Appendix for Financial Frictions and Fluctuations in Volatility *

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This appendix contains five sections. Section 1 provides details for the comparative statics exercise performed in the simple example. Section 2 discusses extending the model to allow firms to default on the wages for managers. Section 3 describes the firm-level and aggregate data. Section 4 contains the details of the computational algorithm. Finally, Section 5 reports the results for our model with a lower labor elasticity.

1 Comparative Statics Exercise for Volatility

To illustrate the effects of increasing volatility on the labor choice of firms in the simple example of Section 2, we consider the case in which $\ln(z)$ follows a normal distribution, $N(\mu, \sigma^2)$. We assume that $b = 0$ and use the demand function $p(z, \ell) = zY^{1/\eta}\ell^{-\alpha/\eta}$ and the threshold $p(\hat{z}, \ell)\ell^{\alpha} - w\ell = 0$ to rewrite the first-order condition

$$E(p(z,\ell)|z \geq \hat{z})\alpha\ell^{\alpha-1} = \frac{\eta}{\eta-1}\left[w + V\frac{\pi_z(\hat{z})}{1-\Pi_z(\hat{z})}\frac{d\hat{z}}{d\ell}\right] \tag{1}$$

as

$$E\left[z \geq \frac{w}{A}\ell^{1-\theta}\right]A\theta\ell^{\theta-1} - w = \frac{(1-\theta)wV}{A\ell^{\theta}}\frac{\pi_z\left(\frac{w}{A}\ell^{1-\theta}\right)}{(1-\Pi_z(\frac{w}{A}\ell^{1-\theta}))}, \tag{2}$$

*The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

where $\theta = \alpha\left(\frac{\eta-1}{\eta}\right)$ and $A = Y^{1/\eta}$. To express these distributions as standard normals, use the fact that $E(z) = e^{\mu+\sigma^2/2}$ and write condition (2) as

$$e^{\mu+\sigma^2/2}\Phi\left(\frac{\mu+\sigma^2-\ln\left(\frac{w}{A}\ell^{1-\theta}\right)}{\sigma}\right)A\theta\ell^{\theta-1} - w = \frac{(1-\theta)wV}{A\ell^{\theta}}h\left(\frac{\ln\left(\frac{w}{A}\ell^{1-\theta}\right)-\mu}{\sigma}\right), \tag{3}$$

where Φ and h are the cumulative distribution function and hazard for the standard normal distribution.

We want to consider the effects of a mean-preserving spread of the distribution. To do so, we set $E(z) = 1$, which implies that $\mu = -\sigma^2/2$. The first-order condition (3) becomes

$$\Phi\left(\frac{\sigma^2/2-\ln\left(\frac{w}{A}\ell^{1-\theta}\right)}{\sigma}\right)A\theta\ell^{\theta-1} - w = \frac{(1-\theta)wV}{A\ell^{\theta}}h\left(\frac{\ln\left(\frac{w}{A}\ell^{1-\theta}\right)+\sigma^2/2}{\sigma}\right). \tag{4}$$

To evaluate how labor ℓ changes with volatility σ, we totally differentiate condition (4) and get an expression for $d\ell/d\sigma$:

$$d\ell/d\sigma = \frac{\frac{(1-\theta)wV}{A\ell^{\theta}}h_{\sigma}(.) - \Phi_{\sigma}(.)\theta\ell^{\theta-1}}{[\Phi_{\ell}(.)A\theta\ell^{\theta-1} + \Phi(.)A\theta(\theta-1)\ell^{\theta-2}] + \theta\frac{(1-\theta)wV}{A\ell^{\theta+1}}h(.) - \frac{(1-\theta)wV}{A\ell^{\theta}}h_{\ell}(.)}. \tag{5}$$

Using (4) for the bottom of equation (5), we find, after some simplification, that

$$d\ell/d\sigma = \frac{\frac{(1-\theta)wV}{A\ell^{\theta}}h'(x)\frac{dx}{d\sigma} - \theta\ell^{\theta-1}\phi(y)\frac{dy}{d\sigma}}{\phi(y)\frac{dy}{d\ell}A\theta\ell^{\theta-1} + \Phi(y)A\ell^{\theta-2}(\theta(2\theta-1)) - \frac{\theta}{\ell}w - \frac{(1-\theta)wV}{A\ell^{\theta}}h'(x)\frac{dx}{d\ell}}, \tag{6}$$

where $x = (\ln\left(\frac{w}{A}\ell^{1-\theta}\right) + \sigma^2/2)/\sigma$ and $y = \left(\sigma^2/2 - \ln\left(\frac{w}{A}\ell^{1-\theta}\right)\right)/\sigma$.

We will show that under the following assumption, this derivative is negative.

Assumption 1 $\theta < 1/2$ and $\{V, \sigma\}$ satisfy

$$V \geq \frac{A^{1/(1-\theta)}\left\{\Phi(\sigma)\exp(\sigma^2/2)\theta - 1\right\}}{h(0)\left(\exp(\sigma^2/2)w\right)^{\theta/(1-\theta)}(1-\theta)} \tag{7}$$

for given A and w.

Proposition 1 *Under Assumption 1, $d\ell/d\sigma < 0$ so that labor declines as volatility increases.*

Proof. Consider the expression in (6). First, note that $dy/d\ell < 0$, $dx/d\ell > 0$, and recall that the derivative of the hazard for a standard normal satisfies $h'(x) > 0$. Sufficient conditions for $d\ell/d\sigma < 0$ are that $\theta < 1/2$ and that $dy/d\sigma < 0$ and $dx/d\sigma > 0$ where

$$dy/d\sigma = \frac{1}{2} + \frac{\ln\left(\frac{w}{A}\ell^{1-\theta}\right)}{\sigma^2} \text{ and } dx/d\sigma = -\frac{\ln\left(\frac{w}{A}\ell^{1-\theta}\right)}{\sigma^2} + \frac{1}{2}. \tag{8}$$

With these sufficient conditions, the bottom of equation (6) is negative and the top is positive.

Now, since $dy/d\sigma < 0$ implies that $dx/d\sigma > 0$, we need only to show that $dy/d\sigma < 0$. We first show that under Assumption 1,

$$\ln\left(\frac{w}{A}\ell^{1-\theta}\right) \leq -\sigma^2/2, \tag{9}$$

which will imply that the default probability $\delta = \Phi\left(\frac{\ln\left(\frac{w}{A}\ell^{1-\theta}\right)+\sigma^2/2}{\sigma}\right) \leq 1/2$.

To show that condition (9) holds, we need to show that the optimal labor is not too large, that is,

$$\left(\frac{w}{A}\ell^{1-\theta}\right) \leq \exp(-\sigma^2/2) \tag{10}$$

or

$$\ell \leq \left(\frac{A}{w}\exp(-\sigma^2/2)\right)^{\frac{1}{1-\theta}}. \tag{11}$$

Let $\bar{\ell} = \left(\frac{A}{w}\exp(-\sigma^2/2)\right)^{\frac{1}{1-\theta}}$.

Next, we show that when $\theta < 1/2$, $d\ell/dV < 0$. Totally differentiating the first-order condition and using $\theta < 1/2$ gives

$$d\ell/dV = \frac{\frac{(1-\theta)w}{A\ell^{\theta}}h(.)}{\phi(y)\frac{dy}{d\ell}A\theta\ell^{\theta-1} + \Phi(y)A\ell^{\theta-2}(\theta(2\theta-1)) - \frac{\theta}{\ell}w - \frac{(1-\alpha)wV}{\ell^{\alpha}}h'(x)\frac{dx}{d\ell}} < 0. \tag{12}$$

Note that $d\ell/dV < 0$ because the top of the numerator of this expression is positive and each term in the bottom is negative.

Thus, we can find a sufficiently high continuation value, denoted V_{min}, so that for all $V \geq V_{min}$ the optimal labor is sufficiently low, that is, $\ell < \bar{\ell}$. We define V_{min} to be the continuation value so that the optimal labor is given by $\bar{\ell}$. After some manipulations, we find that

$$V_{min} = \frac{A^{\frac{1}{1-\theta}}\left[\Phi\left(\sigma\right)\exp\left(\frac{\sigma^2}{2}\right)\theta - 1\right]}{h\left(0\right)\left[\exp\left(\frac{\sigma^2}{2}\right)w\right]^{\frac{\theta}{\theta-1}}\left(1-\theta\right)}.$$

Specifically, since $V \geq V_{\min}$, $dy/d\sigma < 0$ and hence $d\ell/d\sigma < 0$. *Q.E.D*

2 Wages for Managers

In the main text, we assumed that firms always pay the managers' wages. Here we allow firms to default on the managers' wages. Let $w_{mt}(S_{t-1})$ be the face value of wages offered to managers. Defaulting firms pay managers first, then workers, then debt. Hence, defaulting firms pay their managers in full if

$$\kappa_t \leq \bar{\kappa}_m(S_t, z_{t+1}, \ell_{t+1}, b_{t+1}) = p_{t+1}(S_t, z_{t+1}, \ell_{t+1})\ell_{t+1}^{\alpha} - w_{mt+1}(S_t), \tag{13}$$

and they pay managers $\max\{p_t\ell_t^\alpha - \kappa, 0\}$ otherwise. The face value of wages offered to managers $w_{mt}(S_{t-1})$ adjusts with the aggregate state so that managers earn their value in home production $\bar{w}_m$. That is, the face value $w_{mt}(S_{t-1})$ that managers earn satisfies

$$\bar{w}_m = \int \pi_\sigma(\sigma_t|\sigma_{t-1})\pi_z(z_t|z_{t-1},\sigma_{t-1}) \left[\int_{\kappa\in\Omega_{mR}} w_{mt}(S_{t-1})\Upsilon_{t-1}d\Phi(\kappa) + \int_{\kappa\in\Omega_{mD}} \max\{p_t\ell_t^\alpha - \kappa, 0\}\Upsilon_{t-1}d\Phi(\kappa)\right]$$

Here we let Ω_{mR} denote the set of states such that the managers are paid their face value, so that

$$\Omega_{mR}(S_{t-1}, S_t, z_t, x_{t-1}, z_{t-1}) = \{\kappa : \kappa \le \kappa_t^* \text{ or } \kappa \le \bar{\kappa}_{mt}\},$$

and we let Ω_{mD} be the complementary set of states. In the quantitative model with bounded supports on shocks, the firms always repay in full the managers' wage and $w(S_{t-1}) = \bar{w}_m$.

3 Data

We describe in detail the firm-level and aggregate data as well as the definitions of all variables.

3.1 Firm-level data

We use firm-level data on U.S. publicly traded firms from the Compustat database.

Sample of firms: We drop financial firms (SIC code between 6000 and 6799) and public administration firms (SIC code greater than or equal to 9000). We also drop firm-quarter observations with negative sales. We keep firms with at least 100 quarters of observations since 1970Q1. We use the observations since 1985Q1 and have a resulting unbalanced panel with 2258 firms.

3.2 Firm-level variables

Spread: From Compustat we obtain for each firm and quarter its S&P credit rating. We translate the S&P credit rating to a Moody's credit rating using a standard scale, as in Johnson (2003)[1]. From Moody's we obtain spread time series for each of the 17 credit ratings from Aaa to Caa. These data are monthly covering the period from 1/31/1991 to 4/30/2013. We then proxy the firm's spread using the average monthly Moody's spread for that credit rating for the given quarter.

Growth: To calculate the variable Sales Growth, we compute the ratio of change in sales relative to the corresponding quarter in the previous year to the average sales in those two quarters. For each firm, we compute growth_t=(saleq_t-saleq_(t-4))/0.5(saleq_t+saleq_(t-4))).

Leverage: To calculate the variable Leverage, we first calculate a ratio of total debt, defined as the sum of current liabilities and long-term debt (dlcq+dlttq) to the average quarterly sales (saleq). The average of quarterly sales is taken over the eight previous quarters (including the

[1] Johnson, Richard. "An Examination of rating agencies' actions around the investment-grade boundary." FRB of Kansas City Research Paper 03-01 (2003).

current quarter). We then winsorize the ratio at the 1st and the 99th percentiles and divide it by four.

Debt Purchases: To calculate the variable Debt Purchases, we first calculate the ratio of the change in total debt relative to the corresponding quarter in the previous year to average quarterly sales. We then winsorize the ratio at the 1st and the 99the percentiles.

Equity Payouts: To calculate the variable Equity Payouts, we first calculate the equity payout, which is equal to the purchase of common and preferred stock (prstkcy), minus the sale of common and preferred stock (sstky), plus the total value of dividends paid (cshoq, common shares outstanding, times dvpspq, dividends per share paid). We then calculate the ratio of the sum of equity payouts over the four previous quarters (including the current quarter) to the average quarterly sales and winsorize the ratio at the 1st and the 99th percentiles.

3.3 Aggregate variables

Output is the HP-filtered log of real GDP (billions of chained 2009 dollars and seasonally adjusted, NIPA) for the period Q1 1985 to Q1 2013.

Employment is the HP filtered log of total hours worked from the BLS for the period 1985Q1 to 2013Q1.

IQR is the interquartile range of Sales Growth from Compustat.

Spread is the median Spread across all firms from Compustat.

Debt Purchases/Output is calculated as an average across the four previous quarters (including the current quarter) of the ratio of Debt Increases (seasonally adjusted, flow of funds) to nominal GDP.

Equity Payouts/Output is calculated as an average across the four previous quarters (including the current quarter) of the ratio of Equity Payouts (seasonally adjusted, flow of funds) to nominal GDP.

Employment (rel Output) in Table 5 of the main paper is calculated as a ratio of the standard deviation of Employment (logged and HP filtered) to Output (logged and HP filtered).

4 Computational Details

We describe the computational algorithm we use to compute our model.

4.1 Definition of equilibrium

An equilibrium consists of firms' value function $V(S, z, x)$, firms' labor choice $\ell(S, z, x)$, firms' debt choice $b(S, z, x)$, firms' default cutoff $\kappa^*(S, S', z', \ell', b')$, the borrowing limits $M(S, z)$, the bond price schedule $q(S, z, \ell', b')$, the aggregate output $Y(S)$, wage $w(S)$, and the law of motion of distribution $H(\sigma', S)$ such that

1. Given the aggregate output $Y(S)$, wage $w(S)$, and the law of motion of distribution $H(\sigma', S)$, the borrowing limits $M(S, z)$, and the bond price schedule $q(S, z, \ell', b')$, the functions of $\{V(S, z, x),\ \ell(S, z, x),\ b(S, z, x),\ \kappa^*(S, S', z', \ell', b')\}$ solve the firm's problem. For a state (S, z, x) with $x + M(S, z) < 0$, so that the budget set is empty, $V(S, z, x) = 0$ and otherwise

$$V(S,z,x) = \max_{\ell',b'}\ x + q(S,z,\ell',b')b' + \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \int_{z'} \int^{\kappa^*} V(S',z',x')d\Phi(\kappa)dF(z'|z,\sigma) \quad (14)$$

subject to

$$x + q(S,z,\ell',b')b' \geq 0, \quad (15)$$

$$M(S,z) - q(S,z,\ell',b')b' \leq F_m(S,z) \quad (16)$$

$$x' = z'Y(S)^{\frac{1}{\gamma}}(\ell')^{\frac{\alpha(\gamma-1)}{\gamma}} - w(S)\ell' - b' - \kappa \quad (17)$$

$$\kappa^*(S,S',z',\ell',b') = z'Y(S)^{\frac{1}{\gamma}}(\ell')^{\frac{\alpha(\gamma-1)}{\gamma}} - w(S)\ell' - b' + M(S',z') \quad (18)$$

$$S' = (\sigma', H(\sigma', S)) \quad (19)$$

For notational simplicity, let

$$W(S,z,\ell',b') = \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \int_{z'} \int^{\kappa^*} V(S',z',x')d\Phi(\kappa)dF(z'|z,\sigma)$$

and write (14) equivalently as

$$V(S,z,x) = \max_{\ell',b'}\ x + q(S,z,\ell',b')b' + W(S,z,\ell',b') \quad (20)$$

subject to (15)–(19).

2. The bond price schedule ensures that lenders break even,

$$q(S,z,\ell',b') = \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \int_{z'} \Phi\left(\kappa^*(S,S',z',\ell',b')\right) dF(z'|z,\sigma) \quad (21)$$

and the borrowing limits satisfy

$$M(S,z) = \max_{\ell',b'} q(S,z,\ell',b')b'. \quad (22)$$

3. The consumer problem is to solve

$$V^H\left(A_t, S_{t-1}\right) = \max_{L_t}\left\{\sum_{\sigma_t}\pi_\sigma(\sigma_t|\sigma_{t-1})\max_{C_t(\sigma_t),\{A_{t+1}(\sigma_{t+1})\}}\left[\frac{C_t(\sigma_t)^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\nu}}{1+\nu} + \beta V^H(S_t)\right]\right\} \tag{23}$$

subject to the budget constraint, for each σ_t,

$$C_t(\sigma_t) + \sum_{\sigma_{t+1}} Q(\sigma_{t+1}|\sigma_t)A_{t+1}(\sigma_{t+1},\sigma_t) = W_t(S_{t-1})L_t + A_t(\sigma_t) + D_t(\sigma_t, S_{t-1}),$$

where A_t is the vector of assets $\{A_t(\sigma_t)\}$. The first-order condition for the consumption–labor choice is

$$-\frac{\sum_{\sigma_t}\pi_\sigma(\sigma_t|\sigma_{t-1})U_L(C_t(\sigma_t), L_t)}{\sum_{\sigma_t}\pi_\sigma(\sigma_t|\sigma_{t-1})U_C(C_t(\sigma_t), L_t)} = W_t(S_{t-1}). \tag{24}$$

Using the envelope condition and $Q(\sigma_{t+1}|\sigma_t) = \beta\pi(\sigma_{t+1}|\sigma)$, and using the additive separability of utility, the first-order condition for consumption implies $C_t(\sigma_t) = C_{t+1}(\sigma_{t+1})$ all σ_{t+1}, so that consumption is constant, say $C_t(\sigma_t) = \bar{C}$ for all t. Using the functional form for utility, the first-order condition (24) reduces to

$$\bar{C}^\sigma L_t(\sigma_{t-1})^v = W_t(S_{t-1}). \tag{25}$$

4. Aggregate wages and the face value of wages are related by $W(S_{t-1})L(S_{t-1}) =$

$$\begin{aligned}&W(S_{t-1})L(S_{t-1}) = \\ &w(S_{t-1})\int \pi_\sigma(\sigma_t|\sigma_{t-1})\pi(z_t|z_{t-1},\sigma_{t-1})\int_{\kappa\in\Omega_R} d\Phi(\kappa)\ell_t\Upsilon(z_{t-1},x_{t-1}) + \\ &\int \pi_\sigma(\sigma_t|\sigma_{t-1})\pi(z_t|z_{t-1},\sigma_{t-1})\int_{\kappa\in\Omega_D}\max\{p_t\ell_t^\alpha - w_{mt} - \kappa, 0\}d\Phi(\kappa)\Upsilon(z_{t-1},x_{t-1}).\end{aligned}$$

5. The market clearing conditions for labor and output are

$$L(S) = \int \ell(S,z,x)\Upsilon(z,x)$$

$$Y(S) = \left[\int z\ell^{\alpha\frac{\gamma-1}{\gamma}}(S,z,x)\Upsilon(z,x)\right]^{\frac{\gamma}{\gamma-1}}. \tag{26}$$

6. The law of motion of distribution is consistent with the policy functions of firms, households, and shocks.

4.2 Algorithm overview

In order to solve the individual firm's optimization problem, the firm needs to forecast next period's wage $w(S)$ and next period's output $Y(S)$, and it needs a transition law for the aggregate state. In practice, it is infeasible to include the entire measure Υ in the state. Instead, we follow a version of Krusell and Smith (1998) to approximate the forecasting rules for the firm. We do so by approximating the distribution of firms Υ with lags of aggregate shocks, $(\sigma_{-1},\ \sigma_{-2},\ \sigma_{-3},\ k)$ where k records how many periods the aggregate shocks have been unchanged. Here $k = 1, \ldots, \bar{k}$ and $\bar{k}$ is the upper bound on this number of periods. We set $\bar{k} = 9$. In a slight abuse of notation, we use $S = (\sigma,\ \sigma_{-1},\ \sigma_{-2},\ \sigma_{-3},\ k)$ in the rest of this description of the algorithm to denote our approximation to the aggregate state. The law of motion of (our approximation to) the aggregate state is given by $H(\sigma', S) = (\sigma', \sigma, \sigma_{-1}, \sigma_{-2}, k')$ with $k' = k + 1$ if $\sigma' = \sigma = \sigma_{-1} = \sigma_{-2}$ and 0 otherwise. Given our parameterization for $\sigma = \{\sigma_L, \sigma_H\}$ and $\bar{k} = 9$, the total number of points for the mutually exclusive aggregate states S is 32.

We start with an initial guess of two arrays for the aggregate wages and output, $w^0(S)$ and $Y^0(S)$, referred to as *aggregate rules*. We then solve the model with two loops: an inner and an outer loop. In the inner loop, taking as given the current set of aggregate rules, we iteratively solve each firm's optimization problem until convergence. In the outer loop, taking as given the converged decisions from the inner loop, we start with a distribution of firms $\Upsilon_0(z, x)$ and simulate the economy for T periods. In each period t, we record firms' labor choice $\{\ell_{t+1}(z, x)\}$, borrowing $\{b_{t+1}(z, x)\}$, and default decisions $\{\iota_t(z, x)\}$. Moreover, we use (25) and (26) to construct new guesses $w_{t+1}(S)$ and $Y_{t+1}(S)$ for the aggregate rules. We then repeat the procedure until the arrays of aggregate output and wages converge.

4.2.1 Inner loop

Before we solve the inner loop, we discretize the idiosyncratic productivity shock $z(\sigma_{-1})$ using the Gaussian quadrature method. The discretization of this shock consists of 12 productivity points for each level of volatility σ_{-1} and transition matrices $\pi_z(z'|z(\sigma_{-1}), \sigma)$ that depend on σ_{-1} and σ. The idiosyncratic state x is discretized into 15 endogenous grids that depend on the shocks z and the aggregate state S. The state space for the firm's problem has $\#S \times \#Z \times \#X = 5{,}760$ grid points. We also discretize the revenue shock κ into 100 points using the Gaussian quadrature method and use it to evaluate the integrals in the firm's future value.

In the loop, taking as given the current set of aggregate rules, say, $w(S) = w^k(S)$ and $Y(S) = Y^k(S)$, we first construct the bond price schedule and borrowing limits recursively. We then solve firms' decision rules.

Borrowing Limits We start with an initial guess for the borrowing limits $M^0(S, z)$ that is looser than the actual borrowing limit. We set grids for $\{\ell', b'\}$, with 32 points for ℓ' and 64 points for b'. The grid for ℓ' is set around the frictionless choice of labor and depends on $\{S, z\}$. The grid for b' is endogenous and depends on $\{S, z, \ell'\}$. We update the grid on b' with every iteration of the borrowing limit. With these choices, the resulting array for $q(S, z, \ell', b')$ has $\#L \times \#B \times \#Z \times \#S = 786,432$ grid points.

Given $M^0(S, z)$, we construct the associated default cutoff

$$\kappa^{*0}(S, S', z', \ell', b') = z'Y(S)^{\frac{1}{\gamma}}(\ell')^{\frac{\alpha(\gamma-1)}{\gamma}} - w(S)\ell' - b' + M^0(S', z') \tag{27}$$

and the associated bond price schedule

$$q^0(S, z, \ell', b') = \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \int_{z'} \Phi[\kappa^{*0}(S, S', z', \ell', b')] dF(z'|z, \sigma). \tag{28}$$

In the first step of the iteration, we update the borrowing limit to $M^1(S, z)$ using

$$M^1(S, z) = \max_{\ell', b'} \; q^0(S, z, \ell', b')b' \qquad \text{for each } (S, z)$$

and then construct the associated default cutoff array $\left\{\kappa^{*1}(S, S', z', \ell', b')\right\}$ and bond price schedule array $\left\{q^1(S, z, \ell', b')\right\}$ using the analogs of (27) and (28).

We continue this process iteratively until the constructed sequence of borrowing limit arrays $\{M^n(S, z)\}$ converge. We then record the associated arrays of default cutoffs and bond price schedules, denoted $\{\kappa^*(S, S', z', \ell', b')\}$ and $\{q(S, z, \ell', b')\}$ which we hold fixed during each iteration of the firm decision rules that we describe next.

Firm Decision Rules Given the converged borrowing limits and associated default cutoffs and bond price schedule, we solve for the firms' decision rules iterating over a combination of policy functions and value functions. For each grid point, we solve a system of two nonlinear equations by interpolating over the policy functions using a multivariate finite element method. We use the Intel Fortran compiler using the IMSL routine DNEQNF.

Let $\gamma(S, z, x)$ and $\mu(S, z, x)$ denote the multipliers on the nonnegative equity payout condition (15), denoted *NEP*, and the manager deviation condition (16), denoted MD, respectively. The following first-order conditions characterize firms' optimization problem (14):

$$\begin{aligned} (1 + \gamma + \mu)\frac{\partial q(S, z, \ell', b')b'}{\partial \ell'} + \frac{\partial W(S, z, \ell', b')}{\partial \ell'} &= 0 \\ (1 + \gamma + \mu)\frac{\partial q(S, z, \ell', b')b'}{\partial b'} + \frac{\partial W(S, z, \ell', b')}{\partial b'} &= 0, \end{aligned} \tag{29}$$

where

$$\frac{\partial q(S,z,\ell',b')b'}{\partial \ell'} = \beta b' \sum_{\sigma'} \pi(\sigma',\sigma) \left[\sum_{i=1}^{N_z} \pi_z(z_i|z,\sigma)\phi(\kappa^*(S,S',z_i,\ell',b'))[\alpha_h z_i Y^{\frac{1}{\gamma}}(S)(\ell')^{\alpha_h-1} - w(S)] \right]$$

$$\frac{\partial q(S,z,\ell',b')b'}{\partial b'} = \beta \sum_{\sigma'} \pi(\sigma',\sigma) \left[\sum_{i=1}^{n} \pi_z(z_i|z,\sigma) \left[\Phi(\kappa^*(S,S',z_i,\ell',b')) - b'\phi(\kappa^*(S,S',z_i,\ell',b')) \right] \right]$$

and

$$\begin{aligned}
\frac{\partial W}{\partial \ell'} &= \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} \int^{\kappa^*(S,S',z_i,\ell',b')} \left[(1+\gamma(S',z_i,x'))\alpha_h z_i Y(S)^{\frac{1}{\gamma}} \ell^{\alpha_h-1} \right] d\Phi(\kappa)\pi_z(z_i|z,\sigma) \\
&- \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} \int^{\kappa^*(S,S',z_i,\ell',b')} \left[1+\gamma(S',z_i,x') \right] w(S) d\Phi(\kappa)\pi_z(z_i|z,\sigma) \\
&+ \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} V(x^*(\ell',b',z_i,S),z',S')\kappa_\ell^*(S,S',z_i,\ell',b')\pi_z(z_i|z,\sigma)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial W}{\partial b'} &= -\beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} \int_0^{\kappa^*(S,S',z_i,\ell',b')} \left[1+\gamma(S',z_i,x') \right] d\Phi(\kappa)\pi_z(z_i|z,\sigma) \\
&+ \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} V(x^*(\ell',b',z_i,S),z_i,S')\kappa_b^*(S,S',z_i,\ell',b')\pi_z(z_i|z,\sigma),
\end{aligned}$$

where $\alpha_h = \alpha\frac{(\eta-1)}{\eta}$, $\kappa_\ell^* = \frac{\partial\kappa^*}{\partial\ell'}$, and $\kappa_b^* = \frac{\partial\kappa^*}{\partial b'}$.

In the iterations to solve for the firm decision rules, we iterate on a set of arrays of grids $\{X(S,z)\}$ where

$$X(S,z) = \{x_1,\ldots,x_N\},$$

where the set of points $\{x_1,\ldots,x_N\}$ varies with (S,z). Let $\{X^0(S,z)\}$ denote the initial guess on the array of grids. We also begin with an initial guess for the multiplier function $\{\gamma^0(S,z,x)\}$ on the NEP condition and the value function $\{V^0(S,z,x)\}$. Both the multiplier functions and the value functions are defined not just on the grid but also for all values of x in a range $[-M(S,z),\infty]$ as we interpolate between the grid points.

For each iteration n, given the array of grids $\{X^n(S,z)\}$, the multipliers $\{\gamma^n(S,z,x)\}$ and the value function $\{V^n(S,z,x)\}$ from the previous iteration, we solve for the updated array of grids $\{X^{n+1}(S,z)\}$, the multiplier function $\{\gamma^{n+1}(S,z,x)\}$ and the value function $\{V^{n+1}(S,z,x)\}$ in two steps. In these steps, we use the fact that for all cash-on-hand levels x greater than some cutoff level $\hat{x}(S,z)$ the NEP is not binding and the decision rules for labor $\ell'(S,z,x)$ and debt $b'(S,z,x)$ do not vary with x. We refer to the associated values of labor and debt as the *nonbinding levels of labor and debt* and denote them by $\hat{\ell}(S,z)$ and $\hat{b}(S,z)$. So given $\gamma^n(S,z,x)$ and the value function

$\{V^n(S,z,x)\}$, we proceed as follows.

1. Solve for the cutoff $\hat{x}(S,z)$ by solving for the values $\hat{\ell}(S,z)$ and $\hat{b}(S,z)$. To do so, we impose that the NEP condition is not binding.

   (a) Assume the manager deviation condition MD is also not binding and solve for the *tentative* solutions $\ell'_{tent}(S,z)$ and $b'_{tent}(S,z)$. The tentative solutions solve the following two equations in ℓ' and b':

   $$\frac{\partial q(S,z,\ell',b')b'}{\partial \ell'} + \frac{\partial W(S,z,\ell',b')}{\partial \ell'} = 0$$
   $$\frac{\partial q(S,z,\ell',b')b'}{\partial b'} + \frac{\partial W(S,z,\ell',b')}{\partial b'} = 0$$

   We then check whether the constructed tentative solutions satisfy the manager deviation condition. If so, then we set $\ell'_f(S,z) = \ell'_{tent}(S,z)$ and $b'_f(S,z) = b'_{tent}(S,z)$. If not, we continue to step (b).

   (b) If we reach this step, we know that the manager deviation condition is binding. We thus impose the MD condition with equality and define $\hat{\ell}(S,z)$ and $\hat{b}(S,z)$ as the solution to

   $$M(S,z) - q(S,z,\ell',b')b' = F_m(S,z)$$
   $$\frac{\partial q(S,z,\ell',b')b'}{\partial b'}\frac{\partial W(S,z,\ell',b')}{\partial \ell'} - \frac{\partial q(S,z,\ell',b')b'}{\partial \ell'}\frac{\partial W(S,z,\ell',b')}{\partial b'} = 0,$$

   where this last equation is derived by combining the two first-order conditions in (29) and eliminating the multipliers.

   (c) Construct the grid $\left\{X^{n+1}(S,z)\right\} = \{x_1, x_2, \ldots, x_N\}$ by setting

   $$x_1 = -M(S,z) \text{ and } x_N = \hat{x}(S,z).$$

   That is, we know that if the cash-on-hand x is so low that even if the firm borrows the maximum amount $M(S,z)$, the associated dividends $d = x + M(S,z)$ is negative, the firm will default. We also know that if the cash-on-hand is sufficiently high, so that $x \geq \hat{x}(S,z)$, the optimal decisions will be given by the nonbinding levels of labor and debt $\hat{\ell}(S,z)$ and $\hat{b}(S,z)$. We then choose a set of intermediate points $\{x_2, \ldots, x_{N-1}\}$.

2. Solve for decisions at the intermediate points.

   We claim that at any of these intermediate points with $-M(S,z) < x < \hat{x}(S,z)$, the MD condition is not binding. To prove this claim, we note that since $x < \hat{x}(S,z)$, then $-x > -\hat{x}(S,z)$, and the firm must borrow more at x than at $\hat{x}$ to keep dividends nonnegative. Since at $\hat{x}$ the nonnegative equity payout condition binds, implying $d = 0$, then from $d = \hat{x} + qb'$

and the manager deviation condition, we know that

$$-\hat{x}(S,z) = q(S,z,\hat{\ell},\hat{b})\hat{b} \geq M(S,z) - F_m(S,z),$$

so

$$-x \geq M(S,z) - F_m(S,z).$$

Thus, for each intermediate point $x \in \{x_2, \ldots, x_{N-1}\}$, the NEP condition is binding and the manager deviation condition is not binding. For each such x, we solve ℓ', b' from the two equations

$$\frac{\partial q(S,z,\ell',b')b'}{\partial b'} \frac{\partial W(S,z,\ell',b')}{\partial \ell'} - \frac{\partial q(S,z,\ell',b')b'}{\partial \ell'} \frac{\partial W(S,z,\ell',b')}{\partial b'} = 0$$
$$x + q(S,z,\ell',b')b' = 0.$$

Let the solution be $\ell'(S,z,x)$ and $b'(S,z,x)$. We then compute the new multiplier γ^{n+1} from

$$\gamma^{n+1}(S,z,x) = -\frac{\frac{\partial W}{\partial \ell'}}{\frac{\partial q(S,z,\ell',b')b'}{\partial \ell'}} - 1,$$

where the derivatives are evaluated at the solution $\ell'(S,z,x)$ and $b'(S,z,x)$.

3. We then update the value function to V^{n+1} using

$$\begin{aligned} V^{n+1}(S,z,x) = \; & x + q(S,z,\ell',b')b' \\ & + \beta \sum_{\sigma} \pi(\sigma'|\sigma) \sum_{i} \int^{\kappa^*(S,S',z',\ell',b')} V^n(S',z',x') d\Phi(\kappa) \pi(z_i|z,\sigma), \end{aligned}$$

where ℓ' and b' are shorthand notations for $\ell(S,z,x)$ and $b(S,z,x)$.

4. Iterate. We iterate steps 1-3 until the multipliers $\gamma^n(S,z,x)$ and the value functions $V^n(S,z,x)$ converge.

4.2.2 Outer loop

We simulate the model for T periods. For each period, the economy has an aggregate state of $S_t = (\sigma_t, \sigma_{t-1}, \sigma_{t-2}, \sigma_{t-3}, k_t)$. We set a time-varying grid of $X_t(z)$ of 80 points. The time-varying distribution of firms $\Upsilon_t(z,x)$ is an array of $12 \times 80 = 960$ points.

1. For each firm in the distribution Υ_t, we define the *default decision* $\iota_t(z,x)$ by $\iota_t(z,x) = 1$ if and only if $x \leq -M(S_t,z)$. For nondefaulting firms, we calculate their labor choice $\ell_{t+1}(z,x)$ and debt choice $b_{t+1}(z,x)$ by interpolating the decision rules $\ell(S_t,z,x)$ and $b(S_t,z,x)$ from

the inner loop. Summing over default decisions of firms, we get the total mass of exiting firms

$$E_t = \sum_{z,x \in X_t(z)} \iota_t(z,x)\Upsilon_t(z,x).$$

2. We find the labor and debt of new entrants $(\ell^e_{t+1}(\omega), b^e_{t+1}(\omega))$. A new entrant can enter if it draws a sufficiently low entry cost in that

$$\omega \leq M(S_t, z^e).$$

We assume that from the measure of potential entrants with $\omega \leq M(S_t, z_e)$, a subset is chosen randomly so that the measure of entering firms equals E_t. Upon entry, the labor choice and new borrowing are given by $\ell^e_{t+1}(\omega, z) = \ell(S_t, z, -\omega)$, $b^e_{t+1}(\omega, z) = b(S_t, z, -\omega)$, respectively.

3. Next period's output is given by

$$Y_{t+1} = \left[\sum_{(z,x)} (1-\iota_t(z,x)) z\ell_{t+1}(z,x)^{\alpha_h}\Upsilon_t(z,x) + E_t \int_{\omega \leq M(S_t,z^e)} z^e(\ell^e_{t+1}(z^e,\omega))^{\alpha_h} d\Omega(\omega)\right]^{\frac{\gamma}{\gamma-1}},$$

next period's labor is given by

$$L_{t+1} = \sum_{(z,x)} (1-\iota_t(z,x))\ell_{t+1}(z,x)\Upsilon_t(z,x) + E_t \int_{\omega \leq M(S_t,z^e)} \ell^e_{t+1}(z^e,\omega) d\Omega(\omega),$$

and next period's wage is given by

$$W_t(S_{t-1}) = \bar{C}^\sigma L_t(\sigma_{t-1})^v. \tag{30}$$

Since the choices of x_{t+1} vary smoothly with the shocks at t and we record only a finite number of grid points x_i for $i = 1, ..., N$, when updating the distribution Υ we need to assign the mass for any (z', x_{t+1}) to points on the grid (z', x_i). We do so by allocating the mass for any x_{t+1} to the two closest grid points x_{i-1} and x_i where $x_{i-1} \leq x_{t+1} \leq x_i$ in proportion to how close the point is to each. Specifically, let $\Lambda(x_i, x_{t+1})$ be the probability that the choice of x_{t+1} is assigned to x_{i-1} or x_i :

$$\Lambda(x_i, x_{t+1}) = \frac{x_{t+1} - x_{i-1}}{x_i - x_{i-1}} \text{ and } \Lambda(x_{i-1}, x_{t+1}) = 1 - \Lambda(x_i, x_{t+1}),$$

and $\Lambda(x_i, x_{t+1}) = 0$ if $x_{t+1} \notin [x_{i-1}, x_{i+1}]$. Then next period's distribution $\Upsilon_{t+1}(z', x_i)$ for x_i on $X_{t+1}(z')$ is given by

$$\Upsilon_{t+1}(z', x_i) = \sum_{x \in X_t(z), z} \left\{\int^{\kappa^*_{t+1}(z',z,x)} \Lambda(x_i, x_{t+1}(z,x,z',\kappa')) d\Phi(\kappa')\right\} \pi(z'|z,\sigma_t)\Upsilon_t(z,x),$$

where x_{t+1} and κ^*_{t+1} are given by

$$x_{t+1}(z,x;z',\kappa') = z'Y(S_t)^{\frac{1}{\gamma}}\ell_{t+1}(z,x)^{\alpha_h} - w(S_t)\ell_{t+1}(z',x) - b_{t+1}(z,x) - \kappa',$$
$$\kappa^*_{t+1}(z',z,x) = \kappa^*(S_t,\sigma_{t+1},S_{t+1},z',\ell'(S_t,z,x),b'(S,z,x)).$$

We finally project the simulated values for wages and output on a set of dummy variables corresponding to the state S. We use the fitted values as the new aggregate rules $w(S) = w^{k+1}(S)$ and $Y(S) = Y^{k+1}(S)$.

4. We iterate the outer loop until the aggregate rules converge.

4.3 Accuracy checks

Here we report the accuracy checks for the baseline result. We compute standard distance measures across two iterations as follows. For any array $f(\nu)$ over grid ν, we follow Judd's textbook[2] and compute the *distance* across iterations n and $n-1$ as $\frac{\left(\sum_\nu \left(f^n(\nu)-f^{n-1}(\nu)\right)^2\right)^{1/2}}{1+\left(\sum_\nu f^{n-1}(\nu)^2\right)^{1/2}}$. The borrowing limit $M(S,z)$ distance is 10^{-8}. The distance for the stacked policy functions $\gamma(S,z,x)$ and $V(S,z,x)$ is 10^{-6}. In terms of the aggregate rules, $Y(S)$ and $w(S)$, the distance is 10^{-5}. The maximum Euler equation error in the firms' optimization problem is 10^{-8}.

[2] Judd, Kenneth. 1998. *Numerical Methods in Economics.* MIT Press.

5 Results with Lower Labor Elasticity

Here we show results with a labor elasticity of 1.

Table 1: Lower Labor Elasticity: Firm Distributions

	Percentile		
	25	50	75
Data (%)			
Spread	1	1.3	2.1
Growth	-9	0	11
Leverage	9	26	62
Debt purchases	-10	0	21
Equity payouts	-4	0	12
Benchmark (%)			
Spread	1.1	2.8	6.3
Growth	-7	0	9
Leverage	25	29	33
Debt purchases	-14	0	16
Equity payouts	-19	0	23
Lower Elasticity (%)			
Spread	0.9	2.6	6.0
Growth	-7	0	9
Leverage	25	29	33
Debt purchases	-14	0	16
Equity payouts	-13	0	23

See Table 3 notes in the main text.

Table 2: Lower Labor Elasticity: Firm Correlations

Median Corr. with Leverage (%)	Data	Benchmark	Lower Labor Elasticity
Spread	10	20	20
Growth	9	28	28
Debt purchases	45	59	60
Equity payouts	-5	13	13

See Table 4 notes in the main text.

Table 3: Lower Labor Elasticity: Business Cycles

	Data	Benchmark	Lower Labor Elasticity
St. Deviations (%)			
Output	1.13	0.97	0.75
Employment (rel output)	1.26	1.31	1.22
IQR	3.50	3.62	3.71
Spread	1.10	0.91	0.97
Debt purchases/output	2.51	2.83	2.63
Equity payouts/output	1.76	2.74	2.60
Corr. with Output (%)			
Employment	81	94	90
IQR	-27	-45	-38
Spread	-31	-33	-32
Debt purchases/output	75	21	12
Equity payouts/output	45	18	27

See Table 5 notes in the main text.