_{\tau_n,z}\, dz + e^{-ru} p_u \left(\Delta_{\tau_n,u} - \Delta_{\tau_n,u^-}\right) - \left(p_{u^+} + \eta\right) \Delta_{\tau_n,u}.$$

The first integral account for the trading gains over the interval $[\tau_n \wedge t, u)$. The second term account for the trading gains at time $u$. The last term accounts for the last purchase at time $u^+$, with a limit order to buy at price $p_{u^+} + \eta$. Note that we do not need to account for the re-sale of $\Delta_{\tau_n,\tau_{n+1}}$ at time $\tau_{n+1}$, because when $\tau_{n+1} > u$ we have $\Delta_{\tau_n,\tau_{n+1}} = 0$. Next, using the integration by part formula for Lebesgue-Stieltjes integral (see, for instance, Theorem 6.2.2 in Carter and Van Brunt, 2000) and keeping in mind that $p_z$ is continuous over $[t \vee \tau_n, u)$ and that $p_u = p_{u^-}$, the above expressions simplifies to:

$$\begin{aligned}
&\int_{\tau_n \vee t}^{u} e^{-rz} \left(r p_z - \dot{p}_z\right) \Delta_{\tau_n,z}\, dz + \left(p_u - p_{u^+} - \eta\right) \Delta_{\tau_n,u} \\
= &\int_{\tau_n}^{\tau_{n+1}} e^{-rz} \left(r p_z - \dot{p}_z\right) \Delta_{\tau_n,z}\, dz + \left(p_u - p_{u^+} - \eta\right) \Delta_{\tau_n,u},
\end{aligned}$$

where the equality follows because $\Delta_{\tau_n,z} = 0$ for $z \notin [t, u]$. This is also consistent with (IX.17) given that $\tau_n \leq u$ and $\tau_{n+1} > u$.

**If $\tau_{n+1} = u$.** Then, the trading gains are:

$$\int_{[\tau_n \vee t, u)} e^{-rz} p_z d\Delta_{\tau_n,z}\, dz + e^{-ru} p_u \left(\Delta_{\tau_n,u} - \Delta_{\tau_n,u^-}\right) - p_u \Delta_{\tau_n,u}.$$

where the last term arises because of our accounting convention that assets are sold at time $\tau_{n+1} = u$. Proceeding as before we also arrive to equation (IX.17).

**If $\tau_n \leq u$ and $\tau_{n+1} < u$.** Then, the trading gains are given by:

$$\int_{[\tau_n \vee t, \tau_{n+1}]} e^{-rz} p_z d\Delta_{\tau_n, z} \, dz - p_{\tau_{n+1}} \Delta_{\tau_n, \tau_{n+1}},$$

and the same calculations lead to (IX.17).

## IX.5 Proof for Section VII

### IX.5.1 Proof of Lemma VII.1

If there is some time $t$ in a spot such that $\dot{p}_t < 0$ and $1 - rp_t + \dot{p}_t < 0$. Then, by our maintained assumptions on the price path there is some interval $[u_1, u_3]$ around $t$ such that these inequalities hold for all $z$ in the interval. Now pick some $u_2 \in (u_1, u_3)$. We show:

**RIX.11.** *For almost all $\omega \in \Omega$:*

1. *If $\tau_{u_2} \in [u_1, u_2]$, then $q_{\tau_{u_2}, u_2} = 0$.*
2. *If $\tau_{u_2} \in (0, u_1)$, then $q_{\tau_{u_2}, u_2} = q_{\tau_{u_2}, u_1} = q_{\tau_{u_1}, u_1}$.*

The first point means that, if a trader has an information event after $u_1$, then he holds zero at $u_2$. Otherwise, by the second point, the trader has the same holding at $u_1$ and $u_2$. Taken together, these two properties contradict market clearing at time $u_2$.

**A preliminary calculation before proving Result IX.11.** Consider a deviation $\hat{q}$ such that, for all $\omega$ and all information times $\tau_n$, $\hat{q}_{\tau_n,z} = q_{\tau_n,z}$ for all $z \notin [u_1, u_3)$, and $\hat{q}_{\tau_n,z}$ is constant and less than $q_{\tau_n,z}$ for $z$ in $[u_1, u_3)$. The realized utility of the deviation is:

$$\begin{aligned}
\Delta V &= \sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n+1}} e^{-rz} \left[ v(\hat{q}_{\tau_n,z}) - v(q_{\tau_n,z}) - (rp_z - \dot{p}_z)\left(\hat{q}_{\tau_n,z} - q_{\tau_n,z}\right) \right] dz \\
&\geq \sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n+1}} e^{-rz} \left[ -1 + rp_z - \dot{p}_z \right] \left[ q_{\tau_n,z} - \hat{q}_{\tau_n,z} \right] dz \\
&\geq \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \leq u_2\}} \mathbb{I}_{\{\tau_{n+1} \geq u_3\}} \int_{u_2}^{u_3} e^{-rz} \left[ -1 + rp_z - \dot{p}_z \right] \left[ q_{\tau_n,z} - \hat{q}_{\tau_n,z} \right] dz \\
&\geq \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \leq u_2\}} \mathbb{I}_{\{\tau_{n+1} \geq u_3\}} \int_{u_2}^{u_3} e^{-rz} \left[ -1 + rp_z - \dot{p}_z \right] dz \times \left[ q_{\tau_n,u_2} - \hat{q}_{\tau_n,(u_1 \vee \tau_n)} \right]
\end{aligned} \tag{IX.19}$$

where, as usual, we omit the dependence of $v(\,\cdot\,)$ on $\theta$ to simplify notations. The second line follows because of our maintained assumption that $q_{\tau_n,z} \geq \hat{q}_{\tau_n,z}$ and by noting that marginal utility is bounded above by 1. The third line follows by noting that the integrand in the second line is positive since $q_{\tau_n,z} - \hat{q}_{\tau_n,z} \geq 0$ for $z \in [u_1, u_3)$, and equal to zero for all $z \notin [u_1, u_3)$. Therefore, we obtain a lower bound after multiplying term $n$ by $\mathbb{I}_{\{\tau_n \leq u_2\}} \mathbb{I}_{\{\tau_{n+1} \geq u_3\}}$, and integrating over the smaller interval $[u_2, u_3] \subseteq [\tau_n, \tau_{n+1}]$. The fourth line follows by noting that, since the price is decreasing over $[u_1, u_3)$, only limit buy orders can be executed: $q_{\tau_n,z}$ is increasing and so for $z \in [u_2, u_3]$ it is greater than $q_{\tau_n,u_2}$. On the other hand, $\hat{q}_{\tau_n,z}$ is constant for $z \in [u_1, u_3)$ by construction. Now apply the law of iterated expectations: take first expectations of each term on the right–hand side of (IX.19) conditional on $\mathcal{F}_{\tau_n}$, and then take unconditional expectations. We find that $\mathbb{E}[\Delta V] \geq \mathbb{E}[X]$, where $X = \sum_{n=1}^{\infty} X_n$ and $X_n$ is the expectation of term $n$ in (IX.19) conditional on $\mathcal{F}_{\tau_n}$. To calculate $X$, note that, because the information process has independent increments and is independent from the type process, the indicator function $\mathbb{I}_{\{\tau_{n+1} \geq u_3\}} = \mathbb{I}_{\{\tau_{n+1}-\tau_n \geq u_3 - \tau_n\}}$ is independent from $\mathcal{F}_{\tau_n}$. Also, by definition of a holding plan, both $q_{\tau_n,u_2}$ and $\hat{q}_{\tau_n,(u_1 \vee \tau_n)}$ are measurable with respect to $\mathcal{F}_{\tau_n}$. Therefore:

$$X = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \leq u_2\}} e^{-\rho(u_3-\tau_n)} \int_{u_2}^{u_3} e^{-rz} \left[-1 + rp_z - \dot{p}_z\right] dz \times \left[q_{\tau_n,u_2} - \hat{q}_{\tau_n,(u_1 \vee \tau_n)}\right]$$

**Proof of point 1 of Result IX.11.** Consider, then, the set, for $k \geq 1$:

$$C_k = \left\{ \omega \in \Omega \,:\, \tau_k \in [u_1, u_2] \text{ and } q_{\tau_k,u_2} > 0 \right\}$$

and the following deviation. Whenever the information process jumps at $\tau_n \in [u_1, u_3)$, hold zero asset until $u_3$, and revert the your original plan at time $u_3$. This can be achieved by placing a limit order to buy $q_{\tau_n,u_3}$ at time $u_3$, and keeping all the limit orders to be executed after $u_3$ the same. By construction, the deviation coincides with the original holding plan for $z \notin [u_1, u_3)$, and is equal to zero for $z \in [u_1, u_3)$. Thus the preliminary calculation applies. Note also that, in the random variable $X$ of the preliminary calculation, if $\tau_n < u_1$, then $\hat{q}_{\tau_n,u_2} = q_{\tau_n,u_2}$ and term $n$ is zero. On the other hand, if $\tau_n \in [u_1, u_2]$, then $\hat{q}_{\tau_n,u_2} = 0$ and term $n$ is positive . Therefore,

$$X = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in [u_1,u_2]\}} \times e^{-\rho(u_3-\tau_n)} \int_{u_2}^{u_3} e^{-rz} \left[-1 + rp_z - \dot{p}_z\right] dz \times q_{\tau_n,u_2}.$$

One sees that $X$ is positive and strictly positive for $\omega \in C_k$. But if the original plan is optimal we must have that $0 \geq \mathbb{E}\left[\Delta V\right] \geq \mathbb{E}\left[X\right]$. Thus $\mathbb{E}\left[X\right] = 0$. Consequently, since $X \geq 0$, $X = 0$ almost surely. But $X > 0$ for $\omega \in C_k$. It thus follows that $P(C_k) = 0$. Now $C = \left\{\omega \in \Omega \,:\, \tau_{u_2} \in [u_1, u_2] \text{ and } q_{\tau_{u_2},u_2} > 0\right\}$ belongs to the union of the $C_k$'s, and so $P(C) = 0$ as well.

**Proof of point 2 of Result IX.11.** Consider then, the set, for $k \geq 1$:

$$C_k = \left\{ \omega \in \Omega \,:\, \tau_k \in (0, u_1) \wedge q_{\tau_k,u_2} > q_{\tau_k,u_1} \right\}$$

and the following deviation, $\hat{q}$:

- Whenever under the original plan, $q$, you submit a limit order to buy at price $p_z$, to be executed at some time $z \in [u_1, u_3)$, replace it by a limit buy order with limit price $p_{u_3}$, executed at time $u_3$. This is feasible because the price is strictly decreasing at $u_3$, and is strictly below the price at any $z \in [u_1, u_3)$. Therefore, if it possible to submit a limit order to buy at $p_z$, to be executed at $z \in [u_1, u_3)$, then by the price priority rule a limit order to buy at price $p_{u_3}$ is to be executed at time $u_3$.
- Whenever under the original plan, $q$, you cancel some previously submitted limit order to buy at

price $p_z$, executed at $z \in [u_1, u_3)$ then, under the deviation, cancel the corresponding quantity of limit buy orders at price $p_{u_3}$. To see why this is feasible, note the following. If under the original plan, $q$, you can cancel an outstanding order to buy at price $p_z$ executed at time $z$, then by the time priority rule it must be the case that, under the deviation, $\hat{q}$, the corresponding "replacement" order at price $p_{u_3}$, executed at time $u_3$, is still outstanding. So you can cancel it as well.

By construction, $\hat{q}_{\tau_n,z}$ coincides with $q_{\tau_n,z}$ for $z \notin [u_1, u_3)$. For $z \in [u_1, u_3)$, $\hat{q}_{\tau_n,z}$ remains constant, equal to $q_{\tau_n,u_1}$ if $\tau_n \leq u_1$, and to $q_{\tau_n,\tau_n}$ otherwise. Therefore, the preliminary calculation applies. Multiplying each term in $X$ by $\mathbb{I}_{\{\tau_n<u_1\}}$, which is less than one, we find that:

$$X = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n<u_1\}} e^{-\rho(u_3-\tau_n)} \int_{u_2}^{u_3} e^{-rz} \left[-1 + rp_z - \dot{p}_z\right] dz \times \left[q_{\tau_n,u_2} - \hat{q}_{\tau_n,u_1}\right]$$

Clearly this lower bound of $X$ is positive and strictly positive for $\omega \in C_k$, and the usual argument implies that $P(C_k) = 0$ Next, note that $C = \left\{\omega \in \Omega \,:\, \tau_u \in (0, u_1) \text{ and } q_{\tau_{u_2},u_2} > q_{\tau_{u_2},u_1}\right\}$ belongs to the unions of the $C_k$'s, and therefore $P(C) = 0$.

### IX.5.2 Proof of Lemma VII.2

Before proving Lemma VII.2 we need to establish two preliminary results.

**A first preliminary result.** The first preliminary result is:

**R IX.12.** *Assume that $p_z > 1/r$ for all $z \geq t > 0$, and that there is a flat spot $(u_1, u_3)$ with $u_1 \geq t$. Then, for all $u_2 \in (u_1, u_3)$, if $\tau_{u_2} \in (0, u_1)$ then $q_{\tau_{u_2},u_2} = q_{\tau_{u_2},u_3}$.*

This shows that the investor wants to keep his holding constant over $(u_1, u_3)$. The intuition is clear: it is always better to sell just before $u_1$ because the asset is "over-priced" relative to its utility flow, and there is no capital gain to make during the flat spot.

To prove Result IX.12 we exhibit a profitable deviation when $\tau_{u_2} < 1$ and $q_{\tau_{u_2},u_2} \neq q_{\tau_{u_2},u_3}$.

**Utility of a deviation.** Consider a change in asset holding from $q_{\tau_n,z}$ to $\hat{q}_{\tau_n,z}$ such that: $q_{\tau_n,z} \geq \hat{q}_{\tau_n,z}$, $q_{\tau_n,z} = \hat{q}_{\tau_n,z}$ for all $z \notin [u_1, u_3]$, and $q_{\tau_n,z} - \hat{q}_{\tau_n,z}$ is decreasing in $z \in [u_1, u_3]$. Keeping in mind that $p_z = p_{u_1}$ for $z \in [u_1, u_3]$, the realized change in utility is greater than:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n+1}} e^{-rz} \left(q_{\tau_n,z} - \hat{q}_{\tau_n,z}\right) \left(-1 + rp_z - \dot{p}_z\right) dz \\
= &\left(rp_{u_1} - 1\right) \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \leq u_3\}} \int_{u_1 \vee \tau_n}^{u_3} e^{-rz} \mathbb{I}_{\{\tau_{n+1} \geq z\}} \left(q_{\tau_n,z} - \hat{q}_{\tau_n,z}\right) dz.
\end{aligned}$$

Under our assumption that $q_{\tau_n,z} - \hat{q}_{\tau_n,z} \geq 0$ and is decreasing in $z$, this is greater than:

$$(rp_{u_1} - 1)\sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n < u_1\}}(q_{\tau_n,u_2} - \hat{q}_{\tau_n,u_2}) \int_{u_1}^{u_2} \mathbb{I}_{\{\tau_{n+1} \geq z\}} e^{-rz}\, dz.$$

Taking expectations conditional on $\mathcal{F}_{\tau_n}$ in each term we find that the net utility of the deviation is greater than $\mathbb{E}[X]$, where:

$$X = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n < u_1\}}(q_{\tau_n,u_2} - \hat{q}_{\tau_n,u_2}) e^{\rho\tau_n} \int_{u_1}^{u_2} e^{-(r+\rho)z}\, dz$$

**The deviation.** Note first that since $u_1$ is the start of a flat spot and $p_{u_1} > 1/r$, then the price is strictly increasing just before $u_1$. Therefore, if $\tau_n < u_1$, then the asset holding plan $q_{\tau_n,z}$ cannot contain limit buy orders executed during $[u_1, u_3]$, and is thus decreasing. Consider, then, for $n \geq 1$:

$$C_k = \Big\{\omega \in \Omega \,:\, \tau_k \in (0, u_1) \;\text{ and } q_{\tau_k,u_2} > q_{\tau_k,u_3}\Big\},$$

and the following deviation.

- Whenever you have an information event $\tau_n \in (0, u_1)$ and submit a limit sell order executed during $[u_1, u_3]$, cancel that limit sell order and replace it by a limit sell order for the same quantity executed just before $u_1$. Subsequently, if under the original plan you cancel a limit sell order executed during $[u_1, u_3]$ then, under the deviation, cancel the same quantity of corresponding limit sell orders executed at time $u_1$. By construction $\hat{q}_{\tau_n,z} = q_{\tau_n,u_3}$ for $\tau_n \in (0, u_1)$ and $z \in [u_1, u_3]$. Moreover, since prior to $u_1$ it is only possible to submit limit sell orders executed during $[u_1, u_3]$, then it must be that, for $\tau_n < u_1$, $q_{\tau_n,z}$ is decreasing in $z \in [u_1, u_3]$, and therefore $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = q_{\tau_n,z} - q_{\tau_n,u_3}$ must be decreasing as well.

- Whenever you have an information event at some time $\tau_n \in [u_1, u_3]$ when limit sell orders are executed, then under the original plan the asset holding must be decreasing over $[\tau_n, u_3]$. In that you keep all your limit orders executed after $u_3$ the same and chooses $\hat{q}_{\tau_n,z} = q_{\tau_n,u_3}$ for $z \in [\tau_n, u_3]$. As before, $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = q_{\tau_n,z} - q_{\tau_n,u_3}$ is decreasing.

- Lastly, the first time you an information during $[u_1, u_3]$ and limit buy orders at price $p_{u_1}$ are executed, then under both the original plan and under the deviation, all the previously submitted limit sell orders at price $p_{u_1}$ must have been executed. You can then revert to the original plan. Clearly, in that case, $q_{\tau_n,z} - \hat{q}_{\tau_n,z} = 0$ and is thus decreasing.

Note that by construction, asset holdings before and after $u_3$ are the same in the deviation as in the original plan. This deviation allows to apply the preliminary calculation. Note that $X \geq 0$ and $X > 0$ for all $\omega \in C_k$. The usual argument then implies that $P(C_k) = 0$. Then let $C = \big\{\omega \in \Omega \,:\, \tau_{u_2} \in$

$(0, u_1)$ and $q_{\tau_{u_2}, u_2} > q_{\tau_{u_2}, u_3}\}$, and note that $C$ belongs to the union of the $C_k$'s, and thus $P(C) = 0$ as well.

**A second preliminary result.** To state the second preliminary result, we define, for any $t \leq u$:

$$\begin{aligned} f(t,u) &\equiv \int_t^u e^{-(r+\rho)z} \left(1 - rp_z + \dot{p}_z\right) dz \\ &= \int_t^u e^{-(r+\rho)z} \left(1 - (r+\rho)p_z + \dot{p}_z\right) dz + \rho \int_t^u e^{-(r+\rho)z} p_z \, dz \\ &= \left(\frac{1}{r+\rho} - p_t\right) e^{-(r+\rho)t} - \left(\frac{1}{r+\rho} - p_u\right) e^{-(r+\rho)u} + \rho \int_t^u e^{-(r+\rho)z} p_z \, dz. \end{aligned}$$

Clearly, the function $f(t,u)$ is continuous and, as $u \to \infty$, it converges to

$$f(t,\infty) = \left(\frac{1}{r+\rho} - p_t\right) e^{-(r+\rho)t} + \rho \int_t^\infty e^{-(r+\rho)z} p_z \, dz. \tag{IX.20}$$

**RIX.13.** *Assume that $p_t \to p_\infty > 1/r$. Then, for all $T \geq 0$, there exists $t > T$ such that $f(t,u) < 0$ for all $u > t$.*

Consider that $p_t \to p_\infty > 1/r$. We first note that $f(t,\infty) < 0$ as long as $t$ is large enough. This follows because, as $t \to \infty$,

$$f(t,\infty)e^{(r+\rho)t} \to \left(\frac{1}{r+\rho} - p_\infty\right) + \frac{\rho}{r+\rho} p_\infty = \frac{1 - rp_\infty}{r+\rho} < 0,$$

since $p_\infty > 1/r$.

Now, by the above remark, for any $T \geq 0$, we can pick some $t > T$ such that $f(t,\infty) < 0$. If $f(t,u) < 0$ for all $u > t$, we are done. Otherwise, let $t'$ be the last time greater than $t$ such that $f(t,t') \geq 0$. By continuity $f(t,t') = 0$ and $f(t,u) < 0$ for all $u > t'$. Then, by equation (IX.20),

$$f(t',u) = f(t,u) - f(t,t') = f(t,u) < 0$$

for all $u > t'$, and we are done.

**Concluding the proof of Lemma VII.2.** Suppose that the price eventually becomes greater than $1/r$.

There are two cases to consider. The first case is when the price is constant and greater than $1/r$ at all times. Then, clearly, no investor want to hold the asset at any time, and the result follows. In the other case, if the price is not constant at all time, then there there must be an increasing spot where $p_z > 1/r$.[15] Then, by Corollary VII.1, $\dot{p}_z \geq 0$ for all $z$ following this increasing spot. and thus

[15] Indeed, either $p_0 > 1/r$ and the result follows, or $p_0 \leq 1/r$ and then the price has to increase above $1/r$.

$p_z \to p_\infty > 1/r$. This allows to apply Result RIX.13: we can find some time $t$ after the increasing spot such that $p_t > 1/r$, $f(t,\infty) < 0$ and $f(t,u) < 0$ for all $u > t$. If the time $t$ delivered by Result RIX.13 lies in the interior of some flat spot replace it by the lower bound of the flat spot. Clearly, since $1 - rp_z < 0$ over the flat spot, the lower bound of the flat spot also also satisfies $f(t,u) < 0$ for all $u > t$. Note that, by construction, the price is strictly increasing in a left neighborhood of $t$.

Let $T < t$ be the lower bound of the increasing spot to the left of $t$. By construction, to the left of $T$, the price is either flat or strictly decreasing. To the right of $T$, the price is strictly increasing, and increasing forever after. This implies that no limit buy order submitted before $T$ can be executed after $T$. Therefore, for all $\tau_n < T$, $q_{\tau_n,z}$ has to be decreasing in $z \geq T$.

Now consider information events occurring at $\tau_n \in [T,t]$. Because the price is strictly increasing just before $t$ and increasing after, it follows that the asset holding plan $q_{\tau_n,z}$ can only contain limit sell orders for $z \geq t$, and is thus decreasing in $z$. Moreover, by Result RIX.12, the holding plan $q_{\tau_n,z}$ remains constant during all flat spots after $t$: this implies in particular that all the limit sell orders contained in $q_{\tau_n,z}$ are executed during increasing spots. Now we claim that, almost surely, $q_{\tau_n,z} = 0$ for all $z \geq t$. Indeed, consider:

$$C_n = \left\{ \omega \in \Omega \,:\, \tau_n \in [T,t] \;\text{ and }\; q_{\tau_n,z} > 0 \text{ for some } z \geq t \right\},$$

and the following deviation. If the $n$-th information event occurs during $[T,t]$, place a limit sell order to sell all the assets just before $t$, and revert to the original plan at the next information event – such a one-stage deviation is feasible because, by Corollary VII.1, all limit sell orders after $t$ are executed during increasing spots. As usual, we apply the law of iterated expectations and calculate the expected net utility of the deviation by first conditioning with respect to $\mathcal{F}_{\tau_n}$. We find that the expected net utility of the deviation is greater than $\mathbb{E}[X]$, where:

$$\begin{aligned}
X =& - \mathbb{I}_{\{\tau_n \in [T,t]\}} e^{\rho \tau_n} \left[ \int_t^\infty e^{-(r+\rho)u} (1 - rp_u + \dot{p}_u) q_{\tau_n,u} \, du \right] \\
=& - \mathbb{I}_{\{\tau_n \in [T,t]\}} e^{\rho \tau_n} \left[ \int_t^\infty \frac{\partial f}{\partial u}(t,u) q_{\tau_n,u} \, du \right] \\
=& - \mathbb{I}_{\{\tau_n \in [T,t]\}} e^{\rho \tau_n} \left[ \int_t^\infty f(t,u) \, dq_{\tau_n,u} + f(t,\infty) q_{\tau_n,\infty} \right],
\end{aligned}$$

where the last line follows by integration by part after noting that $f(t,t) = 0$.[16]

Now, $X$ is always positive given that $f(t,u) < 0$, $f(t,\infty) < 0$ and $q_{\tau_n,u}$ is decreasing. It is strictly positive for $\omega \in C$. Indeed, there is two cases to consider. Either $q_{\tau_n,\infty} > 0$ and the result follows. Or $q_{\tau_n,\infty} = 0$ and the result follows because $f(t,u)$ is negative everywhere and, for all $\omega \in C_n$, $q_{\tau_n,u}$ must be strictly decreasing somewhere in $[t,\infty)$. But the expected utility of the deviation has to be negative, implying that $\mathbb{E}[X] = 0$ and thus that $X = 0$ almost everywhere. Thus, $P(C_n) = 0$, which

[16] Note also that the last integral is well defined: indeed, $f(t,u)$ is bounded, and because $q_{\tau_n,u}$ is bounded and decreasing over $[t,\infty)$, $u \mapsto -q_{\tau_n,u}$ is a finite measure over $[t,\infty)$.

establishes the claim.

The above shows that traders whose last information event occurred during $(0, T)$ hold less asset at time $t$ than at time $T$, while traders whose last information event occurred during $[T, t]$ hold nothing at time $t$. Our usual argument then implies that the market cannot clear at time $t$.

### IX.5.3 Proof of Corollary VII.3

There are two cases to consider, depending on whether, at $\tau_{u_2}$, the investor can submit limit buy or limit sell orders executed during $[\tau_{u_2}, u_2)$

Suppose first that the price is strictly decreasing over $[u_1, u_3]$, or that it is flat. Suppose that $\tau_k \in [u_1, u_2)$ and that, at time $\tau_k$, there are limit buy orders at price $p_{u_2}$ in the book. In that case, at time $\tau_k$, the trader can only submit limit buy orders executed during $(\tau_k, u_2]$. Formally, we consider the set of events, for $k \geq 1$:

$$C_{Bk} = \Big\{ \omega \in \Omega \,:\, \tau_k \in [u_1, u_2) \;\text{ and }\; \theta_{\tau_k} = h \;\text{ and }\; q_{\tau_k, u_2} < 1$$
$$\text{and, at time } \tau_k, \text{ you can only submit limit buy orders executed during } (\tau_k, u_2] \Big\}$$

and the deviation consisting in buying one unit of asset at $\tau_k$ and holding it forever, i.e. $\hat{q}_{\tau_k, z} = 1$ for all $z \geq \tau_k$. Note that, since the investor can only submit buy orders at $\tau_k$, his asset holding must be weakly increasing during $(\tau_k, u_2]$, so $q_{\tau_k, z} \leq q_{\tau_k, u_2} < 1$ for $z \in [\tau_k, u_2)$. The net utility of the deviation writes:

$$\begin{aligned} \Delta V = {} & \mathbb{I}_{\{\tau_k \in [u_1, u_2)\}} \sum_{n=k}^{\infty} \int_{\tau_n}^{\tau_{n+1}} e^{-rz} \Big[ \big(1 - \min\{q_{\tau_n, z}, 1\}\big)\big(1 - rp_z + \dot{p}_z\big) \\ & \qquad + \big(q_{\tau_k, z} - \min\{q_{\tau_k, z}, 1\}\big)\big(rp_z - \dot{p}_z\big) \Big] \, dz \\ \geq {} & \mathbb{I}_{\{\tau_k \in [u_1, u_2)\}} \mathbb{I}_{\{\tau_{k+1} > u_2\}} \int_{\tau_k}^{u_2} e^{-rz} \big(1 - \min\{q_{\tau_k, u_2}, 1\}\big)\big(1 - rp_z + \dot{p}_z\big) \, dz \end{aligned}$$

where the inequality holds because all integrands in the initial expression for $\Delta V$ are positive: so we obtain a lower bound by ignoring terms $n \neq k$, and then multiplying by the indicator function $\mathbb{I}_{\{\tau_{k+1} > u_2\}}$. Taking expectations conditional on $\mathcal{F}_{\tau_k}$, we find that $\mathbb{E}[\Delta V] \geq \mathbb{E}[X]$, where

$$X = \mathbb{I}_{\{\tau_k \in [u_1, u_2)\}} e^{-\rho(u_2 - \tau_k)} \int_{\tau_k}^{u_2} e^{-rz} \big(1 - \min\{q_{\tau_k, u_2}, 1\}\big)\big(1 - rp_z + \dot{p}_z\big) \, dz.$$

Clearly, $X \geq 0$ and $X > 0$ for $\omega \in C_{Bk}$, implying as usual that $P(C_{Bk}) = 0$.

Now consider the second case: the price is strictly increasing over $[u_1, u_3]$, or it is flat and the information process jumps for the $k$-th time at time $\tau_k \in [u_1, u_2)$ when there are limit sell orders at price $p_{u_2}$. In that case, at time $\tau_k$, the investor can only submit limit sell orders executed during

$(\tau_k, u_2]$. Consider then, for $k \geq 1$:

$$C_{Ak} = \Big\{ \omega \in \Omega \,:\, \tau_k \in [u_1, u_2) \;\text{ and }\; \theta_{\tau_k} = h \;\text{ and }\; q_{\tau_k, u_2} < 1$$
$$\text{and, at time } \tau_k, \text{ you can only submit limit sell orders executed during } (\tau_k, u_2] \Big\}$$

and the deviation consisting in buying one unit of asset at $\tau_k$ and holding it forever, i.e. $\hat{q}_{\tau_k, z} = 1$ for all $z \geq \tau_k$. Note that, since the investors can only submit sell orders at $\tau_k$, his asset holding must be weakly decreasing during $[u_2, u_3]$, so $q_{\tau_k, z} \leq q_{\tau_k, u_2} < 1$ for $z \in [u_2, u_3]$. As in the first case, the utility of the deviation is:

$$\begin{aligned} \Delta V = & \, \mathbb{I}_{\{\tau_k \in [u_1, u_2)\}} \sum_{n=k}^{\infty} \int_{\tau_n}^{\tau_{n+1}} e^{-rz} \Big[ \big(1 - \min\{q_{\tau_n, z}, 1\}\big)\big(1 - r p_z + \dot{p}_z\big) \\ & \qquad\qquad + \big(q_{\tau_k, z} - \min\{q_{\tau_k, z}, 1\}\big)\big(r p_z - \dot{p}_z\big) \Big] \, dz \\ \geq & \, \mathbb{I}_{\{\tau_k \in [u_1, u_2)\}} \mathbb{I}_{\{\tau_{k+1} > u_3\}} \int_{u_2}^{u_3} e^{-rz} \big(1 - \min\{q_{\tau_k, u_2}, 1\}\big)\big(1 - r p_z + \dot{p}_z\big) \, dz. \end{aligned}$$

Taking expectations conditional on $\mathcal{F}_{\tau_k}$ we obtain that $\mathbb{E}[\Delta V] \geq \mathbb{E}[X]$, where

$$X = \mathbb{I}_{\{\tau_k \in [u_1, u_2)\}} e^{-\rho(u_3 - \tau_k)} \int_{u_2}^{u_3} e^{-rz} \big(1 - \min\{q_{\tau_k, u_2}, 1\}\big)\big(1 - r p_z + \dot{p}_z\big) \, dz.$$

As usual $X$ is positive and strictly positive for $\omega \in C_{Ak}$, implying that $P(C_{Ak}) = 0$.

Lastly, in the case the order book is empty at time $\tau_k$, we consider:

$$C_{0k} = \Big\{ \omega \in \Omega \,:\, \tau_k \in [u_1, u_2) \;\text{ and }\; \theta_{\tau_k} = h \;\text{ and }\; q_{\tau_k, u_2} < 1$$
$$\text{and, at time } \tau_k, \text{ you cannot submit limit orders executed during } (\tau_k, u_2] \Big\}$$

and we can apply either one of the argument above to argue that $P(C_{0k}) = 0$.

To conclude, we note that $C = \big\{ \omega \in \Omega \,:\, \tau_{u_2} \in [u_1, u_2) \;\text{ and }\; \theta_{\tau_{u_2}} = h \;\text{ and }\; q_{\tau_{u_2}, u_2} < 1 \big\}$ is included in the union of the $C_{Bk}$, $C_{Ak}$, and $C_{0k}$, and therefore that $P(C) = 0$.

### IX.5.4 Proof of Lemma VII.3

To complete the proof in case 1, it remains to show two points: $(i)$ high-valuation investors who have an information event during $[u_1, u_2]$ hold more than one unit at $u_3$; $(ii)$ market clearing at time $u_3$ is not satisfied.

To complete the proof in case 2, it remains to show Lemma VII.4.

**Proof of point** $(i)$**.** As usual, we consider:

$$C_k = \Big\{\omega \in \Omega \,:\, \tau_k \in [u_1, u_2], \quad \text{and} \quad \theta_{\tau_k} = h, \quad \text{and} \quad q_{\tau_k, u_3} < 1\Big\},$$

and the deviation consisting in submitting a limit buy order just before time $u_2$ in order to hold one unit forever from then on. The utility of the deviation is:

$$\begin{aligned}\Delta V &= \mathbb{I}_{\{\tau_k \in [u_1,u_2)\}} \sum_{n=k}^{\infty} \int_{\tau_n \vee u_2}^{\tau_{n+1} \vee u_2} e^{-rz} \Big[ \big(1 - \min\{q_{\tau_n,z}, 1\}\big)\big(1 - rp_z + \dot{p}_z\big) \\ &\qquad\qquad + \big(q_{\tau_k,z} - \min\{q_{\tau_k,z}, 1\}\big)\big(rp_z - \dot{p}_z\big)\Big]\, dz \\ &\geq \mathbb{I}_{\{\tau_k \in [u_1,u_2)\}} \mathbb{I}_{\{\tau_{k+1} \geq u_4\}} \int_{u_2}^{\tau_{k+1}} e^{-rz} \big(1 - \min\{q_{\tau_k,z}, 1\}\big)\big(1 - rp_z + \dot{p}_z\big)\, dz,\end{aligned}$$

where the inequality is obtained as before, after noting that all the intregrands in $\Delta V$ are positive, and so we obtain a lower bound by ignoring terms and multiplying everything by the indicator function $\mathbb{I}_{\{\tau_{k+1} > u_4\}}$.

Now if the price is strictly increasing over $[u_2, u_4]$, we continue the lower-bound calculation as follows:

$$\Delta V \geq \mathbb{I}_{\{\tau_k \in [u_1,u_2)\}} \mathbb{I}_{\{\tau_{k+1} \geq u_4\}} \int_{u_3}^{u_4} e^{-rz} \big(1 - \min\{q_{\tau_k,u_3}, 1\}\big)\big(1 - rp_z + \dot{p}_z\big)\, dz,$$

where the inequality follows by integrating over the smaller interval $[u_3, u_4]$ and by noting that since the price is strictly increasing over $[u_3, u_4]$, $z \mapsto q_{\tau_k,z}$ is decreasing over $[u_3, u_4]$. We then follow the usual reasoning: we take expectations conditional on $\mathcal{F}_{\tau_k}$ and obtain that $\mathbb{E}\left[\Delta V\right] \geq \mathbb{E}\left[X\right]$ where:

$$X = \mathbb{I}_{\{\tau_k \in [u_1,u_2)\}} e^{-\rho(u_4 - \tau_k)} \int_{u_3}^{u_4} e^{-rz} \big(1 - \min\{q_{\tau_k,u_3}, 1\}\big)\big(1 - rp_z + \dot{p}_z\big)\, dz,$$

which is positive and strictly positive for $\omega \in C_k$. As usual, this implies that $P(C_k) = 0$, and that $P(C) = 0$ where $C = \big\{\omega \in \Omega \,:\, \tau_{u_3} \in [u_1, u_2], \quad \text{and} \quad \theta_{\tau_{u_3}} = h, \quad \text{and} \quad q_{\tau_{u_3},u_3} < 1\big\}$.

Lastly, suppose that the price is flat over $[u_2, u_3]$. Keeping in mind that the price is strictly decreasing over $[u_1, u_2]$, we note that any limit order to sell at the flat-spot price, $p_{u_2}$, submitted at $\tau_k \in [u_1, u_2]$ is executed before $u_2$. Therefore, $q_{\tau_{u_2},z}$ is increasing over $[u_2, u_3]$. This allows to write:

$$\Delta V \geq \mathbb{I}_{\{\tau_k \in [u_1,u_2)\}} \mathbb{I}_{\{\tau_{k+1} \geq u_4\}} \int_{u_2}^{u_3} e^{-rz} \big(1 - \min\{q_{\tau_k,u_3}, 1\}\big)\big(1 - rp_z + \dot{p}_z\big)\, dz.$$

and the same reasoning as above leads to $P(C_k) = 0$ and $P(C) = 0$, where $C = \big\{\omega \in \Omega \,:\, \tau_{u_3} \in [u_1, u_2], \quad \text{and} \quad \theta_{\tau_{u_3}} = h, \quad \text{and} \quad q_{\tau_{u_3},u_3} < 1\big\}$

**Proof of point** $(ii)$**.** The market clearing condition at time $u_3$ is:

$$
\begin{aligned}
0 &= \int_0^{u_3} \mathbb{E}_0 \left[q_{z,u_3} - s\right] e^{-\rho(u_3-z)}\, dz \\
&= \int_0^{u_1} \mathbb{E}_0 \left[q_{z,u_3} - s\right] e^{-\rho(u_3-z)}\, dz + \int_{u_1}^{u_3} \mathbb{E}_0 \left[q_{z,u_3} - s\right] e^{-\rho(u_3-z)}\, dz \\
&\geq \underbrace{\int_0^{u_1} \mathbb{E}_0 \left[q_{z,u_1} - s\right] e^{-\rho(u_3-z)}\, dz}_{=0} + \int_{u_1}^{u_3} \mathbb{E}_0 \left[q_{z,u_3} - s\right] e^{-\rho(u_3-z)}\, dz \\
&\geq \int_{u_1}^{u_3} (\mu_{hz} - s)\, e^{-\rho(u_3-z)}\, dz > 0,
\end{aligned}
$$

which is a contradiction. In the above manipulations, we used three facts. First, $q_{z,u_3} \geq q_{z,u_1}$ for all $z \leq 1$; second, the market clears at time $u_1$; and third, $q_{z,u_3} \geq 1$ for all $z \in [u_1, u_3]$ such that $\theta_z = 1$.

**Proof of Lemma VII.4.** Suppose the price is continuously differentiable in a neighborhood of $t > T_s$, with $\dot{p}_t < 0$ and $1 - rp_t + \dot{p}_t \neq 0$. Because $1 - rp_t + \dot{p}_t \geq 0$ by Corollary VII.2, it must be that $1 - rp_t + \dot{p}_t > 0$. Moreover, because the price is continuously differentiable in a neighborhood of $t$, and since $t > T_s$, these strict inequalites hold in an interval $[u_1, u_3]$ around $t$, with $u_1 > T_s$. Now, by Corollary VII.3, for all $u_2 \in (u_1, u_3)$, all high-valuation traders whose information process jumps during $[u_1, u_2)$ almost surely hold more than one unit at time $u_2$. Then, the market clearing condition at time $u_2$ writes as:

$$
\begin{aligned}
0 &= \int_0^{u_2} \mathbb{E}_0 \left[q_{z,u_2} - s\right] e^{-\rho(u_2-z)}\, dz \\
&= \int_0^{u_1} \mathbb{E}_0 \left[q_{z,u_2} - s\right] e^{-\rho(u_2-z)}\, dz + \int_{u_1}^{u_2} \mathbb{E}_0 \left[q_{z,u_2} - s\right] e^{-\rho(u-z)}\, dz \\
&\geq e^{-\rho(u_2-u_1)} \underbrace{\int_0^{t} \mathbb{E}_0 \left[q_{z,t} - s\right] e^{-\rho(u_1-z)}\, dz}_{=0} + \underbrace{\int_{u_1}^{u_2} \left[\mu_{hz} - s\right] e^{-\rho(u_2-z)}\, dz}_{>0} > 0,
\end{aligned}
$$

which is a contradiction. When moving from the second to the third line, we used three facts. First, since the price is strictly decreasing over $[u_1, u_2]$, asset holding plan are strictly increasing during this interval, and hence $q_{z,u_2} \geq q_{z,u_1}$ for all $z \leq u_1$. Second, the market clears at time $u_1$, implying that the first term on the right-hand side of the third line is zero. And third, $q_{z,u_2} \geq 0$ and, from Corollary VII.3, $q_{z,u_2} \geq 1$ if $\theta_z = h$.

### IX.5.5 Proof of Lemma VII.5

Consider some time $u_2 \in (T_s, T_f)$ and let $[u_1, u_3]$ be the maximal spot $[T_s, \infty)$ where $u_2$ belongs. Suppose $u_2$ is not a boundary point of the spot, i.e., $u_2 \in (u_1, u_3)$.

Keep in mind that, by Lemma VII.3, $\dot{p}_t \geq 0$ during $[u_1, u_3]$ and, by Lemma VII.2 and by definition of $T_f$, $p_t < 1/r$ during $[u_1, u_3]$. Therefore $1 - rp_t + \dot{p}_t > 0$ and Corollary VII.3 applies: all high-valuation traders with an information event during $[u_1, u_2]$ hold more than one unit at time $u_2$.

If $u_1 = T_s$, we are done with the lemma. If $u_1 < T_s$, consider traders with an information event during $(T_s, u_1)$. We start by noting that these traders' asset holding are decreasing over $[u_1, u_3]$: this is obvious if $[u_1, u_3]$ is an increasing spot. If $[u_1, u_3]$ is a flat spot, then there is an increasing spot to the left of $u_1$. Together with the fact that $\dot{p}_t \geq 0$, this implies that $p_t < p_{u_1}$ for all $t \in (T_s, u_1)$. Therefore, any limit order to buy at price $p_{u_1}$ submitted at $\tau_k \in (T_s, u_1)$ is executed immediately, implying in turns that, for any $\tau_k \in (T_s, u_1)$, $q_{\tau_k, z}$ is decreasing over $[u_1, u_3]$. Now, as usual, consider:

$$C_k = \Big\{\omega \in \Omega \,:\, \tau_k \in (T_s, u_1) \ \text{ and } \ \theta_{\tau_k} = h \ \text{ and } \ q_{\tau_k, u_2} < 1\big\},$$

and consider the deviation consisting in buying one unit at any $\tau_n \in (T_s, u_1)$ and holding it forever after. Then, proceeding as in the proof of Corollary VII.3, the expected net utility of the deviation satisfies $\mathbb{E}\left[\Delta V\right] \geq \mathbb{E}\left[X\right]$, where

$$X = \mathbb{I}_{\{\tau_k \in (T_s, u_1)\}} e^{-\rho(u_3 - \tau_k)} \int_{u_2}^{u_3} e^{-rz} \big(1 - \min\{q_{\tau_k, u_2}, 1\}\big)\big(1 - rp_z + \dot{p}_z\big)\, dz.$$

Clearly, $X \geq 0$ and $X > 0$ for $\omega \in C_k$. Our usual argument then implies that $P(C_k) = 0$ and $P(C) = 0$, where $C = \big\{\omega \in \Omega \,:\, \tau_{u_2} \in (T_s, u_1) \ \text{ and } \ \theta_{\tau_{u_2}} = h \ \text{ and } \ q_{\tau_{u_2}, u_2} < 1\big\}$.

### IX.5.6 Proof of Lemma VII.6

Assume that $T_f = \infty$ and pick some $t > T_s$. Lemma VII.5 shows that, at almost all times $u \in (t, \infty)$, for all $z \in (t, u)$, high–valuation trader who experience an information event at time $z$ hold more than one unit at time $u$, implying that $\mathbb{E}\left[q_{z,u} \,|\, \theta_z = h\right] \geq 1$. Plugging this into the market clearing condition, we obtain:

$$\begin{aligned}
0 &= \int_0^u \rho e^{-\rho(u-z)} \Big\{ (1 - \mu_{hz}) \mathbb{E}_0\left[q_{z,u} \,|\, \theta_z = \ell\right] + \mu_{hz} \mathbb{E}_0\left[q_{z,u} \,|\, \theta_z = h\right] - s \Big\}\, dz \\
&\geq -s \int_0^t \rho e^{-\rho(u-z)}\, dz + \int_t^u \rho e^{-\rho(u-z)} \left(\mu_{hz} - s\right)\, dz \\
&\geq -s e^{-\rho u}\left(e^{\rho t} - 1\right) + \left(\mu_{ht} - s\right) s \left(1 - e^{-\rho(u-t)}\right).
\end{aligned}$$

But the right-hand side converges to $\mu_{ht} - s > 0$ as $u \to \infty$ which is a contradiction of market clearing.

### IX.5.7 Proof of Lemma VII.7

Towards a contradiction assume that $rp_t - \dot{p}_t < 1 - \delta$. Because the price is continuously differentiable in a neighborhood of $t$ there is some neighborhood $[u_1, u_3]$ around $t$ such that $rp_z - \dot{p}_z < 1 - \delta$ and $\dot{p}_z > 0$ for all $z \in [u_1, u_3]$. We show that:

**RIX.14.** *Consider some time $u_2 \in (u_1, u_3)$. Then, for almost all $\omega \in \Omega$:*

*1. If $\tau_{u_2} \in [u_1, u_2]$, then $q_{\tau_{u_2},u_2} \geq 1$.*

*2. If $\tau_{u_2} \in (0, u_1)$, then $q_{\tau_{u_2},u_2} \geq q_{\tau_{u_1},u_1}$.*

Clearly, this result contradicts market clearing at time $u_2$.

**Proof of point 1 in Result IX.14.** Consider:

$$C_k \equiv \Big\{\omega \in \Omega \,:\, \tau_k \in [u_1, u_2] \;\text{ and } q_{\tau_k,u_2} < 1\Big\}$$

and the following deviation. Whenever your information process jumps during $[u_1, u_2]$ you choose to hold less than one unit of asset, choose instead to hold one unit until time $u_3$, and reverts to your original holding plan afterwards. That is, if $\tau_n \in [u_1, u_2)$ and $q_{\tau_n,\tau_n} < 1$, switch to $\hat{q}_{\tau_n,z} = 1$ for all $z \in [\tau_n, u_3)$, and place an order to sell $1 - q_{\tau_n,u_3}$ at time $u_3$. If the information process jumps again after $u_2$, just revert to the previous holding. This is feasible given that the price is strictly increasing over $[u_1, u_3]$: you can always re-submits the orders that were supposed to be executed during $[u_2, u_3]$.

If $\tau_n \in [u_1, u_2]$, your new holding plan is equal to $q_{\tau_n,z} + \max\{1 - q_{\tau_n,z}, 0\}$ for $z \in [\tau_n, u_3)$, and otherwise the holding plan is the same as before. Keeping in mind that the marginal utility of holding $q < 1$ asset is bounded below by $1 - \delta$, we find that the net utility of this deviation is greater than

$$\begin{aligned}\Delta V &\geq \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in [u_1,u_2]\}} \int_{\tau_n}^{u_3} e^{-rz} \mathbb{I}_{\{\tau_{n+1} \geq z\}} \left(1 - \delta - rp_z + \dot{p}_z\right) \max\{1 - q_{\tau_n,z}, 0\}\, dz \\ &\geq \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in [u_1,u_2]\}} \int_{u_2}^{u_3} \mathbb{I}_{\{z \leq \tau_{n+1}\}} e^{-rz} \left(1 - \delta - rp_z + \dot{p}_z\right) \max\{1 - q_{\tau_n,u_2}, 0\}\, dz,\end{aligned}$$

where the second line follows because asset holding are decreasing over $[u_1, u_3]$. After taking expectations conditional on $\mathcal{F}_{\tau_n}$, one finds that $\mathbb{E}\left[\Delta V\right] \geq \mathbb{E}\left[X\right]$, where

$$X = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in [u_1,u_2]\}} \int_{u_2}^{u_3} e^{-rz - \rho(z - \tau_n)} \left(1 - \delta - rp_z + \dot{p}_z\right) \max\{1 - q_{\tau_n,u_2}, 0\}\, dz.$$

Given the maintained assumption that $rp_t - \dot{p}_t < 1 - \delta$, the right-hand side is always positive, and strictly positive for all $\omega \in C_k$. As usual this implies that $P(C_k) = 0$ and $P(C) = 0$, where $C \equiv \big\{\omega \in \Omega \,:\, \tau_{u_2} \in [u_1, u_2] \;\text{ and } q_{\tau_u,u_2} < 1\big\}$.

**Proof of point 2 in Result IX.14.** Consider:

$$C_k = \Big\{\omega \in \Omega \,:\, \tau_k \in (0, u_1) \;\text{ and }\; q_{\tau_k,u_2} < q_{\tau_k,u_1}\Big\}$$

and the following deviation. Whenever you have an information event at time $\tau_n \in (0, u_1)$, and a plan such that $q_{\tau_n,u_2} < q_{\tau_n,u_1}$, cancel all limit sell orders executed during $[u_1, u_3)$, and replace them by limit sell orders executed at time $u_3$. At your first information event in $[u_1, u_3)$, if any, revert to your original holding plan. This is feasible because the price is strictly decreasing, implying that you can always re-submits the limit sell orders that were supposed to be executed during $[u_1, u_3)$. This deviation results in the holding plan $\hat{q}_{\tau_n,z} = q_{\tau_n,u_1}$ for $z \in [u_1, u_3)$, if $\tau_n \in (0, u_1)$ and $q_{\tau_n,u_2} < q_{\tau_n,u_1}$. Otherwise, $\hat{q}_{\tau_n,z} = q_{\tau_n,z}$. Thus, the utility of the deviation is bounded below by:

$$\begin{aligned}
\Delta V &\geq \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in (0,u_1) \text{ and } q_{\tau_n,u_2} < q_{\tau_n,u_1}\}} \int_{u_1}^{u_3} e^{-rz} \mathbb{I}_{\{\tau_{n+1} \geq z\}} \left(1 - \delta - rp_z + \dot{p}_z\right) \left(q_{\tau_n,u_1} - q_{\tau_n,z}\right) dz \\
&\geq \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in (0,u_1) \text{ and } q_{\tau_n,u_2} < q_{\tau_n,u_1}\}} \int_{u_2}^{u_3} e^{-rz} \mathbb{I}_{\{\tau_{n+1} \geq z\}} \left(1 - \delta - rp_z + \dot{p}_z\right) dz \times \left(q_{\tau_n,u_1} - q_{\tau_n,u_2}\right)
\end{aligned}$$

where the second line follows because the price is strictly increasing and thus asset holdings must be decreasing over $[u_1, u_3]$. The same reasoning as before implies that $P(C_k) = 0$.

### IX.5.8 Proof of Lemma VII.8 and Lemma VII.9

**Preliminary results** Suppose that the price is continuously differentiable and strictly decreasing in some left neighborhood of some $u_1$, then is flat until some $u_2 \geq u_1$, and then is continuously differentiable and strictly increasing in some right neighborhood of $u_2$. Let

$$q_u^* \equiv \left[\frac{1 - rp_u + \dot{p}_u}{\delta(1 - \mu_{hu})}\right]^{1/\sigma} \tag{IX.21}$$

$$q_{t,u}^* \equiv \min\left\{(1 - \mu_{ht})^{1/\sigma} q_u^*, 1\right\}. \tag{IX.22}$$

If $p_u$ has a kink at $u_1$ and $u_2$, let $q_u^* \equiv 1/2\left(q_{u^+}^* + q_{u^-}^*\right)$. We then show that:

**RIX.15** (Ideal Holding)**.** *Consider some $t \leq u$ such that $\dot{p}_u$ exists. Then $q_{t,u}^*$ maximizes*

$$\mathbb{E}_t\left[v(\theta_u, q) \,|\, \theta_t = \ell\right] - (rp_u - \dot{p})q. \tag{IX.23}$$

Indeed, this objective is concave and equal to:

$$\text{if } q \leq 1 : q - \delta\frac{1 - \mu_{hu}}{1 - \mu_{ht}}\frac{q^{1+\sigma}}{1+\sigma} - \left(rp_u - \dot{p}_u\right)q \tag{IX.24}$$

$$\text{if } q > 1 : 1 - \delta\frac{1 - \mu_{hu}}{1 - \mu_{ht}}\frac{1}{1+\sigma} - \left(rp_u - \dot{p}_u\right)q. \tag{IX.25}$$

and the result follows by taking first-order conditions.

**RIX.16.** *There is $u_1' < u_1$ and $u_2' > u_2$ such that*

- $q_u^* < q_{u_1}^*$ *for all* $u \in [u_1', u_1)$*;*
- $q_u^* > q_{u_1}^*$ *for all* $u \in (u_1, u_2']$*;*

- $p_{u_1'} = p_{u_2'}$.

Let us start with $u_1'$ in the first bullet point. There are two cases to consider. First, if the price has a kink at $u_1$, then the result is obvious because $q^*_u$ jumps up at $u_1$ and is continuous in a left-neighborhood of $u_1$. If the price is differentiable at $u_1$, then since $u_1$ is a local minimum we have $\dot{p}_{u_1} = 0$. Thus, for $u$ in a left neighborhood of $u_1$, we have:

$$1 - rp_u + \dot{p}_u < 1 - rp_{u_1} + 0 = 1 - rp_{u_1} + \dot{p}_{u_1},$$

and, at the same time $1 - \mu_{hu} > 1 - \mu_{hu_1}$. Taken together, these two inequalities imply the result.

Now turn to $u_2'$. If the price has a kink at $u_1$, then the result is obvious because $\dot{p}_u$ jumps up at $u_1$ and $1/(1 - \mu_{hu})$ is strictly increasing. If the price has a flat spot, $u_1 < u_2$, then the result follows because, for $u \in (u_1, u_2)$,

$$\frac{1 - rp_u + \dot{p}_u}{1 - \mu_{hu}} = \frac{1 - rp_{u_1}}{1 - \mu_{hu}}$$

is strictly increasing, so $q^*_{u_2^-} > q^*_{u_1}$. Then, continuing after $u_2$, $q^*_u$ either jumps up at $u_2$, or is continuous. In either case, one can find a right neighborhood of $u_2$ where $q^*_u > q^*_{u_1}$. Lastly, if the price has no flat spot and no kink at $u_1$, then, for $u$ in a right neighborhood of $u_1$, because $\dot{p}_u \geq 0$:

$$\frac{1 - rp_u + \dot{p}_u}{1 - \mu_{hu}} \geq \frac{1 - rp_u}{1 - \mu_{hu}}.$$

with an equality for $u = u_1$. After taking derivative of the right-hand side at $u = u_1$, using $\dot{p}_{u_1} = 0$ and $\dot{\mu}_{hu_1} > 0$, one finds that the right-hand side is strictly increasing at $u = u_1$, and the result follows.

Since $p_t$ is strictly increasing in $[u_1', u_1)$ and $(u_2, u_2']$ we can without loss of generality pick $u_1'$ and $u_2'$ such that $p_{u_1'} = p_{u_2'}$.

**Holdings before $u_1$.** We start by establishing an upper bound on low-valuation investor holdings at $u_1$:

**R IX.17** (A bound on low-valuation holdings at $u_1$)**.** *Suppose $\tau_{u_1} \in [u_1', u_1)$, $\theta_{\tau_{u_1}} = \ell$. Then, for all $u \in [u_1', u_1)$, $q_{\tau_{u_1},u} \leq q^*_{\tau_{u_1},u_1}$, almost surely.*

To see this, fix some $\varepsilon > 0$ and consider the event such that the property of the Result is violated for $u \in [u_1', u_1 - \varepsilon)$:

$$C_{\varepsilon,k} = \left\{\omega \in \Omega \,:\, \tau_k \in [u_1', u_1), \;\text{ and }\; \theta_{\tau_k} = \ell, \;\text{ and }\; \exists\, u \in [u_1', u_1 - \varepsilon) \text{ s.t. } q_{\tau_k,u} > q^*_{\tau_k,u_1}\right\},$$

and the following deviation. Whenever the information process jumps at some time $\tau_n \in [u_1', u_1 - \varepsilon)$ when the investor has a low valuation, we let $\phi_n$ be the earliest time in $[\tau_n, u_1 - \varepsilon)$ such that $q_{\tau_n,u} \geq q^*_{\tau_n,u_1}$, with $\phi_n = u_1 - \varepsilon$ if no such time exists. Then, stop buying for all $u \in [\phi_n, u_1 - \varepsilon)$, hold the quantity $q^*_{\tau_n,u_1}$, and revert to your original holding plan at $u_1-\varepsilon$: this is achieved by canceling part of the limit buy orders that you had planed to execute at $\phi_n$, canceling all the limit buy orders that would have been executed during $(\phi_n,u_1-\varepsilon)$, and placing a limit order to buy so as to hold quantity $q_{\tau_n,u_1-\varepsilon}$ at time $u_1-\varepsilon$. Note that this is feasible because any limit buy order you cancel during the decreasing spot $[u_1',u_1)$ can be re-submitted at the next information event time.

The change in utility is:

$$\Delta V = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in [u_1',u_1-\varepsilon) \text{ and } \theta_{\tau_n}=\ell\}} \\ \times \int_{\phi_n}^{u_1-\varepsilon} e^{-rz}\mathbb{I}_{\{\tau_{n+1}\geq z\}}\left[v(\theta_z,q^*_{\tau_n,u_1}) - v(\theta_z,q_{\tau_n,z}) - (rp_z-\dot{p}_z)\left(q^*_{\tau_n,u_1}-q_{\tau_n,z}\right)\right]dz.$$

Taking the expectation of term $n$ conditional on $\mathcal{F}_{\tau_n}$, we find that $\mathbb{E}_0\left[\Delta V\right]\geq \mathbb{E}_0\left[X\right]$, where

$$X = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in [u_1',u_1-\varepsilon) \text{ and } \theta_{\tau_n}=\ell\}} \\ \times \int_{\phi_n}^{u_1-\varepsilon} e^{-rz}e^{-\rho(z-\tau_n)}\left[\mathbb{E}_{\tau_n}\left[v(\theta_z,q^*_{\tau_n,u_1})\right] - \mathbb{E}_{\tau_n}\left[v(\theta_z,q_{\tau_n,z})\right] - (rp_z-\dot{p}_z)\left(q^*_{\tau_n,u_1}-q_{\tau_n,z}\right)\right]dz.$$

Now by Result IX.16 $q^*_{\tau_n,z}<q^*_{\tau_n,u_1}$, by construction $q^*_{\tau_n,u_1}\leq q_{\tau_n,z}$ and because asset holding plan are increasing $q_{\tau_n,z}\leq q_{\tau_n,u_1}$. Taken these inequalities together gives:

$$q^*_{\tau_n,z}<q^*_{\tau_n,u_1}\leq q_{\tau_n,z}\leq q_{\tau_n,u_1}. \tag{IX.26}$$

Now keep in mind that, given $\theta_n=\ell$, $\mathbb{E}_{\tau_n}\left[v(\theta_z,q)\right]=\mathbb{E}\left[v(\theta_z,q)\,|\,\theta_{\tau_n}=\ell\right]$, the function (IX.23) we studied earlier. Because this function is concave and, by Result IX.15, maximized at $q^*_{\tau_n,u_1}$, it follows that the integrand in each terms of $X$ is positive. Clearly, $X$ is strictly positive for all $\omega\in C_\varepsilon$ since, in that case $q^*_{\tau_n,u_1}<q_{\tau_n,z}$ for $z\in(\phi_n,u_1-\varepsilon)$. As usual, this implies that $P(C_{\varepsilon,k})=0$ and $P(C_\varepsilon)=0$, where $C_\varepsilon=\left\{\omega\in\Omega\,:\,\tau_{u_1}\in[u_1',u_1),\;\text{ and }\;\theta_{\tau_{u_1}}=\ell,\;\text{ and }\;\exists u\in[u_1',u_1-\varepsilon)\text{ s.t. }q_{\tau_{u_1},u}>q^*_{\tau_{u_1},u_1}\right\}$. The final result then follows by letting $\varepsilon$ go to zero.

**Holding after $u_1$.** Next, we show:

**R IX.18** (Holdings of low-valuation after $u_1$ with no flat spot)**.** *Consider the case without flat spot, i.e., $u_1=u_2$. Suppose $\tau_{u_1}\in[u_1',u_1)$, $\theta_{\tau_{u_1}}=\ell$, and $q^*_{\tau_{u_1},u_1}<1$. Then $q_{\tau_{u_1},u_1}>q^*_{\tau_{u_1},u_1}$ almost surely.*

Consider the event such that the property of the Result is violated for $u\in[u_1,u_2')$:

$$C_k=\left\{\omega\in\Omega\,:\,\tau_k\in[u_1',u_1),\;\text{ and }\;\theta_{\tau_k}=\ell,\;\text{ and }\;q^*_{\tau_k,u_1}<1,\;\text{ and }\;q_{\tau_k,u_1}\leq q^*_{\tau_k,u_1}\right\},$$

and the following deviation. Whenever the investor has an information event at some $\tau_n\in[u_1',u_1)$ with a low valuation, if $q_{\tau_n,u_1}\leq q^*_{\tau_n,u_1}$, then he increases his asset holding by some small $y_n$ during

$[u_1, u_2')$. The deviation is implemented by submitting a limit buy order at price just above $p_{u_1}$ for $y_n > 0$ unit of the asset and a limit order to sell back these $y_n$ unit of the asset at time $u_2'$. The first part of the deviation is feasible since the price is strictly decreasing over $[\tau_n, u_1]$; the second part of the deviation is also feasible since $u_2'$ is the first time after $\tau_n$ such that the price crosses $p_{u_2'}$ from below. If the information process jumps again before $u_2'$, then the investor reverts to the original holding plan by canceling the limit order to buy $y_n$ executed at time $u_1$ (if the information event is before $u_1$), or selling $y_n$ (if the information event is after $u_1$), and canceling the limit buy order executed at $u_2'$. The change in utility is:

$$\Delta V = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in [u_1', u_1] \text{ and } \theta_{\tau_n} = \ell \text{ and } q_{\tau_n, u_1} \leq q^*_{\tau_n, u_1}\}}$$
$$\times \int_{u_1}^{u_2'} e^{-rz} \mathbb{I}_{\{\tau_{n+1} \geq z\}} \left\{ v(\theta_z, q_{\tau_n, z} + y_n) - v(\theta_z, q_{\tau_n, z}) - (rp_z - \dot{p}_z) y \right\} dz.$$

Taking expectation of term $n$ in the sum conditional on $\tau_n$ we obtain that $\mathbb{E}_0 [\Delta V] = \mathbb{E}_0 [X]$, where

$$X = \sum_{n=1}^{\infty} \mathbb{I}_{\{\tau_n \in [u_1', u_1] \wedge \theta_{\tau_n} = \ell \wedge q_{\tau_n, u_1} \leq q^*_{\tau_n, u_1}\}}$$
$$\times \int_{u_1}^{u_2'} e^{-rz} e^{-\rho(z - \tau_n)} \left\{ \mathbb{E}_{\tau_n} \left[ v(\theta_z, q_{\tau_n, z} + y_n) - v(\theta_z, q_{\tau_n, z}) \right] - (rp_z - \dot{p}_z) y_n \right\} dz,$$

where used the independence of the information event time $\tau_{n+1}$ and the preference type $\theta_z$ processes. Using that for $z \in [u_1, u_2')$, $q_{\tau_n, z} \leq q_{\tau_n, u_1} \leq q^*_{\tau_n, u_1} < 1$ and that $v(q)$ is concave for $q < 1$, we obtain that for $y_n$ small enough each integral is greater than

$$y_n \times \int_{u_1}^{u_2'} e^{-rz} e^{-\rho(z - \tau_n)} \left\{ \mathbb{E}_{\tau_n} \left[ v_q(\theta_z, q^*_{\tau_n, u_1} + y_n) \right] - (rp_z - \dot{p}_z) \right\} dz.$$

When $y_n$ goes to zero, each integral converges to:

$$\int_{u_1}^{u_2'} e^{-(r+\rho)z} \left[ \mathbb{E}_{\tau_n} \left[ v_q(\theta_z, q^*_{\tau_n, u_1}) \right] - (rp_z - \dot{p}_z) \right] dz.$$

When the indicator function of term $n$ in $X$ is not zero, we have $q^*_{\tau_n, u_1} < 1$ and, by Result IX.16, $q^*_{\tau_n, z} > q^*_{\tau_n, u_1}$. Because (IX.23) is concave and uniquely maximized at $q^*_{\tau_n, z}$, and because it has a continuous and strictly decreasing derivative for $q < 1$, it follows that, for $z \in (u_1, u_2')$, $\mathbb{E}_{\tau_n} \left[ v_q(\theta_z, q^*_{\tau_n, u_1}) \right] - (rp_z - \dot{p}_z) > 0$. Thus, when the indicator function of term $n$ in $X$ is not zero, then at time $\tau_n$ it is possible to pick $y_n$ small enough so that term $n$ is strictly positive. It then follows that $X$ is positive, and strictly positive for $\omega \in C$, implying in turns that $P(C) = 0$.

**R IX.19** (Holdings of low-valuation after $u_1$ with flat spot). *Consider the case with flat spot, i.e., $u_1 < u_2$, and assume that investors find it optimal to follow Markovian holding plans. Then the result of Result IX.18 continues to hold.*

The proof is almost identical to that of Result IX.18, except that the deviation is slightly different. Suppose the investor has an information event at some $\tau_n \in [u_1', u_1)$ with a low valuation, and plans to hold $q_{\tau_n,u_1} \leq q^*_{\tau_n,u_1} < 1$. There are cases to consider.

The first case is when the original holding plan does not have any limit buy order executed during the flat spot, $(u_1, u_2]$. Then, as before, the investor increases the asset holding by some small $y_n$ during $[u_1, u_2')$ by submitting a limit buy order at price just above $p_{u_1}$ for $y_n > 0$ unit of the asset and a limit order to sell back these $y$ unit of the asset at time $u_2'$. If the information process jumps again before $u_2'$ then the investor reverts to the original holding plan.

The second case is when the original holding plan has some limit buy order executed during the flat spot. Let $\phi_n \in (u_1, u_2]$ be earliest execution time of the collection of limit buy orders executed during $(u_1, u_2)$. Then the investor increases the asset holding by some small $y_n$ during $[u_1, \phi_n)$ by submitting a limit buy order at price just above $p_{u_1}$ for $y_n > 0$ unit of the asset and reducing the size of the limit order to buy at time $\phi_n$ by the same amount. If there is an information event before $u_2'$ then the investor reverts to the original holding plan. A potential difficulty in doing so is that, in this deviation, the investor cancels some limit orders that were supposed to be executed at time $\phi_n$, and it may not be possible to re-submit these orders because of the time priority rule. This is where we use the assumption that the holding plan is Markovian: when the investor has an information event, the original continuation holding plan only depends on the type at the information event, so it is the same as if the investor had an information event for the first time. Thus it is always possible to revert to the original holding plan: it suffices to cancel all previously submitted limit order and behave "as if" it was the first jump of the information process. The rest of the proof is identical after replacing $u_2'$ by $\phi_n$.

**R IX.20** (High-valuation holdings after $u_1$)**.** *Suppose* $\tau_{u_1} \geq u_1'$, $\theta_{\tau_{u_1}} = h$*. Then, for all* $u \in (u_1, u_2)$, $q_{\tau_{u_1},u} \geq 1$*, almost surely.*

To see this, consider, for any $u \in (u_1, u_2')$,

$$C_k = \left\{ \omega \in \Omega \,:\, \tau_k \in [u_1', u_1), \quad \text{and} \quad \theta_{\tau_k} = h, \quad \text{and} \quad q_{\tau_k,u} < 1 \right\}.$$

Note that since holdings are increasing over $[u_1, u_2')$, we also have that $q_{\tau_k,u_1} < 1$. Now make the following deviation. Whenever the investor has an information event at time $\tau_n \in [u_1', u_1)$ and plans on $q_{\tau_n,u_1} < 1$, he switches to a holding plan that holds one units forever after time $u_1$. This is made feasible by submitting a limit order to buy $(1 - q_{\tau_n,u_1})$ units at time $u_1$, and keep that unit position forever after. After switching to this new plan, the change in utility flow at time $z > u_1$ can be written:

$$\max\{1 - q_{\tau_z,z}, 0\}\,(1 - rp_z + \dot{p}_z) + \max\{q_{\tau_z} - 1, 0\}\,(rp_z - \dot{p}_z)\,,$$

which is always positive given that $\dot{p}_z \geq 0$ and $p_z < 1/r$ during $(u_1, u_2')$. Clearly, this utility flow is always positive, and it is strictly positive if $\omega \in C_k$ and if there are no information event during $[u_1, u]$.

Thus,

$$\Delta V \geq \mathbb{I}_{\{\omega \in C_k \text{ and } \tau_{k+1} > u\}} \left(1 - q_{\tau_k, u}\right) \int_{u_1}^{u} \left(1 - r p_z + \dot{p}_z\right) e^{-rz} \, dz.$$

Taking expectations conditional on $\mathcal{F}_{\tau_k}$, we obtain that the expected change in utility is greater than $\mathbb{E}[X]$, where

$$X = e^{-\rho(u - \tau_k)} \mathbb{I}_{\{\omega \in C_k\}} \left(1 - q_{\tau_{u_1}, u}\right) \int_{u_1}^{u} \left(1 - r p_z + \dot{p}_z\right) e^{-rz} \, dz,$$

Clearly this is positive and strictly positive whenever $\omega \in C_k$, which as usual implies that $P(C_k) = 0$ and $P(C) = 0$, where $C = \left\{\omega \in \Omega \,:\, \tau_{u_1} \in [u_1', u_1), \text{ and } \theta_{\tau_{u_1}} = h, \text{ and } q_{\tau_{u_1}, u} < 1\right\}$.

**Concluding the proof when $q^*_{u_1, u_1} < 1$.** If $q^*_{u_1, u_1} < 1$, then we can choose $u_1'$ close enough to $u_1$ so that $q^*_{u_1', u_1} < 1$. Therefore for all $\tau_{u_1} \geq u_1'$ we have $q^*_{\tau_{u_1}, u_1} < 1$ and Results IX.17, IX.18 and IX.19 apply: if $\tau_{u_1} \geq u_1'$ and $\theta_{\tau_{u_1}} = \ell$, then $q_{\tau_{u_1}, u} \leq q^*_{\tau_{u_1}, u_1}$ for $u < u_1$ and $q_{\tau_{u_1}, u_1} > q^*_{\tau_{u_1}, u_1}$ almost surely. Otherwise, $q_{\tau_{u_1}, u_1} \geq q_{\tau_{u_1}, u}$. Now consider the market clearing condition (8), page 15 in BHW, at $u_1$ (re-scaled by $e^{\rho u_1}/\rho$), and subtract the corresponding market clearing condition at $u \in [u_1', u_1]$ (re-scaled by $e^{\rho u}/\rho$). One obtains:

$$\begin{aligned}
0 &= \int_0^{u_1'} e^{\rho t} \mathbb{E}\left[q_{t, u_1} - q_{t, u}\right] dt + \int_{u_1'}^{u} e^{\rho t} \mathbb{E}\left[q_{t, u_1} - q_{t, u}\right] dt + \int_{u}^{u_1} e^{\rho t} \mathbb{E}\left[q_{t, u_1} - s\right] dt \\
0 &\geq \int_{u_1'}^{u} e^{\rho t} \mathbb{E}\left[\left(q_{t, u_1} - q_{t, u}\right) \mathbb{I}_{\{\theta_t = \ell\}}\right] dt + \int_{u}^{u_1} e^{\rho t} \mathbb{E}\left[q_{t, u_1} - s\right] dt \\
0 &\geq \int_{u_1'}^{u} e^{\rho t} \mathbb{E}\left[\left(q_{t, u_1} - q^*_{t, u_1}\right) \mathbb{I}_{\{\theta_t = \ell\}}\right] dt - \int_{u}^{u_1} e^{\rho t} s \, dt,
\end{aligned}$$

where, in the second line, we used the fact that asset holding have to be increasing over $[u_1', u_1]$, and so $q_{t, u_1} \geq q_{t, u}$ for all $t \leq u$. In the third line, we used Result IX.17. Now the right-hand side of the third line is made up of two terms. Because, by Results IX.18 and IX.19, $q_{t, u_1} - q^*_{t, u_1} > 0$, the first term is strictly positive and increasing. The second term, on the other hand, goes to zero as $u$ goes to $u_1$. Therefore, for $u$ close enough to $u_1$, the right-hand side is strictly positive, which is a contradiction.

**Concluding the proof when $q^*_{u_1, u_1} = 1$** Now suppose that $q^*_{u_1, u_1} = 1$. Because $q^*_{t, u_1}$ is bounded above by 1 and weakly decreasing in $t$, it follows that $q^*_{\tau_{u_1}, u_1} = 1$ for all $\tau_{u_1} \geq u_1'$. It is then straightforward to adapt the proof of Results IX.18 and IX.19 to show that, if $\tau_{u_1} \geq u_1'$, then $q_{\tau_{u_1}, u_1} = 1$ almost surely. Otherwise increasing the asset holding by a small amount between $u_1$ and $u_2'$ (or $\phi_n$) is a strictly profitable deviation.

Together with Result IX.20 this implies that the asset demand originating from investors whose information process jumps during $[u_1', u_1]$ is greater than 1 times the measure of investors whose information process jumps during that time period: $\int_{u_1'}^{u_1} \rho e^{-\rho(u_1 - z)} \, dz$. But the supply of asset is $s$

times this measure of investor and $s < 1$, and no limit sell orders can be executed during $(u_1', u_1)$ since the price is strictly decreasing. Clearly, this means that demand exceed supply, and contradicts market clearing.

### IX.5.9 Proof of Lemma VII.11

Consider first the set:

$$C_k = \Big\{(u,\omega) \in [0,T_f) \times \Omega \,:\, \tau_k \in (0,u] \;\text{ and } \theta_{\tau_k} = h \text{ and } q_{\tau_k,u} \neq 1\Big\}$$

and the deviation consisting in holding one unit whenever the investor has an information event before $T_f$ with a high valuation. The change in flow utility at time $u$ is:

$$(1 - rp_u + \dot{p}_u) \max\{1 - q_{\tau_k,u}, 0\} + (rp_u - \dot{p}_u) \max\{q_{\tau_k,u} - 1, 0\}.$$

Note that, because the price is weakly increasing and strictly less than $1/r$ for $u \in [0,T_f)$, it follows that $1-rp_u+\dot{p}_u > 0$. Also, by Lemma VII.7, $rp_u-\dot{p}_u > 0$. Taken together, these imply that the change in utility flow is positive, and strictly positive for all $(u,\omega) \in C_k$. As usual, this implies that the set $C_k$ is of measure zero, as well as the set $C = \big\{(u,\omega) \in [0,T_f) \times \Omega \,:\, \tau_u \in (0,u] \;\text{ and } \theta_{\tau_u} = h \text{ and } q_{\tau_u,u} \neq 1\Big\}$

Next consider the set:

$$C_k = \Big\{(u,\omega) \in [T_f,\infty) \times \Omega \,:\, \tau_k \in (0,u] \;\text{ and }\; \theta_{\tau_k} = h \;\text{ and }\; q_{\tau_k,u} > 1\Big\}$$

and proceeding exactly as before shows that this set is of measure zero.

### IX.5.10 Proof of Lemma VII.12

Consider the set

$$C_k = \Big\{(u,\omega) \in [T_f,\infty) \times \Omega \,:\, \tau_k \in (0,u], \;\text{ and }\; \theta_{\tau_k} = \ell, \;\text{ and }\; q_{\tau_k,u} > 0\Big\}$$

and the deviation consisting in holding zero from $t \vee T_f$, and continuing to hold zero as long as the investor keep having jumps of his information process with a low-valuation. If he has an information event with a high valuation, he switches to the optimal strategy of buying one unit if the information event time is less than $T_f$. This deviation is feasible. Suppose indeed that the information event time is before $T_f$. Then since $T_f$ is by construction the first time that the price reaches $1/r$, the price must be strictly increasing in a left-neighborhood of $T_f$, implying that the low-valuation investor can always submit a limit order executed just before $T_f$. If, on the other hand, the information event time is after

$T_f$ the deviation is implemented by selling all the holding. The net utility of the deviation is:

$$\Delta V = \sum_{n=1}^{\infty} \mathbb{I}_{\{\theta_{\tau_n}=\ell\}} \int_{\tau_n}^{\tau_{n+1}} \mathbb{I}_{\{z \geq T_f\}} e^{-rz} \left(-v_q(\theta_z, q_{\tau_n,z}) + 1\right) q_{\tau_n,z}\, dz.$$

For each term in the sum, we take expectations conditional on $\mathcal{F}_{\tau_n}$ and obtain that $\mathbb{E}_0\left[\Delta V\right] = \mathbb{E}\left[X\right]$ where

$$X = \sum_{n=1}^{\infty} \mathbb{I}_{\{\theta_{\tau_n}=\ell\}} \int_{\tau_n}^{\infty} \mathbb{I}_{\{z \geq T_f\}} e^{-rz} e^{-\rho(z-\tau_n)} \left(-\mathbb{E}_{\tau_n}\left[v_q(\theta_z, q_{\tau_n,z})\right] + 1\right) q_{\tau_n,z}\, dz.$$

The integrand is positive, and strictly positive for all $(\omega, u) \in C$, implying that $C_k$ is of measure zero, as is the set $C = \big\{(u,\omega) \in [T_f, \infty) \times \Omega \,:\, \tau_u \in (0,u], \text{ and } \theta_{\tau_u} = \ell, \text{ and } q_{\tau_u,u} > 0\big\}$.

### IX.5.11 Proof of Lemma VII.13

First note that any flat spot before $T_f$ has to be followed by an increasing spot since the price has to increase up to $1/r$ by time $T_f$. Consider, then, a flat spot $[T_1, T_2)$, and let $\mathcal{S} \subseteq [T_1, T_2)$ denote the set of times where the holding plan of low–valuation trader solves the maximization problem of Lemma VII.10. We know from the lemma that this set of time has full measure. We first show:

**R IX.21.** *Suppose that for some $t \in \mathcal{S}$, a limit buy order at price $p_t$ is executed at some $u \in (t, T_2)$. Then, in a Markov equilibrium, for a low-valuation trader the information event at time $t$, $q_{t,t} < 1$ implies that $q_{t,u} > q_{t,t} > 0$. Moreover, $q_{t,t}$ satisfies:*

$$\int_t^u e^{-(r+\rho)z} \left[1 - \frac{1-\mu_{hz}}{1-\mu_{ht}} \delta q_{t,t}^{\sigma} - (rp_z - \dot{p}_z)\right] dz = 0 \tag{IX.27}$$

Note first that, by definition of a Markov equilibrium, a trader who has an information event at time $t$ with $\theta_t = \ell$ behaves as if she has an information event for the first time, i.e. as if he had no limit order outstanding in the book. Since a limit sell order at price $p_t$ is executed immediately at time $t$, the only way she can change her asset holding during $[t, T_2)$ is by submitting a limit buy order at price $p_t$, which ends up executed at time $u$. In other words, the trader's holding plan is either constant over $[t, T_2)$, or constant over $[t, u)$ and $[u, t_2)$, with $q_{t,t} < q_{t,u}$. Next, note that since $v_q(0) = 1 > rp_z$, the trader finds it optimal to hold $q_{t,t} > 0$. Second, the trader finds it optimal to submit a limit buy order executed at time $u$, i.e. $q_{t,u} > q_{t,t}$. Otherwise, suppose holdings were constant in $[t, T_2)$, and consider the deviation consisting in reducing holding by a marginal unit at $t$ and submitting a limit buy order for that marginal unit unit at $u$. The utility of the deviation must be negative, which by Lemma VII.10 leads to:

$$F(u) \geq 0, \quad \text{where} \quad F(v) \equiv \int_t^v e^{-(r+\rho)z} \left[1 - \frac{1-\mu_{hz}}{1-\mu_{ht}} \delta q_{t,t}^{\sigma} - (rp_z - \dot{p}_z)\right] dz.$$

Note that, because $\dot{p}_z = 0$ in $[t, u]$, $e^{(r+\rho)v}F'(v)$ is a strictly increasing function of $v$. Moreover, $F(t) = 0$. So if $e^{(r+\rho)t}F'(t) \geq 0$, we have that $e^{(r+\rho)v}F'(v) > 0$ and thus $F'(v) > 0$ for all $v > t$. Or $e^{(r+\rho)t}F'(t) < 0$ and $F(u) \geq 0$ requires that $F'(u) > 0$ – otherwise $e^{(r+\rho)v}F'(v) \leq 0$ and thus $F'(v) \leq 0$ for all $v \in [t, u]$ with a strict inequality close to $v = t$, which would imply that $F(u) < 0$. In both cases $F'(v) > 0$ for all $v \in [u, T_2]$, which implies that

$$F(T_2) - F(u) = \int_u^{T_2} e^{-(r+\rho)z} \left[1 - \frac{1 - \mu_{hz}}{1 - \mu_{ht}} \delta q_{t,t}^{\sigma} - (rp_z - \dot{p}_z)\right] dz > 0.$$

meaning that the trader can improve her utility by submitting a limit buy order for a marginal unit executed at $u$ and a limit order to sell this marginal unit at time $T_2$.

The equality (IX.27) follows from considering two deviations: *i*) reducing holdings by a marginal unit at $t$, and submitting a limit buy order for that marginal unit at $u$, and *ii*) increasing holdings by a marginal unit at $t$, and reducing the limit buy order $q_{t,u} - q_{t,t} > 0$ by a marginal unit executed at time $u$.

Next, we show that:

**RIX.22.** *Suppose that, at some time $t_1 \in [T_1, T_2)$ a limit order to buy at the flat-spot price is executed at time $u_1 > t_1$. Then, there is a positive measure set of times $\mathcal{T} \subseteq (t, u)$ such that, for a low–valuation trader with information event during $\mathcal{T}$, $q_{t,t} < 1$.*

Otherwise, since the equilibrium is Markov, low-valuation traders with an information event at almost all $\in (t_1, u_1)$ choose $q_{t,t} \geq 1$. Besides, at all times $t \in (t_1, u_1)$ there is a limit buy order outstanding in the book – otherwise, by time priority, a limit order submitted at time $t_1$ would be executed strictly before $u_1$. Thus no limit sell order are executed because otherwise this would violate volume maximization, and asset holdings cannot be decreasing. Therefore, for all $t_1 < t \leq u < u_1$ and all low–valuation tradersm $q_{t,u} = 1$, and for all $t \leq t_1 < u < u_1$, $q_{t,u} \geq q_{t_1,u}$. Moreover, for all high–valuation traders we know from Lemma VII.11 that $q_{t,u} = 1$. Taken together, this contradicts market clearing.

**R IX.23.** *Suppose that at some time $t_1 \in \mathcal{S}$, a low-valuation trader submits a limit order to buy executed at time $u_1 > t_1$. Then, for all times $t_2 \in (t_1, u_1)$, limit orders at price $p_{t_2}$ are executed at times $u_2 > u_1$.*

First, by time priority, given that $p_{t_2} = p_{t_1}$, we must have that the execution time $u_2$ is greater than $u_1$ and is increasing the submission time $t_2$. To show that the inequality is strict, suppose that, for some $t_2 \in (t_1, u_1)$, the execution time of a limit buy order at price $p_{t_2} = p_{t_1}$ is $u_2 = u_1$. Then by time priority this is also the case of all $t_2' \in (t_1, t_2)$. Now consider the positive set of times $\mathcal{T} \subseteq (t_1, t_2)$, given by Result IX.22, such that, for a low–valuation trader, $q_{t,t} < 1$. It then follows from Result IX.21 that all low-valuation traders with an information event at time $t \in \mathcal{T} \cap \mathcal{S}$ find it optimal to increase their holdings by a strictly positive amount at the common execution time $u_1$ of their limit-buy orders, i.e., there is an atom of limit buy orders executed at time $u_1$. For other traders, asset holding plan have to be increasing since there is a limit buy order in the book and, therefore, limit sell orders are executed immediately. Using the same argument as in Section IX.5.8 leads to a contradiction of market clearing. Namely, if we consider the market clearing condition at time $u_1$ (rescaled by $e^{\rho u_1}/\rho$), and subtract the market clearing condition at time $u \in (t_2, u_1)$ (rescaled by $e^{\rho u}/\rho$), we obtain:

$$0 = \int_0^u e^{\rho t}\mathbb{E}\left[q_{t,u_1} - q_{t,u}\right] dt + \int_u^{u_1} e^{\rho t}\mathbb{E}\left[q_{t,u_1} - s\right] dt$$
$$0 \geq \int_0^u e^{\rho t}\mathbb{E}\left[q_{t,u_1} - q_{t,u_1^-}\right] dt - \int_u^{u_1} e^{\rho t} s\, dt,$$

since asset holdings are increasing over $(t, u_1)$ and, as a result, $q_{t,u} \leq q_{t,u_1^-}$. The first integral on the right–hand side is a strictly positive function since the integrand is positive, and strictly positive for all the low–valuation traders in $\mathcal{T} \cap \mathcal{S}$ who submit a limit buy order executed at $u_1$. Since the integrand is positive, it is also weakly increasing. Letting $u \to u_1$ leads to the conclusion that the right-hand side is strictly positive, a contradiction.

Finally, the next Result implies that limit buy orders are not consistent with Markovian holding plan:

**RIX.24.** *Suppose that, in a Markov equilibrium at some time $t_1 \in \mathcal{S}$, a low-valuation trader ("trader A") submits a limit buy order at price $p_{t_1}$ executed at time $u_1 \in (t_1, T_2)$. Then, there is a positive measure set of $\mathcal{T} \in (t_1, u_1)$ such that, if trader A has another information event at time $t_2 \in \mathcal{T}$ with a low-valuation, her optimal holding plan differs from the holding plan of a low-valuation trader with an information event at time $t_2$ for the first time ("trader B").*

Let $\mathcal{T}$ be the intersection of $\mathcal{S}$ with the positive measure subset of $(t_1, u_1)$, given by Result IX.22. Consider some $t_2 \in \mathcal{T}$. By construction, for a low-valuation trader, $q_{t_2,t_2} < 1$. By result IX.23, a limit order to buy at price $p_{t_2}$ submitted at time $t_2$ is executed at time $u_2 > u_1$.

**Suppose first that** $u_2 < T_2$**.** We argue that trader $A$ can improve on $B$'s holding plan by using the limit order he submitted earlier at time $t_1$, to be executed at time $u_1$. Suppose indeed that traders $A$ and $B$ have the same optimal holding plan, $q_{t_2,u}$, and consider the following deviation for trader $A$: decrease the asset holding $q_{t_2,t_2}$ by a marginal unit and submit a limit order to buy this marginal unit at time $u_1$. The marginal value of this deviation is equal to $F(u_1$, where

$$F(u) \equiv -\int_{t_1}^{u} e^{-(r+\rho)z}\left[1 - \frac{1-\mu_{hz}}{1-\mu_{ht_2}}\delta q^{\sigma}_{t_2,t_2} - (rp_z - \dot{p}_z)\right] dz > 0.$$

Obviously, $F(t_1) = 0$. Moreover, by (IX.27), $F(u_2) = 0$ as well. Next, observe that $e^{(r+\rho)u}F'(u)$ is as strictly decreasing function of $u$, and $F(t_1) = F(u_2) = 0$. Therefore, in the interval $[t_1, u_2]$, the function $F'(u)$ must be first strictly positive and then strictly negative, and the function $F(u)$ has to be strictly positive over $(t_1, u_2)$. Hence, $F(u_1) > 0$, meaning that trader $A$ has a profitable deviation.

**Consider next that a limit buy order submitted at time** $t_2$ **at price** $p_{t_2} = p_{t_1}$ **is not executed before,** $T_2$**, the end of the flat spot.** For trader $B$, the value of increasing asset holdings by a marginal unit at $t_2$ and re-selling with a limit sell order at $T_2$ is negative:

$$\int_{t_2}^{T_2} e^{-(r+\rho)z} \left[1 - \frac{1-\mu_{hz}}{1-\mu_{ht_2}} \delta q^{\sigma}_{t_2,t_2} - (rp_z - \dot{p}_z)\right] dz \leq 0.$$

But then again, the function multiplying the discount factor is a strictly increasing function of $z$. It has to be strictly negative at $z = t_2$ or else the above integral would be strictly positive. So, over $[t_2, T_2]$, it is either always strictly negative or strictly negative and then strictly positive. Either way, this implies that for $u_1 < T_2$:

$$-\int_{t_2}^{u_1} e^{-(r+\rho)z} \left[1 - \frac{1-\mu_{hz}}{1-\mu_{ht_2}} \delta q^{\sigma}_{t_2,t_2} - (rp_z - \dot{p}_z)\right] dz > 0,$$

meaning that trader $A$ can improve on the candidate equilibrium holding plan by reducing his holding by a marginal unit at time $t_2$ and submitting a limit order to buy this marginal unit at time $u_1$, with the help of her previously submitted limit buy order.

### IX.5.12 Proof of Lemma VII.15

We prove each point in turn:

**Proof of $q_{t,u} \leq 1$.** First suppose $q_{t,u} > 1$ for some $u$. Then, define $T_1 = \inf\{u \geq t \,:\, q_{t,u} \leq 1\}$ and consider the deviation $\hat{q}_{t,u} = 1$ for $u \leq T_1$ and $\hat{q}_{t,u} = q_{t,u}$ for $u > T_1$. By construction, $\hat{q}_{t,u}$ is decreasing and satisfies $\hat{q}_{t,u} < q_{t,u}$ for $u < T_1$. The net change in utility flow is zero for $u > T_1$ and, for $u < T_1$:

$$(rp_u - \dot{p}_u)\,(q_{t,u} - 1) > 0,$$

because $rp_u - \dot{p}_u \geq 1 - \delta > 0$, and thus the deviation is profitable.

**Proof of $q_{t,u} = 1$ if $\theta_t = h$ and $u < T_f$.** Suppose that $q_{t,u} < 1$ for some $u \in [t, T_f)$. Then $T_1 = \inf\{u \geq t \,:\, q_{t,u} < 1\} < T_f$. Consider then the deviation $\hat{q}_{t,u} = 1$ for all $u$. The change in utility flow is zero for $u < T_1$ and, for $u > T_f$. For $u \in (T_1, T_f)$:

$$(1 - rp_u + \dot{p}_u)\,(1 - q_{t,u}) > 0,$$

because $\dot{p}_u \geq 0$ and $p_u < 1/r$. The deviation is thus profitable. Together with the fact that $p_u = 1/r$ for $u \geq T_f$, this clearly implies that any asset holding plan such that $q_{t,u} = 1$ for $u < T_f$ is optimal.

**Proof of $q_{t,u} = 0$ if $\theta_t = \ell$ and $u > T_f$.** Suppose that $q_{t,u} > 0$ for some $u > T_f$. Then $T_0 = \inf\{u \geq t \,:\, q_{t,u} = 0\} > T_f$. Consider then the deviation $\hat{q}_{t,u} = q_{t,u}$ for $u \leq T_f$ and $\hat{q}_{t,u} = 0$ for $u > T_f$. The expected utility flow of the deviation is zero for $u < T_f$ and $u > T_0$. For $u \in (T_f \vee t, T_0)$, because of strict concavity at $q = 0$, it is strictly greater greater than $(v_q(0) - rp_z)\, q_{t,u} = 0$, since $v_q(0) = 1$ and $rp_z = 1$.

**Proof that $q_{t,u}$ maximizes the relaxed objective (VII.3).** Suppose it did not. Then, for

$u \in [t, T_f)$, we can replace $q_{t,u}$ by a plan that achieves a higher value in the objective, and for $u \geq T_f$ keep $q_{t,u} = 0$. Clearly, the resulting holding plan is decreasing and achieves a higher value.s

### IX.5.13 Proof of Lemma VII.16

The objective is concave because the integrand is concave. It is also continuous because the integrand has uniformly bounded derivatives. The constraint set is evidently convex. To show that it is closed, consider some convergence sequence $q_{t,u}^{(n)} \to q_{t,u}^{(\infty)}$ of elements of the constraint set. The limit, $q_{t,u}^{(\infty)}$, belongs to $L^2([t, T_f])$, and thus to $L^1([t, T_f])$. Therefore, by Theorem 7.11 in Rudin (1974), $\int_t^u q_{t,z}^{(\infty)}\, dz$ is differentiable for all $u$ in a set $S$ of full measure. Now for any $u < u'$ in $S$ and any small enough $\varepsilon$, we have:

$$\frac{1}{\varepsilon} \int_u^{u+\varepsilon} q_{t,z}^{(n)}\, dz \geq \frac{1}{\varepsilon} \int_{u'}^{u'+\varepsilon} q_{t,z}^{(n)}\, dz,$$

since each element of the sequence is decreasing almost everywhere. Taking the limit as $n$ goes to infinity, we obtain that:

$$\frac{1}{\varepsilon} \int_u^{u+\varepsilon} q_{t,z}^{(\infty)}\, dz \geq \frac{1}{\varepsilon} \int_{u'}^{u'+\varepsilon} q_{t,z}^{(\infty)}\, dz.$$

Now since $u$ and $u'$ are in $S$, the function $\int_t^v q_{t,z}^{(\infty)}\, dz$ is differentiable at $v = u$ and $v = u'$. This allows us to take the limit as $\varepsilon$ goes to zero. And we find $q_{t,u}^{(\infty)} \geq q_{t,u'}^{(\infty)}$. A similar reasoning yields that $q_{t,u}^{(\infty)} \in [0, 1]$ for all $u \in S$.

Given the properties established above we can apply Proposition 1.2, page 35, Chapter II of Eckland and Téman (1987) and assert that a solution exists. Let us denote this solution by $q_{t,u}^*$. Note that $q_{t,u}^*$ is, by construction, decreasing almost everywhere instead of everywhere. Consider, however,

$$\begin{aligned}
&u \in S : \hat{q}_{t,u}^* = q_{t,u}^* \\
&u \notin S : \hat{q}_{t,u}^* = \sup\{q_{t,u}^* \,:\, z \in S \text{ and } z \geq u\}.
\end{aligned}$$

One easily verify that $\hat{q}_{t,u}^*$ is decreasing everywhere and belong to $[0, 1]$.

### IX.5.14 Proof of Lemma VII.17

Consider a flat spot $[t_1, t_2)$ and an investor who has an information event at time $t < t_2$ with a low valuation. We denote $u_0 = \max\{t, t_1\}$ and we want to show that, in the relaxed problem, $q_{t,u}$ is constant over $[u_0, t_2)$.

By contradiction assume that there exists some $u \in [u_0, t_2)$ such that $q_{t,u} < q_{t,u_0}$. Consider the "ideal" asset holding $q_{t,u}^*$ defined in equation (IX.22). One sees easily that $q_{t,u}^*$ solves the problem of maximizing:

$$\mathbb{E}_t\left[v(\theta_z, q)\right] - (rp_z - \dot{p}_z)\, q. \tag{IX.28}$$

subject to $q \in [0,1]$, . This ideal holding is $q^*_{t,u}$ is weakly increasing during the flat spot, because $\dot{p}_z = 0$. Define the first time in $[u_0, t_2)$ such that the asset holding goes below the ideal asset holding: $u_1 = \inf\{u \in [u_0, t_2) : q_{t,u} \leq q^*_{t,u}\}$, with the convention that $u_1 = t_2$ if the set is empty. Since $q_{t,u}$ is weakly decreasing and $q^*_{t,u}$ is weakly increasing during the flat spot, we have that

$$u \in [u_0, u_1) \,:\, q_{t,u} \geq q_{t,u_1} \geq q^*_{t,u_1} \geq q^*_{t,u} \tag{IX.29}$$

$$u \in (u_1, u_2) \,:\, q_{t,u} \leq q_{t,u_1^+} \leq q^*_{t,u_1} \leq q^*_{t,u}. \tag{IX.30}$$

Consider now the following deviation: $\hat{q}_{t,u} = q^*_{t,u_1}$ if $u \in [u_0, t_2)$ and $\hat{q}_{t,u} = q_{t,u}$ otherwise. From equation (IX.29) and (IX.30), the deviation is weakly closer to the ideal asset holding than in the original plan. Moreover, since we have assumed that $q_{t,u_0} > q_{t,u}$, for some $u \in [u_0, t_2)$, it is strictly closer to the ideal asset holding on for a strictly positive measure set of times. Because the objective (IX.28) is hump-shaped with a unique maximum at $q^*_{t,u}$, the expected utility of the deviation is strictly higher that of the original plan.

### IX.5.15 Proof of Lemma VII.21

First note that, following the same steps as for Lemma 1, page 14 in BHW, one finds that the planner's objective can be written:

$$W(q) = \mathbb{E}\left[\int_0^\infty e^{-rt}\int_t^\infty e^{-(r+\rho)(u-t)}\left\{\mathbb{I}_{\{\theta_t=\ell\}}\mathbb{E}_t\big[v(\theta_u, q_{t,u})\big] + \mathbb{I}_{\{\theta_t=h\}}\mathbb{E}_t\big[v(\theta_u, q_{t,u})\big]\right\} dt\, du\right]$$

Consider, then, two solutions of the planning problem, $q_{t,u}$ and $q'_{t,u}$ and let:

$$C = \left\{(t,u,\omega) \in \mathbb{R}^2_+ \times \Omega \,:\, t \leq u \;\text{ and }\; \theta_t = \ell \;\text{ and }\; q_{t,u} \neq q'_{t,u}\right\}.$$

Then if one consider a convex combination $\hat{q}_{t,u} = \lambda q_{t,u} + (1-\lambda)q'_{t,u}$. This new allocation is clearly feasible. Moreover, by concavity of the flow utility:

$$\mathbb{E}_t\left[v(\theta_u, \hat{q}_{t,u})\right] \geq \lambda\mathbb{E}_t\left[v(\theta_u, q_{t,u})\right] + (1-\lambda)\mathbb{E}_t\left[v(\theta_u, q'_{t,u})\right]$$

with a strict inequality for $(t,u,\omega) \in C$ since when $\theta_t = \ell$, the expected utility $\mathbb{E}_t\left[v(\theta_u, q)\right]$ is a strictly concave function of $q \in [0,1]$. Thus, the set $C$ has to be of measure zero, or else $\hat{q}_{t,u}$ would achieve a higher value in the planner's objective.

But we know from Proposition 2, page 19 in BHW, that the BHW-LOE asset holding plan solves the planning problem. Together with the above, this means that in all planning solutions, the asset holdings of low valuation investors are (for almost all $(t,u,\omega)$) the same as in the LOE of BHW. Now turn to the holdings of high-valuation investors. Integrating the market–clearing condition, $\mathbb{E}\left[q_{\tau_u,u} - s\right] = 0$, over $u \in [0, T_f]$, we find

$$\mathbb{E}\left[\int_0^{T_f} e^{-ru} \int_0^u \rho e^{-\rho(u-t)} \left\{ q_{t,u}\mathbb{I}_{\{\theta_t=h\}} + q_{t,u}^{\text{BHW–LOE}}\mathbb{I}_{\{\theta_t=\ell\}} - s \right\} dt\right] = 0.$$

But the same equation holds in the BHW–LOE with $q_{t,u}\mathbb{I}_{\{\theta_t=h\}}$ being replaced by $\mathbb{I}_{\{\theta_t=h\}}$. Thus

$$\mathbb{E}\left[\int_0^{T_f} e^{-ru} \int_0^u \rho e^{-\rho(u-t)} \mathbb{I}_{\{\theta_t=h\}} \left\{ 1 - q_{t,u} \right\} dt\, du\right] = 0$$

Now in the planning problem we restricted ourselves to $q_{t,u} \in [0,1]$, meaning that the integrand has to be positive. Then, for the above equality holds it must be the case that for almost all $(t,u,\omega)$ such that $0 < t \le u \le T_f$, and $\theta_t = h$, $q_{t,u} = 1$.

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